Decompositions in algebra

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Cover

The cover of this thesis features abstract sunflowers, each of which is constructed as the Voronoi tesselation of a spiral of points that is subsequently projected onto the surface of revolution of a hyperbolic spiral. The image was created in the Python API of Blender (v3.5.1) using the SciPy library (v1.10.1). It is produced by the following code, which is optimized for compactness.

```
import bpy, bmesh, numpy as np; from scipy.spatial import Voronoi
nm,a2,col=np.linalg.norm,np.arctan2,bpy.data.collections.new('c')
def make_cell(vs,fs,cs,s):
  r,i=[a if a >= 0 else len(cs) for a in s],len(vs)
  l=[cs[a] for a in s if a>=0]; c=sum(l)/len(l); cs+=[c*3/nm(c)]
  x=sorted([(nm(cs[a]),cs[a]) for a in r]); d=(x[0][1]+x[-1][1])/2
  t=min(nm(d)*.2,.15); t=0 if t<.03 else t
  l=[(1-t)*cs[a]+t*c for a in r]; vs+=[list(d)+[0]]
  for c,d in zip(l,l[1:]+[l[0]]):
    for t in np.linspace(0,1,max(2,int(10*nm(c-d))))[1:]:
      vs+=[list(t*d+(1-t)*c)+[0]]; fs+=[[i,len(vs)-1,len(vs)]]
  fs[-1][-1]=i+1; return (vs,fs,cs)
def make_object(name,mesh,colors,location,cuts):
  l,m=bpy.data.meshes.new(name),bpy.data.materials.new(name)
  m.use_nodes=True; n=m.node_tree.nodes
  x=[n.get('Material_Output')]+list(map(lambda s:n.new('ShaderNode'
    +s),2*['BsdfDiffuse']+['MixShader','NewGeometry']))
  for i,a,j,b in [(0,0,3,0),(3,1,1,0),(3,2,2,0),(3,0,4,6)]:
    m.node_tree.links.new(x[i].inputs[a],x[j].outputs[b])
  for i in [0,1]: x[i+1].inputs[0].default_value=colors[i]
  l.from_pydata(mesh[0],[],mesh[1]); l.update()
  o=bpy.data.objects.new(name,l); o.data.materials.append(m)
  col.objects.link(o); o.location,m=(location,0,0),bmesh.new()
  m.from_mesh(o.data); bmesh.ops.subdivide_edges(m,edges=m.edges,
    use_grid_fill=1, cuts=cuts); m.to_mesh(o.data); o.data.update()
  for v in o.data.vertices:
    t=nm(v.co); p=(v.co/t)*(1.-np.sin(4*t)/4/t)
    v.co=[p[0],p[1],-np.cos(4*t)/4/t]
def mkflower(offset,rad=.65,exp=.6,cell=350):
  p=2*np.pi; ct=lambda a,r:r*np.array([np.cos(p*a),np.sin(p*a)])
  points=[ct(.6180339*c,rad*(c/cell)**exp) for c in range(cell)]
  v=Voronoi(points); m=[[[],[],list(v.vertices)] for i in [0,1]]
  for j in filter(lambda j:j+3>2<len(v.regions[j]),v.point_region):
    s=v.regions[j]; l=[m[1][2][a] for a in s if a>=0]; c=sum(l)/len(l)
    f=lambda:[a2(*(m[1][2][i]-c if i>=0 else c)[:2]) for i in s]
    k=np.argmin(f()); s=[s[(i+k)%len(s)] for i in range(len(s))]
    if all(x<=y for (x,y) in zip(f(),f()[1:])): s.reverse()
    b=all(i>=0 for i in s) and nm(c)<=.75; m[b]=make_cell(*m[b],s)
  make_object('p',m[0],((1,.72,0,1),(1,.45,0,1)),offset,25)
  make_object('s',m[1],((.7,.16,0,1),(.11,.45,0,1)),offset,7)
c,d=bpy.context.scene.collection,bpy.data.cameras.new('v')
d.lens=55; v=bpy.data.objects.new('v',d); c.children.link(col)
v.location,v.rotation_euler=(3.1,1.6,5.6),(.55,-.55,2.65)
c.objects.link(v); mkflower(0); mkflower(-3.1); mkflower(-6.2)
mkflower(-9.3); mkflower(-12.4); mkflower(-15.5);
```

Tables

Table 1: A table of minimal polynomials f_α together with a minimal polynomial g_β of degree 2 over $\mathbb{Q}(\alpha)$ such that $(\beta, \alpha-\beta)$ is a non-trivial decomposition of α.

