Decompositions in algebra

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CHAPTER 2

Hilbert lattices

2.1 Introduction

This chapter is based on [18]. A set of $\mathbb{R}$-linearly independent vectors $\{b_1, \dots, b_k\}$ of some Euclidean vector space such as $\mathbb{R}^n$ gives rise to a discrete subgroup

$$\Big\{ \sum_{i=1}^{k} x_i b_i \,\Big|\, x_1, \dots, x_k \in \mathbb{Z} \Big\},$$

which we call a lattice. In particular, a lattice is a free abelian group of finite rank. In preparation of Chapter 3 we study a generalization of lattices that includes 'infinite rank lattices'. These will be the discrete subgroups of Hilbert spaces, which we call *Hilbert lattices*, and they include the 'Euclidean lattices' as a special case. We will primarily generalize existing theory from the finite dimensional case, and highlight the things that fail to generalize.

Theorem 2.3.13. *Every countable subgroup of a Hilbert lattice is free.*

Whether or not all Hilbert lattices are free themselves is still an open problem. Let Λ be a Hilbert lattice. An *orthogonal decomposition* of Λ is a decomposition $\{\Lambda_i\}_{i \in I}$ of Λ as abelian group, as defined in the Preliminaries, such that $\langle \Lambda_i, \Lambda_j \rangle = \{0\}$ for all distinct $i, j \in I$. The collection of orthogonal decompositions of Λ inherit the structure of a category. We say an orthogonal decomposition is *universal* if it is an initial object in this category.

Theorem 2.5.4. *Every Hilbert lattice has a universal orthogonal decomposition.*

Let Λ be a Hilbert lattice in a Hilbert space $\mathcal{H}$. The *Voronoi cell* of Λ is the set

$$\operatorname{Vor}(\Lambda) = \{ z \in \mathcal{H} \mid (\forall\, x \in \Lambda \setminus \{0\}) \ \|z\| < \|z - x\| \},$$

i.e. the set of all points which have the origin as their unique closest lattice point. It is almost a 'fundamental domain' for Λ.

Theorem 2.6.9. *Let Λ be a Hilbert lattice in a Hilbert space $\mathcal{H}$ and consider the natural map $\mathcal{H} \to \mathcal{H}/\Lambda$. Its restriction to* $\operatorname{Vor}(\Lambda)$ *is injective and for all* $\varepsilon > 0$ *its restriction to* $(1+\varepsilon)\operatorname{Vor}(\Lambda)$ *is surjective.*

A *decomposition* of $z \in \Lambda$ is a pair $(x, y) \in \Lambda^2$ such that $x + y = z$ and $\langle x, y \rangle \geq 0$. We say $x \in \Lambda$ is *indecomposable* or *Voronoi relevant* if it has precisely 2 decompositions, i.e. $(0, x)$ and $(x, 0)$ are the only decompositions and $x \neq 0$. One can interpret the Voronoi cell as the intersection of half spaces $H_x = \{z \in \mathcal{H} \mid \|z\| < \|z - x\|\}$, but we do not need all $x \in \Lambda \setminus \{0\}$ to carve out $\mathrm{Vor}(\Lambda)$. For example, $H_x \cap H_{2x} = H_x$.

Theorem 2.6.11. *Let Λ be a Hilbert lattice in a Hilbert space $\mathcal{H}$. Then there exists a unique set $S \subseteq \Lambda \setminus \{0\}$ which is minimal with respect to inclusion such that* $\mathrm{Vor}(\Lambda) = \{z \in \mathcal{H} \mid (\forall\, x \in S)\ \|z\| < \|z - x\|\}$*, and S equals the set of indecomposable vectors.*

2.2 Inner products and Hilbert spaces

Definition 2.2.1. Let $R \subseteq \mathbb{C}$ be a subring. An *R-norm* on an R-module M is a map $\|\cdot\|\colon M \to \mathbb{R}_{\geq 0}$ that satisfies:

(*Absolute homogeneity*) For all $x \in M$ and $a \in R$ we have $\|ax\| = |a| \cdot \|x\|$;

(*Triangle inequality*) For all $x, y \in M$ we have $\|x + y\| \leq \|x\| + \|y\|$;

(*Positive-definiteness*) For all non-zero $x \in M$ we have $\|x\| \in \mathbb{R}_{>0}$.

A *normed R-module* is an R-module M together with an R-norm on M. For normed R-modules M and N an R-module homomorphism $f\colon M \to N$ is called an *isometric map* if $\|x\| = \|f(x)\|$ for all $x \in M$. The isometric maps are the morphisms in the category of normed R-modules.

Note that an isometric map is injective, but not necessarily surjective.

Definition 2.2.2. Let $R \subseteq \mathbb{C}$ be a subring and M be an R-module. An *R-inner product* on M is a map $\langle\,\cdot\,,\,\cdot\,\rangle\colon M^2 \to \mathbb{C}$ that satisfies:

(*Conjugate symmetry*) For all $x, y \in M$ we have $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(*Left linearity*) For all $x, y, z \in M$ and $a \in R$ we have

$$\langle x + ay, z\rangle = \langle x, z\rangle + a\langle y, z\rangle;$$

(*Positive-definiteness*) For all non-zero $x \in M$ we have $\langle x, x\rangle \in \mathbb{R}_{>0}$.

We say it is a *real* inner product if $\langle M, M\rangle \subseteq \mathbb{R}$, which implies $R \subseteq \mathbb{R}$ when $M \neq 0$. An *R-inner product space* is an R-module together with an R-inner product. For R-inner product spaces M and N a morphism is an R-module homomorphism $f\colon M \to N$ for which there exists an R-module homomorphism $f^*\colon N \to M$ such that $\langle f(x), y\rangle = \langle x, f^*(y)\rangle$ for all $x \in M$ and $y \in N$. This f^* is unique if it exists, and we call it the *adjoint* of f.

Remark 2.2.3. An R-inner product space M comes with an R-norm given by $\|x\| = \sqrt{\langle x, x\rangle}$, which in turn induces a metric $d(x, y) = \|x - y\|$ and a topology. One can then speak about the completeness of M with respect to this metric.

Lemma 2.2.4. *Suppose $R \subseteq \mathbb{C}$ is a subring and M is a real R-inner product space. Then the induced norm satisfies the* parallelogram law*: For all $x, y \in M$ we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \quad \square$$

The following is an exercise in many standard texts.

Theorem 2.2.5 (Jordan–von Neumann [26])**.** *Let $R \subseteq \mathbb{Q}$ be a subring, M an R-module and suppose a map $\|\cdot\|\colon M \to \mathbb{R}_{\geq 0}$ satisfies positive-definiteness and the parallelogram law. Then $\|\cdot\|$ is an R-norm on M induced by a real R-inner product $\langle\cdot,\cdot\rangle\colon M^2 \to \mathbb{R}$ given by*

$$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2).$$

Proof. Note that taking $x = y = 0$ in the parallelogram law shows $2\|0\|^2 = 4\|0\|^2$, hence $\|0\| = 0$. For all $x \in M$ we have $\|x+x\|^2 = 2\|x\|^2 + 2\|x\|^2 - \|x-x\|^2 = 4\|x\|^2$, hence $\langle x, x\rangle = \frac{1}{2}(\|2x\|^2 - 2\|x\|^2) = \|x\|^2$. It now suffices to show that $\langle\cdot,\cdot\rangle$ is an inner product, as $\|\cdot\|$ is then the associated norm as in Remark 2.2.3. Clearly $\langle\cdot,\cdot\rangle$ satisfies conjugate symmetry and positive definiteness, so it remains to prove left linearity. It suffices to show for all $x \in M$ that $x \mapsto \langle x, z\rangle$ is $\mathbb{Z}$-linear: Since R is in the field of fractions of $\mathbb{Z}$, any $\mathbb{Z}$-linear map to $\mathbb{R}$ is also R-linear. Let $x, y, z \in M$ and note that $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$. By the parallelogram law we have

$$\begin{aligned} 2\|y+z\|^2 + 2\|x\|^2 - \|-x+y+z\|^2 &= \|x+y+z\|^2 \\ = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2. \end{aligned}$$

so

$$\begin{aligned} 2\|x+y+z\|^2 + \|-x+y+z\|^2 + \|x-y+z\|^2 \\ = 2\|x+z\|^2 + 2\|y+z\|^2 + 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

Applying this equation also with z replaced by $-z$, we obtain

$$\begin{aligned} 8\langle x+y, z\rangle &= 2\|x+y+z\|^2 - 2\|x+y-z\|^2 \\ &= 2\|x+z\|^2 + 2\|y+z\|^2 - 2\|x-z\|^2 - 2\|y-z\|^2 \\ &= 8\langle x, z\rangle + 8\langle y, z\rangle, \end{aligned}$$

as was to be shown. We conclude that $\langle\cdot,\cdot\rangle$ is an R-inner product. $\square$

Inner product spaces over $\mathbb{Z}$ or $\mathbb{Q}$ can be extended to $\mathbb{R}$ in a 'canonical' way. This can best be expressed in a categorical sense in terms of universal morphisms. We proceed as in Chapter III of [37].

