Pair correlation function of the one-dimensional Riesz gas
Beenakker, C.W.J.

Pair correlation function of the one-dimensional Riesz gas

C. W. J. Beenakker
Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, Netherlands

(Received 6 December 2022; accepted 9 February 2023; published 27 February 2023)

A method from random-matrix theory is used to calculate the pair correlation function of a one-dimensional gas of $N \gg 1$ classical particles with a power-law repulsive interaction potential $u(x) \propto |x|^{-s}$ (a so-called Riesz gas). An integral formula for the covariance of single-particle operators is obtained, which generalizes known results in the limits $s \to -1$ (Coulomb gas) and $s \to 0$ (log-gas). As an application, we calculate the variance of the center of mass of the Riesz gas, which has a universal large-N limit that does not depend on the shape of the confining potential.

DOI: 10.1103/PhysRevResearch.5.013152

I. INTRODUCTION

The one-dimensional (1D) Riesz gas [1–4] describes N classical particles that move on a line (the x axis) with a repulsive interaction potential $u(x)$ of the form

$$u(x) = \begin{cases} \operatorname{sign}(s)|x|^{-s} & \text{for } s > -2, \\ -\ln|x| & \text{for } s = 0. \end{cases} \tag{1.1}$$

The particles are prevented from moving off to infinity by a confining potential $V(x)$.

The cases $s = 0$ and $s = -1$ are also referred to as log-gas and Coulomb gas, respectively. The log-gas plays a central role in random-matrix theory (RMT) [5]. Experimentally, an interaction potential with an adjustable exponent $s \lesssim 1$ has been realized in a chain of trapped ion spins [6].

Thermal averages $\langle \cdots \rangle$, at inverse temperature β, are defined with respect to the Gibbs measure

$$P(x_1, x_2, \ldots x_N) = Z^{-1} \exp\left(-\beta\left[J\sum_{i<j=1}^{N} u(x_i - x_j) + \sum_{i=1}^{N} V(x_i)\right]\right), \tag{1.2}$$

where Z normalizes the distribution to unity. The interaction strength is parameterized by $J > 0$. The average density is

$$\rho(x) = \left\langle \sum_{i=1}^{N} \delta(x - x_i) \right\rangle, \tag{1.3}$$

normalized to the particle number,

$$\int_{-\infty}^{\infty} \rho(x)\, dx = N. \tag{1.4}$$

Much is known about the dependence of $\rho(x)$ on the range of the interaction [7–10], in the three regimes $s > 1$ (short-range repulsion), $-1 < s < 1$ (weakly long-range repulsion), and $-2 < s < -1$ (strongly long-range repulsion). In what follows we consider the second regime, $-1 < s < 1$, which includes the log-gas at $s = 0$ and the Coulomb gas in the limit $s \to -1$.

For $N \gg 1$ the density has a compact support, which may consist of multiple disjunct intervals. Considering a single interval (a, b), an end point may be N independent, fixed by a hard wall, or it may be N dependent, freely adjustable in a smooth potential. The density vanishes as a power law at a free end point, while it diverges (with an integrable singularity) at a fixed end point.

For example, in the case of a quadratic confinement $V(x) \propto x^2$, there are two free N-dependent end points at $b = -a \propto N^{1/(2+s)}$ and the density profile is [9]

$$\rho(x) \propto (b^2 - x^2)^{(s+1)/2}, \quad |s| < 1. \tag{1.5}$$

The density profile becomes flat in the Coulomb gas limit, while the log-gas has the Wigner semicircle law [11]. Alternatively, if we set $V \equiv 0$ and confine the Riesz gas by a hard wall at $x = a$ and $x = b$, then the density diverges on approaching a fixed end point [10],

$$\rho(x) \propto (x - a)^{(s-1)/2}(b - x)^{(s-1)/2}, \quad |s| < 1. \tag{1.6}$$

Knowledge of the density allows us to calculate by integration the average $\langle F \rangle$ of a single-particle observable $F = \sum_{i=1}^{N} f(x_i)$ (also known as a "linear statistic"). For $N \gg 1$ we can restrict the integral to the interval (a, b),

$$\langle F \rangle = \int_a^b \rho(x) f(x)\, dx. \tag{1.7}$$

Going beyond the first moment, Flack, Majumdar, and Schehr recently obtained [12] a strikingly simple formula for the large-N limit of the variance of F in the Coulomb case [13],

$$\mathrm{Var}\, F = \frac{1}{2\beta J}\int_a^b [df(x)/dx]^2\, dx \quad \text{for } s=-1. \tag{1.8}$$

The corresponding formula in the RMT case $s=0$ is known [14,15], but results for other values of the interaction parameter s are not known. The aim of this paper is to provide that information for the entire range $-1<s<1$.

