**Dynamics of coupled quantum systems**
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# 4 Early post-quench perturbative expansion

## 4.1. Early time expansion

We study the early time evolution of two, initially independent, subsystems (A and B) after coupling them with an instantaneous quench at $t = 0$:

$$\hat{H}(t) = \hat{H}_A \otimes \mathbb{1}_{d_B} + \mathbb{1}_{d_A} \otimes \hat{H}_B + \theta(t)\hat{H}_{int} \tag{4.1a}$$

$$\hat{H}_{int} = \sum_{IK} \lambda_{IK}\, \hat{\Psi}_I \otimes \hat{\Gamma}_K \tag{4.1b}$$

Before the quench ($t < 0$), each subsystem is governed by the Hamiltonians $\hat{H}_A$ and $\hat{H}_B$ respectively, and the identity matrix action on the complementary Hilbert space encodes their independence. The subscript on the identity operators indicates the dimensionality of the respective Hilbert space: $d_A = \dim\mathcal{H}_A$ and $d_B = \dim\mathcal{H}_B$. Initially, we consider a generic interaction Hamiltonian $\hat{H}_{int}$ (4.1b), given in a tensor-product basis of the individual Hilbert spaces $\hat{\Psi}_I \in \mathcal{H}_A$ and $\hat{\Gamma}_K \in \mathcal{H}_B$ and later we will demonstrate our results on two specific models, SYK and Mixed Field Ising.

Before the quench, the whole system is prepared in a tensor product state of the two individual subsystems:

$$\rho_0 = \rho_A \otimes \rho_B, \tag{4.2}$$

and the post-quench time evolution is given by the unitary transformation with the full interacting Hamiltonian which can be expanded as a time series with operator-valued coefficients $\hat{R}_n$:

$$\rho(t) = e^{-iHt}\rho_0 e^{iHt} = \left(1 - iHt - \frac{1}{2}H^2t^2 \dots\right)\rho_0\left(1 + iHt - \frac{1}{2}H^2t^2 \dots\right) = \\ = \sum_{n=0} \frac{t^n}{n!}\hat{R}_n \tag{4.3a}$$

$$R_0 = \rho_0 \quad ; \quad \hat{R}_{n+1} = i[\hat{R}_n, \hat{H}] \tag{4.3b}$$

By explicitly computing the first few terms in the expansion (4.3a), a recursive relation appears that determines all of the $\hat{R}_n$ operators (4.3b). However, in this letter, we are interested in the evolution of the reduced density matrices which, similarly to $\rho$, can be expressed as a time series by simply tracing out the complementary subsystem from $\hat{R}_n$. So, to obtain the reduced density matrix $\rho_A(t)$ of subsystem A one traces out the subsystem B, and tracing out subsystem A yields $\rho_B(t)$.

$$\rho_A(t) = \mathrm{Tr}_B(\rho(t)) = \sum_{n=0} \frac{t^n}{n!}\hat{A}_n \quad ; \quad \hat{A}_n = \mathrm{Tr}_B(\hat{R}_n) \tag{4.4a}$$

$$\rho_B(t) = \mathrm{Tr}_A(\rho(t)) = \sum_{n=0} \frac{t^n}{n!}\hat{B}_n \quad ; \quad \hat{B}_n = \mathrm{Tr}_A(\hat{R}_n) \tag{4.4b}$$

Once the reduced density matrices are obtained, we can compute the time evolution of any observable in each individual subsystem, for example the behavior of the energy $E_A(t)$ is:

$$E_A(t) = \mathrm{Tr}_A(\rho_A(t)\hat{H}_A) = \sum_{n=0}^{N} \frac{t^n}{n!}\,\mathrm{Tr}_A(\hat{A}_n\hat{H}_A) = \sum_n \frac{e_n}{n!}t^n. \tag{4.5}$$

Despite the compactness of the expansion, evaluating the operators $\hat{R}_n$ and subsequently $\hat{A}_n$ or $\hat{B}_n$ is rather tedious for $n > 2$, so we will restrict our study to only the second-order expansion of the energy of the subsystem A:

$$E_A(t) = \mathrm{Tr}_A(\rho_A(t)\hat{H}_A) = e_0 + e_1 t + \frac{e_2}{2}t^2 + \mathcal{O}(t^3) \tag{4.6}$$

The particular initial states we are interested in — thermal or energy eigenstates — commute with their respective Hamiltonians (e.g. $[\rho_A(0), H_A] = 0$) leading to many vanishing terms in the coefficients $\hat{A}_n$ (4.7), when compared to the coefficients from a general state (4.28). We will drop the time $t = 0$ argument $\rho_A(0)$ from here on, and implicitly will mean the density matrix at $t = 0$ when no argument is given $\rho_A \equiv \rho_A(0)$.

$$\hat{A}_0 = \rho_A; \tag{4.7a}$$

$$\hat{A}_1 = i\sum_{IK}\lambda_{IK}[\rho_A, \hat{\Psi}_I]\,\mathrm{Tr}_B(\rho_B\,\hat{\Gamma}_K); \tag{4.7b}$$

$$\begin{aligned}\hat{A}_2 = i^2\Bigg\{ &\sum_{IK}\lambda_{IK}[[\rho_A, \hat{\Psi}_I], \hat{H}_A]\,\mathrm{Tr}_B(\rho_B\,\hat{\Gamma}_K) + \\ &+ \sum_{IK}\sum_{K'I'}\lambda_{IK}\lambda_{I'K'}\Big([\rho_A\,\hat{\Psi}_I, \hat{\Psi}_{I'}]\,\mathrm{Tr}_B(\rho_B\,\hat{\Gamma}_K\,\hat{\Gamma}_{K'}) - \\ &- [\hat{\Psi}_I\,\rho_A, \hat{\Psi}_{I'}]\,\mathrm{Tr}_B(\rho_B\,\hat{\Gamma}_{K'}\,\hat{\Gamma}_K)\Big)\Bigg\};\end{aligned} \tag{4.7c}$$

Additionally, when studying the time evolution of observables that commute with the Hamiltonian the coefficients in their expansion simplify even further. This condition is trivially satisfied for the energy of the subsystem $E_A(t)$ and using the relation (4.40) we can compute the first three coefficients:

$$e_0 = \mathrm{Tr}_A(\rho_A \hat{H}_A) \equiv E_A(0) \tag{4.8a}$$

$$\begin{aligned} e_1 =& i \sum_{IK} \lambda_{IK} \, \mathrm{Tr}_A \left( [\rho_A, \hat{\Psi}_I] \hat{H}_A \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_K) = \\ =& i \sum_{IK} \lambda_{IK} \, \mathrm{Tr}_A \left( [\hat{H}_A, \rho_A] \, \hat{\Psi}_I \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_K) = 0; \end{aligned} \tag{4.8b}$$

$$\begin{aligned} e_2 =& i^2 \Bigg\{ \sum_{IK} \lambda_{IK} \, \mathrm{Tr}_A \left( [[\rho_A, \hat{\Psi}_I], \hat{H}_A] \hat{H}_A \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_K) + \\ &+ \sum_{IK} \sum_{I'K'} \lambda_{IK} \lambda_{I'K'} \Bigg( \mathrm{Tr}_A \left( [\rho_A \, \hat{\Psi}_I, \hat{\Psi}_{I'}] \hat{H}_A \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_K \, \hat{\Gamma}_{K'}) - \\ &- \mathrm{Tr}_A \left( [\hat{\Psi}_I \, \rho_A, \hat{\Psi}_{I'}] \hat{H}_A \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_{K'} \, \hat{\Gamma}_K) \Bigg) \Bigg\} = \\ =& i^2 \Bigg\{ \sum_{IK} \sum_{I'K'} \lambda_{IK} \lambda_{I'K'} \Bigg( \mathrm{Tr}_A \left( \rho_A \, \hat{\Psi}_I [\hat{\Psi}_{I'}, \hat{H}_A] \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_K \, \hat{\Gamma}_{K'}) \\ &- \mathrm{Tr}_A \left( \rho_A [\hat{\Psi}_{I'}, \hat{H}_A] \, \hat{\Psi}_I \right) \mathrm{Tr}_B(\rho_B \, \hat{\Gamma}_{K'} \, \hat{\Gamma}_K) \Bigg) \Bigg\}. \end{aligned} \tag{4.8c}$$

Naturally, the time-independent contribution $e_0$ is equal to the pre-quench energy and the first term is zero due to the aforementioned vanishing commutators. Therefore, we need to determine only the $e_2$ term, which is presented in the next section for two specific models.