$q(\alpha)$	f_α	g_β
2.2844	$x^3 - x^2 + 3x + 1$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 + 1)$
2.5198	$x^3 + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2$
2.6178	$x^3 + 2x^2 + 4x + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2$
2.7246	$x^3 + 2x + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2$
2.7850	$x^3 - 2x^2 + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2$
2.8582	$x^3 + 2x^2 - x + 2$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 + \alpha)$
2.8657	$x^3 - 2x^2 + 3x + 2$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 - \alpha)$
2.9073	$x^3 + 2x^2 - 2x + 1$	$x^2 - \alpha x - \alpha$
2.9379	$x^3 + 2x^2 + x + 4$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 + \alpha)$
2.9380	$x^3 - x^2 - x + 5$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 - 1)$
2.9516	$x^3 + x^2 + x + 5$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 + 1)$
3	$x^3 + x^2 - 4x + 1$	$x^2 - (\alpha + 1)\, x + 1$
3.1102	$x^3 + x^2 - 2x + 4$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 + \alpha)$
3.2714	$x^3 + 2x^2 + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2$
3.2849	$x^3 - 2x^2 + 2x + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2$
3.2981	$x^3 + 2x^2 - 2x + 2$	$x^2 - (\alpha - 1)\, x - (\alpha - 1)$

$q(\alpha)$	f_α	g_β
3.3321	$x^3 + 3x + 5$	$x^2 - \alpha x + \frac{1}{3}(\alpha^2 + \alpha + 1)$
3.3333	$x^3 - 2x^2 - 3x + 1$	$x^2 - (\alpha - 1)\, x + 1$
3.3378	$x^3 + x^2 - 4x + 3$	$x^2 - (\alpha - 1)\, x - (\alpha - 1)$
3.3853	$x^3 + 4x + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2 + 1$
3.4436	$x^3 - 2x^2 - x + 6$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 - \alpha)$
3.5716	$x^3 + 2x^2 + 6x + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2 + 1$
3.6432	$x^3 + 2x^2 + 6x + 2$	$x^2 - (\alpha + 1)\, x - 1$
3.6830	$x^3 + x^2 - 4x + 4$	$x^2 - \alpha x + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha - 1$
3.6839	$x^3 - x^2 - 3x + 7$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 - 1)$
3.7159	$x^3 + x^2 + 3x + 7$	$x^2 - \alpha x + \frac{1}{2}(\alpha^2 + 1)$
3.7362	$x^3 + 2x^2 - 3x + 2$	$x^2 - \alpha x - \alpha$
3.7962	$x^3 + 2x^2 + 6x + 1$	$x^2 - (\alpha - 1)\, x - \alpha - 1$
3.8854	$x^3 - 3x + 7$	$x^2 - \alpha x + \frac{1}{3}(\alpha^2 - \alpha + 1)$
3.9111	$x^3 - 2x^2 + 5x + 2$	$x^2 - (\alpha - 1)\, x - 1$
3.9938	$x^3 + x^2 - 5x + 4$	$x^2 - (\alpha + 1)\, x + 1$
4	$x^3 - 2x^2 - 4x + 1$	$x^2 - \alpha x + \alpha$

Table 2: A table of minimal polynomials f_α together with a minimal polynomial g_β of degree greater than 2 over $\mathbb{Q}(\alpha)$ such that $(\beta, \alpha-\beta)$ is a non-trivial decomposition of α.

