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Detecting Bell Correlations in Multipartite Non-Gaussian Spin States

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We expand the toolbox for studying Bell correlations in multipartite systems by introducing permutationally invariant Bell inequalities (PIBIs) involving few-body correlators. First, we present around twenty families of PIBIs with up to three- or four-body correlators, that are valid for an arbitrary number of particles. Compared to known inequalities, these show higher noise robustness, or the capability to detect Bell correlations in highly non-Gaussian spin states. We then focus on finding PIBIs that are of practical experimental implementation, in the sense that the associated operators require collective spin measurements along only a few directions. To this end, we formulate this search problem as a semidefinite program that embeds the constraints required to look for PIBIs of the desired form.

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Some correlations arising from quantum physics cannot be explained within the paradigm of local realism, and are thus called nonlocal [1]. These are detected via the violation of a so-called Bell inequality [2], tested in practice through a Bell experiment [3]. Besides their fundamental interest, nonlocal correlations are the resource enabling device-independent quantum information processing tasks, such as quantum key distribution [4], randomness amplification [5], or self-testing [6]. Although much research has focused on few-partite scenarios, mostly bipartite [7,8], nonlocal correlations also appear naturally in the multipartite regime [9,10] and, in particular, in physically relevant many-body systems [11–13]. With mild additional assumptions, multipartite nonlocality can be revealed in experimentally practical ways, and take the name of Bell correlations [14,15].

Detection of Bell correlations is of great interest, as they are related to quantum critical points [16], metrology [17], open quantum systems [18], and bosonic systems at finite temperature [19,20], and provide an avenue to quantify device-independent entanglement and Bell correlation depth [21–23]. However, the available inequalities are scarce, because a complete characterization is an intractable task [24]. An approach that finds a good compromise between expressivity and complexity is to focus on Bell inequalities with particular symmetries and low-order correlators [11,12,25,26]. In turn, this reduces the experimental requirements to reveal Bell correlations from them. A paradigmatic example is the use of two-body, permutationally invariant Bell inequalities (PIBIs) to detect a class of Gaussian states known as spin-squeezed states [14,15].

Despite all this progress, so far only PIBIs with up to two-body correlators are known, which poses a fundamental limit on their applicability. It is thus of great interest to find PIBIs involving higher-order moments of physical observables, such that Bell correlations can be detected in larger classes of states and with higher noise tolerance. In particular, these tools would enable the study of Bell correlations in non-Gaussian states [27], which cannot be characterized by only second moments.

In this Letter, we present around twenty new PIBIs involving three- and four-body correlators, and illustrate that compared to known PIBIs they provide an advantage in terms of noise robustness and sensitivity to non-Gaussian states. Moreover, we provide a general framework to derive new PIBIs with high-order correlators under the constraint of being experimentally practical, in the sense that they can be tested by performing collective spin measurements along only a few directions. This is based on a semidefinite program (SDP) that allows us to find Bell correlation witnesses of a desired ansatz.

Preliminaries.—We consider the multipartite Bell experiment in which N observers, labeled as $i = 1, 2, \ldots, N$, perform one of the two measurements $M_0^{(i)}, M_1^{(i)}$, on their part of the system, and obtain one of the two possible outcomes ± 1. Correlations among parties are characterized by the correlators $\langle M_{j_1}^{(i_1)} \cdots M_{j_k}^{(i_k)} \rangle$. However, to reduce the complexity of characterizing multipartite correlations, we

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restrict ourselves to permutationally invariant (PI) observables, namely

$$\mathcal{S}_{j_1\cdots j_k} = \sum_{\substack{1\le i_1<\cdots<i_k\le N\\ +\text{perm}}} \langle M_{j_1}^{(i_1)}\cdots M_{j_k}^{(i_k)}\rangle. \quad (1)$$

Here, $k = 1, \ldots, K$ indicates the order of the PI correlator, and $j_l = 0, 1$ is the measurement setting for $l = 1, \ldots, k$.