## 4.2. Examples

In this section, we derive the early time evolution of the energy $E_A(t)$, to the second order in time, for two models the $\mathrm{SYK}_4$ and Mixed Field Ising model (MFI). The analytical expressions help us understand why the energy bump in the $\mathrm{SYK}_4$ happens for any temperature $T_A$, and provide an insight into the emergence of the critical temperature $T_c(h_x, h_z)$ in the MFI that marks the disappearance of the energy bump for $T_A > T_c$.

### 4.2.1. SYK

First, we consider two, initially decoupled SYK dots of size $N_A$ and $N_B$ governed by the following Hamiltonians:

$$\hat{H}_A = -\sum_{j=1}^{N_A} J^A_{j_1,j_2,j_3,j_4} \hat{\psi}_{j_1}\hat{\psi}_{j_2}\dots\hat{\psi}_{j_4} \quad ; \quad \hat{H}_B = -\sum_{l=1}^{N_B} J^B_{l_1,l_2\dots l_4} \hat{\chi}_{l_1}\hat{\chi}_{l_2}\hat{\chi}_{l_3}\hat{\chi}_{l_4} \tag{4.9}$$

$$\langle J^\alpha_{j_1,j_2,j_3,j_4} J^\alpha_{j_1,j_2,j_3,j_4}\rangle_J = \frac{J_\alpha^2}{N^3}, \quad \alpha \in \{A, B\} \tag{4.10}$$

We prepare the system in a tensor product state (4.2) then, at $t = 0$, we quench couple both SYKs with a two-point interaction Hamiltonian with random interactions:

$$\hat{H}_{int} = i\sum_{ij} \lambda_{ij}\hat{\psi}_i\hat{\gamma}_c \otimes \hat{\chi}_j \tag{4.11a}$$

$$\langle \lambda_{ij}\lambda_{i'j'}\rangle_\lambda = \frac{\lambda^2}{N_B}\delta_{ii'}\delta_{jj'} \tag{4.11b}$$

Here, $\hat{\gamma}_c$ is proportional to the product of all Majorana fields in $A$ and, as explained in Appendix 4.B, it is necessary for proper anti-commutation relations between the two subsystems. Substituting this interaction Hamiltonian in (4.8) we get the second order coefficient of $E_A(t)$:

$$\begin{aligned} e_2 = i^4 \sum_{ii'}^{N_A}\sum_{jj'}^{N_B} \lambda_{ij}\lambda_{i'j'} \Big( &\mathrm{Tr}_A \left(\rho_A \hat{\psi}_i\hat{\gamma}_c[\hat{\psi}_{i'}\hat{\gamma}_c, \hat{H}_A]\right) \mathrm{Tr}_B(\rho_B\hat{\chi}_j\hat{\chi}_{j'}) - \\ &- \mathrm{Tr}_A\left(\rho_A[\hat{\psi}_{i'}\hat{\gamma}_c, \hat{H}_A]\hat{\psi}_i\hat{\gamma}_c\right) \mathrm{Tr}_B(\rho_B\hat{\chi}_{j'}\hat{\chi}_j)\Big). \end{aligned} \tag{4.12}$$

For SYK-like interactions, the coefficients $e_n$ simplify even further upon disorder-averaging which, due to the independence between the inter-dot and intra-dots couplings, can either be averaged simultaneously or one after the other. Here, we take the latter route and initially average $e_2$ over the inter-dot coupling $\lambda_{ij}$:

$$\begin{aligned}
\langle e_2\rangle_\lambda =&\frac{\lambda^2}{N_B}\sum_i^{N_A}\Bigg(\mathrm{Tr}_A\left(\rho_A\hat{\psi}_i\hat{\gamma}_c[\hat{\psi}_i\hat{\gamma}_c,\hat{H}_A]\right)-\\
&-\mathrm{Tr}_A\left(\rho_A[\hat{\psi}_i\hat{\gamma}_c,\hat{H}_A]\hat{\psi}_i\hat{\gamma}_c\right)\Bigg)\sum_j^{N_B}\mathrm{Tr}_B(\rho_B\hat{\chi}_j\hat{\chi}_j)\\
=&\frac{\lambda^2}{N_B}\sum_i^{N_A}\Bigg(\mathrm{Tr}_A\left(\rho_A\hat{\psi}_i\hat{\gamma}_c[\hat{\psi}_i\hat{\gamma}_c,\hat{H}_A]\right)-\mathrm{Tr}_A\left(\rho_A[\hat{\psi}_i\hat{\gamma}_c,\hat{H}_A]\hat{\psi}_i\hat{\gamma}_c\right)\Bigg)\frac{N_B}{2}\\
=&\frac{\lambda^2}{2}\sum_i^{N_A}\mathrm{Tr}_A\left(\rho_A\left[\hat{\psi}_i\hat{\gamma}_c,[\hat{\psi}_i\hat{\gamma}_c,\hat{H}_A]\right]\right)=-\frac{\lambda^2}{2}2\cdot 4\,\mathrm{Tr}_A\left(\rho_A\hat{H}_A\right)\\
=&-4\lambda^2E_A(0).
\end{aligned}\tag{4.13}$$

In the second row we used the Majorana identity $\hat{\chi}_j^2=1/2$. The double commutator on the last line is evaluated in Appendix 4.C.2 with result (4.50) for $q=4$. Lastly, by disorder-averaging over the intra-dot couplings $\langle\cdot\rangle_J$:

$$\langle e_0\rangle=\langle E_A(0)\rangle_J \tag{4.14a}$$
$$\langle e_1\rangle=0; \tag{4.14b}$$
$$\langle e_2\rangle=-4\lambda^2\langle E_A(0)\rangle_J, \tag{4.14c}$$

we notice that the first two non-zero coefficients depend only on the initial energy of the analyzed subsystem and the interaction constant $\lambda$. Using these coefficients in (4.6), we obtain the averaged energy of the subsystem $A$ up to the second order in time:

$$\langle\Delta E(t)\rangle=\frac{\langle e_2\rangle}{2}t^2=-2\lambda^2\langle E_A(0)\rangle_J t^2. \tag{4.15}$$

In Fig. 4.1, we compare this expression with the energy obtained from a numerical time evolution of two equally-sized SYKs ($N_A=N_B$) with $J_A=J_B\equiv J$ when the quench happens from two independent thermal states at temperatures $T_A=0.5J$ and $T_B=0.1J$. We observe that (4.15) qualitatively matches the early time behavior of $\langle E_A(t)\rangle_J$. In order to quantify how well the analytical expression explains the behavior of the numerical results, we fit the data from the early-time interval to a quadratic model (4.16) and study the ratio between the two coefficients $a_2/e_2$.

$$f(t)=a_0+\frac{a_2}{2}t^2 \tag{4.16}$$

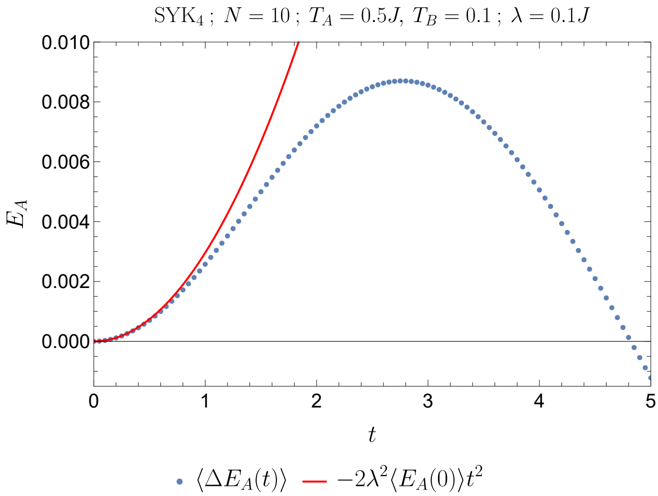


**Figure 4.1.** Early time evolution of $\langle \Delta E_A(t) \rangle$ computed numerically (blue dots) and analytically from (4.15) (red line). Numerical results are obtained from a quench of two equally sized ($N_A = N_B = 10$) SYK$_4$ dots.

