Dynamics of coupled quantum systems

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1 Introduction

1.1. Two roads to thermal equilibrium

Soon after the discovery of quantum mechanics, it was realized that the probability-preserving unitary evolution of the wave function clashed with the entropy production needed to explain thermalization. The straightforward resolution was to make the notion of an environmental heat bath, or more general any environment, a more principled one. So-called open quantum systems indeed thermalize beautifully though they sidestep many subtle issues intrinsic to quantum mechanics. Thermalization in these systems is not much different from Boltzmann's original fully classical picture of chaotic mixing due to microscopic collisions.[1]

This all changed rather abruptly with the numerical findings of Rigol, Dunjko, and Olshanii. Simulating the unitary time evolution of a finite number of bosonic quantum excitations, they surprisingly found that the density matrix of this system rapidly becomes nearly indistinguishable from a thermal density matrix [1]. Mathematically this could not be the exact thermal density matrix, but the difference was exponentially small. It turned out that this rapid near-indistinguishability of a unitarily evolved many-body density matrix from a thermal one was already predicted independently by Deutsch [2] and Srednicki [3], who coined it the *Eigenstate Thermalization Hypothesis* (ETH). In the simplest of terms, their result boils down to the sleight of hand that in a quantum system the system itself can act as a heat bath due to its parametrically larger Hilbert space.[2] Physically what ETH really emphasizes is that a specific quantum aspect — entanglement — does not only play an important role in thermalization but can actually *overpower* the semi-classical Boltzmann point of view.

This poses a deeper question. Given a dynamical many-body system, there are now two drastically different ways for this system to approach equilibrium.

[1] Recall that even in classical physics there is an inherent paradox between thermalization and Liouville's theorem that the volume of phase-space is preserved under classical Hamiltonian evolution. The extreme sensitivity to initial conditions inherent in chaotic systems resolves this paradox. In quantum systems, it is the connection with the heat-bath that allows an initially fully determined initial state to mix and be considered an ensemble average.

[2] A system of N particles has a $6N$-dimensional classical phase space, but a $\sim e^N$-dimensional quantum Hilbert space.

Though not so much voiced in the literature [4], this has triggered a consistently recurring debate on whether a system thermalizes classically (Boltzmann) or quantum-mechanically (ETH) and what determines this (see e.g. [5] for an attempt to unify the descriptions). Nearly non-interacting theories (dilute gases) clearly ought to be an example of the former. By contrast strongly interacting theories or more precisely densely entangled theories should exhibit ETH. But where is the boundary?

This is not only a philosophical question. Experiments on cold atoms have already passed the threshold of the number of constituents where ETH should apply and it has conclusively been seen [6, 7, 8, 9, 10, 11]. These same experiments have also provided a controlled avenue into non-equilibrium phenomena. Far-from-equilibrium physics is often cast aside in introductory physics. The intuition is that the extreme sensitivity to initial conditions in any rapid change makes it intractable, non-universal and therefore not interesting. Certainly, this last statement does not hold water: there is an enormous amount of interesting far-from-equilibrium physics (see e.g. [12]).

This thesis will explore these questions by studying the time evolution of quantum systems after a rapid change: a so-called quench, with focus on the early-time non-equilibrium behavior and the transition to late-time classical thermalization. With this method, we probed the differences in the early stage of thermalization between systems with a chaotic spectrum like the Mixed Field Ising, ones that additionally display exponential out-of-time-ordered correlation growth, as is the case with the Sachdev-Ye-Kitaev model, and quantum integrable systems. This deep dive into the early time evolution of those systems provided us with enlightening intuition that proved crucial for discerning between quantum and classical features that are intertwined during the evolution of a quantum system far from equilibrium. It also helped us solve the paradox of temperature rise in a hotter system when coupled to a colder bath, which was initially identified in [13] based on the energy increase of the hotter system. Interestingly enough, similar behavior shows up in other physical systems during the early stages of their thermalization. For example, it was reported in studies of evaporative black hole formation [14] which, on the gravity side, is explained by the averaged null energy condition [15] and in this thesis was related to the von Neumann and relative entropy of the quantum system, which paves a way for a future holographic connection in the same spirit as [16]. Another case in point is when the systems under consideration are charged, chaotic, and strongly entangled, e.g. charged SYKs whose energy and charge transport, on the road to thermalization, display features subtly related to the linear in T resistivity of strange metals [17, 18]. While it might seem unusual, it is in accord with other results that use SYK chains as toy models for studying high T_c superconductors [19, 20].