Definition 2.2.6. Let $\mathcal{C}$ be a category. An object U of $\mathcal{C}$ is called *universal* if for each object X of $\mathcal{C}$ there exists a unique morphism $U \to X$ in $\mathcal{C}$.

Definition 2.2.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and Z an object of $\mathcal{D}$. A *universal morphism* from Z to F is a pair (X, η) with X an object of $\mathcal{C}$ and $\eta \in \mathrm{Hom}_{\mathcal{D}}(Z, F(X))$ such that for all objects Y of $\mathcal{C}$ and every $g \in \mathrm{Hom}_{\mathcal{D}}(Z, F(Y))$ there exists a unique $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for which $F(f) \circ \eta = g$. Equivalently, for all objects Y of $\mathcal{C}$ and morphisms $g\colon Z \to F(Y)$ we have the following diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{\eta} & F(X) & \qquad & X \\ & \searrow^{g} & \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle !f} \\ & & F(Y) & & Y \end{array}$$

For a reader familiar with category theory we remark that, if for a functor $F\colon \mathcal{C} \to \mathcal{D}$ every object Z of $\mathcal{D}$ has a universal morphism to F, then F is a *right adjoint* functor.

Example 2.2.8. We will give a concrete example of a universal morphism.

1. Let k be a field. Consider the forgetful functor F from the category of k-vector spaces to the category of abelian groups, i.e. the functor that sends a k-vector space to its underlying abelian group. Now let Z be an abelian group. We take $X = k \otimes_{\mathbb{Z}} Z$, which is a k-vector space, and $\eta\colon Z \to F(X)$ the map $z \mapsto 1 \otimes z$. Because F is a forgetful functor, as will always be the case in our applications, we may omit it in the notation for simplicity and state that η is a morphism $Z \to X$ of abelian groups.

Now let $g\colon Z \to Y$ be a morphism of abelian groups, and take $f\colon X \to Y$ to be the morphism $a \otimes z \mapsto a \cdot g(z)$ of k-vector spaces. Then $(f \circ \eta)(z) = f(1 \otimes z) = g(z)$ for all $z \in Z$, so $f \circ \eta = g$. Suppose f' also satisfies $f' \circ \eta = g$. Then $(f - f') \circ \eta = 0$. Since $\eta(Z)$ generates X as a k-vector space we obtain $f - f' = 0$, so f is unique and (X, η) is a universal morphism.

Since (X, η) is universal the vector space X corresponding to Z is 'uniquely unique', meaning that any other universal morphism (X', η') induces a unique isomorphism $\varphi\colon X \to X'$ such that $\varphi \circ \eta = \eta'$.

Note that η need not be injective. For $k = \mathbb{Q}$ it is only injective when A is torsion-free. Then η can be thought of as a canonical embedding.

2. Similarly, we can consider a forgetful functor F from the category of $\mathbb{Q}$-inner product spaces to the category of $\mathbb{Z}$-inner product spaces. The underlying universal morphism (X, η) is the same as before, and we equip X with the inner product we extend $\mathbb{Q}$-bilinearly from Z. To show that

this inner product is positive definite we use that A is torsion-free, being a $\mathbb{Z}$-inner product space.

Definition 2.2.9. A *Hilbert space* is an $\mathbb{R}$-module $\mathcal{H}$ equipped with a real $\mathbb{R}$-inner product such that $\mathcal{H}$ is complete with respect to the induced metric. The morphisms of Hilbert spaces are the isometric maps.

Theorem 2.2.10 (Theorem 3.2-3 in [29])**.** *Let F be the forgetful functor from the category of Hilbert spaces to the category of $\mathbb{Q}$-inner product spaces. Then every $\mathbb{Q}$-inner product space V has an injective universal morphism to F, and a morphism $f\colon V \to \mathcal{H}$ for some Hilbert space $\mathcal{H}$ is universal precisely when f is injective and the image of f is dense in $\mathcal{H}$.* □

The Hilbert space constructed for V in Theorem 2.2.10 can be obtained as the topological completion of V with respect to the metric induced by the inner product, and the inner product is extended continuously.

Definition 2.2.11. For a set B and $p \in \mathbb{R}_{>0}$ we define the $\mathbb{R}$-vector space

$$\ell^p(B) = \left\{ (x_b)_b \in \mathbb{R}^B \;\middle|\; \begin{array}{l} x_b = 0 \text{ for all but countably many } b \in B \\ \text{and } \sum_{b\in B} |x_b|^p < \infty \end{array} \right\}$$

and $\|x\|_p = (\sum_{b\in B} |x_b|^p)^{1/p}$ for all $x = (x_b)_b \in \ell^p(B)$.

Theorem 2.2.12 (Minkowski's inequality, Theorem 1.2-3 in [29])**.** *For any set B and $p \in \mathbb{R}_{\geq 1}$ the map $\|\cdot\|_p$ is an $\mathbb{R}$-norm on $\ell^p(B)$.* □

Lemma 2.2.13 (Example 3.1-6 in [29])**.** *For any set B the space $\ell^2(B)$ is a Hilbert space with inner product given by $\langle x, y\rangle = \sum_{b\in B} x_b \cdot y_b$ for $x = (x_b)_b, y = (y_b)_b \in \ell^2(B)$, such that $\langle x, x\rangle = \|x\|_2^2$.* □

Lemma 2.2.14. *Let $n \in \mathbb{Z}_{\geq 1}$, $x \in \mathbb{R}^n$ and let $0 < p \leq q$ be real. Then we have*

$$\|x\|_q \leq \|x\|_p \quad \textit{and} \quad n^{-1/p} \cdot \|x\|_p \leq n^{-1/q} \cdot \|x\|_q.$$

Proof. Clearly we may assume $x \neq 0$. For the first inequality, consider $y = x/\|x\|_p$. Then $|y_i| \leq 1$ for all i, from which $|y_i|^q \leq |y_i|^p$ follows. Now

$$\|y\|_q^q = \sum_{i=1}^n |y_i|^q \leq \sum_{i=1}^n |y_i|^p = \|y\|_p^p = 1.$$

Hence $\|x\|_q/\|x\|_p = \|y\|_q \leq 1$, as was to be shown. For the second inequality, note that $x \mapsto x^{q/p}$ is a convex function on $\mathbb{R}_{\geq 0}$. We have by Jensen's inequality (Theorem 7.3 in [7]) that

$$\|x\|_q^q = \sum_{i=1}^n |x_i|^q = \sum_{i=1}^n |x_i^p|^{q/p} \geq n \left(\frac{1}{n} \sum_{i=1}^n |x_i|^p \right)^{q/p} = n^{1-q/p} \|x\|_p^q,$$

so $n^{-1/p} \cdot \|x\|_p \le n^{-1/q} \cdot \|x\|_q$. □

Definition 2.2.15. Let $\mathcal{H}$ be a Hilbert space. A subset $S \subseteq \mathcal{H}$ is called *orthogonal* if $0 \notin S$ and $\langle x, y \rangle = 0$ for all distinct $x, y \in S$. The *orthogonal dimension* of $\mathcal{H}$, written $\operatorname{orth\,dim} \mathcal{H}$, is the cardinality of a maximal orthogonal subset of $\mathcal{H}$.

That the orthogonal dimension is well-defined, i.e. that maximal orthogonal subsets of a given Hilbert space have the same cardinality follows from Proposition 4.14 in [6].

Theorem 2.2.16 (Theorem 5.4 in [6])**.** *Let $\mathcal{H}$ be a Hilbert space and B a set. Then the Hilbert spaces $\mathcal{H}$ and $\ell^2(B)$ are isomorphic if and only if the cardinality of B equals* $\operatorname{orth\,dim} \mathcal{H}$*.* □

In particular, up to isomorphism every Hilbert space is of the form $\ell^2(B)$ for some set B.

2.3 Hilbert lattices

Definition 2.3.1. A *Hilbert lattice* is an abelian group Λ together with a map $q \colon \Lambda \to \mathbb{R}$, which we then call the *square-norm* of Λ, that satisfies:

(*Parallelogram law*) For all $x, y \in \Lambda$ we have

$$q(x+y) + q(x-y) = 2q(x) + 2q(y);$$

(*Positive packing radius*) There exists an $r \in \mathbb{R}_{>0}$ such that $q(x) \ge r$ for all non-zero $x \in \Lambda$.

We write $\mathrm{P}(\Lambda) = \inf\{q(x) \mid x \in \Lambda \setminus \{0\}\}$.

The following lemma gives an equivalent definition of a Hilbert lattice.