For that purpose one needs the connected pair correlation function

$$R(x,y) = \left\langle \sum_{i,j=1}^{N}\delta(x-x_i)\delta(y-x_j)\right\rangle - \rho(x)\rho(y), \tag{1.9}$$

which gives the variance upon integration,

$$\mathrm{Var}\, F = \int_a^b dx \int_a^b dy\, R(x,y) f(x) f(y). \tag{1.10}$$

In RMT there are basically two methods to compute $R(x,y)$. The method of orthogonal polynomials [16] applies to specific confining potentials (typically linear or quadratic) and then gives results for any N. The alternative method of functional derivatives [17] (equivalently, the method of loop equations [18]) takes the large-N limit, but then works generically for any form of confinement. Since the latter method does not assume a logarithmic repulsion, it is the method of choice in what follows.

II. PAIR CORRELATION FUNCTION

For $N\gg 1$ the pair correlation function oscillates rapidly on the scale of the interparticle spacing $\delta x \simeq (b-a)/N$. These oscillations are irrelevant for the computation of the variance of an observable that varies smoothly on the scale of δx, so that in the large-N limit it is sufficient to know the smoothed correlation function.

The method of functional derivatives starts from the exact representation

$$R(x,y) = -\frac{1}{\beta}\frac{\delta\rho(x)}{\delta V(y)}. \tag{2.1}$$

The variation of the density is to be carried out at constant particle number,

$$\int_a^b \delta\rho(x)\, dx = 0. \tag{2.2}$$

The integration interval (a,b) is the support of the smoothed particle density $\rho(x)$ in the large-N limit. (We assume that the confining potential produces a support in a single interval.)

In the regime $-1<s<1$ of a weakly long-ranged repulsion, and for $J \gtrsim 1/\beta$, variations in the smoothed density ρ and in the confining potential V are related by the condition of mechanical equilibrium [9,19],

$$J\int_a^b u(x-y)\delta\rho(y)\, dy + \delta V(x) = \text{constant}, \; a<x<b. \tag{2.3}$$

Taking the derivative with respect to x we have a singular integral equation,

$$J(|s|+\delta_{0,s})\mathcal{P}\int_a^b dy\, \delta\rho(y)\frac{\mathrm{sign}(x-y)}{|x-y|^{s+1}} = \frac{d}{dx}\delta V(x),$$
$$a<x<b, \tag{2.4}$$

which we need to invert in order to obtain the functional derivative (2.1). (The symbol $\mathcal{P}$ indicates the principal value of the integral.)

For $-1<s<1$ the general solution to Eq. (2.4) is given by the Sonin inversion formula [20–22] (see Appendix A),

$$J(|s|+\delta_{0,s})\delta\rho(x) = C_{\delta V}[(x-a)(b-x)]^{s_-} - C_1\mathcal{S}_{\delta V}(x), \tag{2.5a}$$

$$\mathcal{S}_{\delta V}(x) = -(x-a)^{s_+}\frac{d}{dx}\int_x^b dt\, \frac{(t-x)^{s_-}}{(t-a)^s}\frac{d}{dt} \times \int_a^t dy\, (y-a)^{s_-}(t-y)^{s_+}\frac{d}{dy}\delta V(y), \tag{2.5b}$$

$$C_1 = \frac{\sin(\pi s_+)\Gamma(s+1)}{\pi s_+\Gamma(s_+)^2}, \quad s_\pm = (s\pm 1)/2. \tag{2.5c}$$

The coefficient $C_{\delta V}$ is fixed by the constraint (2.2),

$$C_{\delta V} = \frac{C_1}{C_2}\int_a^b dx\, \mathcal{S}_{\delta V}(x), \tag{2.6a}$$

$$C_2 = \int_a^b dx\, [(x-a)(b-x)]^{s_-} = \frac{(b-a)^s\sqrt{\pi}\,\Gamma(s_+)}{2^s\,\Gamma(1+s/2)}. \tag{2.6b}$$

The function $\Gamma(x)$ is the usual gamma function.