By considering N parties and at most Kth-order PI correlators, the set of classical correlations form a polytope $\mathbb{P}^S_{N,K}$. The vertices of this polytope are identified with the correlations originating from deterministic local hidden variable models (LHVM), for which it holds $\langle M_{j_1}^{(i_1)}\cdots M_{j_k}^{(i_k)}\rangle = \langle M_{j_1}^{(i_1)}\rangle\cdots\langle M_{j_k}^{(i_k)}\rangle$ and $\langle M_j^{(i)}\rangle = \pm 1$ for all i, j. From these vertices, it is possible to derive a dual description of $\mathbb{P}^S_{N,K}$ in terms of the linear inequalities (i.e., PIBIs) defining its facets. Measurement statistics lying outside $\mathbb{P}^S_{N,K}$ indicate the presence of Bell correlations. For details about this framework we refer the reader to Refs. [11,12].

In a scenario with $N < 20$ and $K \lesssim 3$ it is relatively simple to list all vertices of $\mathbb{P}^S_{N,K}$ and to obtain from them the full list of facet inequalities. For larger N and K, however, this approach is unfeasible. Therefore, to find PIBIs that allow the detection of Bell correlations in many-body systems, we might rely on the following method. For a given K, we characterize $\mathbb{P}^S_{N,K}$ for a few small values of N. Then, we look for patterns in the inequalities that appear, and use them to conjecture families of PIBIs valid for arbitrary N. Finally, we prove the conjectured families by demonstrating that they cannot be violated by LHVM.

Until now, only a couple of PIBIs valid for arbitrary N have been known, and only for $K = 2$ [11,28]. Of particular relevance is the inequality

$$I_2 \equiv -2\mathcal{S}_0 + \frac{1}{2}\mathcal{S}_{00} - \mathcal{S}_{01} + \frac{1}{2}\mathcal{S}_{11} + 2N \ge 0, \quad (2)$$

which enabled the experimental detection of Bell correlations in spin-squeezed BECs [14] and cold atomic ensembles [15].

Third-order Bell inequalities.—We start with considering the case $K = 3$. Remarkably, by computing the polytope $\mathbb{P}^S_{N,3}$ for all $N < 12$, we were able to identify around twenty families of Bell inequalities (see Supplemental Material [29], Sec. V). As we will see, an interesting family among these is

$$\begin{aligned} I_3 \equiv{} & -12(N-1)\mathcal{S}_0 - 12(N-1)\mathcal{S}_1 + 3(N-2)\mathcal{S}_{00} + 6N\mathcal{S}_{01} \\ & + 3(N-2)\mathcal{S}_{11} - 2\mathcal{S}_{000} - 3\mathcal{S}_{001} + \mathcal{S}_{111} \\ & + 12N(N-1) \ge 0, \end{aligned} \quad (3)$$

which we have proven to be valid for all N in Supplemental Material [29], Sec. I.

As all the inequalities we consider here involve two measurement settings and two outcomes, Jordan's lemma guarantees that their maximum quantum violation can be achieved by qubit measurements [37,38]. To this end, we identify $M_j^{(i)} = \vec{u}_j \cdot \vec{\sigma}^{(i)}$ ($j = 0, 1$) for the ith party, where $\vec{\sigma}$ is the vector of Pauli matrices. Even if not necessarily optimal nor required by the inequality, we assume that the same pair of observables is chosen by all parties, i.e., $M_j^{(i)} = M_j$. Since local rotations are irrelevant here, we can simplify further our discussion by choosing $M_0 = \sigma_z$ and $M_1(\theta) = \sin(\theta)\sigma_x + \cos(\theta)\sigma_z$, where $\theta \in [0, \pi]$. With this definition, the correlators Eq. (1) can be written as operators

$$\hat{\mathcal{S}}_{j_1\cdots j_k}(\theta) = \sum_{\substack{1\le i_1<\cdots<i_k\le N\\ +\text{perm}}} M_{j_1}^{(i_1)}(\theta) \otimes \cdots \otimes M_{j_k}^{(i_k)}(\theta), \quad (4)$$

so that the Bell inequalities (2) and (3) can now be understood as Bell operators $\hat{I}_2(\theta)$ and $\hat{I}_3(\theta)$, respectively.