Lemma 2.3.2. *A Hilbert lattice Λ with square-norm q is a discrete $\mathbb{Z}$-inner product space with inner product given by*

$$(x, y) \mapsto \frac{1}{2}(q(x+y) - q(x) - q(y)).$$

Conversely, every discrete $\mathbb{Z}$-inner product space M is a Hilbert lattice with square norm given by $x \mapsto \langle x, x \rangle$.

Proof. The first statement is Theorem 2.2.5 with the observation that the positive packing radius implies non-degeneracy and discreteness. The second statement is Lemma 2.2.4 with the observation that discreteness implies a positive packing radius. □

Example 2.3.3. Consider for some $n \in \mathbb{Z}_{\geq 0}$ the vector space $\mathbb{R}^n$ with the standard inner product. If $\Lambda \subseteq \mathbb{R}^n$ is a discrete subgroup, then Λ is a Hilbert lattice when q is given by $x \mapsto \|x\|^2$.

Example 2.3.4. Let B be a set. Then

$$\mathbb{Z}^{(B)} = \{(x_b)_b \in \mathbb{Z}^B \mid x_b = 0 \text{ for all but finitely many } b\}$$

is a Hilbert lattice in $\ell^2(B)$ when q is given by $x \mapsto \|x\|^2$. In fact, any discrete subgroup of $\ell^2(B)$ is a Hilbert lattice.

Example 2.3.5. The infimum defining $\mathrm{P}(\Lambda)$ of a Hilbert lattice Λ need not be attained. Certainly if $\Lambda = 0$ we have that $\mathrm{P}(\Lambda) = \infty$ is not attained. For an example of a non-degenerate Λ consider the following. For a set I and a map $f\colon I \to \mathbb{R}_{\geq 0}$ we write Λ^f for the group $\mathbb{Z}^{(I)}$ together with the map $q((x_i)_i) = \sum_{i \in I} f(i)^2 x_i^2$. Note that $\inf\{q(x) \mid x \in \Lambda^f \setminus \{0\}\} = \inf\{f(i)^2 \mid i \in I\}$, so Λ^f is a Hilbert lattice if and only if $\inf\{f(i) \mid i \in I\} > 0$. We now simply take $f\colon \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ given by $n \mapsto 1 + 1/n$.

Lemma 2.3.6. *Let Λ be a Hilbert lattice with square-norm q. Then any subgroup $\Lambda' \subseteq \Lambda$ is a Hilbert lattice when equipped with the square-norm $q|_{\Lambda'}$.* □

Theorem 2.3.7. *Let F be the forgetful functor from the category of Hilbert spaces to the category of $\mathbb{Z}$-inner product spaces. Then every $\mathbb{Z}$-inner product space L has an injective universal morphism η to F. For every $\mathbb{Z}$-inner product space L, Hilbert space $\mathcal{H}$ and injective morphism $f\colon L \to \mathcal{H}$ we have that f is universal if and only if $\mathbb{Q} \cdot f(L)$ is dense in $\mathcal{H}$, and L is a Hilbert lattice if and only if $f(L)$ is discrete in $\mathcal{H}$.*

It follows from this theorem that the Hilbert lattices are, up to isomorphism, precisely the discrete subgroups of Hilbert spaces. Hence Theorem 2.3.7 allows us to assume without loss of generality that a Hilbert lattice is a discrete subgroup of a Hilbert space.

Proof. The first and second statement are just a combination of Example 2.2.8.2 and Theorem 2.2.10, while the third is trivial when taking the equivalent definition of Lemma 2.3.2. □

Remark 2.3.8. Let Λ be a Hilbert lattice in a Hilbert space $\mathcal{H}$ and suppose that Λ is finitely generated. Then $\mathbb{R}\Lambda$ is a finite dimensional $\mathbb{R}$-inner product space and thus complete. It follows that $\Lambda \to \mathbb{R}\Lambda$ is a universal morphism because $\mathbb{Q}\Lambda$ is dense in $\mathbb{R}\Lambda$. Since $\mathbb{R}\Lambda$ is finite dimensional, Λ is a lattice in the classical sense: a discrete subgroup of a Euclidean vector space.

Lemma 2.3.9. *Let Λ be a Hilbert lattice in a Hilbert space $\mathcal{H}$. Then the natural map $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}$ is injective.*

Proof. To show $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}$ is injective we may assume by Lemma 2.3.6 without loss of generality that Λ is finitely generated, as any element in the kernel is also in $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda'$ for some finitely generated sublattice $\Lambda' \subseteq \Lambda$. Write $V = \mathbb{R}\Lambda \subseteq \mathcal{H}$. We may choose an $\mathbb{R}$-basis for V in Λ, and let Λ' be the group generated by this basis. Then $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda' \to V$ is an isomorphism. As Λ is discrete in V, also Λ/Λ' is discrete in V/Λ'. Now V/Λ' is compact, so the quotient Λ/Λ' is finite. Then $\Lambda \subseteq \frac{1}{n}\Lambda'$, where $n = \#(\Lambda/\Lambda')$. Now the natural map $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}$ is injective because it is the composition of the map $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \to \mathbb{R} \otimes_{\mathbb{Z}} (\frac{1}{n}\Lambda') = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda'$, which is injective since $\mathbb{R}$ is flat over $\mathbb{Z}$, and the map $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda' \to V$, which is injective by construction. □

Proposition 2.3.10. *Let Λ be a Hilbert lattice in a Hilbert space $\mathcal{H}$ and suppose Λ is finitely generated as $\mathbb{Z}$-module. Then Λ has a $\mathbb{Z}$-basis and any $\mathbb{Z}$-basis is $\mathbb{R}$-linearly independent.*

Proof. Since Λ is finitely generated and torsion free, it is clear that Λ is free. By Lemma 2.3.9, any $\mathbb{Z}$-linearly independent subset of Λ is $\mathbb{R}$-linearly independent. □

Proposition 2.3.11. *Suppose Λ is a Hilbert lattice in a Hilbert space $\mathcal{H}$ and let $\Lambda' \subseteq \Lambda$ be a finitely generated subgroup. Let $\pi\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto the orthogonal complement of Λ'. Then for each $0 \leq t < \frac{1}{4}\mathrm{P}(\Lambda)$ there are only finitely many $z \in \pi\Lambda$ such that $q(z) \leq t$, and $\pi\Lambda$ is a Hilbert lattice.*

Proof. Suppose that $\pi\Lambda$ contains infinitely many points z with $q(z) \leq t$, or equivalently there exists some infinite set $S \subseteq \Lambda$ such that $\pi|_S$ is injective and $q(\pi(x)) \leq t$ for all $x \in S$. Consider the map $\tau\colon \mathcal{H} \to \mathbb{R}\Lambda'$, the complementary projection to π. As $(\mathbb{R}\Lambda')/\Lambda'$ is compact, there must exist distinct $x, y \in S$ such that $q(\tau(x) - \tau(y) + w) < \mathrm{P}(\Lambda) - 4t$ for some $w \in \Lambda'$. Then

$$\begin{aligned} 0 < q(x - y + w) &= q(\pi(x - y)) + q(\tau(x - y) + w) \\ &< 2\big(q(\pi(x)) + q(\pi(y))\big) + \mathrm{P}(\Lambda) - 4t \leq \mathrm{P}(\Lambda), \end{aligned}$$

a contradiction. Hence there are only finitely many $z \in \pi\Lambda$ such that $q(z) \leq t$. To verify that $\pi\Lambda$ is a Hilbert lattice it suffices to show that it is discrete in $\mathcal{H}$, which follows from the previous by taking any non-zero value for t. □

Lemma 2.3.12. *Let Λ be a Hilbert lattice which is finitely generated as $\mathbb{Z}$-module and let $S \subseteq \Lambda$ be a set of vectors that forms a basis for $\Lambda \cap (\mathbb{R}S)$. Then there exists a basis $B \supseteq S$ of Λ.*

Proof. Let π be the projection onto the orthogonal complement of S. Then $\pi\Lambda$ is a Hilbert lattice by Proposition 2.3.11 with a basis B_π by Proposition 2.3.10. Now choose for every $b_\pi \in B_\pi$ a lift $b \in \Lambda$ and let T be the set of those elements. It is easy to show that $B = S \cup T$ is a basis of Λ. □

Theorem 2.3.13. *Every countable subgroup of a Hilbert lattice is free.*

We call an abelian group for which all its countable subgroups are free an *almost free* abelian group.