$q(\alpha)$	f_α	g_β
2.9240	x^3+5	$x^4-2\alpha x^3+2\alpha^2x^2+5x-\alpha$
3.0103	x^3-x^2+5	$x^6-3\alpha x^5+(4\alpha^2+1)x^4+(-3\alpha^2-2\alpha+15)x^3+(3\alpha^2-7\alpha-6)x^2+(\alpha^2+\alpha+5)x-\alpha+1$
3.2595	x^3+2x^2-x+3	$x^4-2\alpha x^3+(2\alpha^2+\alpha-1)x^2+(\alpha^2+3)x-\alpha+1$
3.2624	x^3-2x^2+3x+3	$x^4-2\alpha x^3+(2\alpha^2-\alpha+1)x^2+(-\alpha^2+2\alpha+3)x-\alpha$
3.3019	x^3+6	$x^6-3\alpha x^5+4\alpha^2x^4+18x^3-8\alpha x^2+2\alpha^2x+1$
3.3378	x^3+x+6	$x^4-2\alpha x^3+(\frac{3}{2}\alpha^2-\frac{1}{2}\alpha)x^2+(\frac{1}{2}\alpha^2+\frac{1}{2}\alpha+3)x+1$
3.3656	x^3-2x^2+4x+2	$x^4-2\alpha x^3+(2\alpha^2-\alpha+2)x^2+(-\alpha^2+2\alpha+2)x-\alpha$
3.3709	x^3-x^2+2x+5	$x^6-3\alpha x^5+4\alpha^2x^4+(-3\alpha^2+6\alpha+15)x^3+(-\alpha^2-10\alpha-5)x^2+3\alpha^2x+3$
3.3863	x^3+x^2-3x+4	$x^8-4\alpha x^7+7\alpha^2x^6+(7\alpha^2-21\alpha+28)x^5+(18\alpha^2-30\alpha+17)x^4+(19\alpha^2-30\alpha+32)x^3+(13\alpha^2-24\alpha+19)x^2+(6\alpha^2-9\alpha+8)x+\alpha^2-2\alpha+2$
3.3895	x^3-x^2+6	$x^6-3\alpha x^5+(4\alpha^2+1)x^4+(-3\alpha^2-2\alpha+18)x^3+(3\alpha^2-8\alpha-7)x^2+(\alpha^2+\alpha+6)x-\alpha+1$
3.4190	x^3-2x+6	$x^6-3\alpha x^5+(4\alpha^2-1)x^4+(-4\alpha+18)x^3+(\alpha^2-8\alpha)x^2+2\alpha^2x+1$
3.4338	x^3-x^2+4x+3	$x^8-4\alpha x^7+(7\alpha^2-1)x^6+(-7\alpha^2+31\alpha+21)x^5+(-17\alpha^2-30\alpha-13)x^4+(19\alpha^2-27\alpha-24)x^3+(4\alpha^2+26\alpha+13)x^2+(-5\alpha^2+3\alpha+3)x-2\alpha-1$
3.4438	x^3+x^2+6	$x^{10}-5\alpha x^9+11\alpha^2x^8+(14\alpha^2+84)x^7+(11\alpha^2-69\alpha+68)x^6+(44\alpha^2-36\alpha+30)x^5+(29\alpha^2-5\alpha+99)x^4+(5\alpha^2-24\alpha+42)x^3+(3\alpha^2-6\alpha)x^2-1$
3.4537	x^3+2x+6	$x^6-3\alpha x^5+(4\alpha^2+1)x^4+(4\alpha+18)x^3+(-\alpha^2-8\alpha)x^2+2\alpha^2x+1$

Table 2: Continued.