Given a Bell operator, we search for the optimal measurements and states that maximizes the quantum violation relative to the classical bound, i.e., Q_V^N/β_C^N, where the classical bound β_C^N is the constant term appearing in the inequalities. Bell nonlocality is detected for a state $|\psi\rangle$ if the Bell operator yields a negative expectation value $\langle\psi|\hat{I}_K|\psi\rangle < 0$. As the classical bound is a constant, the maximum quantum violation Q_V^N can thus be identified with the minimum eigenvalue of Bell operator

$$Q_V^N = \min_\theta \min_{|\psi\rangle} \langle\psi|\hat{I}_K|\psi\rangle = \min_\theta \lambda_{\min}(\hat{I}_K), \quad (5)$$

and the associated eigenvector is the state that maximally violates I_K.

In the N-qubit Hilbert space, the dimension of Bell operator scales exponentially with N, making it a challenge to solve the eigenvalue problem Eq. (5) for large N. Fortunately, since the correlators $\mathcal{S}$ have permutation symmetry, it is possible to introduce a symmetry-adapted basis in which to express the Bell operator, such that it block diagonalizes due to Schur-Weyl duality [39,40]. We can then focus the search of nonlocality onto the fully symmetric block $\hat{I}_K^{\text{sym}}$ of size $(N+1)\times(N+1)$, and the maximum quantum violation of $\hat{I}_K$ can then be obtained from the lowest eigenvalue of $\hat{I}_K^{\text{sym}}$ [12]. In Fig. 1 we show Q_V^N/β_C for I_2 and I_3, as a function of N. It is evident the significantly better scaling for the higher-order Bell inequalities I_3 compared to I_2. In the limit $N \to \infty$, it is possible to show through a variational calculation [12] that the relative violation of I_3 tends to $-2\sqrt{3}/9 \approx -0.3849$, which is larger than the value $-1/4$ obtained for I_2. A larger relative violation indicates a higher noise robustness, as well as the possibility to detect Bell correlations in a larger class of states, as we will see with an example in the

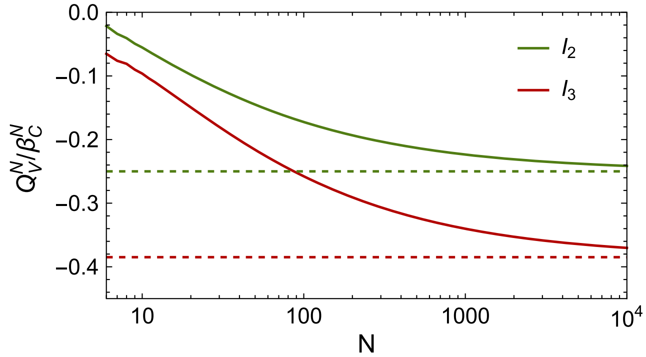

FIG. 1. Maximum relative quantum violation Q_V^N/β_C^N for the third-order Bell inequality I_3 and second-order Bell inequality I_2, as a function of the number of parties N. The two horizontal dashed lines indicate the asymptotic violation for $N \to \infty$, which for I_2 is $-1/4$, and for I_3 is $-2\sqrt{3}/9 \approx -0.3849$.

next paragraph. A similar analysis for the quantum violation of some other third-order Bell inequalities I_3 and a fourth-order Bell inequality I_4 are given in Supplemental Material [29], Secs. V and VI.

Bell correlations in spin-squeezed states.—We now show that I_3 allows us to detect Bell correlations in many-body spin states of experimental relevance. To illustrate this, let us consider the states that can be prepared through the paradigmatic one-axis twisting (OAT) Hamiltonian $\hat{H} = \hbar\chi\hat{S}_z^2$ [41]. The evolution of an initial coherent spin state along the x axis for a time t can be parametrized through the (adimensional) interaction strength $\mu = 2\chi t$, and reads $|\Phi(\mu)\rangle = 2^{-N/2}\sum_{k=0}^{N}\sqrt{\binom{N}{k}}e^{-i(N/2-k)^2\mu/2}|k\rangle$, where $|k\rangle$ represents the N-particle Dicke state with k excitations.