Proof. Let Λ be a Hilbert lattice. By Lemma 2.3.6 suffices to show that if Λ is countable then Λ is free. We may write $\Lambda = \{x_1, x_2, \dots\}$ and let $V_i = \sum_{j=1}^{i} \mathbb{R}x_j$ and $\Lambda_i = V_i \cap \Lambda$. We claim that there exist bases B_i for Λ_i such that $B_i \subseteq B_j$ for all $i \leq j$. Indeed, take $B_0 = \varnothing$ and inductively for Λ_{i+1} note that B_i is a basis for $\Lambda_i = \Lambda_{i+1} \cap V_i$, so that by Lemma 2.3.12 there exists some basis B_{i+1} for Λ_{i+1} containing B_i. Then $B = \bigcup_{i=0}^{\infty} B_i$ is a basis for Λ, so Λ is free. □

Question 2.3.14. We have by Example 2.3.4 and Theorem 2.3.13 two inclusions

$$\begin{aligned}\{\text{free abelian groups}\} &\subseteq \{\text{underlying groups of Hilbert lattices}\}\\ &\subseteq \{\text{almost free abelian groups}\}.\end{aligned}$$

Is one of these inclusions an equality, and if so, which?

Example 2.3.15. There are abelian groups which are almost free but not free. Let X be a countably infinite set and consider the Baer–Specker group $B = \mathbb{Z}^X$. Then by Theorem 21 in [27], we have that B is not free. Since B is a torsion-free $\mathbb{Z}$-module, so is any countable subgroup, which is then free by Theorem 16 in [27], i.e. B is almost free.

Definition 2.3.16. For a Hilbert lattice Λ we define its *rank* as $\operatorname{rk}\Lambda = \dim_{\mathbb{Q}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$. We will say a Hilbert lattice Λ is of *full rank* in an ambient Hilbert space $\mathcal{H}$ if $\mathbb{Q}\Lambda$ is dense in $\mathcal{H}$.

For free Hilbert lattices Λ we have $\Lambda \cong \mathbb{Z}^{(\operatorname{rk}\Lambda)}$ as abelian group. By Theorem 2.3.7 every Hilbert lattice has a uniquely unique Hilbert space in which it is contained and of full rank.

Lemma 2.3.17. *Let $\mathcal{H}$ be a Hilbert space and let $S, T \subseteq \mathcal{H}$ be subsets such that S is infinite, the $\mathbb{Q}$-vector space generated by S is dense in $\mathcal{H}$ and $\inf\{\|x - y\| \mid x, y \in T,\ x \neq y\} > 0$. Then $\#S \geq \#T$.*

Proof. Because S is infinite, the set S and the $\mathbb{Q}$-vector space V generated by S have the same cardinality. Let $\rho = \inf\{\|x - y\| \mid x, y \in T, x \neq y\}$. Since V is dense in $\mathcal{H}$ we may for each $x \in T$ choose $f(x) \in V$ such that $\|x - f(x)\| < \rho/2$. If $f(x) = f(y)$, then $\|x - y\| = \|(x - f(x)) - (y - f(y))\| \leq \|x - f(x)\| + \|y - f(y)\| < \rho$, so $x = y$. Hence f is injective and we have $\#S = \#V \geq \#T$. □

The following proposition is generalized from proofs by O. Berrevoets and B. Kadets.

Proposition 2.3.18. *If Λ is a Hilbert lattice in a Hilbert space $\mathcal{H}$, then* $\operatorname{rk}\Lambda \leq \operatorname{orth}\dim\mathcal{H}$ *with equality if Λ is of full rank in $\mathcal{H}$.*

Proof. First suppose $\operatorname{rk}\Lambda$ is finite. It follows from Lemma 2.3.9 that $\operatorname{rk}\Lambda = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} \Lambda) = \dim_{\mathbb{R}}(\mathbb{R}\Lambda) \leq \dim_{\mathbb{R}} \mathcal{H}$. If Λ is of full rank, then $\mathbb{R}\Lambda$ is dense in $\mathcal{H}$, but $\mathbb{R}\Lambda$ is complete as it is finite-dimensional, so $\mathbb{R}\Lambda = \mathcal{H}$ and $\operatorname{rk}\Lambda = \dim_{\mathbb{R}} \mathcal{H}$. Lastly, it follows from Theorem 2.2.16 that $\dim_{\mathbb{R}} \mathcal{H} = \operatorname{orth}\dim\mathcal{H}$ when $\dim_{\mathbb{R}} \mathcal{H}$ is finite.

Now suppose $\operatorname{rk}\Lambda$ is infinite and thus $\#\Lambda = \operatorname{rk}\Lambda$. By Theorem 2.2.16 we may assume without loss of generality that $\mathcal{H} = \ell^2(B)$ for some set B of cardinality $\operatorname{orth}\dim\mathcal{H}$, which must be infinite. Observe that the $\mathbb{Q}$-vector space generated by B is dense in $\mathcal{H}$. We may apply Lemma 2.3.17 by discreteness of Λ to obtain $\operatorname{orth}\dim\mathcal{H} = \#B \geq \#\Lambda = \operatorname{rk}\Lambda$, as was to be shown. If $\mathbb{Q}\Lambda$ is dense in $\mathcal{H}$, then we may apply Lemma 2.3.17 since $\|b - c\|^2 = \|b\|^2 + \|c\|^2 = 2$ for all distinct $b, c \in B$ to conclude that $\operatorname{rk}\Lambda = \#\Lambda \geq \#B = \operatorname{orth}\dim\mathcal{H}$, and thus we have equality. □

2.4 Decompositions

Definition 2.4.1. Let Λ be a Hilbert lattice. A *decomposition* of an element $z \in \Lambda$ is a pair $(x, y) \in \Lambda^2$ such that $z = x + y$ and $\langle x, y \rangle \geq 0$. A decomposition (x, y) of $z \in \Lambda$ is *trivial* if $x = 0$ or $y = 0$. We say $z \in \Lambda$ is *indecomposable* if it has exactly two decompositions, i.e. $z \neq 0$ and the only decompositions of z are trivial. Write $\operatorname{dec}(z)$ for the set of decompositions of $z \in \Lambda$ and $\operatorname{indec}(\Lambda)$ for the set of indecomposable elements of Λ.

Indecomposable elements are in the computer science literature often called Voronoi-relevant vectors, for example in [23]. This name is clearly inspired by Theorem 2.6.11.

Example 2.4.2. Let $f\colon I \to \mathbb{R}_{\geq 0}$ be such that Λ^f as in Example 2.3.5 is a Hilbert lattice. We will compute the indecomposables of Λ^f. Let $x = (x_i)_i \in \operatorname{indec}(\Lambda^f)$ and write e_i for the i-th standard basis vector. Note that

x must be *primitive*, i.e. not be of the form ny for any $y \in \Lambda^f$ and $n \in \mathbb{Z}_{>1}$, because otherwise $\langle y, (n-1)y\rangle = (n-1)\langle y, y\rangle > 0$ shows $(y, (n-1)y)$ is a non-trivial decomposition. If x_i and x_j are non-zero for distinct $i, j \in I$, then $q(x - e_i x_i) + q(e_i x_i) = q(x)$ and we have a non-trivial decomposition of x. Hence $x = \pm e_i$ for some $i \in I$. Note that e_i is indeed indecomposable for all $i \in I$: Any decomposition $(x, y) \in \mathrm{dec}(e_i)$ must have $|x_i| + |y_i| = 1$ and $x_j = y_j = 0$ for $i \neq j$, so $x = 0$ or $y = 0$. As $z \mapsto -z$ is an isometry of Λ^f, we have that $-e_i$ is indecomposable as well. Hence $\mathrm{indec}(\Lambda^f) = \{\pm e_i \mid i \in I\}$.

Lemma 2.4.3. *Let Λ be a Hilbert lattice and let $x, y, z \in \Lambda$. Then the following are equivalent:*
(i) *The pair (x, y) is a decomposition of z.*
(ii) *We have $x + y = z$ and $q(x) + q(y) \leq q(z)$.*
(iii) *We have $x + y = z$ and $q(x - z/2) \leq q(z/2)$.*
(iv) *We have $x + y = z$ and $q(z - 2y) \leq q(z)$.*

For a visual aid to this lemma see Figure 2.1.

Proof. (i $\Leftrightarrow$ ii) By bilinearity we have

$$q(z) = \langle x + y, x + y\rangle = q(x) + q(y) + 2\langle x, y\rangle.$$

(ii $\Leftrightarrow$ iii) By the parallelogram law we have

$$q(x) + q(y) = 2q\Big(\frac{x+y}{2}\Big) + 2q\Big(\frac{x-y}{2}\Big) = 2q\Big(\frac{z}{2}\Big) + 2q\Big(x - \frac{z}{2}\Big),$$

so $q(x) + q(y) - q(z) = 2 \cdot [q(x - z/2) - q(z/2)]$. The claim then follows trivially.

(iii $\Leftrightarrow$ iv) Note that $z - 2y = x - y = 2(x/2 - y/2) = 2(x - z/2)$. By the parallelogram law we have $q(2w) = 4q(w)$ for all $w \in \mathbb{Q}\Lambda$, from which this equivalence trivially follows. □

By Lemma 2.4.3, finding decompositions of $z \in \Lambda$ amounts to finding $x \in \Lambda$ sufficiently close to $z/2$.