$q(\alpha)$	f_α	g_β
3.4767	x^3+x^2-2x+5	$x^{14}-7\alpha x^{13}+23\alpha^2x^{12}+(47\alpha^2-94\alpha+235)x^{11}+(200\alpha^2-467\alpha+333)x^{10}+(695\alpha^2-761\alpha+1040)x^9+(1143\alpha^2-1912\alpha+2739)x^8+(1840\alpha^2-3030\alpha+3435)x^7+(2256\alpha^2-3290\alpha+4258)x^6+(1960\alpha^2-3100\alpha+3990)x^5+(1343\alpha^2-2101\alpha+2601)x^4+(667\alpha^2-1022\alpha+1300)x^3+(226\alpha^2-354\alpha+447)x^2+(49\alpha^2-75\alpha+95)x+5\alpha^2-8\alpha+10$
3.4858	x^3+2x^2+2x+6	$x^6-3\alpha x^5+(4\alpha^2+1)x^4+(6\alpha^2+4\alpha+18)x^3+(4\alpha^2-3\alpha+16)x^2+(3\alpha^2-2\alpha+6)x+\alpha^2+2$
3.5833	x^3-2x^2+6	$x^8-4\alpha x^7+(7\alpha^2+\alpha)x^6+(-17\alpha^2+42)x^5+(25\alpha^2-26\alpha-75)x^4+(-14\alpha^2+36\alpha+72)x^3+(-\alpha^2-22\alpha-29)x^2+(4\alpha^2+5\alpha)x-\alpha^2+2$
3.6133	x^3+x^2-x+6	$x^6-3\alpha x^5+(4\alpha^2+\alpha)x^4+(\alpha^2-3\alpha+18)x^3+(\alpha^2-8\alpha-1)x^2+(2\alpha^2+\alpha)x+1$
3.6361	x^3+2x^2-x+4	$x^6-3\alpha x^5+(4\alpha^2-1)x^4+(6\alpha^2-\alpha+12)x^3+(5\alpha^2-8\alpha+11)x^2+(4\alpha^2-4\alpha+4)x+\alpha^2-\alpha+2$
3.6370	x^3-2x^2+3x+4	$x^6-3\alpha x^5+(4\alpha^2+1)x^4+(-6\alpha^2+7\alpha+12)x^3+(3\alpha^2-14\alpha-10)x^2+(2\alpha^2+5\alpha+4)x-\alpha^2+1$
3.6521	x^3+2x^2+5	$x^6-3\alpha x^5+4\alpha^2x^4+(6\alpha^2+15)x^3+(5\alpha^2-7\alpha+13)x^2+(4\alpha^2-3\alpha+5)x+\alpha^2+2$
3.6593	x^3+7	$x^3-2\alpha x^2+\alpha^2x+1$
3.6785	x^3-x^2-x+7	$x^{10}-5\alpha x^9+(11\alpha^2+1)x^8+(-14\alpha^2-18\alpha+98)x^7+(30\alpha^2-68\alpha-81)x^6+(16\alpha^2+27\alpha+140)x^5+(-13\alpha^2-70\alpha+18)x^4+(24\alpha^2+8\alpha-14)x^3+(-8\alpha^2-5\alpha+31)x^2+(2\alpha^2-2\alpha-7)x+1$
3.6878	x^3-x+7	$x^6-3\alpha x^5+4\alpha^2x^4+(-3\alpha+21)x^3+(\alpha^2-9\alpha)x^2+2\alpha^2x+1$
3.6915	x^3+x+7	$x^6-3\alpha x^5+4\alpha^2x^4+(3\alpha+21)x^3+(-\alpha^2-9\alpha)x^2+2\alpha^2x+1$
3.6939	x^3+x^2+x+7	$x^6-3\alpha x^5+(4\alpha^2+\alpha+1)x^4+(\alpha^2+\alpha+21)x^3+(-9\alpha-1)x^2+(2\alpha^2+\alpha)x+1$
3.7123	x^3-2x^2+4x+3	$x^6-3\alpha x^5+4\alpha^2x^4+(-6\alpha^2+12\alpha+9)x^3+(-15\alpha-7)x^2+(4\alpha^2+\alpha)x-\alpha^2+2\alpha+1$
3.7228	x^3+2x^2+x+6	$x^6-3\alpha x^5+4\alpha^2x^4+(6\alpha^2+3\alpha+18)x^3+(4\alpha^2-5\alpha+16)x^2+(3\alpha^2-3\alpha+6)x+\alpha^2+2$