Fig. 4.2 shows that this ratio is close to 1 for a few different temperatures $T_A$, proving the validity of the analytical expression (4.15) and the perturbative approach in general. The error bars on this Fig. 4.2 are computed with the error propagation relation using 99.7% confidence estimators for the errors $\sigma_{a_2}$ and $\sigma_{E_A}$ (3.21).

$$\sigma_{a_2/e_2} = \left| \frac{a_2}{e_2} \right| \sqrt{\left( \frac{\sigma_{a_2}}{a_2} \right)^2 + \left( \frac{\sigma_{E_A}}{\langle E_A(0) \rangle_J} \right)^2} \tag{4.17}$$

It is important to emphasize that for SYK dots with random interaction of the form (4.11a) the time evolution of $E_A(t)$, up to the second order, depends on the temperature $T_A$ only implicitly through $E_A(0)$, and is completely impartial to any parameter of the subsystem $B$. The same holds, the other way around, for $E_B(t)$. From the exposition above, we see that the initial energy rise happens when $\langle E_A(0) \rangle_J < 0$, regardless of the initial temperature $T_B$. Recalling that $\langle E_A(0) \rangle_J < 0$ holds always for the SYK$_4$ [55] explains why the energy increases initially for any temperature $T_A$.

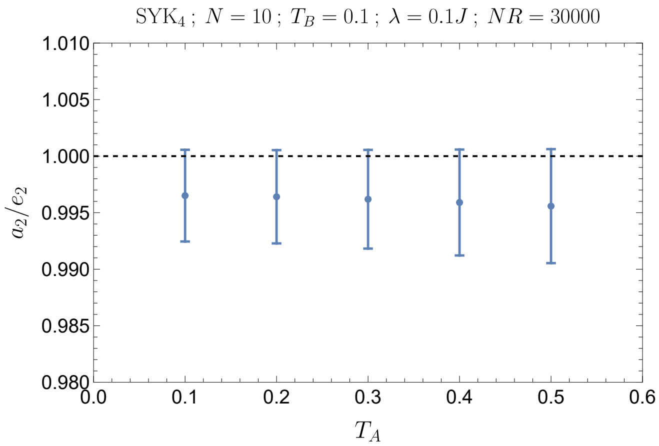


**Figure 4.2.** The ratio of the second order coefficients obtained from a fit of the numerical results $a_2$ and the analytical perturbative expansion $e_2$ with error bars given by the error propagation relation (4.17).

### 4.2.2. Mixed Field Ising

Next, we consider a system composed of two Mixed Field Ising models:

$$H_\alpha = -\sum_{i}^{N_\alpha} \left( J Z_i^\alpha Z_{i+1}^\alpha + g X_i^\alpha + h Z_i^\alpha \right), \qquad \alpha = A, B \tag{4.18}$$

coupled at $t = 0$, by the same quench procedure as before, with an interaction Hamiltonian that connects the last site of $A$ to the first site of $B$:

$$\begin{aligned} H_{\text{int}} = -\lambda_{+-}\hat{\sigma}^+_{N_A} \otimes \hat{\sigma}^-_{1_B} - \lambda_{-+}\hat{\sigma}^-_{N_A} \otimes \hat{\sigma}^+_{1_B} = -\sum_{ab\in\{+,-\}} \lambda_{a,b}\hat{\sigma}^a_{N_A} \otimes \hat{\sigma}^b_{1_B}, \\ \lambda^*_{-+} = \lambda_{+-} \equiv \lambda, \qquad \lambda_{++} = \lambda_{--} = 0. \end{aligned} \tag{4.19}$$

Here, $\sigma^\pm = X \pm iY$ are the ladder operators and we express $H_{int}$ in this particular form so it is readily usable in the general relations (4.8). Then we proceed the same as before, preparing the system in a tensor product of two decoupled subsystems, with density matrices that satisfy $[\rho_\alpha, \hat{H}_\alpha] = 0$, and directly substituting those parameters in (4.8) to obtain the early time evolution of the MFI subsystem $A$. As explained in Sec. 4.1, the time-independent contribution is equal to the pre-quench energy $e_0 = E_A(0)$, the first-order term vanishes $e_1 = 0$ and $e_2$ is evaluated below:

$$e_2 = i^2 \sum_{aa'} \sum_{bb'} \lambda_{ab}\lambda_{a'b'} \Big( \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^a_{N_A}[\hat{\sigma}^{a'}_{N_A}, \hat{H}_A]\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^b_{1_B}\hat{\sigma}^{b'}_{1_B}) - \\ - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^{a'}_{N_A}, \hat{H}_A]\hat{\sigma}^a_{N_A}\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^{b'}_{1_B}\hat{\sigma}^{b}_{1_B}) \Big). \tag{4.20}$$

First we expand the double sum:

$$\begin{aligned} e_2 = i^2 \Big\{ \lambda_{+-}\lambda_{+-} \Big( & \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^+_{N_A}[\hat{\sigma}^+_{N_A}, \hat{H}_A]\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^-_{1_B}\hat{\sigma}^-_{1_B}) - \\ & - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^+_{N_A}, \hat{H}_A]\hat{\sigma}^+_{N_A}\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^-_{1_B}\hat{\sigma}^-_{1_B}) \Big) + \\ +\lambda_{+-}\lambda_{-+} \Big( & \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^+_{N_A}[\hat{\sigma}^-_{N_A}, \hat{H}_A]\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^-_{1_B}\hat{\sigma}^+_{1_B}) - \\ & - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^-_{N_A}, \hat{H}_A]\hat{\sigma}^+_{N_A}\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^+_{1_B}\hat{\sigma}^-_{1_B}) \Big) + \\ +\lambda_{-+}\lambda_{+-} \Big( & \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^-_{N_A}[\hat{\sigma}^+_{N_A}, \hat{H}_A]\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^+_{1_B}\hat{\sigma}^-_{1_B}) - \\ & - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^+_{N_A}, \hat{H}_A]\hat{\sigma}^-_{N_A}\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^-_{1_B}\hat{\sigma}^+_{1_B}) \Big) + \\ +\lambda_{-+}\lambda_{-+} \Big( & \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^-_{N_A}[\hat{\sigma}^-_{N_A}, \hat{H}_A]\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^+_{1_B}\hat{\sigma}^+_{1_B}) - \\ & - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^-_{N_A}, \hat{H}_A]\hat{\sigma}^-_{N_A}\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^+_{1_B}\hat{\sigma}^+_{1_B}) \Big) \Big\}, \end{aligned} \tag{4.21}$$

and notice that the terms on the first and last lines of (4.21) vanish due to identity $\hat{\sigma}^{\pm}\hat{\sigma}^{\pm} = 0$ (4.54). Taking this into account and regrouping the other four terms the expression for $e_2$ simplifies to:

$$e_2 = i^2|\lambda|^2 \Big\{ \left( \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^-_{N_A}[\hat{\sigma}^+_{N_A}, \hat{H}_A]\right) - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^-_{N_A}, \hat{H}_A]\hat{\sigma}^+_{N_A}\right)\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^+_{1_B}\hat{\sigma}^-_{1_B}) + \\ + \left( \mathrm{Tr}_A \left(\rho_A \hat{\sigma}^+_{N_A}[\hat{\sigma}^-_{N_A}, \hat{H}_A]\right) - \mathrm{Tr}_A \left(\rho_A [\hat{\sigma}^+_{N_A}, \hat{H}_A]\hat{\sigma}^-_{N_A}\right)\right) \mathrm{Tr}_B(\rho_B \hat{\sigma}^-_{1_B}\hat{\sigma}^+_{1_B}) \Big\} \tag{4.22}$$