1.2. Quantum thermodynamics and quenches

The role of entanglement or strong quantum rather than classical correlations is more readily encoded in the density matrix of the quantum system than its wavefunction. Given a basis $|\alpha\rangle$ of states in the Hilbert space, a generic state $|\psi\rangle = c_\alpha|\alpha\rangle$ that is normalized $\sum_\alpha c_\alpha c_\alpha^\dagger = 1$ corresponds to the density matrix

$$\rho = |\psi\rangle\langle\psi| = \sum_{\alpha,\beta} c_\alpha c_\beta^\dagger |\alpha\rangle\langle\beta| \tag{1.1}$$

The density matrix can also describe an ensemble of quantum states which is the basis of quantum thermodynamics. For example, if the system can be found in the $i-$th vector state from the set $\{|\psi_i\rangle\}$ with a probability p_i, its state is given by the following density matrix:

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \tag{1.2}$$

As the density matrix consists of a bra and a ket, it can also be seen as an operator. From the Schrödinger equation we directly see that it evolves with time as

$$\rho(t) = \hat{U}_t \rho \hat{U}_t^\dagger \ , \quad \hat{U}_t = e^{-i\hat{H}t/\hbar} \tag{1.3}$$

The expectation value of an observable $\hat{O}$ is computed as a trace of the state times the observable:

$$O = \mathrm{Tr}\left(\rho\hat{O}\right). \tag{1.4}$$

Density matrices are positive operators with trace one ($\mathrm{Tr}(\rho) = \sum_i p_i = 1$), which asserts conservation of probability. As a positive operator, it has a spectral decomposition:

$$\rho = \sum_i \lambda_i \, |i\rangle\langle i| \, , \quad \langle i|i\rangle = 1, \quad \lambda_i \leq 1 \tag{1.5}$$

where the vectors $|i\rangle$ are ortho-normal, and λ_i are real, non-negative eigenvalues of ρ. The quantum systems that can be described by a single vector $|\psi\rangle \in \mathcal{H}$ with the equivalent density matrix representation $\rho = |\psi\rangle\langle\psi|$ are known as pure states. A system represented by an ensemble, i.e. a mixture of pure states $\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$ is accordingly named a mixed state. For the density matrix of a pure state it holds that $\mathrm{Tr}\,\rho^2 = 1$, whereas a mixed state has $\mathrm{Tr}\,\rho^2 < 1$.

Another important difference between those two types of states is their von Neumann entropy:

$$S_{\mathrm{vN}} = -\,\mathrm{Tr}\,(\rho \ln \rho) = -\sum_i \lambda_i \ln \lambda_i, \tag{1.6}$$

where the last expression is obtained from the spectral decomposition of the density matrix. Now, combining (1.6) and (1.5) we immediately deduce that pure states ($\lambda = 1$) have vanishing von Neumann entropy, wheres for mixed states ($\lambda_i < 1$) S_{vN} is a positive quantity.

A quintessential mixed state is the density operator that describes a thermal quantum system at inverse temperature $\beta = 1/T$. Classical systems with constant temperature are described by the canonical ensemble, likewise, quantum thermal states are represented by an ensemble of the Hamiltonian's eigenstates $\{|n\rangle\}$ with probabilities $e^{-\beta E_n}/Z_\beta$:

$$\rho_\beta = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n}\,|n\rangle\langle n| \equiv \frac{1}{Z_\beta}e^{-\beta\hat{H}}, \quad Z_\beta = \text{Tr}\left(e^{-\beta\hat{H}}\right). \tag{1.7}$$

Here, Z_β is the partition function and the free energy of the system is obtained through the well-known relation $F_\beta = -\frac{1}{\beta}\ln Z_\beta$. It is easy to check this is a mixed state, since $\text{Tr}\left(\rho_\beta^2\right) = \sum_n e^{-2\beta E_n}/Z_\beta^2 < 1$. It has von Neumann entropy:

$$S_\beta = -\frac{1}{Z_\beta}\sum_n e^{-\beta E_n}\ln\left(e^{-\beta E_n}/Z_\beta\right) = \beta\sum_n E_n e^{-\beta E_n} - \ln Z_\beta = \beta(E_\beta - F_\beta). \tag{1.8}$$

It is important to note that the von Neuman entropy of a Gibbs state at inverse temperature β is identical to the thermal entropy of a thermal system with internal energy equal to the energy of the state $E_\beta = \text{Tr}\left(\rho_\beta\hat{H}\right)$ and free energy F_β. This is the connection with classical thermodynamics.

Another quantity, important for the discussions in this thesis, is the relative entropy between two density matrices $\rho, \tilde{\rho} \in \mathcal{B}(\mathcal{H})$:

$$D\left(\rho||\tilde{\rho}\right) = \text{Tr}\,\rho\left(\ln\rho - \ln\tilde{\rho}\right) \geq 0. \tag{1.9}$$

which is often used in both quantum information processing [21] and quantum thermodynamics [22] to distinguish between two quantum states and as a measure of the irreversibility of a thermodynamic process [23]. In Chapter 2 and Chapter 3 we use the relative entropy as a measure of how different a certain state is from a corresponding thermal state with the same energy $D(\rho||\rho_\beta)$.