3.7238	$x^3 - x^2 + 2x + 6$	$x^8 - 4\alpha x^7 + 7\alpha^2 x^6 + (-7\alpha^2 + 14\alpha + 42)x^5 + (-4\alpha^2 - 35\alpha - 26)x^4 + (15\alpha^2 + 8\alpha - 6)x^3 + (-6\alpha^2 + 6\alpha + 20)x^2 + (\alpha^2 - 4\alpha - 6)x + \alpha + 1$
3.7345	$x^3 + x^2 - 3x + 5$	$x^3 - 2\alpha x^2 + (\frac{3}{2}\alpha^2 - \frac{1}{2})x + \frac{1}{2}\alpha^2 - \alpha + \frac{3}{2}$
3.7479	$x^3 - x^2 + 7$	$x^6 - 3\alpha x^5 + (4\alpha^2 + 1)x^4 + (-3\alpha^2 - 2\alpha + 21)x^3 + (3\alpha^2 - 9\alpha - 9)x^2 + (\alpha^2 + 2\alpha + 7)x - \alpha$
3.7664	$x^3 - 2x + 7$	$x^6 - 3\alpha x^5 + (4\alpha^2 - 1)x^4 + (-4\alpha + 21)x^3 + (\alpha^2 - 9\alpha)x^2 + 2\alpha^2 x + 1$
3.7771	$x^3 - 2x^2 + x + 6$	$x^6 - 3\alpha x^5 + 4\alpha^2 x^4 + (-6\alpha^2 + 3\alpha + 18)x^3 + (4\alpha^2 - 10\alpha - 15)x^2 + (4\alpha + 6)x - \alpha - 1$
3.7949	$x^3 + 2x + 7$	$x^6 - 3\alpha x^5 + (4\alpha^2 + 1)x^4 + (4\alpha + 21)x^3 + (-\alpha^2 - 9\alpha)x^2 + 2\alpha^2 x + 1$
3.7995	$x^3 + x^2 + 7$	$x^6 - 3\alpha x^5 + (4\alpha^2 + \alpha)x^4 + (\alpha^2 + 21)x^3 + (-9\alpha - 1)x^2 + (2\alpha^2 + \alpha)x + 1$
3.8253	$x^3 + x^2 - 2x + 6$	$x^6 - 3\alpha x^5 + 4\alpha^2 x^4 + (3\alpha^2 - 6\alpha + 18)x^3 + (4\alpha^2 - 10\alpha + 9)x^2 + (3\alpha^2 - 5\alpha + 6)x + \alpha^2 - \alpha + 1$
3.8560	$x^3 + 2x^2 + 2x + 7$	$x^6 - 3\alpha x^5 + (4\alpha^2 + 1)x^4 + (6\alpha^2 + 4\alpha + 21)x^3 + (4\alpha^2 - 4\alpha + 19)x^2 + (3\alpha^2 - 3\alpha + 7)x + \alpha^2 + 2$
3.8739	$x^3 - x^2 + x + 7$	$x^3 - 2\alpha x^2 + (\frac{3}{2}\alpha^2 + \frac{1}{2})x - \frac{1}{2}\alpha^2 + \frac{5}{2}$
3.9466	$x^3 - 2x^2 + 7$	$x^6 - 3\alpha x^5 + (4\alpha^2 - 1)x^4 + (-6\alpha^2 + 2\alpha + 21)x^3 + (4\alpha^2 - 10\alpha - 19)x^2 + (\alpha^2 + 5\alpha + 7)x - \alpha^2 - \alpha + 1$
3.9928	$x^3 + 2x^2 - x + 5$	$x^6 - 3\alpha x^5 + (4\alpha^2 - 1)x^4 + (6\alpha^2 - \alpha + 15)x^3 + (5\alpha^2 - 9\alpha + 13)x^2 + (4\alpha^2 - 4\alpha + 5)x + \alpha^2 - \alpha + 1$
3.9948	$x^3 - x^2 + 3x + 6$	$x^6 - 3\alpha x^5 + (4\alpha^2 + 1)x^4 + (-3\alpha^2 + 7\alpha + 18)x^3 + (-\alpha^2 - 12\alpha - 9)x^2 + (3\alpha^2 + 3\alpha)x - \alpha^2 + \alpha + 1$

Table 3: A table of minimal polynomials f_α together with a polynomial g over $\mathbb{Q}(\alpha)$ such that $g(x+\alpha/2)$ is exponentially bounded at radius $\sqrt{q(\alpha/2)}$, proving α is indecomposable.