The pair correlation function $R(x,y)$ is obtained from Eq. (2.1) as a distribution, defined by its action on a test function $g(y)$ upon integration over y. Using the functional-derivative identities

$$\int_a^b dy\, \frac{\delta\mathcal{S}_{\delta V}(x)}{\delta V(y)} g(y) = \mathcal{S}_g(x), \tag{2.7a}$$

$$\int_a^b dy\, \frac{\delta C_{\delta V}}{\delta V(y)} g(y) = \frac{C_1}{C_2}\int_a^b dx\, \mathcal{S}_g(x), \tag{2.7b}$$

we find the following expression:

$$J\beta(|s|+\delta_{0,s})\int_a^b dy\, R(x,y) g(y) = C_1\mathcal{S}_g(x) - [(x-a)(b-x)]^{s_-}\frac{C_1}{C_2}\left(\int_a^b dx\, \mathcal{S}_g(x)\right). \tag{2.8}$$

III. COVARIANCE OF SINGLE-PARTICLE OBSERVABLES

Equation (2.8) provides a formula for the covariance of the two observables $F=\sum_{i=1}^N f(x_i)$ and $G=\sum_{i=1}^N g(x_i)$,

$$\mathrm{CoVar}(F,G) = \langle FG\rangle - \langle F\rangle\langle G\rangle = \int_a^b dx\int_a^b dy\, R(x,y) f(x) g(y). \tag{3.1}$$

The covariance is given by integrals over $f(x)$ and $\mathcal{S}_g(x)$,

$$J\beta(|s|+\delta_{0,s})\,\mathrm{CoVar}\,(F,G) = C_1\int_a^b dx\, f(x)\mathcal{S}_g(x) - \frac{C_1}{C_2}\left(\int_a^b dx\,\mathcal{S}_g(x)\right) \times \int_a^b dx\,[(x-a)(b-x)]^{s_-} f(x). \tag{3.2}$$

The variance of a single observable then follows from $\mathrm{Var}\,F = \mathrm{CoVar}(F,F)$.

For general functions f, g the integrals in Eq. (3.2) may be carried out numerically (see Appendix A). Closed-form expressions can be obtained for polynomial functions. It is convenient to shift the origin of the coordinate system so that $(a,b)\rightarrow(0,L)$. We define $X_p = \sum_i x_i^p$, $p \geqslant 1$, and obtain the covariance

$$J\beta(|s|+\delta_{0,s})\,\mathrm{CoVar}\,(X_p,X_q) = \frac{2\pi L^{p+q+s}\Gamma(s)}{\Gamma(-p-s_-)\Gamma(p+s+1)\Gamma(-q-s_-)\Gamma(q+s+1)} \times \frac{pqs\sin(\pi s_+)}{(p+q+s)(\cos[\pi(p+q+s)]+\cos[\pi(p-q)])}. \tag{3.3}$$

For the variance this reduces to

$$J\beta(|s|+\delta_{0,s})\,\mathrm{Var}\,X_p = \frac{2\pi L^{2p+s}p^2 s\Gamma(s)\sin(\pi s_+)}{(2p+s)\Gamma(-p-s_-)^2\Gamma(p+s+1)^2(1+\cos[\pi(2p+s)])}. \tag{3.4}$$

We have checked that the general formula (3.3) agrees with the known formulas in the Coulomb gas limit [12] (see also Appendix B),

$$J\beta\,\mathrm{CoVar}\,(F,G) = \frac{1}{2}\int_0^L \frac{df(x)}{dx}\frac{dg(x)}{dx}\,dx, \quad \text{for } s=-1, \Rightarrow J\beta\,\mathrm{CoVar}\,(X_p,X_q) = \frac{pqL^{p+q-1}}{2(p+q-1)}, \tag{3.5}$$

and in the log-gas limit [23,24],

$$J\beta\,\mathrm{CoVar}\,(F,G) = \frac{1}{\pi^2}\mathcal{P}\int_0^L dx\int_0^L dy\,\frac{g(y)df(x)/dx}{y-x}\sqrt{\frac{x(L-x)}{y(L-y)}}, \quad \text{for } s=0,$$
$$\Rightarrow J\beta\,\mathrm{CoVar}\,(X_p,X_q) = \frac{\pi L^{p+q}}{(p+q)\Gamma\left(\frac{1}{2}-p\right)\Gamma(p)\Gamma\left(\frac{1}{2}-q\right)\Gamma(q)\cos(\pi p)\cos(\pi q)}. \tag{3.6}$$