Additionally, using (4.57), (4.58) and the relation between the ladder operators and Pauli matrices $\hat{\sigma}^{\pm}\hat{\sigma}^{\mp} = 2(1 - \hat{Z})$ we can express this coefficient in terms of one-point and two-point functions of the spin operators:

$$
\begin{aligned}
e_2 =& 2i^2|\lambda|^2\Bigg\{ \left( \mathrm{Tr}_A \left( \rho_A \left[ \hat{\sigma}^-_{N_A}, [\hat{\sigma}^+_{N_A}, \hat{H}_A] \right] \right) + \mathrm{Tr}_A \left( \rho_A \left[ \hat{\sigma}^+_{N_A}, [\hat{\sigma}^-_{N_A}, \hat{H}_A] \right] \right) \right) + \\
&+ \left( \mathrm{Tr}_A \left( \rho_A \left\{ \hat{\sigma}^-_{N_A}, [\hat{\sigma}^+_{N_A}, \hat{H}_A] \right\} \right) - \mathrm{Tr}_A \left( \rho_A \left\{ \hat{\sigma}^+_{N_A}, [\hat{\sigma}^-_{N_A}, \hat{H}_A] \right\} \right) \right) \mathrm{Tr}_B(\rho_B \hat{Z}_{1_B}) \Bigg\} \\
=& 2i^2|\lambda|^2 \Bigg( \left( -16J\langle \hat{Z}_{N-1}\hat{Z}_N \rangle_A - 8h^x_N \langle \hat{X}_N \rangle_A - 16h^z_N \langle \hat{Z}_N \rangle_A \right) + \\
&+ \left( 16J\langle \hat{Z}_{N-1} \rangle_A + 16h^z_N \right) \langle \hat{Z}_{1_B} \rangle_B \Bigg) \\
=& -32i^2|\lambda|^2 \Bigg( J\left( \langle \hat{Z}_{N-1}\hat{Z}_N \rangle_A - \langle \hat{Z}_{N-1} \rangle_A \langle \hat{Z}_{1_B} \rangle_B \right) + \\
&+ \frac{1}{2} h^x_N \langle \hat{X}_N \rangle_A + h^z_N \left( \langle \hat{Z}_N \rangle_A - \langle \hat{Z}_{1_B} \rangle_B \right) \Bigg).
\end{aligned}
\tag{4.23}
$$

With this, we have solved the early time behavior of $E_A(t)$ up to the second order in time, however, there are no analytical relations for the temperature dependence of the one-point and two-point functions at arbitrary field strengths $(h_x, h_z)$. Therefore, in order to evaluate $e_2$, we numerically compute the thermal expectation values in the last line of (4.23).

In Chapter 3, using numerical time evolution of the whole system, we discovered that when $A$ is coupled to an equivalent MFI at temperature $T_B = 0.1$ there is no energy increase in the classical case $h_x = 0$, and on the other extreme, the increase appears for any temperature $T_A$ when the system is at the critical point $h_x = 1, h_z = 0$. Interestingly, moving slightly away from $h_x = 1, h_z = 0$ a finite critical temperature $T_c$ emerges above which the early time energy increase disappears, but below which it is present. The height of the numerically obtained energy bump $E_m$ for those three examples is depicted in Figure 4.3 (left), and the critical temperature for the particular case $h_x = 1$ and $h_z = 0.05$ is $T_c \simeq 77.845J$.

The newly derived analytical relation for the energy (4.6) predicts the existence of an initial energy increase when $e_2 > 0$ in the case under consideration when $e_1 = 0$. Now, using the expansion for $e_2$ in (4.23), we compute and show in Fig. 4.3 (right) the $e_2$ value as a function of $T_A$ for the three particular cases elaborated above. These results confirm our previous findings on the existence of the energy increase. For any $T_A$, $e_2 < 0$ in the classical case, $e_2 > 0$ at the quantum critical

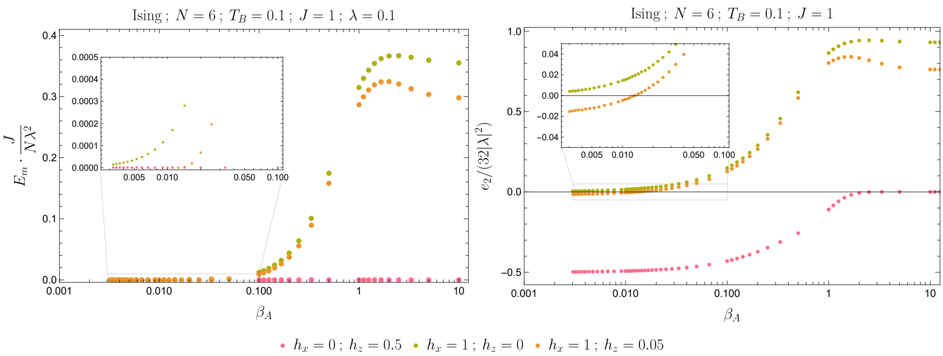


**Figure 4.3.** Existence of the early time energy bump in the classical ($h_x = 0, h_z = 1$), quantum critical ($h_x = 1, h_z = 0$) and arbitrary case with ($h_x = 1, h_z = 0.05$). On the left, the height of the energy bump $E_m$ is presented with $E_m = 0$ indicating its absence. The right panel presents the $e_2$ coefficient (4.23), for the same three models. The energy bump disappears when $e_2 < 0$.

point and when the system is tuned slightly away from it the coefficient changes from a positive sign when $T_A < 77.5$ to negative for $T_A > 77.5$.

To show that this match between the numerical results and the analytical expression is not limited to these three special cases we apply the same reasoning to three other models with results presented in Fig. 4.4. As before, the left panel displays the height of the energy bump $E_m$, which goes to zero when the bump disappears. On the right panel, which plots the second coefficient $e_2$, one notices that it turns negative exactly at the same temperature for which $E_m \to 0$.

This match in the critical temperature demonstrates the equivalence and validity of our two approaches and allows us to use the early time expansion to understand the early time behavior and in particular the quantum energy ruse in the hot system $A$. Note that for the MFI we can write the second coefficient (4.23) as a difference of two separate contributions, $e_2^A$ that depends only on the subsystem under consideration $A$ and $e_2^{AB}$ which depends on both $A$ and $B$, therefore being sensitive on the temperature $T_B$:

$$e_2 = 32|\lambda|^2 \left(e_2^A - e_2^{AB}\right), \tag{4.24a}$$

$$e_2^A = J\langle \hat{Z}_{N-1}\hat{Z}_N\rangle_A + \frac{h_N^x}{2}\langle \hat{X}_N\rangle_A + h_N^z\langle \hat{Z}_N\rangle_A; \tag{4.24b}$$

$$e_2^{AB} = \left(J\langle \hat{Z}_{N-1}\rangle_A + h_N^z\right)\langle \hat{Z}_{1_B}\rangle_B. \tag{4.24c}$$

Here we see that, similar to the SYK model, the result for $e_2^A$ depends only on the

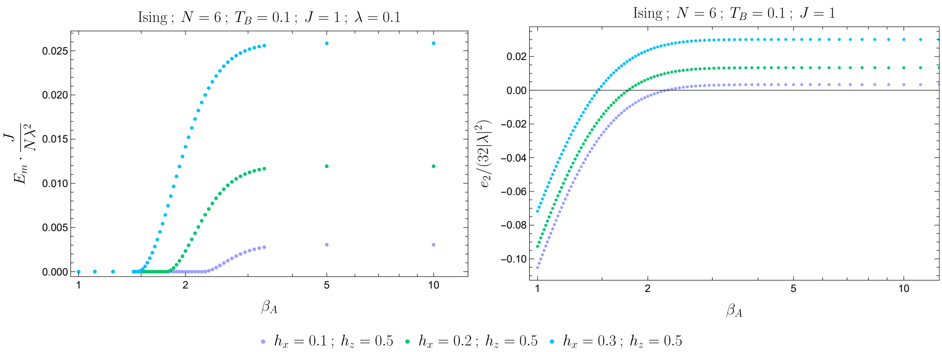


**Figure 4.4.** Disappearance of the energy bump, for three different models, computed by a time evolution of the full model (left) and from the exact thermodynamic evaluation of the $e_2$ coefficient (right).

properties of the subsystem under consideration $A$. However, unlike before, the additional term $e_2^{AB}$ depends also on the subsystem $B$, therefore being sensitive to the temperature $T_B$. Since both of these terms are positive, the energy bump disappears ($e_2 < 0$) in the temperature regime where $e_2^A < e_2^{AB}$, as shown on Fig. 4.5.

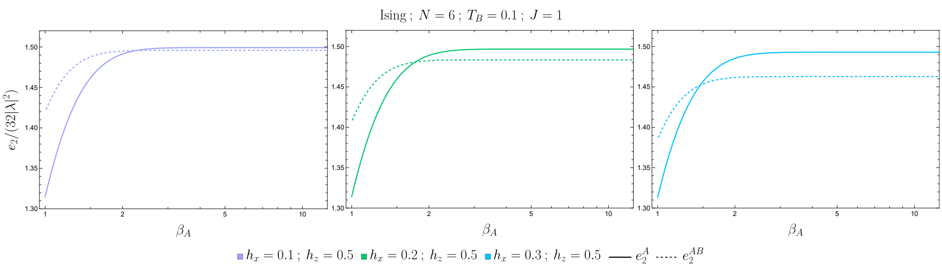


**Figure 4.5.** Those plots present the $e_2^A$ (full line) and the $e_2^{AB}$ (dashed line) terms of the $e_2$ coefficient (4.24), for the same models as on Fig. 4.4.