The full potential of the density matrix approach comes to light when studying composite quantum systems, as a descriptive tool for the subsystems. Imagine a system in a state ρ_{AB} that is composed of two subsystems A and B. If we are only interested in, or have access to, the subsystem A, we can study it through its reduced density matrix ρ_A, obtained by tracing out B:

$$\rho_A = \text{Tr}_B\left(\rho_{AB}\right) \tag{1.10}$$

Naturally, for two non-interacting systems the full density matrix is a tensor product state $\rho_{AB} = \rho_A \otimes \rho_B$, and the reduced density matrix is equal to the individual state (e.g. $\mathrm{Tr}_B(\rho_{AB}) = \rho_A \,\mathrm{Tr}_B(\rho_B) = \rho_A$), where we used that density operators are trace 1. In general, for interacting subsystems, the full state is not a tensor product and the reduced density matrix encodes information about the correlations between the two subsystems which are reflected in their von Neumann entropies (e.g. $S_A = -\,\mathrm{Tr}\,(\rho_A \ln \rho_A)$). The simplest example is the entangled state between two quantum spins $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)$. Tracing over the second spin one finds that the reduced density matrix $\rho_A = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$ has von Neumann entropy $S_{\rho_A} = \ln(2)$. This is why sometimes S_{vN} is called entanglement entropy. However, one has to be extremely careful with this nomenclature, for, as we've seen above, the von Neumann entropy also captures the thermal entropy of the system and there is no practical way of distinguishing those two contributions. Only at zero temperature, in the absence of the thermal contribution according to the third law of thermodynamics, is the von Neumann entropy proportional to the entanglement between the subsystems.

Such composite systems are the focus of this thesis. More precisely we consider situations where initially ($t < 0$) both subsystems A and B do not interact and are governed by their respective Hamiltonians $\hat{H}_A$ and $\hat{H}_B$. Additionally, we shall assume that each of them has fully been made to relax to independently to thermal equilibrium at temperature T_A and T_B, respectively, and the whole system is in a decoupled state $\rho_0 = \rho_{T_A} \otimes \rho_{T_B}$. Then, at time $t = 0$ we quench the system by instantaneously turning on an interaction between A and B, modeled with the Hamiltonian H_{int}, and for times ($t \geq 0$) the evolution of the combined system is governed with the following Hamiltonian:

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{int} \tag{1.11}$$

Using concepts introduced in this section we study what happens to each subsystem after the quench. Analyzing the interplay between thermal fluxes, correlations, charged currents, and the exchange of information between A and B we strive to explain the post-quench non-equilibrium dynamics, the time-scale for the transition between quantum and conventional hydrodynamic behavior and propose experimental applications of the discovered quantum features.

1.3. Quantum Chaos "Quantum chaology"

In the introduction we pointed out the difference between the thermalization mechanisms in classical systems and quantum systems. Classical Boltzmannian thermalization relies on the assumption of chaos; thermalization in closed quantum systems on eigenstate thermalization. Nevertheless, some notion of mixing or information scrambling as it has been recently called, must also occur in quantum

systems. For quantum systems, there are two notions of chaos. One is defined by the early time evolution of quantum systems that display exponential growth of an observable known as out-of-time-correlator (OTOC) Section 1.3.1, analogously to exponentially divergent trajectories in classical phase space. The other definition is based on some of the Hamiltonian's spectral properties, and their relation to random matrices, as exposed in Section 1.3.2 which is significant for understanding the late-time evolution of quantum systems and especially their thermalization, in light of the Eigenstate Thermalization Hypothesis (ETH). The natural time-scale associated with the two viewpoints distinguishes them, but also provides a window on a unifying viewpoint [5]. We shall not pursue this question directly, but for this thesis it is useful to know the underlying thoughts concepts and concepts in more detail and we briefly review them here.

1.3.1. Exponential growth of OTOC

The defining feature of classically chaotic dynamics is the exponential sensitivity of the system on the initial conditions. Namely, an infinitesimal change in $\delta q^j(0)$ can result in exponentially diverging trajectories (1.12), known as the butterfly effect. For a Hamiltonian systems, where dynamics is encoded in Poisson brackets, this is defined by:

$$|\{q^i(t), p^j(0)\}_c| = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \approx \left|\frac{\delta q^i(t)}{\delta q^j(0)}\right| \sim e^{\lambda t} \tag{1.12}$$

The rate of exponential growth is determined by the Lyapunov coefficient λ, which is a property of the system. Going to quantum systems, this Poisson bracket can be generalized to a commutator to construct an observable $C(t)$ such that it captures the early time chaotic behavior from the correlations between two reasonably local Hermitian operators V and W, on which one has a free choice as long as they're not conserved charges and are simple [24].