IV. VARIANCE OF THE CENTER OF MASS

By way of illustration, we compute the variance of the center of mass $M = N^{-1}\sum_{i=1}^N x_i$ of the Riesz gas. Equation (3.4) for $p=1$ gives

$$\mathrm{Var}\,M = \frac{C_s\sqrt{\pi}}{2^{s+3}\Gamma\left(\frac{1}{2}-s/2\right)\Gamma(2+s/2)}, \quad C_s = \frac{L^{s+2}}{N^2 J\beta}\frac{1}{|s|+\delta_{0,s}}. \tag{4.1}$$

The dimensionless coefficient C_s is N independent if the system is scaled at constant interparticle spacing $\delta x = L/N$ with interaction strength $J \propto N^s$.

The dependence of $\mathrm{Var}\,M$ on the interaction exponent s is plotted in Fig. 1. At the upper limit $s\rightarrow 1$ we find $C_s^{-1}\mathrm{Var}\,M \rightarrow 0$, to leading order in $1/N$. This is consistent with the fact that the repulsion is short range for $s>1$, so we would expect the positions of the particles to fluctuate independently. The variance of the center of mass (rescaled by C_s) would then be of order $1/N$.

The theory applies to the interval $-1 < s < 1$ of a weakly long-range repulsion. The dashed curve in Fig. 1 is the analytical continuation of Eq. (4.1) to the interval $-2 < s < -1$ of a strongly long-range repulsion. We surmise that the formula remains valid in that regime.

V. CONCLUSION

In conclusion, we have computed the pair correlation function of a 1D system of classical particles with a long-range power-law repulsion. In the thermodynamic limit (particle number N and system size L to infinity at fixed interparticle spacing $\delta x = L/N$), and upon smoothing

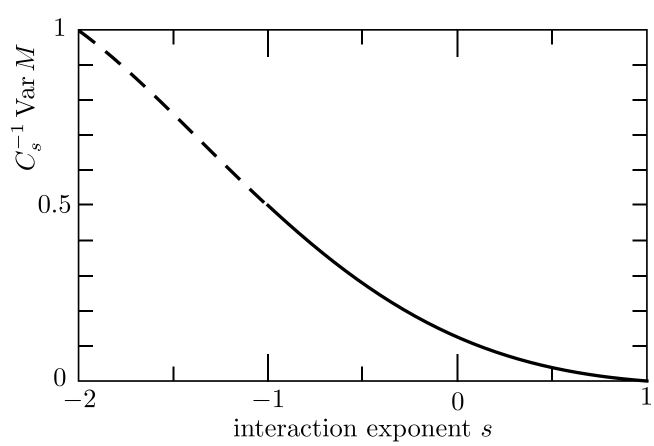

FIG. 1. Variance of the center of mass of the Riesz gas, computed from Eq. (4.1). The calculation applies to the interval $|s| < 1$, the dashed curve is the analytical continuation of Eq. (4.1) to smaller s.

over δx, the pair correlation function becomes a *universal* function of the power law exponent $s \in (-1, 1)$—independent of the shape of the confining potential $V(x)$ for a given single-interval support (a, b) of the average density. So it does not matter if the Riesz gas is confined to the interval $(0, L)$ by a soft parabolic potential or by a hard-wall confinement—the density fluctuations are the same even though the average density profile $\rho(x)$ is very different in the two cases [10].

Our result (3.2) for the covariance of a pair of single-particle observables generalizes old results for a logarithmic repulsion [14,15] and a very recent result for a linear repulsion [12]. We rely on a solution of an integral equation that requires $|s| < 1$ (weakly long-range regime), but the method of functional derivatives that we have used can be applied also outside of this interval. The independence of the pair correlation function $R(x, y) = -\beta^{-1}\delta\rho(x)/\delta V(y)$ on the shape of the confining potential follows directly from the linearity of the relation between ρ and V, so we expect a universal result also for $-2 < s < -1$. In the short-range regime $s > 1$, in contrast, the ρ–V relation is nonlinear [9] and no universal answer is expected.

ACKNOWLEDGMENT

The author receives funding from the European Research Council (Advanced Grant No. 832256).