## 4.3. Conclusion

In this Chapter we presented how to expand the post-quench time evolution of the density matrix into a time series and how to obtain from there the time evolution of the subsystem-reduced density matrices perturbatively in the time $t$ since the quench. We have derived a general expression for the first three coefficients of the expansion and used it to analyze the early time behavior of the subsystems energies for two distinct models. Namely, in order to compare results we studied the same models as Chapter 3: SYK as a representative of a strongly interacting highly entangled, and chaotic model and the Mixed Field Ising in its classical, fully quantum quantum critical, and mixed quantum-classical regimes.

We have shown that this analytical approach not only reproduces the results from our study on a numerical time evolution but it provides an explanation for the omnipresence of the bump in the SYK and its disappearance above a critical temperature in the MFI models. The peculiar nature of the Majorana SYK conspires in such a way that the first three coefficients of the energy expansion are completely independent of the other subsystem resulting in an energy increase even when the analyzed subsystem is at a higher temperature than the other one. On the other hand, the second term $e_2$ of the MFI expansion depends on both subsystems leading to the appearance of the critical temperature $T_c$.

Continuing this analytic approach to derive the third and fourth coefficients of the energy expansion might give access to the time at which the maximum in the bump appears. This would be useful in understanding how the system transitions from this early-time quantum behavior to the late-time evaporation. Additionally, one expects a universal behavior of the energy coefficients in the SYK setup and it would be interesting to see what other thermodynamic quantities appear in those higher-order terms. We leave this for future work.

# 4.A. General State Expansion

*Series expansion of the density matrix* — Here we represent the time-dependent density matrix, after an instantaneous quench (4.1b), as a time series, similar to the main text, under the assumption of a separable initial state $\rho_0 = \rho_A \otimes \rho_B$ composed of arbitrary subsystem states $\rho_\alpha$:

$$\begin{aligned}\rho(t) =& e^{-iHt}\rho_0 e^{iHt} = \left(1 - iHt - \frac{1}{2}H^2t^2 \dots\right)\rho_0\left(1 + iHt - \frac{1}{2}H^2t^2 \dots\right) = \\ =& \sum_{n=0} \frac{t^n}{n!} R_n\end{aligned} \tag{4.25a}$$

$$R_0 = \rho_0 \quad ; \quad R_{n+1} = i[R_n, H] \tag{4.25b}$$

Where $R_n$ are operator-valued coefficients and the first four are given below:

$$\hat{R}_0 = \rho_0 \tag{4.26a}$$

$$\hat{R}_1 = i[\hat{R}_0, H] = i\left([\rho_1, H_1] \otimes \rho_2 + \rho_1 \otimes [\rho_2, H_2] + [\rho_0, H_{int}]\right) \tag{4.26b}$$

$$\hat{R}_2 = i[\hat{R}_1, H] = i\Big([\hat{R}_1, H_1 \otimes \mathbb{1}_{d_2}] + [\hat{R}_1, \mathbb{1}_{d_1} \otimes H_2] + [R_1, H_{int}]\Big) \tag{4.26c}$$

$$\hat{R}_3 = i[R_2, H] = i\Big([\hat{R}_2, H_1 \otimes \mathbb{1}_{d_2}] + [\hat{R}_2, \mathbb{1}_{d_1} \otimes H_2] + [R_2, H_{int}]\Big) \tag{4.26d}$$

Expanding the commutators we obtain:

$$\hat{R}_0 = \rho_0 \tag{4.27a}$$

$$\begin{aligned}\hat{R}_1 =& i[\hat{R}_0, \hat{H}] = i[\rho_A, \hat{H}_A] \otimes \rho_B + i\rho_A \otimes [\rho_B, \hat{H}_B] + \\ &+ i\sum_{IK} \lambda_{IK}[\rho_A \otimes \rho_B, \hat{\Psi}_I \otimes \hat{\Gamma}_K]\end{aligned} \tag{4.27b}$$

$$\begin{aligned}\hat{R}_2 =& i[\hat{R}_1, \hat{H}] = i[\hat{R}_1, \hat{H}_A \otimes \mathbb{1}_{d_B}] + i[\hat{R}_1, \mathbb{1}_{d_A} \otimes \hat{H}_B] + i\sum_{IK} \lambda_{IK}[\hat{R}_1, \hat{\Psi}_I \otimes \hat{\Gamma}_K] \\ =& i^2\Bigg\{\Big[[\rho_A, \hat{H}_A], \hat{H}_A\Big] \otimes \rho_B + \rho_A \otimes \Big[[\rho_B, \hat{H}_B], \hat{H}_B\Big] + 2[\rho_A, \hat{H}_A] \otimes [\rho_B, \hat{H}_B] + \\ &+ \sum_{IK} \lambda_{IK}\bigg(\Big[[\rho_A \otimes \rho_B, \hat{\Psi}_I \otimes \hat{\Gamma}_K], \hat{H}_A\Big] + \Big[[\rho_A \otimes \rho_B, \hat{\Psi}_I \otimes \hat{\Gamma}_K], \hat{H}_B\Big] + \\ &+ \Big[[\rho_A, \hat{H}_A] \otimes \rho_B, \hat{\Psi}_I \otimes \hat{\Gamma}_K\Big] + \Big[\rho_A \otimes [\rho_B, \hat{H}_B], \hat{\Psi}_I \otimes \hat{\Gamma}_K\Big]\bigg) \\ &+ \sum_{IK}\sum_{I'K'} \lambda_{IK}\lambda_{I'K'}\Big[[\rho_A \otimes \rho_B, \hat{\Psi}_I \otimes \hat{\Gamma}_K], \hat{\Psi}_{I'} \otimes \hat{\Gamma}_{K'}\Big]\Bigg\}\end{aligned} \tag{4.27c}$$

$$\hat{R}_3 = i[\hat{R}_2, \hat{H}] = i[\hat{R}_2, \hat{H}_A \otimes \mathbb{1}_{d_B}] + i[\hat{R}_2, \mathbb{1}_{d_A} \otimes \hat{H}_B] + i \sum_{IK} \lambda_{IK} [\hat{R}_2, \hat{\Psi}_I \otimes \hat{\Gamma}_K] \ . \tag{4.27d}$$

Next, we present the operator-valued coefficients $\hat{A}_n$ (4.28) which define the time evolution of the reduced density matrix $\rho_A(t)$ (4.4a). Note that many terms vanish due to (4.39a).

$$\hat{A}_0 = \mathrm{Tr}_B(\hat{R}_0) = \mathrm{Tr}_B(\rho_0) = \rho_A; \tag{4.28a}$$

$$\hat{A}_1 = \mathrm{Tr}_B(\hat{R}_1) = i \Big\{ [\rho_A, \hat{H}_A] + \sum_{IK} \lambda_{IK} [\rho_A, \hat{\Psi}_I] \,\mathrm{Tr}_B(\rho_B \,\hat{\Gamma}_K) \Big\}; \tag{4.28b}$$

$$\begin{aligned} \hat{A}_2 = \mathrm{Tr}_B(\hat{R}_2) = i^2 \Big\{ & \Big[[\rho_A, \hat{H}_A], \hat{H}_A\Big] + \\ & + \sum_{IK} \lambda_{IK} \Big( \Big( \Big[[\rho_A, \hat{\Psi}_I], \hat{H}_A\Big] + \Big[[\rho_A, \hat{H}_A], \hat{\Psi}_I\Big] \Big) \mathrm{Tr}_B(\rho_B \,\hat{\Gamma}_K) + \\ & + [\rho_A, \hat{\Psi}_I] \,\mathrm{Tr}_B([\rho_B, \hat{H}_B], \hat{\Gamma}_K) \Big) \\ & + \sum_{IK} \sum_{K'I'} \lambda_{IK} \lambda_{I'K'} \Big( [\rho_A \,\hat{\Psi}_I, \hat{\Psi}_{I'}] \,\mathrm{Tr}_B(\rho_B \,\hat{\Gamma}_K \,\hat{\Gamma}_{K'}) - \\ & - [\hat{\Psi}_I \,\rho_A, \hat{\Psi}_{I'}] \,\mathrm{Tr}_B(\rho_B \,\hat{\Gamma}_{K'} \,\hat{\Gamma}_K) \Big) \Big\}. \end{aligned} \tag{4.28c}$$