$$\begin{aligned} C(t) =& -\operatorname{Tr}\left(\rho^{1/2}[V(t), W(0)]\rho^{1/2}[V(t), W(0)]\right) = \\ =& 2\operatorname{Tr}\left(V(t)\rho^{1/2}V(t)W\rho^{1/2}W\right) - \operatorname{OTOC}\left(t - \frac{i\beta}{4}\right) - \operatorname{OTOC}\left(t + \frac{i\beta}{4}\right) \end{aligned} \tag{1.13a}$$

$$\operatorname{OTOC}(t) = \operatorname{Tr}\left(\rho^{1/4}V(t)\rho^{1/4}W\rho^{1/4}V(t)\rho^{1/2}W\right) \tag{1.13b}$$

Here ρ is the thermal density matrix of the system. To avoid confusion with quantum interference, one has squared the amplitude [24] and obtains an estimator of the effect an initial perturbation $W(0)$ has on a measurement of $V(t)$ at a time t. The choice of the particular regularization ($\rho^{1/4} = \left(e^{-\beta H}/Z\right)^{1/4}$) in the $OTOC(t)$ was initially proposed on the basis of hermiticity [25], and later

was proven to have an even more profound reasoning as it most closely reflects physical microscopic chaos [26]. In systems with a small parameter $\epsilon \ll 1$, the communtator-squared $C(t) \sim \epsilon^2 e^{2\lambda t}$ will have exponential growth in time with an exponent λ, which is usually used as an indication of chaotic behavior. In known systems $\epsilon = \hbar$ or $\epsilon = 1/N$ with N the number of (field theory) degrees of freedom. Different from classical physics, this exponential growth continues up until the scrambling time $t_* \sim \beta \log 1/\epsilon$, It was shown in [25] that, under some physically motivated assumptions, the rate at which a given system can scramble information is bounded from above:

$$\lambda \leq \frac{2\pi k_B}{\hbar \beta}. \tag{1.14}$$

Systems that saturate this bound are known as "fast scramblers" and the Sachdev-Ye-Kitaev model, which we shall review shortly, is one of them and is the main protagonist of this thesis.

1.3.2. Spectral Chaos

The other definition of quantum chaos is inspired by similarities between the spectral properties of Random Matrices (RM) and certain Hamiltonians. We know that for large classical systems the exact knowledge of the position and momentum of each degree of freedom, even if theoretically possible, is practically intractable so we study those systems using statistical mechanics. Similarly, for quantum systems, where the Hilbert space scales exponentially with the degrees of freedom, the exact determination of each eigenstate and corresponding eigenvalue is unfeasible, and of limited usefulness, hence one studies their statistical properties. The origins of this idea are in Wigner's research on large nuclei, and then it advanced into one of the most useful approaches in understanding many-body quantum systems. This development was substantiated by the fact that in the middle of the spectrum, spectral properties (e.g. nearest neighbor spacing, spectral form factor) of many-body Hamiltonians effectively resemble those of random matrices. To better understand this line of reasoning, we introduce some relevant concepts of Random Matrix Theory (RMT) [27, 28, 4] and will comment on how they relate to chaotic quantum systems.

We are interested in square $N \times N$ matrices $\mathcal{H}$ drawn from a probability distribution function $P(\mathcal{H})$. Broadly speaking, there are three different Gaussian ensembles based on the invariance properties of the probability (1.15a). One of them is the Gaussian Orthogonal Ensemble (GOE), named due to its invariance under orthogonal transformation, which samples real matrices and is used for modeling Hamiltonians with time-reversal symmetry. Next is the Gaussian Unitary Ensemble (GUE) which samples Hermitian matrices that can represent a generic Hamiltonian without time reversal or rotational symmetry. The third class is the

Gaussian symplectic ensemble (GSE) which is useful for studying Hamiltonians with time-reversal symmetry but broken rotational symmetry. Each of the Gaussian ensembles (1.15b) is denoted by their Dyson index $\beta = 1$, $\beta = 2$, and $\beta = 4$ respectively [29, 30, 31], which counts the number of real degrees of freedom per matrix element.

$$P(\mathcal{H})d\mathcal{H} = P(\mathcal{H}')d\mathcal{H}' \quad ; \quad \mathcal{H}' = \mathcal{W}^{-1}\mathcal{H}\mathcal{W} \tag{1.15a}$$

$$P_\beta(\mathcal{H}) = \frac{1}{Z_\beta}\exp\left\{-\beta\frac{N}{4}\operatorname{Tr}\mathcal{H}^2\right\} \tag{1.15b}$$