To investigate Bell correlations in the OAT states $|\Phi(\mu)\rangle$, we compute I_2 and I_3 as a function of μ. For this, we need to minimize the associated Bell operators over the measurement directions M_j at every μ, which in the case of a given state have to be both parametrized over the full sphere as $M_j = \vec{u}_j \cdot \vec{\sigma}$, with $\vec{u}_j = [\cos(\phi_j)\sin(\theta_j), \sin(\phi_j)\sin(\theta_j), \cos(\theta_j)]$. This yields Bell operators $\hat{I}_K(\phi_0, \theta_0, \phi_1, \theta_1)$ that are now functions of four angles.

Concretely, we express the Bell operators in the fully symmetric subspace in terms of collective spin operators, make use of the analytic results for their expectation values for spin OAT states (see Supplemental Material [29], Secs. II and III), and then minimize the lowest eigenvalue of the operator over the measurement directions. The violations we obtain are shown in Fig. 2 for $N = 50$, where we can observe that I_3 outperforms I_2 by reaching larger relative violation Q_V^N/β_C^N as well as detecting Bell correlations over a wider squeezing range, and thus for a larger class of states. To investigate the noise robustness, we consider OAT states mixed with white noise as $\rho(\eta,\mu) = \eta|\Phi(\mu)\rangle\langle\Phi(\mu)| + (1-\eta)\mathbf{I}/(N+1)$. In Fig. 2 we plot the minimum η for observing a PIBI violation,

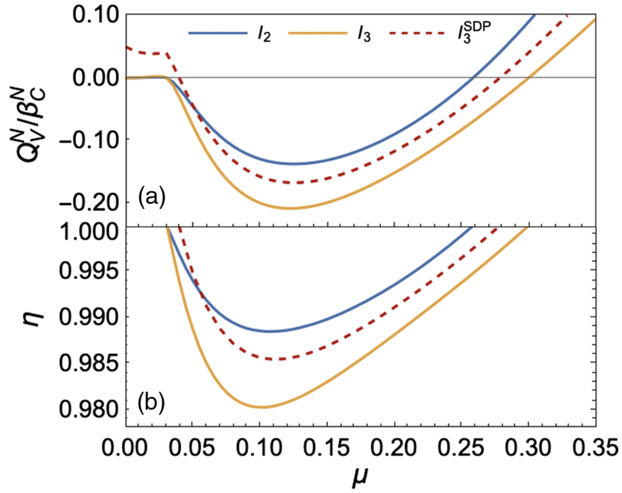

FIG. 2. (a) Relative quantum violation of the PIBIs I_2 (blue) and I_3 (orange) for $N = 50$ spin OAT states $|\psi(\mu)\rangle$ as a function of μ. An advantage over I_2 can also be found for I_3^{SDP} (red dashed), which requires us to measure only one third moment of the collective spin. (b) For mixed states $\rho(\eta,\mu) = \eta|\Phi(\mu)\rangle\langle\Phi(\mu)| + (1-\eta)\mathbf{I}/(N+1)$, the minimum purity η required to violate each PIBI.

and show that high-order inequalities detect Bell correlations with higher noise tolerance.

Finding practical high-order Bell inequalities.—We have shown, taking I_3 as an example, that high-order Bell inequalities can outperform their low-order counterparts and allow us to robustly detect Bell correlations in states that are routinely investigated experimentally. However, we note that the associated Bell operator often requires us to measure high-order moments of the collective spin along several directions, because we have, e.g. (see Supplemental Material [29], Sec. II),

$$\mathcal{S}_{001} = \frac{8}{3}\langle\hat{S}_{\vec{n}}\hat{S}_{\vec{n}}\hat{S}_{\vec{m}} + \hat{S}_{\vec{n}}\hat{S}_{\vec{m}}\hat{S}_{\vec{n}} + \hat{S}_{\vec{m}}\hat{S}_{\vec{n}}\hat{S}_{\vec{n}}\rangle + \cdots. \quad (6)$$