APPENDIX A: SONIN INVERSION FORMULA

We summarize results from Refs. [20–22] on the solution of the singular integral equation

$$\mathcal{P}\int_a^b dy\, S(y)\frac{\text{sign}(x-y)}{|x-y|^{s+1}} = g'(x), \quad x \in (a, b), \quad |s| < 1. \tag{A1}$$

The homogenous integral equation, with zero on the right-hand side, has the solution

$$S_0(x) = [(x-a)(b-x)]^{s_-}. \tag{A2}$$

The Sonin inversion formula gives two particular solutions to the inhomogeneous integral equation,

$$S_-(x) = -C_1(x-a)^{s_-}\frac{d}{dx}\int_x^b dt\,\frac{(t-x)^{s_+}}{(t-a)^s}\frac{d}{dt}\int_a^t dy\,(y-a)^{s_+}(t-y)^{s_-}g'(y), \tag{A3}$$

$$S_+(x) = C_1(x-a)^{s_+}\frac{d}{dx}\int_x^b dt\,\frac{(t-x)^{s_-}}{(t-a)^s}\frac{d}{dt}\int_a^t dy\,(y-a)^{s_-}(t-y)^{s_+}g'(y). \tag{A4}$$

For the general solution we take either one of these two particular solutions and add it to an arbitrary multiple of the homogeneous solution,

$$S(x) = S_\pm(x) + \text{constant} \times S_0(x), \tag{A5}$$

where "constant" means independent of x. Since, by construction, the difference of two particular solutions solves the homogeneous integral equation, it does not matter for the general solution, which particular solution we choose.

In the main text we chose $S_+(x)$. This has the benefit over $S_-(x)$ that the derivatives can be eliminated upon partial integration,

$$\int_a^b dx\, f(x)S_+(x) = -C_1\int_a^b dx\,[f'(x)(x-a) + s_+f(x)](x-a)^{s_-} \times \int_x^b dt\,\frac{(t-x)^{s_-}}{(t-a)^s}\int_a^t dy\, s_+(y-a)^{s_-}(t-y)^{s_-}g'(y). \tag{A6}$$

The three definite integrals of Eq. (A6) are in a form that can be evaluated numerically, without the need to take derivatives.

APPENDIX B: COULOMB GAS LIMIT

The Coulomb gas limit $s = -1$ can be obtained from the general formulas for $|s| < 1$ by means of the identities [25]

$$\lim_{\epsilon\searrow 0}\int_x^b \frac{\epsilon f(y)}{(y-x)^{1-\epsilon}}dy = f(x), \tag{B1a}$$

$$\lim_{\epsilon\searrow 0}\int_a^b \frac{\epsilon f(x)}{[(b-x)(x-a)]^{1-\epsilon}}dx = \frac{f(a)+f(b)}{b-a}. \tag{B1b}$$

We start from the derivative-free representation (A6) of the integrals in Eq. (3.2), and apply Eq. (B1) first to the integral over y,

$$I_1(t) = \lim_{s\searrow -1}\int_a^t dy\, s_+(y-a)^{s_-}(t-y)^{s_-}g'(y) = \frac{g'(t)+g'(a)}{t-a}, \tag{B2}$$

then to the integral over t,

$$I_2(x) = \lim_{s\searrow -1}\int_x^b dt\,\frac{(t-x)^{s_-}}{(t-a)^s}I_1(t) = \frac{1}{s_+}[g'(x)+g'(a)], \tag{B3}$$

and finally to the integral over x,

$$I_3 = \lim_{s\searrow -1}\int_a^b dx\,[f'(x)(x-a)+s_+f(x)](x-a)^{s_-}I_2(x) = \frac{1}{s_+}\int_a^b dx\, f'(x)[g'(x)+g'(a)] + \frac{2}{s_+}f(a)g'(a). \tag{B4}$$

Moreover, since $1/C_2 \to \frac{1}{2}s_+(b-a)$ for $s \to -1$, we have

$$\lim_{s\searrow -1} \frac{1}{C_2} \int_a^b dx\,[(x-a)(b-x)]^{s_-} f(x) = \frac{1}{2}[f(b)+f(a)]. \tag{B5}$$

Using also $C_1 = \frac{1}{2}s_+ + \mathcal{O}(s+1)^2$ we thus arrive at

$$\lim_{s\searrow -1} C_1 \int_a^b dx\, f(x)\mathcal{S}_g(x)$$
$$= \frac{1}{2}\int_a^b dx\, f'(x)[g'(x)+g'(a)] + f(a)g'(a), \tag{B6a}$$

$$\lim_{s\searrow -1} \frac{C_1}{C_2}\left(\int_a^b dx\,\mathcal{S}_g(x)\right)\int_a^b dx\,[(x-a)(b-x)]^{s_-} f(x)$$
$$= \frac{1}{2}g'(a)[f(b)+f(a)]. \tag{B6b}$$

Substitution of Eq. (B6) into Eq. (3.2) gives

$$\lim_{s\searrow -1} \mathrm{CoVar}\,(F,G) = \frac{1}{2J\beta}\int_a^b dx\, f'(x)g'(x), \tag{B7}$$

in accord with Ref. [12].