## 4.B. Proper anti-commuting interactions between coupled SYK dots

There are different ways to study two $N$-Majorana SYK dots. For example, one can take $2N$ Majoranas and model the subsystems through the interactions. However, we want to have manifestly separate subsystems so we will generate two Hilbert spaces using the techniques from [115]. First dot has $N_1 = 2K_1$ Majoranas denoted with $\psi_i$ and the second has $N_2 = 2K_2$ Majoranas denoted with $\chi_j$, living in their respective Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\begin{aligned} N_1 = 2K_1: \quad \tilde{\psi}_i \in \mathcal{H}_1, \quad \dim \mathcal{H}_1 = 2^{K_1}; \quad \{\tilde{\psi}_i, \tilde{\psi}_j\} = \delta_{i,j}\mathbb{1} \quad \tilde{\psi}_i^2 = \frac{1}{2}\mathbb{1} \\ [\tilde{\psi}_i, \tilde{\psi}_j] = 2\tilde{\psi}_i\tilde{\psi}_j - \delta_{ij} = -2\tilde{\psi}_j\tilde{\psi}_i + \delta_{ij} \end{aligned} \tag{4.29a}$$

$$[\tilde{\psi}_i, \tilde{\psi}_j\gamma_c] = \{\tilde{\psi}_i, \tilde{\psi}_j\}\gamma_c = \delta_{ij}\gamma_c \quad ; \quad (\psi_i\gamma_c)^2 = -\psi_i^2\gamma_c^2 = -\frac{1}{2}$$

$$N_2 = 2K_2: \quad \tilde{\chi}_i \in \mathcal{H}_2, \quad \dim \mathcal{H}_2 = 2^{K_2}; \quad \{\tilde{\chi}_i, \tilde{\chi}_j\} = \delta_{i,j}\mathbb{1} \quad \tilde{\chi}_i^2 = \frac{1}{2}\mathbb{1} \tag{4.29b}$$

$$h_1 = -\sum_{j=1}^{N_1} J^{(1)}_{j_1 j_2 j_3 j_4} \tilde{\psi}_{j_1} \tilde{\psi}_{j_2} \tilde{\psi}_{j_3} \tilde{\psi}_{j_4} \quad ; \quad h_2 = -\sum_{l=1}^{N_2} J^{(2)}_{l_1 l_2 l_3 l_4} \tilde{\chi}_{l_1} \tilde{\chi}_{l_2} \tilde{\chi}_{l_3} \tilde{\chi}_{l_4} \tag{4.30}$$

The whole Hilbert space consists of the two dots $\mathcal{H}$ has $N = 2K = N_1 + N_2$ Majoranas in total:

$$\begin{aligned} N = N_1 + N_2 : \quad \psi_i, \chi_j \in \mathcal{H}, \quad \dim \mathcal{H} = 2^K \\ \{\psi_i, \psi_j\} = \delta_{i,j}\mathbb{1}, \quad \{\chi_i, \chi_j\} = \delta_{i,j}\mathbb{1}, \quad \{\psi_i, \chi_j\} = 0 \end{aligned} \tag{4.31}$$

In order to generate those Majoranas we need to recall that Majorana operators are closely related to the Clifford algebra of dimension $n = 2k$:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1} \quad \Rightarrow \quad \gamma_i^2 = \mathbb{1} \tag{4.32}$$

$$\gamma_c = (-i)^k \gamma_1 \gamma_2 \ldots \gamma_n, \quad \gamma_c^\dagger = \gamma_c; \quad \gamma_c^2 = \mathbb{1} \quad ; \quad \{\gamma_i, \gamma_c\} = 0 \tag{4.33}$$

## 4.B.1. Numerical implementation of Majoranas

Now we would like to generate the two dots from Clifford algebra $\{\Gamma_i\}$. We will consider the simple case when $N_1 = N_2$ so $N = 2N_1 = 4K_1$, which would be easy to extend to a system of asymmetric dots. The Clifford algebra $\{\Gamma_i\}$ can be written in terms of subsystem algebra $\{\gamma_i\}$:

$$\gamma_i \in \mathcal{H}_1, \quad \dim \mathcal{H}_1 = 2^{K_1} \tag{4.34a}$$

$$\Gamma_i = \gamma_i \otimes \mathbb{1}, \quad i \in \{1, 2, \ldots N_1\} \tag{4.34b}$$

$$\Gamma_{N_1+j} = \gamma_c \otimes \gamma_j, \quad j \in \{1, 2, \ldots N_2\} \tag{4.34c}$$

The Majorana operators are obtained by renormalizing $\{\Gamma_i\}$:

$$\Gamma_1, \Gamma_2 \ldots \Gamma_N \tag{4.35a}$$

$$\psi_i = \frac{1}{\sqrt{2}}\Gamma_i = \frac{1}{\sqrt{2}}\gamma_i \otimes \mathbb{1} \equiv \tilde{\psi}_i \otimes \mathbb{1}, \quad i \in \{1, 2, \ldots N_1\} \tag{4.35b}$$

$$\chi_j = \frac{1}{\sqrt{2}}\Gamma_{N_1+j} = \frac{1}{\sqrt{2}}\gamma_c \otimes \gamma_j \equiv \gamma_c \otimes \tilde{\chi}_j, \quad j \in \{1, 2, \ldots N_2\}. \tag{4.35c}$$

The noteworthy part is the appearance of the matrix $\gamma_c$ that ensures anticommutation between the Majoranas in the two dots:

$$\{\psi_i, \psi_j\} = \frac{1}{2}\{\gamma_i, \gamma_j\} \otimes \mathbb{1} = \delta_{ij}\mathbb{1} \otimes \mathbb{1} \tag{4.36a}$$

$$\{\chi_i, \chi_j\} = \frac{1}{2}\gamma_c^2 \otimes \{\gamma_i, \gamma_j\} \equiv \frac{1}{2}\mathbb{1} \otimes \{\gamma_i, \gamma_j\} = \delta_{ij}\mathbb{1} \otimes \mathbb{1} \tag{4.36b}$$

$$\{\psi_i, \chi_j\} = \frac{1}{2}\{\gamma_i, \gamma_c\} \otimes \gamma_j = 0 \tag{4.36c}$$

In this basis, before the quench, the two Hamiltonians are manifestly decoupled :

$$\begin{aligned} H_1 =& - \sum_{i_1,\dots i_4} J_{i_1\dots i_4}\psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4} \\ =& - \sum_{i_1,\dots i_4} J_{i_1\dots i_4}\tilde{\psi}_{i_1}\tilde{\psi}_{i_2}\tilde{\psi}_{i_3}\tilde{\psi}_{i_4} \otimes \mathbb{1}^4 = h_1 \otimes \mathbb{1} \end{aligned} \tag{4.37a}$$

$$\begin{aligned} H_2 =& - \sum_{l_1,\dots l_4} J_{l_1\dots l_4}\chi_{l_1}\chi_{l_2}\chi_{l_3}\chi_{l_4} \\ =& - \gamma_c^4 \otimes \sum_{l_1,\dots l_4} J_{l_1\dots l_4}\tilde{\chi}_{l_1}\tilde{\chi}_{i_2}\tilde{\chi}_{i_3}\tilde{\chi}_{i_4} \otimes \mathbb{1}^4 = \mathbb{1} \otimes h_2 \end{aligned} \tag{4.37b}$$

$$H_{int} = i\sum_{xy}\lambda_{xy}\psi_x\chi_y = \sum_{xy}\lambda_{xy}\tilde{\psi}_x\gamma_c \otimes \tilde{\chi}_y \tag{4.37c}$$

# 4.C. Operators relations

In this appendix we gather some useful relations needed for coefficients derivation in the early time expansion.