Where Z_β is an ensemble-dependent normalization constant. Using the Gaussian PDFs (1.15b) one can derive the joint probability distribution of the eigenvalues $\{\lambda\} = \{\lambda_1, \lambda_2 \dots \lambda_N\}$:

$$P_\beta(\{\lambda\}) = \frac{1}{Z_\beta}\exp\left\{-\beta\frac{N}{4}\sum_i \lambda_i^2 + \beta\sum_{i<j}\log\left|\lambda_i - \lambda_j\right|\right\}, \tag{1.16}$$

which is a useful object for studying their spectral properties. We note here that the logarithmic term acts as a repulsive potential between two eigenstates, effectively preventing coincident eigenstates which is the mathematical explanation behind the famous level repulsion in quantum mechanics. Integrating (1.16) over all but one eigenvalue the density of states can be derived [27], which in the limit $N \to \infty$ simplifies to the Wigner semicircle distribution:

$$\rho(\lambda) = \lim_{N\to\infty}\int P(\lambda, \lambda_2, \dots \lambda_N)\prod_{i=2}^{N} d\lambda_i = \begin{cases}\frac{1}{\pi}\sqrt{1-\lambda^2} & |\lambda| \le 1\\ 0 & |\lambda| > 1\end{cases}. \tag{1.17}$$

Another important spectral property is the distribution of the normalized level spacing between two adjacent eigenstates $s = (\lambda_n - \lambda_{n-1})/\langle s\rangle$, where $\langle s\rangle = \langle\lambda_n - \lambda_{n-1}\rangle$. In the limit $N \to \infty$ its probability distribution — first surmised by Wigner — can be analytically computed [32].

$$P_\beta(s) = \begin{cases}\frac{\pi}{2}s e^{-\frac{s^2\pi}{4}} & \beta = 1\\ \frac{32}{\pi^2}s^2 e^{-\frac{s^2 4}{\pi}} & \beta = 2\\ \frac{2^{18}}{3^6\pi^3}s^4 e^{-\frac{s^2 64}{9\pi}} & \beta = 4\end{cases} \tag{1.18}$$

The turnover between the polynomial rise at low s and the exponential decay at large s shows the level repulsion between states whose eigenvalues are closer together than the average.

As mentioned before, large nuclei were the initial testing ground for the RMT applicability of to quantum many-body systems, however, the connection between random matrix theory and quantum chaos was made by Bohigas, Giannoni and Schmidt [33]. Studying the spectrum of a quantum particle in a Sinai billiard potential, they discovered that in the semi-classical limit, the nearest neighbor statistics match the GOE ($\beta = 1$) Wigner surmise . This observation led them to conjecture that this is more general: the nearest neighbor statistics of quantum systems with a classically chaotic counterpart can be described with RMT. Afterward, this conjecture has also been used the other way around, namely as a diagnosis of quantum chaos even for systems that don't have good classical analogs, like quantum chains, and lattice fermions [34, 35]. The corollary of this conjecture was made by Berry and Tabor, who observed that in quantum systems that are non-chaotic in the classical limit the statistic of s exhibits Poisson statistics instead of Wigner-Dyson statistics [36]. [3]

It is this connection between chaos and RMT that underlies the novel quantum mechanism of eigenstate thermalization. When talking about quantum thermalization, we specifically refer to the thermalization of observables. More precisely, if we prepare a system, with a Hamiltonian $\hat{H}$, in a nonstationary state $|\psi\rangle$, with well-defined mean energy $\langle\psi|\,\hat{H}\,|\psi\rangle = E$, an observable $\hat{O} = \sum_i O_i\,|i\rangle\langle i|$ thermalizes if, under the time evolution of the system, it relaxes to the microcanonical expectation value:

$$\lim_{t\to\infty}\left\langle\psi(t)\right|\hat{O}\left|\psi(t)\right\rangle = \sum_{n\in\{E-\Delta E,E+\Delta E\}}\langle n|\,\hat{O}\,|n\rangle = \langle\hat{O}\rangle_{\text{microcan.},E}, \tag{1.19}$$

and remains close to it, meaning temporal fluctuations around the microcanonical value are negligible. If the quantum system is isolated and evolves unitarily with Hamiltonian $\hat{H}$, the puzzle is how to square this expectation with the exact expression in terms of energy eigenstates $\left|\psi(0)\right\rangle = \sum_n c_n|n\rangle$.

$$\left\langle\psi(t)\right|\hat{O}\left|\psi(t)\right\rangle = \sum_n |c_n|^2 O_{nn} + \sum_{n\neq m} c_m^* c_n e^{i(E_m-E_n)t} O_{mn} \tag{1.20}$$

For an arbitrary quantum system, there is no real reason to expect the second term to be small. If the Hamiltonian is a random matrix, however, its eigenstates are practically random orthogonal vectors in any arbitrary basis ($\overline{\langle m|i\rangle\,\langle j|n\rangle} = \delta_{mn}\delta_{ij}1/d$) [38], where the overline denotes averaging over the random eigenstates $|n\rangle$ and $|m\rangle$, and d is the dimension of the Hilbert space. In this case, the off-

[3] Neither the Bohigas-Gianonni-Schmidt nor the Berry-Tabor conjecture are strictly true. Counterexamples are known to both, see the previous sentence and [37, 4].