$q(\alpha)$	f_α	g
2	x^3-2x^2-x+1	$\left(\alpha^2-2\alpha\right)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\alpha^2-\alpha-1)$
2.0347	x^3+x^2+3x+2	$x^2\cdot(x-\alpha)^2\cdot(x^2-\alpha x-1)$
2.0780	x^3-2x^2+x+2	$(x-1)\cdot(x-\alpha+1)\cdot x^2\cdot(x-\alpha)^2$
2.0801	x^3+3	$x^2\cdot(x-\alpha)^2\cdot(x^4-2\alpha x^3+2\alpha^2x^2-3x+\alpha)$
2.0826	x^3+2x^2+3x+3	$\left(\alpha^2+\alpha+1\right)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2)$
2.0872	x^3-x^2-x+3	$(\alpha+1)\cdot x\cdot(x-\alpha)\cdot(x^4-2\alpha x^3+\left(2\alpha^2-\alpha+1\right)x^2+(-2\alpha+3)\,x+\frac{1}{2}\alpha^2-\alpha+\frac{1}{2})$
2.0905	x^3-x^2-2x+3	$\alpha\cdot x\cdot(x-\alpha)\cdot(x^4-2\alpha x^3+\left(2\alpha^2-1\right)x^2+\left(-\alpha^2-\alpha+3\right)x+\frac{1}{3}\alpha^2-\frac{1}{3}\alpha-\frac{2}{3})$
2.0967	x^3+x^2+x+3	$\alpha\cdot x^2\cdot(x-\alpha)^2\cdot(x^4-2\alpha x^3+\left(\frac{5}{3}\alpha^2-\frac{1}{3}\alpha-\frac{1}{3}\right)x^2+\left(\alpha^2+\alpha+2\right)x+1)$
2.1102	x^3+x^2+2x+3	$x\cdot(x-\alpha)\cdot(x^4-2\alpha x^3+2\alpha^2x^2+\left(\alpha^2+2\alpha+3\right)x+1)$
2.1279	x^3-x+3	$x^2\cdot(x-\alpha)^2\cdot(x^4-2\alpha x^3+2\alpha^2x^2+(-\alpha+3)\,x-\alpha)$
2.1390	x^3+x+3	$2\cdot x\cdot(x-\alpha)\cdot(x^6-3\alpha x^5+\left(4\alpha^2+\frac{1}{2}\right)x^4+(2\alpha+9)\,x^3+\left(-\frac{1}{2}\alpha^2-4\alpha\right)x^2+\alpha^2x+\frac{1}{2})$
2.1626	x^3-x^2+3	$(\alpha-1)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2)$
2.1815	x^3-2x^2-x+3	$x\cdot(x-\alpha)\cdot(x^2-\alpha x+1)$
2.1904	x^3-x^2+2x+2	$(\alpha-1)\cdot x\cdot(x-\alpha)\cdot(x^4-2\alpha x^3+\frac{3}{2}\alpha^2x^2+\left(-\frac{1}{2}\alpha^2+\alpha+1\right)x-\frac{1}{4}\alpha^2-\frac{1}{2})$
2.1973	x^3+2x^2+2x+3	$(\alpha+1)\cdot x^2\cdot(x-\alpha)^2\cdot(x^4-2\alpha x^3+2\alpha^2x^2+\left(2\alpha^2+2\alpha+3\right)x+\frac{1}{2}\alpha^2+\frac{1}{2}\alpha+\frac{3}{2})$
2.2230	x^3+2x^2+4x+2	$\left(\alpha^2+\alpha+1\right)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2+\frac{1}{3})\cdot(x^2-\alpha x+\alpha^2+\alpha+2)$
2.2309	x^3+x^2+3	$(\alpha+1)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2)$
2.2512	x^3-2x+3	$\alpha\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2+\frac{1}{3})\cdot(x^4-2\alpha x^3+\left(\alpha^2-\alpha\right)x^2+\alpha^2x+1)$