## 4.C.1. General relations

$$[AB, C] = [A, C]B + A[B, C] \tag{4.38a}$$

$$\{AB, C\} = A[B, C] + \{A, C\}B \tag{4.38b}$$

$$[\rho_1 \otimes \rho_2, \hat{O}_X \otimes \hat{O}_Y] = \rho_1\hat{O}_X \otimes \rho_2\hat{O}_Y - \hat{O}_X\rho_1 \otimes \hat{O}_Y\rho_2 \tag{4.38c}$$

$$\mathrm{Tr}\left([\hat{A}, \hat{B}]\right) = \mathrm{Tr}\left(\hat{A}\hat{B}\right) - \mathrm{Tr}\left(\hat{B}\hat{A}\right) = 0 \tag{4.39a}$$

$$\begin{aligned} \mathrm{Tr}_2\left([\hat{\alpha} \otimes \hat{\beta}, \hat{O}_X \otimes \hat{O}_Y]\right) =& \hat{\alpha}\hat{O}_X \otimes \mathrm{Tr}\left(\hat{\beta}\hat{O}_Y\right) - \hat{O}_X\hat{\alpha} \otimes \mathrm{Tr}\left(\hat{O}_Y\hat{\beta}\right) = \\ =& [\hat{\alpha}, \hat{O}_X]\,\mathrm{Tr}\left(\hat{\beta}\hat{O}_Y\right) \end{aligned} \tag{4.39b}$$

$$\begin{aligned} \mathrm{Tr}\left([A, B]C\right) = \mathrm{Tr}\left(ABC - BAC\right) =& \mathrm{Tr}\left(ABC - ACB\right) = \\ =& \mathrm{Tr}\big\{A[B, C]\big\} = \mathrm{Tr}\big\{[C, A]B\big\} \end{aligned} \tag{4.40a}$$

$$\mathrm{Tr}\left([[[[R, A_1], A_2], \cdots], A_{n-1}]A_n\right) = \mathrm{Tr}\left(R, [A_1, [A2, [\cdots, [A_{n-1}, A_n]]]]\right) \tag{4.40b}$$

## 4.C.2. Majoranas

*General Majoranas —*

Here, we consider a system with $N$ Majoranas, forming a $d = \dim(\mathcal{H}) = 2^{N/2}$ dimensional Hilbert space $\mathcal{H}$, and we use the same normalization for the Majoranas as in the numerical code.

$$\{\psi_i, \psi_j\} = \delta_{ij} \quad ; \quad \{\psi_c, \psi_j\} = 0 \quad ; \quad \psi_i^2 = \frac{1}{2}\mathbb{1}_d \quad ; \quad \gamma_c^2 = 1 \quad ; \quad (\psi_i\gamma_c)^2 = -\frac{1}{2} \tag{4.41a}$$

$$[\psi_i, \psi_j] = 2\psi_i\psi_j - \delta_{ij} = -2\psi_j\psi_i + \delta_{ij} \quad ; \quad [\psi_i\gamma_c, \psi_j] = -\{\psi_i, \psi_j\}\gamma_c = -\delta_{ij}\gamma_c \tag{4.41b}$$

Next, we present some common commutators of Majorana strings, but first, we introduce a notation for such strings that will help us write more compact expressions, especially for large $q$ SYKs.

$$\Psi_I^{(q)} = \psi_{i_1}\psi_{i_2}\dots\psi_{i_q} \tag{4.42a}$$

$$\Psi_{I_\sigma}^{(q-1)} = \psi_{i_1}\dots\psi_{i_{\sigma-1}}\psi_{i_{\sigma+1}}\dots\psi_{i_q} \tag{4.42b}$$

The easiest is to start with four Majoranas Strings ($q = 4$):

$$\begin{aligned}[\psi_\alpha, \Psi_I^{(4)}] &= [\psi_\alpha, \psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4}] \\ &= (\delta_{i_1\alpha}\psi_{i_2}\psi_{i_3}\psi_{i_4} - \delta_{i_2\alpha}\psi_{i_1}\psi_{i_3}\psi_{i_4} + \delta_{i_3\alpha}\psi_{i_1}\psi_{i_2}\psi_{i_4} - \delta_{i_4\alpha}\psi_{i_1}\psi_{i_2}\psi_{i_3}) \\ &= \sum_{\sigma=1}^{4} \delta_{\alpha i_\sigma}(-1)^{\sigma+1}\Psi_{I_\sigma}^{(3)}\end{aligned} \tag{4.43a}$$

$$\begin{aligned}[\psi_\alpha\gamma_c, \Psi_I^{(4)}] &= [\psi_\alpha\gamma_c, \psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4}] \\ &= -(\delta_{i_1\alpha}\gamma_c\psi_{i_2}\psi_{i_3}\psi_{i_4} + \delta_{i_2\alpha}\psi_{i_1}\gamma_c\psi_{i_3}\psi_{i_4} + \delta_{i_3\alpha}\psi_{i_1}\psi_{i_2}\gamma_c\psi_{i_4} + \delta_{i_4\alpha}\psi_{i_1}\psi_{i_2}\psi_{i_3}\gamma_c) \\ &= -\gamma_c\sum_{\sigma=1}^{4} \delta_{\alpha i_\sigma}(-1)^{\sigma+1}\Psi_{I_\sigma}^{(3)}\end{aligned} \tag{4.43b}$$

Then, those results can be generalized for an arbitrarily sized string with ($q = 2p$):

$$[\psi_\alpha, \Psi_I^{(q)}] = \sum_{\sigma=1}^{q} \delta_{\alpha i_\sigma}(-1)^{\sigma+1}\Psi_{I_\sigma}^{(q-1)} \tag{4.44a}$$

$$[\psi_\alpha\gamma_c, \Psi_I^{(q)}] = -\gamma_c\sum_{\sigma=1}^{q} \delta_{\alpha i_\sigma}(-1)^{\sigma+1}\Psi_{I_\sigma}^{(q-1)} \tag{4.44b}$$

*SYK like interaction —*

Relations obtained in the previous paragraph can be used for commutators of an SYK Hamiltonian with a string of Majoranas. Initially we're interested in a ($q=4$) SYK Hamiltonians (4.9) :

$$\begin{aligned}[\psi_\alpha, H^{(4)}] =& i^2 \sum_{i_1\cdots i_4=1}^{N} J_{i_1 i_2 i_3 i_4}[\psi_\alpha, \psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4}] \\ =& -4i^2 \sum_{i_1 i_2 i_3=1}^{N} J_{i_1 i_2 i_3 \alpha}\psi_{i_1}\psi_{i_2}\psi_{i_3},\end{aligned} \tag{4.45a}$$

$$\begin{aligned}[\psi_\alpha\gamma_c, H^{(4)}] =& i^2 \sum_{i_1\cdots i_4=1}^{N} J_{i_1 i_2 i_3 i_4}[\psi_\alpha\gamma_c, \psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4}] \\ =& 4i^2 \sum_{i_1 i_2 i_3=1}^{N} J_{i_1 i_2 i_3 \alpha}\gamma_c\psi_{i_1}\psi_{i_2}\psi_{i_3}.\end{aligned} \tag{4.45b}$$

Same as before, those commutators can be easily generalized to ($q=2p$) SYK Hamiltonians:

$$[\psi_\alpha, H^{(q)}] = i^{q/2}\sum_{I=1}^{N} J_I[\psi_\alpha, \Psi_I^{(q)}] = -q i^{q/2} \sum_{i_1\cdots i_{q-1}=1}^{N} J_{i_1\cdots i_{q-1}\alpha}\Psi_{I_q}^{(q-1)}, \tag{4.46a}$$

$$[\psi_\alpha\gamma_c, H^{(q)}] = i^{q/2}\sum_{I=1}^{N} J_I[\psi_\alpha\gamma_c, \Psi_I^{(q)}] = q i^{q/2} \sum_{i_1\cdots i_{q-1}=1}^{N} J_{i_1\cdots i_{q-1}\alpha}\gamma_c\Psi_{I_q}^{(q-1)}. \tag{4.46b}$$

When analyzing the time evolution of operators similar expressions to (4.46) appear with an additional sum over the Majorana field ($\psi_\alpha$):

$$\begin{aligned}\sum_{\alpha=1}^{N}\psi_\alpha[\psi_\alpha, H^{(q)}] =& -q i^{q/2} \sum_{i_1\cdots i_{q-1}=1}^{N}\sum_{\alpha=1}^{N} J_{i_1\cdots i_{q-1}\alpha}\psi_\alpha\Psi_{I_q}^{(q-1)} \\ =& q i^{q/2} \sum_{i_1\cdots i_{q-1}=1}^{N}\sum_{\alpha=1}^{N} J_{i_1\cdots i_{q-1}\alpha}\Psi_{I_q}^{(q-1)}\psi_\alpha = qH^{(q)}\end{aligned} \tag{4.47a}$$

$$\begin{aligned}\sum_{\alpha=1}^{N}\psi_\alpha\gamma_c[\psi_\alpha\gamma_c, H^{(q)}] =& q i^{q/2} \sum_{i_1\cdots i_{q-1}=1}^{N}\sum_{\alpha=1}^{N} J_{i_1\cdots i_{q-1}\alpha}\psi_\alpha\gamma_c^2\Psi_{I_q}^{(q-1)} \\ =& -q i^{q/2} \sum_{i_1\cdots i_{q-1}=1}^{N}\sum_{\alpha=1}^{N} J_{i_1\cdots i_{q-1}\alpha}\Psi_{I_q}^{(q-1)}\psi_\alpha = -qH^{(q)},\end{aligned} \tag{4.47b}$$

$$\sum_{\alpha=1}^{N}[\psi_\alpha, H^{(q)}]\psi_\alpha = -\,qH^{(q)}, \tag{4.48a}$$

$$\sum_{\alpha=1}^{N}[\psi_\alpha\gamma_c, H^{(q)}]\psi_\alpha\gamma_c = qH^{(q)}, \tag{4.48b}$$

where we have used ($\gamma_c^2 = 1$) and the fact that the interaction constant ($J_{i_1\cdots i_{q-1}\alpha}$) vanishes in the case of at least two identical indices, hence permuting ($\psi_\alpha$) past the other Majoranas results only in an additional minus sign since $\big((-1)^{q-1} = (-1)^{2p-1} = -1\big)$.