This can pose practical challenges, as estimating many high-order moments requires the collection of large measurement statistics. For this reason, we would like to find inequalities with coefficients satisfying additional constraints, such that the associated operator involves, e.g., only *one* third-order moment. To see the form of such constraint, note that for some measurement direction $\vec{a} = \alpha\vec{m} + \beta\vec{n}$ we have (see Supplemental Material [29], Sec. IV)

$$\langle\hat{S}_{\vec{a}}^3\rangle = \frac{\beta^3}{8}\mathcal{S}_{000} + \frac{\alpha^3}{8}\mathcal{S}_{111} + \frac{3\alpha\beta^2}{8}\mathcal{S}_{001} + \frac{3\alpha^2\beta}{8}\mathcal{S}_{011} + f(\mathcal{S}_0, \mathcal{S}_1), \quad (7)$$

where $f(\mathcal{S}_0, \mathcal{S}_1)$ is a linear function of one-body correlators only, and $\alpha, \beta \in \mathbb{R}$, $|\alpha^2| + |\beta^2| = 1$. Therefore, only inequalities whose coefficients for the third-order correlators are following the pattern of Eq. (7) will result in a Bell operator involving only one third-order moment of the collective spin.

Note here that imposing such constraints as a further projection of the local polytope is not trivial, because of the high nonlinearity in α and β of the coefficients multiplying the correlators. One obtains a different polytope projection for each pair (α, β), which dramatically increases the optimization complexity, as for each projection new families of Bell inequalities need to be found. Moreover, such an approach does not guarantee finding tight inequalities, as these might correspond to tilts of facets.

We thus propose a way to circumvent these difficulties, by developing a method to find practical Bell operators that is based on a hierarchy of semidefinite programs (SDPs) that search for inequalities of a particular form. For the sake of brevity and clarity, in the following we present our method in brief, and show a concrete example for $K = 3$. For a more detailed and general formulation we refer the reader to Supplemental Material [29], Sec. VII.

To illustrate our idea, let us start recalling that for deterministic LHVM the PI correlators Eq. (1) can be written as polynomials in four non-negative integers (a, b, c, d) such that $a + b + c + d = N$ (see Supplemental Material [29], Sec. I). This allows us to parametrize the vertices of $\mathbb{P}^S_{N,K}$ with integer partitions of N, but it also implies that considering (a, b, c, d) to be non-negative reals gives us an outer approximation of $\mathbb{P}^S_{N,K}$ in terms of a semialgebraic set. In the space of PI correlators, the latter is specified by (i) the set of polynomial equalities $f_i(\vec{\mathcal{S}}) = 0$ expressing constraints among correlators, e.g., $\mathcal{S}_{000} = \mathcal{S}_0^3 - [3(\mathcal{S}_0^2 - \mathcal{S}_{00}) - 2]\mathcal{S}_0$, and (ii) the set of four polynomial inequalities $g_i(\vec{\mathcal{S}}) \geq 0$ expressing $(a, b, c, d) \geq 0$ through correlators, e.g., $a \geq 0$ implies $g_1(\vec{\mathcal{S}}) = (\mathcal{S}_0^2 - \mathcal{S}_{00}) + \mathcal{S}_0 + \mathcal{S}_1 + (\mathcal{S}_0\mathcal{S}_1 - \mathcal{S}_{01}) \geq 0$.

Having at hand an outer approximation of $\mathbb{P}^S_{N,K}$ in terms of a semialgebraic set, we can apply known techniques that use SDP hierarchies to test membership to the convex Hull of such a set [42,43]. The first step consists of defining a basis vector, e.g., $\vec{b} = (1, \mathcal{S}_0, \ldots, \mathcal{S}_{111})^T$, from which the moment matrix $\tilde{\Gamma} = \oplus_{i=0}^4 g_i \vec{b} \cdot \vec{b}^T$ is constructed. Then, to check whether the given experimental data $\vec{\mathcal{S}}^* = (\mathcal{S}_0^*, \mathcal{S}_1^*, \ldots)$ is outside the convex Hull approximating $\mathbb{P}^S_{N,K}$ we can linearize $\tilde{\Gamma}$ and write the SDP [44]