Lemma 2.4.4. *Let $z \in \Lambda$ such that $0 < q(z) \leq 2\mathrm{P}(\Lambda)$. Suppose that the latter inequality is strict or $\mathrm{P}(\Lambda)$ is not attained by any vector in Λ. Then $z \in \mathrm{indec}(\Lambda)$.*

Proof. If $(x, y) \in \mathrm{dec}(z)$ is non-trivial, then by Lemma 2.4.3 we have $2\mathrm{P}(\Lambda) \geq q(z) \geq q(x) + q(y) \geq \mathrm{P}(\Lambda) + \mathrm{P}(\Lambda)$ with either the first or last inequality strict, which is a contradiction. Since $z \neq 0$ it follows that z is indecomposable. □

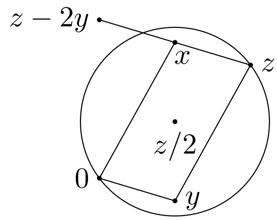

Figure 2.1: A decomposition $z = x + y$

Proposition 2.4.5. *If Λ is a non-zero Hilbert lattice, then every $z \in \Lambda$ can be written as a sum of at most $q(z)/\mathrm{P}(\Lambda)$ indecomposables $z_1, \ldots, z_n \in \Lambda$ such that $\sum_i q(z_i) \le q(z)$.*

Proof. By scaling q we may assume without loss of generality that $\mathrm{P}(\Lambda) = 1$. We apply induction to $\lfloor q(z) \rfloor$. When this equals 0 we have $q(z) < \mathrm{P}(\Lambda)$, so $z = 0$, which we can write as a sum of zero indecomposables. Now suppose $\lfloor q(z) \rfloor \ge 1$ and thus $z \ne 0$. If z is indecomposable then indeed it is the sum of $1 \le \lfloor q(z) \rfloor$ indecomposable, so suppose there is a non-trivial $(x, y) \in \mathrm{dec}(z)$. Then $q(x) + q(y) \le q(z)$ by Lemma 2.4.3 and since $y \ne 0$ also $q(y) \ge \mathrm{P}(\Lambda) = 1$. Hence $\lfloor q(x) \rfloor \le \lfloor q(z) - q(y) \rfloor < \lfloor q(z) \rfloor$, so by the induction hypothesis we may write $x = \sum_i x_i$ with $x_1, \ldots, x_a \in \mathrm{indec}(\Lambda)$ and $a \le q(x)$ such that $\sum_i q(x_i) \le q(x)$. By symmetry we may similarly write y as a sum of at most $q(y)$ indecomposables $y_1, \ldots, y_b$. Hence we can write $z = \sum_i x_i + \sum_i y_i$ as a sum of $a + b \le q(x) + q(y) \le q(z)$ indecomposables such that $\sum_i q(x_i) + \sum_i q(y_i) \le q(x) + q(y) \le q(z)$. The proposition follows by induction. □

Lemma 2.4.6. *Suppose Λ is a Hilbert lattice and $z \in \Lambda$ is the sum of some non-zero $z_1, \ldots, z_n \in \Lambda$ and $n \in \mathbb{Z}_{\ge 2}$. If $\sum_i q(z_i) \le q(z)$, then z has a non-trivial decomposition.*

Proof. We have

$$\sum_{i=1}^{n} \langle z_i, z - z_i \rangle = \sum_{i=1}^{n} \langle z_i, z \rangle - \sum_{i=1}^{n} \langle z_i, z_i \rangle = q(z) - \sum_{i=1}^{n} q(z_i) \ge 0,$$

so $\langle z_i, z - z_i \rangle \ge 0$ for some i. As neither z_i nor $z - z_i$ are 0, we conclude that $(z_i, z - z_i)$ is a non-trivial decomposition of z. □

Proposition 2.4.7. *Let Λ be a Hilbert lattice. The group $\{\pm 1\}$ acts on* $\mathrm{indec}(\Lambda)$ *by multiplication, and the natural map* $\mathrm{indec}(\Lambda)/\{\pm 1\} \to \Lambda/2\Lambda$ *is injective.*

Proof. Note that $\{\pm 1\}$ acts on Λ and thus also on $\operatorname{indec}(\Lambda)$. By (i $\Leftrightarrow$ iv) of Lemma 2.4.3 we have

$$\operatorname{dec}(z) = \left\{ \left(\frac{z-a}{2}, \frac{z+a}{2} \right) \,\middle|\, a \in z + 2\Lambda,\ q(a) \le q(z) \right\}.$$

Let $z \in \operatorname{indec}(\Lambda)$. Then $\{(0,z),(z,0)\} = \operatorname{dec}(z)$, so the only $a \in z + 2\Lambda$ such that $q(a) \le q(z)$ are $a = z$ and $a = -z$. Thus z is a q-minimal element of its coset in $\Lambda/2\Lambda$ and this minimal element is unique up to sign. Consequently, the map $\operatorname{indec}(\Lambda)/\{\pm 1\} \to \Lambda/2\Lambda$ is injective. □

Corollary 2.4.8. *Let Λ be a Hilbert lattice. Then* $\operatorname{indec}(\Lambda)$ *is finite if and only if* $\operatorname{rk}\Lambda < \infty$.

Proof. Recall that $\operatorname{indec}(\Lambda)$ generates Λ by Proposition 2.4.5. Hence if $\operatorname{indec}(\Lambda)$ is finite, then $\operatorname{rk}\Lambda < \infty$. If $\operatorname{rk}\Lambda < \infty$, then Proposition 2.4.7 implies $\#\operatorname{indec}(\Lambda) \le 2 \cdot \#(\Lambda/2\Lambda) = 2^{1+\operatorname{rk}\Lambda} < \infty$. □

The zero coset of $\Lambda/2\Lambda$ is never in the image of the map of Proposition 2.4.7, as any non-zero element of the form $2x$ with $x \in \Lambda$ has a non-trivial decomposition (x,x). A non-zero coset C of $\Lambda/2\Lambda$ can fail to be in the image for two reasons: Either C has no minimal element or a minimal element exists but is not unique up to sign. In the latter case, with $z \in C$ minimal, there exists a $(x,y) \in \operatorname{dec}(z)$ with $x, y \ne 0$ and $q(x) + q(y) = q(z)$ and thus $\langle x, y \rangle = 0$, i.e. z has an orthogonal decomposition. This is exhibited, for example, by the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ with the standard inner product and $z = (1,1)$, where $(-1,1) \in z + 2\mathbb{Z}^2$ gives rise to the orthogonal decomposition $(1,0) + (0,1) = z$. In the former case, $\operatorname{rk}\Lambda$ has to be infinite: If $\operatorname{rk}\Lambda$ is finite, then for any $x \in \Lambda$ there are only finitely many $y \in \Lambda$ with $q(y) \le q(x)$, so q assumes a minimum on any non-empty subset of Λ, in particular C. An example is the following.

Example 2.4.9. We will exhibit a Hilbert lattice Λ and a coset of 2Λ on which q does not attain a minimum. Let $f \colon I \to \mathbb{R}_{\ge 0}$ be such that Λ^f as in Example 2.3.5 is a Hilbert lattice. We define the Λ_2^f to be the sublattice

$$\Lambda_2^f = \ker(\Lambda^f \xrightarrow{\Sigma} (\mathbb{Z}/2\mathbb{Z})) = \left\{ (x_i)_i \in \mathbb{Z}^{(I)} \,\middle|\, \sum_{i \in I} x_i \equiv 0 \bmod 2 \right\}.$$

Consider $f \colon \mathbb{Z}_{\ge 0} \to \mathbb{R}_{\ge 0}$ strictly decreasing, write $f(\infty)$ for its limit, assume $f(\infty) > 0$, and let $\Lambda = \Lambda_2^f$. Let $z = 2e_k \in \Lambda$ for any $k \in \mathbb{Z}_{\ge 0}$. Then for all $y = (y_i)_i \in \Lambda$ we have

$$q(z - 2y) = 4\Big((1 - y_k)^2 f(k)^2 + \sum_{i \ne k} y_i^2 f(i)^2\Big) > 4 f(\infty)^2,$$

since either $y_i^2 \geq 1$ for some $i \neq k$ or y_k is even. However, we also have for $y = e_k + e_i$ that $q(z-2y) = q(2e_i) = 4f(i)^2 \to 4f(\infty)^2$ as $i \to \infty$. Hence $\{q(z-2y) \,|\, y \in \Lambda\}$ does not contain a minimum.

If we eliminate the zero coset in the proof of Corollary 2.4.8 the upper bound on the number of indecomposables becomes $2(2^{\mathrm{rk}(\Lambda)} - 1)$ and this bound is tight. If one took the effort to define a sensible probability measure on the space of all Hilbert lattices of given finite rank, then this upper bound will in fact be an equality with probability 1.