Other related expressions are:

$$\sum_{\alpha=1}^{N}[\psi_\alpha, [\psi_\alpha, H^{(q)}]] = 2qH^{(q)}, \tag{4.49a}$$

$$\sum_{\alpha=1}^{N}[\psi_\alpha\gamma_c[\psi_\alpha\gamma_c, H^{(q)}]] = -2qH^{(q)}. \tag{4.49b}$$

$$\begin{aligned}
&\sum_{\alpha=1}^{N}[\psi_\alpha\gamma_c, [H^{(q)}, [\psi_\alpha\gamma_c, H^{(q)}]]] = \\
&\quad=\sum_{\alpha=1}^{N}\big(2\psi_\alpha\gamma_c H^{(q)}\psi_\alpha\gamma_c H^{(q)} - 2H^{(q)}\psi_\alpha\gamma_c H^{(q)}\psi_\alpha\gamma_c - (\psi_\alpha\gamma_c)^2(H^{(q)})^2 + \\
&\quad\quad + (\psi_\alpha\gamma_c)(H^{(q)})^2(\psi_\alpha\gamma_c) - (\psi_\alpha\gamma_c)(H^{(q)})^2(\psi_\alpha\gamma_c) + (H^{(q)})^2(\psi_\alpha\gamma_c)^2\big) \\
&\quad=2\sum_{\alpha=1}^{N}\big(\psi_\alpha\gamma_c H^{(q)}\psi_\alpha\gamma_c H^{(q)} - H^{(q)}\psi_\alpha\gamma_c H^{(q)}\psi_\alpha\gamma_c\big) \\
&\quad=2\sum_{\alpha=1}^{N}[\psi_\alpha\gamma_c, H^{(q)}\psi_\alpha\gamma_c H^{(q)}] \\
&\quad=2\sum_{\alpha=1}^{N}\big([\psi_\alpha\gamma_c, H^{(q)}]\psi_\alpha\gamma_c H^{(q)} + H^{(q)}\psi_\alpha\gamma_c[\psi_\alpha\gamma_c, H^{(q)}]\big) = 0
\end{aligned} \tag{4.50}$$

### 4.C.3. Pauli

*General Pauli matrices - Single site —*

Note that the following (anti)commutation relations are for same site operators. Operators on different sites act on different spin states, so they commute.

$$X \equiv \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad Y \equiv \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad Z \equiv \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.51}$$

$$\sigma^{\alpha\dagger} = \sigma^\alpha \quad ; \quad \sigma^{\alpha 2} = \mathbb{1}_d \quad ; \quad \alpha, \beta, \gamma \in \{x, y, z\} \tag{4.52a}$$

$$\{\sigma^\alpha, \sigma^\beta\} = 2\delta_{\alpha\beta}\mathbb{1}_d \quad ; \quad [\sigma^\alpha, \sigma^\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma^\gamma \tag{4.52b}$$

From the Pauli matrices ladder operators can be formed

$$\sigma^+ \equiv X + iY = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad ; \quad \sigma^- \equiv X - iY = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tag{4.53}$$

$$\begin{gathered}
\sigma^{\pm\dagger} = \sigma^\mp \quad ; \quad \sigma^\pm\sigma^\pm = 0 \quad ; \quad \sigma^\pm\sigma^\mp = 2\,(1 \pm Z) \\
[\sigma^\pm, X] = \pm 2Z \quad ; \quad [\sigma^\pm, Y] = \mp 2iZ \quad ; \quad [\sigma^\pm, Z] = \mp 2\sigma^\pm \quad ; \quad [\sigma^+, \sigma^-] = 4Z \\
\{\sigma^\pm, X\} = 2\mathbb{1} \quad ; \quad \{\sigma^\pm, Y\} = \pm i\mathbb{1} \quad ; \quad \{\sigma^\pm, Z\} = 0 \quad ; \quad \{\sigma^+, \sigma^-\} = 4\mathbb{1}
\end{gathered}$$

*Mixed Field Ising with position-dependent fields* $(h_i^x, h_i^z)$ *—*

Here we present commutator/anti-commutator relations for the most generic Mixed Field Ising with position-dependent couplings (4.55). Setting the couplings to the same constant at each site one recovers the standard MFI.

$$H_{MFI1} = -J \sum_{i=1}^{N-1+p_f} \hat{Z}_i\hat{Z}_{i+1} - \sum_{i=1}^{N} h_i^x \hat{X}_i - \sum_{i=1}^{N} h_i^z \hat{Z}_i \tag{4.55}$$

$$[\hat{\sigma}_N^-, \hat{H}_{MFI1}] = -\,2J\left(\hat{Z}_{N-1}\hat{\sigma}_N^- + p_f\hat{\sigma}_N^-\hat{Z}_1\right) + 2h_N^x\hat{Z}_N - 2h_N^z\hat{\sigma}_N^- \tag{4.56a}$$

$$[\hat{\sigma}_N^+, \hat{H}_{MFI1}] = -\,2J\left(-Z_{N-1}\hat{\sigma}_N^+ - p_f\hat{\sigma}_N^+ Z_1\right) - 2h_N^x Z_N + 2h_N^z\hat{\sigma}_N^+ \tag{4.56b}$$

$$[\hat{\sigma}_N^+, [\hat{\sigma}_N^-, \hat{H}_{MFI1}]] = -8J\left(\hat{Z}_{N-1}\hat{Z}_N + p_f\hat{Z}_N\hat{Z}_1\right) - 4h_N^x\hat{\sigma}_N^+ - 8h_N^z\hat{Z}_N \tag{4.57a}$$

$$[\hat{\sigma}_N^-, [\hat{\sigma}_N^+, \hat{H}_{MFI1}]] = -8J\left(\hat{Z}_{N-1}\hat{Z}_N + p_f\hat{Z}_N\hat{Z}_1\right) - 4h_N^x\hat{\sigma}_N^- - 8h_N^z\hat{Z}_N \quad (4.57b)$$

$$[\hat{\sigma}_N^-, [\hat{\sigma}_N^+, \hat{H}_{MFI1}]] + [\hat{\sigma}_N^+, [\hat{\sigma}_N^-, \hat{H}_{MFI1}]] = \\ = -16J\left(\hat{Z}_{N-1}\hat{Z}_N + p_f\hat{Z}_N\hat{Z}_1\right) - 8h_N^x\hat{X}_N - 16h_N^z\hat{Z}_N \quad (4.57c)$$

$$\{\hat{\sigma}_N^+, [\hat{\sigma}_N^-, \hat{H}_{MFI1}]\} = -8J\left(\hat{Z}_{N-1} + p_f\hat{Z}_1\right) - 8h_N^z \quad (4.58a)$$

$$\{\hat{\sigma}_N^-, [\hat{\sigma}_N^+, \hat{H}_{MFI1}]\} = 8J\left(\hat{Z}_{N-1} + p_f\hat{Z}_1\right) + 8h_N^z \quad (4.58b)$$

$$\{\hat{\sigma}_N^-, [\hat{\sigma}_N^+, \hat{H}_{MFI1}]\} - \{\hat{\sigma}_N^+, [\hat{\sigma}_N^-, \hat{H}_{MFI1}]\} = 16J\left(\hat{Z}_{N-1} + p_f\hat{Z}_1\right) + 16h_N^z \quad (4.58c)$$