2.3044	x^3+x^2-2x+2	$(\alpha+2)\cdot x^3\cdot(x-\alpha)^3\cdot(x^2-\alpha x+\frac{1}{2}\alpha^2+\frac{1}{2}\alpha)$
2.3681	x^3+2x^2+4x+1	$\left(\alpha^2+\alpha+4\right)\cdot x\cdot(x+1)\cdot(x-\alpha-1)\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{2}\alpha^2+\frac{1}{2}\alpha+\frac{1}{2})\cdot(x^4-2\alpha x^3+\left(\frac{7}{5}\alpha^2-\frac{2}{5}\alpha-\frac{1}{5}\right)x^2+\left(\frac{6}{5}\alpha^2+\frac{9}{5}\alpha+\frac{2}{5}\right)x+\frac{1}{5}\alpha^2+\frac{4}{5}\alpha+\frac{2}{5})$
2.4206	x^3+2x^2+2	$(\alpha+1)\cdot x^2\cdot(x-\alpha)^2\cdot(x^4-2\alpha x^3+\left(\frac{4}{3}\alpha^2-\frac{2}{3}\alpha-\frac{1}{3}\right)x^2+\left(\frac{4}{3}\alpha^2+\frac{1}{3}\alpha+\frac{2}{3}\right)x+\frac{2}{3}\alpha^2+\frac{2}{3}\alpha+\frac{1}{3})\cdot(x^4-2\alpha x^3+\left(2\alpha^2+\alpha\right)x^2+\left(\alpha^2+2\right)x-\alpha)$
2.4239	x^3+2x^2-x+1	$\left(\alpha^2+2\alpha-2\right)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2+\frac{1}{3}\alpha+\frac{1}{3})\cdot(x^4-2\alpha x^3+\left(2\alpha^2+\alpha-1\right)x^2+\left(\alpha^2+1\right)x-\alpha)$
2.4300	x^3-2x^2+2x+2	$(\alpha-1)\cdot x\cdot(x-\alpha)\cdot(x^2-\alpha x+\frac{1}{3}\alpha^2-\frac{1}{3}\alpha+\frac{1}{3})\cdot(x^8-4\alpha x^7+\left(7\alpha^2+\alpha\right)x^6+\left(-17\alpha^2+14\alpha+14\right)x^5+\left(17\alpha^2-35\alpha-24\right)x^4+\left(-2\alpha^2+30\alpha+20\right)x^3+\left(-7\alpha^2-11\alpha-3\right)x^2+\left(5\alpha^2-\alpha-2\right)x-\alpha^2+\alpha+1)$
2.4436	x^3-2x^2+3x+1	$x\cdot(x-\alpha)\cdot(x^4-2\alpha x^3+\left(2\alpha^2-\alpha+2\right)x^2+\left(-\alpha^2+\alpha+1\right)x-1)\cdot(x^6-3\alpha x^5+\left(4\alpha^2-\alpha\right)x^4+\left(-4\alpha^2+9\alpha+3\right)x^3+\left(-2\alpha^2-4\alpha-1\right)x^2+\left(2\alpha^2-3\alpha-1\right)x+\alpha)$
2.4517	x^3+x^2-x+3	$\left(\alpha^2+2\alpha-2\right)\cdot x^2\cdot(x-\alpha)^2\cdot(x^2-\alpha x+\frac{1}{2}\alpha^2-\frac{1}{2})\cdot(x^6-3\alpha x^5+\left(\frac{179}{46}\alpha^2+\frac{6}{23}\alpha+\frac{1}{46}\right)x^4+\left(\frac{52}{23}\alpha^2-\frac{65}{23}\alpha+\frac{192}{23}\right)x^3+\left(\frac{44}{23}\alpha^2-\frac{101}{23}\alpha+\frac{51}{23}\right)x^2+\left(\frac{67}{46}\alpha^2-\frac{16}{23}\alpha+\frac{51}{46}\right)x+\frac{4}{23}\alpha^2-\frac{5}{23}\alpha+\frac{13}{23})$
2.4960	x^3+2x^2+x+3	$\left(3\alpha^2+9\alpha+5\right)\cdot x^3\cdot(x-\alpha)^3\cdot(x^4-2\alpha x^3+\left(\alpha^2-1\right)x^2+\alpha x-\alpha^2-1)\cdot(x^4-2\alpha x^3+\left(2\alpha^2+\alpha\right)x^2+\left(\alpha^2+\alpha+3\right)x-\alpha)\cdot(x^4-2\alpha x^3+\left(\frac{10}{7}\alpha^2-\frac{2}{7}\alpha-\frac{1}{7}\right)x^2+\left(\frac{8}{7}\alpha^2+\frac{4}{7}\alpha+\frac{9}{7}\right)x+\frac{2}{7}\alpha^2+\frac{1}{7}\alpha+\frac{4}{7})^2$
2.5248	x^3-x^2-2x+4	$\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha+1\right)\cdot x\cdot(x-\alpha)\cdot(x-\frac{1}{2}\alpha)^2\cdot(x^2+(-\alpha-1)x+\frac{1}{2}\alpha^2+\frac{1}{2}\alpha)\cdot(x^2-\alpha x+1)\cdot(x^2+(-\alpha+1)x+\frac{1}{2}\alpha^2-\frac{1}{2}\alpha)$
2.5324	x^3+x^2+2x+4	$(-\alpha+1)\cdot x\cdot(x-\alpha)\cdot(x-\frac{1}{2}\alpha)^2\cdot(x^2-\alpha x+\frac{1}{2}\alpha^2)\cdot(x^4-2\alpha x^3+\left(\frac{3}{2}\alpha^2-\frac{1}{2}\alpha\right)x^2+\left(\alpha^2+\alpha+2\right)x+1)$

Table 3: Continued.