2.5 Orthogonal decompositions

The main motivation for considering indecomposable elements is found in the study of decompositions of lattices. In this section we generalize a result of Eichler [11] on the existence of a universal decomposition in lattices to Hilbert lattices.

Recall the definition of a graph and of a decomposition of a module from the Preliminaries. We say a decomposition $\mathcal{M}$ of a module M is *universal* if it is an initial object in this category, i.e. for all decompositions $\mathcal{N}$ of M there exists a unique morphism $\mathcal{M} \to \mathcal{N}$.

Definition 2.5.1. Let Λ be a Hilbert lattice. For a set I, an *I-indexed orthogonal decomposition* of Λ is an I-indexed decomposition $\{\Lambda_i\}_{i\in I}$ of Λ as $\mathbb{Z}$-module such that $\langle \Lambda_i, \Lambda_j \rangle = 0$ for all $i \neq j$, which we write as $\bigoplus_{i\in I} \Lambda_i = \Lambda$. An *orthogonal decomposition* of Λ is an I-indexed orthogonal decomposition for any set I. We say Λ is *orthogonally indecomposable* if $\Lambda \neq 0$ and for all $\Lambda_1, \Lambda_2 \subseteq \Lambda$ such that $\Lambda_1 \oplus \Lambda_2 = \Lambda$ we have $\Lambda_1 = 0$ or $\Lambda_2 = 0$. We interpret the class of orthogonal decompositions of Λ as a full subcategory of the category of decompositions of Λ.

Lemma 2.5.2. *Let $G = (V, E)$ be a graph. Then the connected components of G are pairwise disjoint, and if for $S \subseteq V$ there exist no $\{u, v\} \in E$ such that $u \in S$ and $v \notin S$, then S is a union of connected components.*

Proof. Let $\mathcal{C}$ be the set of $S \subseteq V$ such that there are no $\{u, v\} \in E$ such that $u \in S$ and $v \notin S$, so that the connected components of G become the minimal non-empty elements of $\mathcal{C}$ with respect to inclusion. Note that $\mathcal{C}$ is closed under taking complements, arbitrary unions and arbitrary intersections, i.e. $\mathcal{C}$ is a clopen topology on V. Suppose $S, T \in \mathcal{C}$ are connected components that intersection non-trivially, then $S \cap T \in \mathcal{C}$ is non-empty, so by minimality $S = T$. Hence the connected components are pairwise disjoint.

Now let $S \in \mathcal{C}$ and for all $s \in S$ let $A_s = \{T \in \mathcal{C} \mid s \in T\}$ and $C_s = \bigcap_{T \in A_s} T$, which is an element of A_s. For all $T \in \mathcal{C}$ we either have $T \in A_s$ or $V \setminus T \in A_s$, and thus either $C_s \subseteq T$ or $C_s \cap T = \varnothing$. It follows that no non-empty $T \in \mathcal{C}$ is strictly contained in C_s, i.e. C_s is a connected component of G. As $S \in A_s$ we have $s \in C_s \subseteq S$ and thus $S = \bigcup_{s \in S} C_s$ is a union of connected components. □

The following is a generalization of a theorem due to Eichler [11], although the proof more closely resembles that of Theorem 6.4 in [39].

Theorem 2.5.3. *Let Λ be a Hilbert lattice and $V \subseteq \operatorname{indec}(\Lambda)$ such that V generates Λ as a group. Let G be the graph with vertex set V and with an edge between x and y if and only if $\langle x, y \rangle \neq 0$. Let U be the set of connected components of G and for $u \in U$ let $\Upsilon_u \subseteq \Lambda$ be the subgroup generated by the elements in u. Then $\{\Upsilon_u\}_{u \in U}$ is a universal orthogonal decomposition of Λ.*

As corollary to this theorem we have that Λ is orthogonally indecomposable if and only if G is connected.

Proof. We have $V \subseteq \bigcup_{u \in U} \Upsilon_u$ by Lemma 2.5.2, so $\sum_{u \in U} \Upsilon_u = \Lambda$ by assumption on V. For $u, v \in U$ distinct we have $\langle u, v \rangle = \{0\}$ by definition of G, so $\langle \Upsilon_u, \Upsilon_v \rangle = \{0\}$. We conclude that $\Lambda = \bigoplus_{u \in U} \Upsilon_u$ is an orthogonal decomposition.

To show it is universal, let $\{\Lambda_i\}_{i \in I}$ be a family of sublattices of Λ such that $\bigoplus_{i \in I} \Lambda_i = \Lambda$. Let $x \in \operatorname{indec}(\Lambda)$ and write $x = \sum_{i \in I} \lambda_i$ with $\lambda_i \in \Lambda_i$ for all $i \in I$. If $j \in I$ is such that $\lambda_j \neq 0$, then $\langle \lambda_j, \sum_{i \neq j} \lambda_i \rangle = 0$ and thus $\lambda_j = x$, because otherwise we obtain a non-trivial decomposition of x. Therefore every indecomposable of Λ is in precisely one of the Λ_i. We conclude that the $S_i = \Lambda_i \cap V$ for $i \in I$ are pairwise disjoint and have V as their union. Then by Lemma 2.5.2 every connected component $u \in U$ is contained in precisely one of the S_i, say in $S_{f(u)}$. By definition of the map $f \colon U \to I$ and the Υ_u we have $\bigoplus_{u \in f^{-1}\{i\}} \Upsilon_u \subseteq \Lambda_i$ for all i, and since both the Υ_u and the Λ_i sum to Λ we must have equality for all i. It follows trivially from the construction that f is the unique map $\{\Upsilon_u\}_{u \in U} \to \{\Lambda_i\}_{i \in I}$, and we conclude that $\{\Upsilon_u\}_{u \in U}$ is a universal orthogonal decomposition of Λ. □

We will use this theorem in Section 4.3 to generalize some theorems from [34].

Theorem 2.5.4. *Every Hilbert lattice has a universal orthogonal decomposition.*

Proof. Take $V = \mathrm{indec}(\Lambda)$ in Theorem 2.5.3 and note that it satisfies the conditions to this theorem by Proposition 2.4.5. $\square$

2.6 Voronoi cells

We will generalize the Voronoi cell as defined for classical lattices to Hilbert lattices and extend some known definitions and properties. Some of these definitions relate to the ambient Hilbert space of the Hilbert lattice, which exists and is uniquely unique by Theorem 2.3.7 when we require the $\mathbb{Q}$-vector space generated by the lattice to lie dense in the Hilbert space. In this section we will write $\mathcal{H}_\Lambda$ for this ambient Hilbert space of a Hilbert lattice Λ.

Definition 2.6.1. Let Λ be a Hilbert lattice. The *packing radius* of Λ is

$$\rho(\Lambda) = \inf\{\tfrac{1}{2}\|x-y\| \mid x, y \in \Lambda,\, x \neq y\} = \tfrac{1}{2}\sqrt{\mathrm{P}(\Lambda)},$$

the *covering radius* of Λ is

$$r(\Lambda) = \inf\{b \in \mathbb{R}_{>0} \mid (\forall z \in \mathcal{H}_\Lambda)\, (\exists\, x \in \Lambda)\, \|z - x\| \leq b\}$$

and the *Voronoi cell* of Λ in $\mathcal{H}$ is the set

$$\mathrm{Vor}(\Lambda) = \{z \in \mathcal{H}_\Lambda \mid (\forall x \in \Lambda \setminus \{0\})\ \|z\| < \|z - x\|\}.$$

We call $\rho(\Lambda)$ the packing radius because it is the radius of the largest open sphere $B \subseteq \mathcal{H}_\Lambda$ such that the spheres $x + B$ for $x \in \Lambda$ are pairwise disjoint. Similarly $r(\Lambda)$ is the radius of the smallest closed sphere $B \subseteq \mathcal{H}_\Lambda$ for which $\bigcup_{x\in\Lambda}(x + B) = \mathcal{H}_\Lambda$. Note that $r(\Lambda) = 0$ only for $\Lambda = 0$ by discreteness.

Example 2.6.2. The covering radius of a Hilbert lattice need not be finite. Take Λ^f as in Example 2.3.5 but with $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ diverging to infinity. The lattice point closest to $\frac{1}{3}e_i$ is 0 for all $i \in \mathbb{Z}_{\geq 0}$, so it has distance $\frac{1}{3}f(i)$ to the lattice. Hence $r(\Lambda^f) \geq \sup\{\frac{1}{3}f(i) \mid i \in \mathbb{Z}_{\geq 0}\} = \infty$.

Example 2.6.3. The Voronoi cell does not need to be an open set. Consider the lattice $\Lambda = \Lambda_2^f$ as in Example 2.4.9 with $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ strictly decreasing. Let $i \in \mathbb{Z}_{\geq 0}$ and $\mathrm{A} = (1 + f(\infty)^2 f(i)^{-2})/2$ and write e_i for the i-th standard basis vector. We claim that $\alpha e_i \in \mathrm{Vor}(\Lambda)$ for $\alpha \in \mathbb{R}$ precisely when $|\alpha| \leq \mathrm{A}$, which proves the Voronoi cell is not open.