diagonal terms disappear $\overline{O_{mn}} = 0$, the diagonal ones are state-independent:

$$\overline{O_{nn}} = \sum_i O_i \overline{\langle n|i\rangle \langle i|n\rangle} = \frac{1}{d}\sum_i O_i \equiv \overline{O}, \tag{1.21}$$

$$\overline{O_{nn}^2} = \sum_{ij} O_i O_j \langle i|j\rangle \overline{\langle n|i\rangle \langle j|n\rangle} = \sum_i O_i^2 \overline{\langle n|i\rangle \langle i|n\rangle} = \frac{1}{d}\sum_i O_i^2 \equiv \overline{O^2}, \tag{1.22}$$

and each of the fluctuations is suppressed by the dimension of the Hilbert space [4]. From those results, one can deduce the form of the matrix elements in the basis of a random Hamiltonian, to a leading order in $1/d$, and such that they not only capture the average value but the fluctuations too:

$$O_{mn}^{\mathrm{RMT}} \approx \bar{O}\delta_{mn} + \sqrt{\frac{\overline{O^2}}{d}} R_{mn}. \tag{1.23}$$

More precisely, the fluctuations are encoded in the second term where R_{mn} is a zero mean and unit variance random variable that depends on the ensemble of the random matrix.

The natural suppression of the variance in an RMT inspired the idea that the evolution of an isolated quantum system can mimic thermalization. Of course, in real systems, we know that the Hamiltonian has more structure compared to a random matrix, hence the diagonal and off-diagonal elements have to contain information on the energy of the system and the relaxation time. The generalization of RMT, needed to describe real physical systems, was suggested by Deutsch [2] and Srednicki [3] and is known as the Eigenstate Thermalization Hypothesis (ETH):

$$O_{mn} = O(\overline{E})\delta_{mn} + e^{-\frac{S(\overline{E})}{2}} f_O(\overline{E}, \omega) R_{mn} \ . \tag{1.24}$$

Here $\overline{E} = (E_m + E_n)/2$ and $\omega = E_n - E_m$ are the average energy and energy difference between the two energy levels, $S(E)$ is the thermodynamic entropy and R_{mn} a random number, same as before. The important difference with RMT is that the diagonal term in ETH is equal to the expectation value in the microcanonical ensemble at energy $\overline{E}$, the entropy $S(E)$ also has the appropriate energy dependence and is not just the dimension of the Hilbert space, and a final important requirement is that $O(\overline{E})$ and $f_O(\overline{E}, \omega)$ are smooth functions of their arguments.

Due to the exponential growth of the Hilbert space with the number of degrees of freedom, many quantum systems in isolation have a regime where they exhibit ETH as the physics of their very large N-body Hamiltonian becomes closer and closer to a random one [4]. This mathematical exponential growth is the reflection of the physically new quantum feature of entanglement. In these settings the

scrambling and information loss driving thermalization is therefore of a physically different origin than classical chaos, even though in the abstract it still resides in the impracticality of tracing every bit of information with infinite precision.

1.4. Theoretical Models

We present a brief overview of the theoretical models studied in this thesis. One of the main models we shall use in this thesis to study the dynamics of relaxation and thermalization in an isolated quantum system is the Sachdev Ye Kitaev model (SYK) [39, 40]. This is a strongly interacting quantum mechanical system of N Majorana fermions with all-to-all random couplings. which is exactly solvable in the IR limit when $N \to \infty$. The strongly coupled IR fixed point has an emergent reparametrization symmetry where the effective dynamic is described by the Schwarzian action. The SYK model can be argued to be the ideal (solvable) representative of many-body quantum systems. It displays maximally chaotic behavior, as defined in Section 1.3.1, with an exponentially growing OTOC and a Lyapunov coefficient that, in the deep IR, saturates the bound (1.14), but at its heart it also obeys ETH [41]. These properties are precisely what makes the SYK an interesting playground for studying quantum chaos, quantum gravity and information scrambling. Moreover, as we will briefly describe for completeness in Section 1.5, these properties are also typical for 2D dilaton gravities which describe the near horizon physics of extremal black hole [42, 43, 44]. And a version of SYK built on charged Dirac fermions instead of Majoranas [45, 46, 47] has proved to be a useful toy model for studying strange metals and high-temperature superconductivity [19, 48].

The other model we shall study is the Transverse Field Ising model (TI), which is a set of Pauli matrices on a 1D chain, a well-known model in the study of quantum thermalization [49, 50]. On one hand, it is also solvable: using the Jordan-Wigner (1.46) and Bogoliubov (1.51) transformations can be shown it is equivalent to free fermions (1.52) (Section 1.4.2). On the other hand, it is representative of systems with a second-order quantum phase transition. The quantum phase transition occurs for an infinite chain at zero temperature $T = 0$ when the transverse magnetic field equals the hopping strength $h_x = J$. At this quantum critical point, the system becomes strongly entangled and it has distinct dynamics even at finite temperatures and finite system sizes.