[1] M. Riesz, Intégrales de Riemann-Liouville et potentiels, Acta Sci. Math. Szeged **9**, 1 (1938).
[2] M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math. **81**, 1 (1949).
[3] S. Serfaty, Microscopic description of Log and Coulomb gases, arXiv:1709.04089.
[4] M. Lewin, Coulomb and Riesz gases: The known and the unknown, J. Math. Phys. **63**, 061101 (2022).
[5] P. J. Forrester, *Log-Gases and Random Matrices* (Princeton University Press, Princeton, 2010).
[6] J. Zhang, G. Pagano, P. W. Hess, A. Kyprianidis, P. Becker, H. Kaplan, A. V. Gorshkov, Z.-X. Gong, and C. Monroe, Observation of a many-body dynamical phase transition with a 53-qubit quantum simulator, Nature (London) **551**, 601 (2017).
[7] T. Leblé and S. Serfaty, Large deviation principle for empirical fields of Log and Riesz gases, Invent. Math. **210**, 645 (2017).
[8] D. P. Hardin, T. Leblé, E. B. Saff, and S. Serfaty, Large deviation principles for hypersingular Riesz gases, Constr. Approx. **48**, 61 (2018).
[9] S. Agarwal, A. Dhar, M. Kulkarni, A. Kundu, S. N. Majumdar, D. Mukamel, and G. Schehr, Harmonically Confined Particles with Long-Range Repulsive Interactions, Phys. Rev. Lett. **123**, 100603 (2019).
[10] J. Kethepalli, M. Kulkarni, A. Kundu, S. N. Majumdar, D. Mukamel, and G. Schehr, Harmonically confined long-ranged interacting gas in the presence of a hard wall, J. Stat. Mech. (2021) 103209.
[11] E. P. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels, Math. Proc. Cambridge Philos. Soc. **47**, 790 (1951).
[12] A. Flack, S. N. Majumdar, and G. Schehr, An exact formula for the variance of linear statistics in the one-dimensional jellium model, arXiv:2211.11850.
[13] To compare with Ref. [12] substitute $\beta J \mapsto 2\alpha N$.
[14] M. L. Mehta, *Random Matrices* (Elsevier, Amsterdam, 2004).
[15] C. W. J. Beenakker, Random-matrix theory of quantum transport, Rev. Mod. Phys. **69**, 731 (1997).
[16] M. L. Mehta, On the statistical properties of the level-spacings in nuclear spectra, Nucl. Phys. **18**, 395 (1960).
[17] C. W. J. Beenakker, Universality in the Random-Matrix Theory of Quantum Transport, Phys. Rev. Lett. **70**, 1155 (1993).
[18] J. Ambjørn, J. Jurkiewicz, and Yu. M. Makeenko, Multiloop correlators for two-dimensional quantum gravity, Phys. Lett. B **251**, 517 (1990).
[19] The variational equation (2.3) does not contain a contribution $\propto \rho(b)\delta b$ from a variation of the end point b, for two reasons: Either $\delta b = 0$, for a fixed end point, or $\rho(b) = 0$, for a free end point. Similarly for the end point a.
[20] N. Ya. Sonin, *Studies on Cylinder Functions and Special Polynomials* (Gostekhizdat, Moscow, 1954; in Russian).
[21] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, Philadelphia, 1993).
[22] S. V. Buldyrev, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, Average time spent by Lévy flights and walks on an interval with absorbing boundaries, Phys. Rev. E **64**, 041108 (2001).
[23] C. W. J. Beenakker, Universality of Brézin and Zee's spectral correlator, Nucl. Phys. B **422**, 515 (1994).
[24] F. D. Cunden and P. Vivo, Universal Covariance Formula for Linear Statistics on Random Matrices, Phys. Rev. Lett. **113**, 070202 (2014).
[25] Equation (B1) is a special case of Ramanujan's master theorem, for a proof see https://mathoverflow.net/q/435723/11260.