$$\begin{aligned} \max_{\tilde{\Gamma}} \quad & \lambda \\ \text{s.t.} \quad & \tilde{\Gamma} \succeq 0 \\ & \tilde{\Gamma}_{00} = 1 \\ & \tilde{\Gamma}_{0i} = \lambda(\vec{\mathcal{S}}^*)_i \\ & \tilde{\Gamma}_{ij} = p(\tilde{\Gamma}), \end{aligned} \tag{8}$$

where $p(\tilde{\Gamma})$ is the function expressing the constraints between the entries of $\tilde{\Gamma}$. If SDP (8) returns $\lambda < 1$ we must conclude that $\vec{\mathcal{S}}^*$ lies outside the convex Hull outer approximating $\mathbb{P}^S_{N,K}$, and therefore that this point cannot be described by a LHVM. In this case, the SDP dual to (8) gives us a certificate of nonlocality by providing a PIBI that is violated by $\vec{\mathcal{S}}_3^*$.

To ensure that the dual to (8) provides us with PIBIs that are experimentally practical, we now modify SDP (8) by adding the additional constraints required to obtain Bell operators of the desired form. Note, however, that this is nontrivial, as such constraints are highly nonlinear [cf. Eq. (7)]. For example, asking for a Bell operator involving only one third moment of the collective spin requires imposing the constraints

$$\begin{aligned} (\tilde{\Gamma}_{01}, \ldots, \tilde{\Gamma}_{05}) &= \lambda(\mathcal{S}_0^*, \mathcal{S}_1^*, \ldots, \mathcal{S}_{11}^*) \\ y &= \lambda(\beta^3\mathcal{S}_{000}^* + 3\alpha\beta^2\mathcal{S}_{001}^* + 3\alpha^2\beta\mathcal{S}_{011}^* + \alpha^3\mathcal{S}_{111}^*), \end{aligned} \tag{9}$$

where $y = (\beta^3\tilde{\Gamma}_{06} + 3\alpha\beta^2\tilde{\Gamma}_{07} + 3\alpha^2\beta\tilde{\Gamma}_{08} + \alpha^3\tilde{\Gamma}_{09})$. Therefore, we run SDP (8) with Eq. (9) as an optimization problem over (α, β), in order to find $\min_{\alpha,\beta}(\lambda)$.

Example with OAT states.—Let us go back to the problem of detecting Bell correlations in OAT spin states. We aim to find a third-order PIBI with coefficients such that the resulting Bell operator requires the measurement of only one third moment of the collective spin.

First, we specify the target state $|\Phi(\mu)\rangle$ where to detect Bell correlations, e.g., by choosing $N = 50$ and $\mu = 0.2$. Then, we find a pair of measurement axes $\vec{u}_{1,2}$ such that the resulting list of correlators $\vec{\mathcal{S}}_3^* = (\mathcal{S}_0^*, \mathcal{S}_1^*, \mathcal{S}_{00}^*, \ldots, \mathcal{S}_{111}^*)$ shows Bell correlations. This step can be implemented by using SDP (8) to run the optimization problem $\min_{\vec{u}_1,\vec{u}_2}(\lambda)$. At this point, note that the dual to SDP (8) will provide us a PIBI that is violated by $\vec{\mathcal{S}}_3^*$, but that is in general of difficult experimental implementation. For this reason, we modify SDP (8) to include the constraints Eq. (9) and run the optimization problem $\min_{\alpha,\beta}(\lambda)$. The dual SDP gives now a PIBI in the form

$$I_3^{\text{SDP}} = c_0 + c_1\mathcal{S}_0 + c_2\mathcal{S}_1 + \cdots + c_6(\beta^3\mathcal{S}_{000} + \cdots) \geq 0, \tag{10}$$

where $\vec{c} = (c_0, \ldots, c_5, c_6)$ are the variables dual to $(\tilde{\Gamma}_{00}, \ldots, \tilde{\Gamma}_{05}, y)$, respectively, and whose associated Bell operator involves only one third-order moment of the collective spin. For our target state, we obtain $\alpha/\beta = 41/59$ and $\vec{c} = (1, -0.0055, -0.0141, 0.0046, 0.0099, 0.0051, -56.1412)$. The resulting I_3^{SDP} has a worse noise tolerance than I_3, but it still outperforms I_2, see Fig. 2. This could be improved by searching for a different inequality