The last considered model is a generalization of the TI known as the Mixed Field Ising (MFI) Section (1.4.3), a fruitful playground for, among others, studying quantum thermalization, quantum chaos and information scrambling [51, 52, 53, 54]. Its usefulness comes from the ability, simply by tuning its parameters, to describe different phenomena which range from quantum criticality, through a phase with quantum chaotic level statistics, all the way to a classical Ising model.

This will prove very useful in understanding possible differences between classical and quantum thermalization in Chapter 3.

1.4.1. SYK

The Sachdev-Ye-Kitaev model (SYK) is a quantum mechanical model of N all-to-all interacting fermions with random independent couplings drawn from a Gaussian distribution with zero mean and standard deviation. The latter sets the energy scale of the model. There are two main types of SYK models, based on the type of fermions. One is the Majorana SYK (mSYK), where the fermions are Majorana fields $\{\psi_i, \psi_j\} = \delta_{ij}$, which is usually used for studying fast scramblers, quantum chaos, and black holes through its duality with 2D dilaton gravity. As we shall see, it is the model where energy dynamics alone dominates. The other type is the complex SYK (cSYK) whose constituents are charged Dirac fermions $\{c_i^\dagger, c_j\} = \delta_{ij}$. This type of model includes a conserved $U(1)$-current naturally used for studying phenomena in strongly correlated electron systems. The usefulness of either type of SYK model lies in the formal solvability of its Schwinger-Dyson equations —i.e. the full quantum evolution equations — in the large N-limit. Here we briefly review the derivation of these Schwinger-Dyson equations and its solution for the mSYK, directly from the disorder-averaged partition function. The same result can be obtained by diagrammatic expansion, for which we refer the reader to [55, 56].

We start from the path integral representation of SYK partition function [57]:

$$Z_m = \int \mathcal{D}[\psi] \exp\left\{ i \int dt \left(\frac{i}{2} \sum_j \psi_j \partial_t \psi_j + \sum_{j_1,j_2,j_3,j_4=1}^{N} J_{j_1 j_2 j_3 j_4} \psi_{j_1} \psi_{j_2} \psi_{j_3} \psi_{j_4} \right) \right\}, \tag{1.25}$$

A key part of the SYK model is that the couplings are independent random variables drawn from a Gaussian distribution (1.26) :

$$P(J) = \frac{1}{\sigma_J \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma_J^2} \sum_{j_1<j_2\cdots<j_4} J_{j_1 j_2 j_3 j_4}^2 \right\} \quad ; \quad \sigma_J = \frac{3! J^2}{N^3}, \tag{1.26a}$$

$$\langle J_{j_1 j_2 j_3 j_4} \rangle_J \equiv \int \mathcal{D}J J_{j_1 j_2 j_3 j_4} P(J) = 0 \quad ; \quad \langle J_{j_1 j_2 j_3 j_4} J_{j_1' j_2' j_3' j_4'} \rangle_J = \sigma^2 \delta_{j_1 j_1'} \delta_{j_2 j_2'} \delta_{j_3 j_3'} \delta_{j_4 j_4'}, \tag{1.26b}$$

The physical observables are defined to be the ones that follow after averaging over the ensemble of different couplings (quenched disorder) Formally this has to be done through the replica trick, but for replica diagonal solutions which are the

ones of interest[4], one can just naively include the averaging in the path-integral directly

$$\langle Z_m \rangle_{\text{SYK}} = \int \mathcal{D}J P(J) \int \mathcal{D}[\psi] \exp\left\{ i \int dt S_m \right\} \tag{1.27}$$

$$= K' \int \mathcal{D}[\psi] \exp\left\{ i \int dt \left(\frac{i}{2} \sum_j \psi_j \partial_t \psi_j + \right. \right. \tag{1.28}$$

$$\left. \left. + i \frac{\sigma_J^2}{2 \cdot 4!} \int dt \int dt' \sum_{j_1, j_2, j_3, j_4 = 1}^{N} \psi_{j_1} \psi'_{j_1} \psi_{j_2} \psi'_{j_2} \psi_{j_3} \psi'_{j_3} \psi_{j_4} \psi'_{j_4} \right) \right\} \tag{1.29}$$

where $\psi' \equiv \psi(t')$. One introduces the bilinear field

$$\tilde{G}(t, t') = -\frac{i}{N} \sum_i^N \psi_i(t) \psi_i(t') \tag{1.30}$$

through a Lagrange multiplier $\Sigma(t, t')$ (1.32), that imposes this identity

$$1 = \int \mathcal{D}\tilde{G} \int \mathcal{D}\tilde{\Sigma} \exp\left\{ i^2 \frac{N}{2} \int dt \int dt' \tilde{\Sigma}(t, t') \left(\tilde{G}(t', t) + \frac{i}{N} \sum_i^N \psi_i(t') \psi_i(t) \right) \right\} \tag{1.31}$$