$q(\alpha)$	f_α	g
2.5426	x^3+x^2+x+4	$\alpha \cdot x^3 \cdot (x-\alpha)^3 \cdot (x^4 - 2\alpha x^3 + 2\alpha^2 x^2 + \left(\alpha^2+\alpha+4\right) x - \alpha + 1) \cdot (x^4 - 2\alpha x^3 + \left(\frac{3}{2}\alpha^2 - \frac{1}{2}\alpha - \frac{1}{2}\right) x^2 + \left(\alpha^2+\alpha+2\right) x + 1)^2$
2.5601	x^3-x+4	$\left(-\frac{1}{2}\alpha^2 - \frac{3}{2}\alpha\right) \cdot x \cdot (x-\alpha) \cdot (x^4 - 2\alpha x^3 + \left(\frac{3}{2}\alpha^2 - \frac{1}{2}\alpha\right) x^2 + \left(\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha + 2\right) x - \frac{1}{2}\alpha + \frac{1}{2}) \cdot (x^4 - 2\alpha x^3 + \left(\frac{3}{2}\alpha^2 + \frac{1}{2}\alpha\right) x^2 + \left(-\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha + 2\right) x - 1) \cdot (x^4 - 2\alpha x^3 + \left(\frac{8}{5}\alpha^2 + \frac{1}{5}\alpha - \frac{1}{5}\right) x^2 + \left(-\frac{1}{5}\alpha^2 - \frac{2}{5}\alpha + \frac{12}{5}\right) x + \frac{1}{10}\alpha^2 - \frac{3}{10}\alpha - \frac{1}{5})$
2.6335	x^3+x^2-3x+2	$\left(\alpha^2+2\alpha\right) \cdot x^2 \cdot (x-\alpha)^2 \cdot (x^2 + (-\alpha-1)\, x + 1) \cdot (x^2 + (-\alpha+1)\, x - \alpha + 1) \cdot (x^2 - \alpha x + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha - \frac{1}{2})^2 \cdot (x^4 - 2\alpha x^3 + \left(\frac{3}{2}\alpha^2 - \frac{1}{2}\alpha - \frac{1}{2}\right) x^2 + \left(\alpha^2 - \alpha + 1\right) x + \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha - \frac{1}{2})$
2.6377	x^3-2x^2-x+4	$\left(-\alpha^2+\alpha\right) \cdot x^2 \cdot (x-\alpha)^2 \cdot (x^2 - \alpha x + 1)^2 \cdot (x^2 - \alpha x + \frac{1}{2}\alpha^2 - \frac{1}{2})^2 \cdot (x^8 - 4\alpha x^7 + \left(7\alpha^2 - 1\right) x^6 + \left(-14\alpha^2 - 4\alpha + 28\right) x^5 + \left(18\alpha^2 - 9\alpha - 35\right) x^4 + \left(-8\alpha^2 + 8\alpha + 24\right) x^3 + \left(-\alpha^2 - 3\alpha - 3\right) x^2 + \left(2\alpha^2 - 4\right) x - \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha + 1)$

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Curriculum Vitae

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In 2016 behaalde hij cum laude zijn bachelorgraden in Wiskunde en Informatica aan de Universiteit Leiden. Hij schreef zijn bachelorscriptie, getiteld *"Graph-isomorphism in quasi-polynomial time"*, onder begeleiding van Owen Biesel en Hendrik Jan Hoogeboom. In 2019 verdedigde hij zijn masterscriptie, getiteld *"Algorithms for finding the gradings of reduced rings"*, aan de Universiteit Leiden onder begeleiding van Hendrik Willem Lenstra. Hij behaalde hierbij cum laude zijn mastergraad Wiskunde.

Tijdens zijn bachelor- en masterstudie is hij werkzaam geweest als student-assistent bij de Universiteit Leiden. Ook vormde hij met Onno Bernardus Berrevoets en Wessel Pieter Jacobus van Woerden een team voor verscheidene internationale programmeerwedstrijden. Zij wonnen in 2017 de *"Catalyst Coding Contest"*. In 2019 behaalde hij in een team met Berrevoets en Pim Spelier de finale van de *"International Collegiate Programming Contest"*.

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