Let $x = \sum_j x_j e_j \in \Lambda$ such that $x_i \neq 0$. Then $|x_i| = 1$ or $q(x)/|x_i| \geq |x_i| f(i)^2 > f(i)^2 + f(i+1)^2$. It follows that

$$\inf\left\{\frac{q(x)}{2|x_i|f(i)^2} \,\middle|\, x \in \Lambda,\, x_i \neq 0\right\} = \frac{f(i)^2 + f(\infty)^2}{2f(i)^2} = \mathrm{A}$$

and that the infimum is not attained. By definition $\alpha e_i \in \mathrm{Vor}(\Lambda)$ if and only if $g(x) := q(\alpha e_i - x) - q(\alpha e_i) > 0$ for all $x \in \Lambda \setminus \{0\}$. Note that $g(x) = q(x) - 2\alpha x_i f(i)^2$.

Suppose $|\alpha| \leq \mathrm{A}$ and let $x \in \Lambda \setminus \{0\}$. If $\alpha x_i \leq 0$, then $g(x) \geq q(x) > 0$, so suppose $\alpha x_i > 0$. Then

$$\frac{g(x)}{2|x_i|f(i)^2} = \frac{q(x)}{2|x_i|f(i)^2} - |\alpha| > \mathrm{A} - |\alpha| \geq 0,$$

so $g(x) > 0$. We conclude that $\alpha e_i \in \mathrm{Vor}(\Lambda)$. Conversely, if $|\alpha| > \mathrm{A}$, then

$$\inf\left\{ \frac{g(x)}{2|x_i|f(i)^2} \,\middle|\, x \in \Lambda,\, x_i \neq 0 \right\} = \mathrm{A} - |\alpha| < 0,$$

hence there exists some $x \in \Lambda$ such that $g(x) < 0$ and thus $\alpha e_i \notin \mathrm{Vor}(\Lambda)$.

Definition 2.6.4. Let $\mathcal{H}$ be a Hilbert space and let $S \subseteq \mathcal{H}$ be a subset. We say S is *symmetric* if for all $x \in S$ also $-x \in S$. We say S is *convex* if for all $x, y \in S$ and $t \in [0,1]$ also $(1-t)x + ty \in S$.

Lemma 2.6.5. *Let $\mathcal{H}$ be a Hilbert space. For $X \subseteq \mathcal{H}$ write $\overline{X}$ for the topological closure of X. Then*

1. *The intersection $\bigcap_i S_i$ of convex sets $(S_i)_{i\in I}$ in $\mathcal{H}$ is convex;*
2. *The topological closure $\overline{S}$ of a convex set S in $\mathcal{H}$ is convex;*
3. *For all S in $\mathcal{H}$ open convex, $x \in S$, $y \in \overline{S}$ and $t \in [0,1)$ we have $(1-t)x + ty \in S$;*
4. *For convex open sets $(S_i)_{i\in I}$ in $\mathcal{H}$ with non-empty intersection we have $\overline{\bigcap_i S_i} = \bigcap_i \overline{S_i}$.*

Proof. 1. Trivial.

2. Let $x, y \in \overline{S}$ and let $(x_n)_n$ and $(y_n)_n$ be sequences in S with limit x respectively y. For all $t \in [0,1]$ we have $(1-t)x_n + ty_n \in S$, and since addition and scalar multiplication are continuous also have $(1-t)x + ty = \lim_{n\to\infty}[(1-t)x_n + ty_n] \in \overline{S}$.

3. By translating S we may assume without loss of generality that $(1-t)x + ty = 0$. Since $x \in S$ and S is open there exists some $r_x > 0$ such that the open ball B_x of radius r_x around x is contained in S. For $r_y > 0$ sufficiently small (in fact $r_y = r_x \cdot (1-t)/t$ suffices, see Figure 2.2) it holds that for any z in the open ball B_y of radius r_y around y the line through 0 and z intersects B_x. Taking $z \in B_y \cap S$, which exists because y is in the closure of S, there exists some $w \in B_x \subseteq S$ such that 0 lies on the line segment between w and z. By convexity $0 \in S$ follows, as was to be shown.

4. Since $\bigcap_i \overline{S_i}$ is closed and contains $\bigcap_i S_i$, clearly $\overline{\bigcap_i S_i} \subseteq \bigcap_i \overline{S_i}$. By 1 the set $\bigcap_i S_i$ is convex and by assumption it contains some x. For $t \in [0,1)$

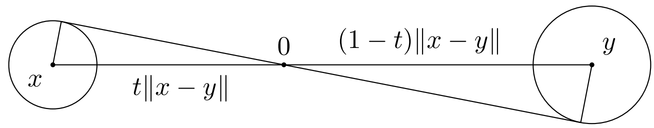

Figure 2.2: Computation of r_y from r_x via similar triangles.

and $y \in \bigcap_i \overline{S_i}$ we have $z_t = (1-t)x + ty \in S_i$ by 3. Thus $z_t \in \bigcap_i S_i$ for all $t \in [0,1)$, so $y = \lim_{t\to 1} z_t \in \overline{\bigcap_i S_i}$, proving the reverse inclusion. □

Lemma 2.6.6. *Let $\mathcal{H}$ be a Hilbert space. Then for $x, y \in \mathcal{H}$ we have $\|y\| \leq \|y-x\|$ if and only if $2\langle x,y\rangle \leq \langle x,x\rangle$, and similarly with $\leq$ replaced by $<$.*

Proof. We have $\|y-x\|^2 - \|y\|^2 = \langle x,x\rangle - 2\langle x,y\rangle$, from which the lemma trivially follows. □

Proposition 2.6.7. *For all Hilbert lattices Λ the set* $\mathrm{Vor}(\Lambda)$ *is symmetric, convex and has topological closure*

$$\overline{\mathrm{Vor}}(\Lambda) := \{z \in \mathcal{H}_\Lambda \,|\, (\forall x \in \Lambda)\ \|z\| \leq \|z-x\|\}.$$

Proof. It follows readily from the definition that $\mathrm{Vor}(\Lambda)$ is symmetric. Now for $x \in \Lambda$ consider $H_x = \{z \in \mathcal{H}_\Lambda \,|\, 2\langle x,z\rangle < \langle x,x\rangle\}$. It is easy to show for all $x \in \Lambda$ that H_x is convex: For $a, b \in H_x$ and $t \in [0,1]$ we have

$$2\langle (1-t)a + tb, x\rangle = (1-t)2\langle a,x\rangle + t2\langle b,x\rangle < (1-t)\langle x,x\rangle + t\langle x,x\rangle = \langle x,x\rangle,$$

so $(1-t)a + tb \in H_x$. As $\mathrm{Vor}(\Lambda)$ is the intersection of all H_x with $x \in \Lambda$ non-zero by Lemma 2.6.6, it follows from Lemma 2.6.5.1 that $\mathrm{Vor}(\Lambda)$ is convex. The H_x are all open, and for x non-zero we have $0 \in H_x$. Hence the topological closure of $\mathrm{Vor}(\Lambda)$ equals $\{z \in \mathcal{H}_\Lambda \,|\, (\forall x \in \Lambda \setminus \{0\})\ \|z\| \leq \|z-x\|\}$ by Lemma 2.6.5.4, from which the proposition follows. □

Example 2.6.8. We do not have in general that $\mathcal{H}_\Lambda = \Lambda + \overline{\mathrm{Vor}}(\Lambda)$ for all Hilbert lattices Λ, as in the finite-dimensional case. Note that $z \in \mathcal{H}_\Lambda$ is in $\Lambda + \overline{\mathrm{Vor}}(\Lambda)$ if and only if the infimum $\inf\{\|z-x\| \,|\, x \in \Lambda\}$ is attained for some $x \in \Lambda$. Consider Example 2.4.9, where we exhibit a lattice Λ and a coset $z + 2\Lambda$ of $\Lambda/2\Lambda$ where $\inf\{q(z+2x) \,|\, x \in \Lambda\}$ is not attained. Equivalently, $\inf\{\|\frac{1}{2}z - x\| \,|\, x \in \Lambda\}$ is not attained, so $\frac{1}{2}z \notin \Lambda + \overline{\mathrm{Vor}}(\Lambda)$.