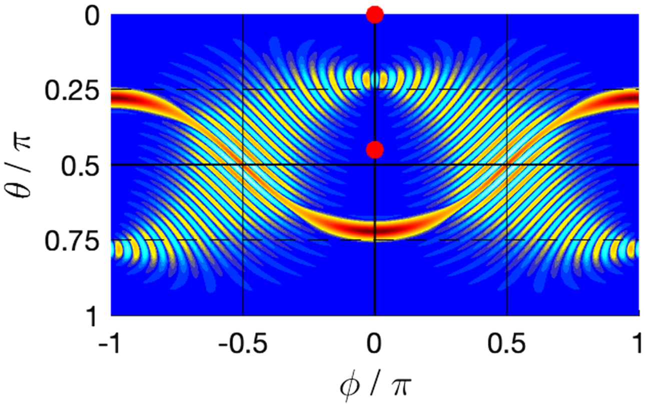

FIG. 3. For $N = 50$, Wigner function of the eigenstate corresponding to the minimum eigenvalue of Bell operator Eq. (12). Red dots indicate the optimal measurement directions $\hat{S}_{\vec{n}}, \hat{S}_{\vec{m}}$, for violating I_4.

for each μ, or by allowing the measurement of two third-order moments.

Fourth-order Bell inequalities.—Following the ideas presented so far, we can now guide our search for experimentally practical high-order PIBIs. For $K = 4$ we find the inequality

$$\begin{aligned} I_4 \equiv\ & 24(N-1)\mathcal{S}_{00} + 48(N-1)\mathcal{S}_{01} + 24(N-3)\mathcal{S}_{11} \\ & + \mathcal{S}_{0000} + 4\mathcal{S}_{0001} + 6\mathcal{S}_{0011} + 4\mathcal{S}_{0111} + \mathcal{S}_{1111} \\ & + 48N(N-1) \geq 0, \end{aligned} \tag{11}$$

which involves two- and four-body correlators. The associate Bell operator reads

$$\hat{I}_4 = 64\hat{S}_{\vec{a}}^4 - 192\hat{S}_{\vec{m}}^2 + 32g(\vec{a},\vec{m})\hat{S}_{\vec{a}}^2 + h(\vec{a},\vec{m}), \tag{12}$$

where g and h are scalar functions of $\vec{a}\cdot\vec{m}$ (see Supplemental Material [29], Sec. VI). Remarkably, note that Eq. (12) involves measurements of the collective spin operator along two directions only. Diagonalizing $\hat{I}_4$ according to Eq. (5) we further conclude that (i) its maximum relative quantum violation increases with N, (ii) the states maximally violating inequality (11) are highly non-Gaussian, and resembling a superposition of OAT states, see Fig. 3. The latter states do not violate I_2, as it is also expected from the fact that they have zero polarization (see Supplemental Material [29], Sec. VIII).

Conclusions.—We addressed the problem of finding multipartite PIBIs involving correlators of order higher than second, and that are of practical experimental implementation in the sense that the associated operators require collective measurements along only a few directions. We propose about twenty new PIBIs valid for arbitrary N that involve up to three-body correlators, and one that involves two- and four-body correlators. From a systematic analysis, we conclude that in general these inequalities indeed outperform the currently known PIBIs, since they show higher noise tolerance and the ability to detect Bell correlations in highly non-Gaussian states. In general, for a PIBI to be experimentally practical, we note that the coefficients of the correlators must satisfy some (nonlinear) constraints. We find that these can be imposed *a priori*, and formulate a SDP that looks for PIBIs resulting in Bell operators of the desired form (e.g., involving only one third moment). Our results can pave the way to studying Bell correlations in non-Gaussian spin states, and to use generalized spin-squeezing parameters as Bell correlation witnesses.

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