Theorem 2.6.9. *Let Λ be a Hilbert lattice and consider the natural map $\mathcal{H}_\Lambda \to \mathcal{H}_\Lambda/\Lambda$. Its restriction to* $\mathrm{Vor}(\Lambda)$ *is injective and for all $\varepsilon > 0$ its restriction to* $(1+\varepsilon)\,\mathrm{Vor}(\Lambda)$ *is surjective.*

Proof. Suppose $x, y \in \mathrm{Vor}(\Lambda)$ are distinct such that $x - y \in \Lambda$. Then

$$\|x\| < \|x - (x-y)\| = \|y\| < \|y - (y-x)\| = \|x\|,$$

which is a contradiction. Hence the map $\mathrm{Vor}(\Lambda) \to \mathcal{H}_\Lambda/\Lambda$ is injective.

Assume without loss of generality that $\mathrm{P}(\Lambda) = 2$. Let $\varepsilon > 0$ and $z \in \mathcal{H}_\Lambda$. Choose $y \in \Lambda$ such that $q(z-y) \leq \varepsilon + \inf\{q(z-w) \mid w \in \Lambda\}$. Suppose $x \in \Lambda \setminus \{0\}$. Then

$$\begin{aligned}(1+\varepsilon)\langle x, x\rangle - 2\langle x, z-y\rangle &= \varepsilon q(x) + (q(z-y-x) - q(z-y)) \\ &\geq \varepsilon q(x) - \varepsilon \geq \varepsilon(\mathrm{P}(\Lambda) - 1) = \varepsilon > 0.\end{aligned}$$

It follows that $2\langle x, (z-y)/(1+\varepsilon)\rangle < \langle x, x\rangle$ for all $x \in \Lambda \setminus \{0\}$, so $(z-y)/(1+\varepsilon) \in \mathrm{Vor}(\Lambda)$ by Lemma 2.6.6. Thus $z \in \Lambda + (1+\varepsilon)\mathrm{Vor}(\Lambda)$, and the map $(1+\varepsilon)\mathrm{Vor}(\Lambda) \to \mathcal{H}_\Lambda/\Lambda$ is surjective. □

Proposition 2.6.10. *For all Hilbert lattices* Λ *the set* $\mathrm{Vor}(\Lambda)$ *contains the open sphere of radius* $\rho(\Lambda)$ *around* $0 \in \Lambda$ *and* $\overline{\mathrm{Vor}}(\Lambda)$ *is contained in the closed sphere of radius* $r(\Lambda)$ *around* 0*.*

Proof. Let $z \in \mathcal{H}_\Lambda$ be such that $\|z\| < \rho(\Lambda)$ and let $x \in \Lambda \setminus \{0\}$. By Cauchy–Schwarz we have $\langle x, z\rangle \leq \|x\| \cdot \|z\| < \|x\| \cdot \frac{1}{2}\|x\| = \frac{1}{2}\langle x, x\rangle$, so $z \in \mathrm{Vor}(\Lambda)$ by Lemma 2.6.6.

Let $z \in \overline{\mathrm{Vor}}(\Lambda)$. For each $r > r(\Lambda)$ there exists $x \in \Lambda$ such that $\|z - x\| \leq r$ by definition of $r(\Lambda)$. Then by Proposition 2.6.7 we have $\|z\| \leq \|z-x\| \leq r$. Taking the limit of r down to $r(\Lambda)$ proves the second inclusion. □

Theorem 2.6.11. *Let* Λ *be a Hilbert lattice. Then there is a unique subset* $S \subseteq \Lambda \setminus \{0\}$ *that is minimal with respect to inclusion such that* $\mathrm{Vor}(\Lambda) = \{z \in \mathcal{H}_\Lambda \mid (\forall\, x \in S)\ \|z\| < \|z - x\|\}$*. This subset is equal to* $\mathrm{indec}(\Lambda)$*.*

Proof. For $S \subseteq \Lambda$ write $V(S) = \{z \in \mathcal{H}_\Lambda \mid (\forall\, x \in S)\ \|z\| < \|z-x\|\}$.

First suppose $V(S) = \mathrm{Vor}(\Lambda)$ for some $S \subseteq \Lambda \setminus \{0\}$. Let $z \in \mathrm{indec}(\Lambda)$ and note that $\frac{1}{2}z \notin \mathrm{Vor}(\Lambda)$ since $\|\frac{1}{2}z\| \geq \|\frac{1}{2}z - z\|$. As $V(S) = \mathrm{Vor}(\Lambda)$ there must be some $x \in S$ such that $\|\frac{1}{2}z\| \geq \|\frac{1}{2}z - x\|$. Hence $(x, z-x)$ is a decomposition of z by Lemma 2.4.3, so $z = x$ since z is indecomposable and $x \neq 0$. We conclude that $z \in S$ and $\mathrm{indec}(\Lambda) \subseteq S$.

It remains to show that $V(\mathrm{indec}(\Lambda)) = \mathrm{Vor}(\Lambda)$. We clearly have that $\mathrm{Vor}(\Lambda) \subseteq V(\mathrm{indec}(\Lambda))$. Suppose $z \in V(\mathrm{indec}(\Lambda))$ and let $x \in \Lambda \setminus \{0\}$. By Proposition 2.4.5 we may write $x = \sum_{i=1}^n x_i$ for some $n \in \mathbb{Z}_{\geq 1}$ and $x_i \in \mathrm{indec}(\Lambda)$ such that $\sum_{i=1}^n \langle x_i, x_i\rangle \leq \langle x, x\rangle$. Then by Lemma 2.6.6 we have

$$2\langle x, z\rangle = \sum_{i=1}^n 2\langle x_i, z\rangle < \sum_{i=1}^n \langle x_i, x_i\rangle \leq \langle x, x\rangle$$

and thus $z \in \mathrm{Vor}(\Lambda)$. We conclude that $\mathrm{Vor}(\Lambda) = V(\mathrm{indec}(\Lambda))$. □

Corollary 2.6.12. *Let Λ be a Hilbert lattice. Then*

$$\overline{\mathrm{Vor}}(\Lambda) = \{z \in \mathcal{H}_\Lambda \mid (\forall x \in \mathrm{indec}(\Lambda))\ \|z\| \leq \|z - x\|\}.$$

Proof. By Lemma 2.6.5.4 we have for $S \subseteq \Lambda$ not containing 0 that $\overline{V}(S) = \{z \in \mathcal{H}_\Lambda \mid (\forall x \in S)\ \|z\| \leq \|z - x\|\}$ is the topological closure of $V(S)$ as defined in the proof of Theorem 2.6.11. The corollary then follows from Proposition 2.6.7 and Theorem 2.6.11. □

Example 2.6.13. For a Hilbert lattice Λ the set $\mathrm{indec}(\Lambda)$ can fail to be the minimum among all sets $S \subseteq \Lambda$ such that $\overline{V}(S) = \{z \in \mathcal{H}_\Lambda \mid (\forall x \in S)\ \|z\| \leq \|z - x\|\}$ equals $\overline{\mathrm{Vor}}(\Lambda)$. We will give a counterexample.

Let $I = \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let $f\colon I \to \mathbb{R}_{\geq 0}$ such that $f|_{\mathbb{Z}_{\geq 0}}$ is strictly decreasing with limit $f(\infty) > 0$. Consider the lattice $\Lambda = \Lambda_2^f$ as in Example 2.4.9. Note that $\mathrm{P}(\Lambda) = 2f(\infty)^2$ and that $\mathrm{P}(\Lambda)$ is not attained by any vector. Hence $2e_\infty \in \Lambda$ is indecomposable by Lemma 2.4.4. Now let $S = \mathrm{indec}(\Lambda) \setminus \{\pm 2e_\infty\}$. We claim that $\overline{V}(S) = \overline{V}(\mathrm{indec}(\Lambda))$, the latter being equal to $\overline{\mathrm{Vor}}(\Lambda)$ by Corollary 2.6.12. It remains to show for all $z = (z_i)_i \in \overline{V}(S)$ that $\|z\| \leq \|z - 2e_\infty\|$ by symmetry.

For all i let $s_i \in \{\pm 1\}$ such that $s_i z_i = |z_i|$. Since f is strictly decreasing there exists some $N \in \mathbb{Z}_{\geq 0}$ such that $f(n) < \sqrt{3} f(\infty)$ for all integers $n \geq N$. For all integers $n \geq N$ we have $s_n e_n + e_\infty \in S$ by Lemma 2.4.4 as $q(s_n e_n + e_\infty) = f(n)^2 + f(\infty)^2 < 4f(\infty)^2 = 2\mathrm{P}(\Lambda)$. Then

$$0 \leq \|z - (s_n e_n + e_\infty)\|^2 - \|z\|^2 = (1 - 2|z_n|)f(n)^2 + (1 - 2z_\infty)f(\infty)^2.$$

As $z \in \mathcal{H}_\Lambda$ we must have $\lim_{n\to\infty} |z_n| = 0$, so taking the limit over the above inequality we get $0 \leq 2(1 - z_\infty)f(\infty)^2$ and thus $z_\infty \leq 1$. But then $\|z - 2e_\infty\|^2 - \|z\|^2 = 4f(\infty)^2(1 - z_\infty) \geq 0$ and we are done.