Inserting this identity into the path integral and after integrating out the Majorana fields, one can write the dynamics of the model in terms of the bilinear fields:

$$\langle Z_m \rangle_{\text{SYK}} = \int \mathcal{D}\tilde{G} \int \mathcal{D}\tilde{\Sigma} \exp\left\{ -\frac{N}{2} \left[-\operatorname{Tr} \log\left(-i(i\partial_t \delta_{tt'} - \tilde{\Sigma}(t, t')) \right) + \right. \right. \tag{1.32}$$

$$\left. \left. + \int dt \int dt' \left(\tilde{\Sigma}(t, t') + \frac{J^2}{4} \tilde{G}^3(t', t) \right) \tilde{G}(t', t) \right] \right\}. \tag{1.33}$$

Upon using the redefinition $\sigma_J^2 = 3! J^2 / N^3$, we notice that in the limit $N \gg 1$, $1/N$ is a small parameter, analogously to $\hbar$, and the path integral is dominated by the saddle point solution. This can be obtained by varying the action with respect to $\tilde{G}$ and $\tilde{\Sigma}$:

$$\frac{\delta S[\tilde{G}, \tilde{\Sigma}]}{\delta \tilde{\Sigma}(t, t')} = 0 \quad \Rightarrow \quad \Sigma(t, t') = i\partial_t \delta_{tt'} - G^{-1}(t, t') \tag{1.34a}$$

$$\frac{\delta S[\tilde{G}, \tilde{\Sigma}]}{\delta \tilde{G}(t', t)} = 0 \quad \Rightarrow \quad \Sigma(t, t') = -\frac{\sigma_J^2 N^3}{3!} G^3(t', t) \tag{1.34b}$$

[4] See [58, 59] for a discussion of replica symmetry breaking in SYK models.

We adopted the notation from [55] where the tilded expressions $\tilde{G}$ and $\tilde{\Sigma}$ label the dynamic fields whereas G, Σ without tildes are the classical solutions of the Schwinger-Dyson equations (1.34) These Schwinger-Dyson equations — or when computed on the doubled time Schwinger-Keldysh contour the Kadanoff-Baym equations, can be solved numerically to study non-equilibrium dynamics in an SYK system [57] as we shall do in Chapter 5. The equilibrium properties, however, can be solved analytically at low temperatures, which are where the power of the SYK model lies. Wick rotating the SD equations to periodic Euclidean time $t = -i\tau = -i(\tau + \beta)$, the first term on the RHS of (1.34a) can be neglected at low temperatures $\beta J \gg 1$ equivalent to the strong coupling regime. This leads to an emergent time reparametrization symmetry $\tau \to f(\tau), \partial_\tau f(\tau) > 0$:

$$G(\tau, \tau') \to [\partial_\tau f(\tau) \partial'_\tau f(\tau')]^{\Delta} G(f(\tau), f(\tau')), \tag{1.35a}$$

$$\Sigma(\tau, \tau') \to [\partial_\tau f(\tau) \partial'_\tau f(\tau')]^{3\Delta} \Sigma(f(\tau), f(\tau')). \tag{1.35b}$$

This indicates that in the deep IR the fermions develop an anomalous conformal dimension $\Delta = \frac{1}{4}$ which yields a scaling solution at zero temperature $\beta \to \infty$:

$$G_c(\tau) = \frac{1}{(4\pi)^{1/4}} \frac{\operatorname{sgn}(\tau)}{|J\tau|^{2\Delta}}. \tag{1.36}$$

The conformal propagator at inverse temperature β can be promptly obtained from the reparametrization $\tau \to \tan \frac{\pi\tau}{\beta}$ and the symmetry transformation (1.35a):

$$G_c^{\beta}(\tau) = \frac{\pi^{1/4}}{\sqrt{2\beta J}} \frac{\operatorname{sgn}\left(\sin \frac{\pi\tau}{\beta}\right)}{|\sin \frac{\pi\tau}{\beta}|^{2\Delta}}, \quad \tau \in \left[-\beta/2, \beta/2\right) \tag{1.37}$$

It is important to note here that these solutions spontaneously break the emergent reparametrization symmetry (1.35a) down to the $SL(2, \mathbb{R})$ group. This is because the scaling solutions (1.37), are invariant only under the reparametrization $\tau \to \frac{a\tau + b}{c\tau + d}$ where $ad - bc = 1$. Additionally, the full reparametrization symmetry is also explicitly broken by $i\partial_\tau \delta_{\tau,\tau'}$, which cannot be neglected once we move away from the deep IR. For this reason, the reparametrization modes are pseudo-Nambu-Goldstone bosons (also known as "soft-mode") with dynamics approximately described by the Schwarzian action:

$$S_{\text{Sch}} = -N \frac{C}{J} \int d\tau \text{Sch}[f, \tau] \quad ; \quad \text{Sch}[f, \tau] = \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right) \tag{1.38}$$

In a similar fashion, the leading contributions ($\sim 1/N$) to the (Euclidean time-

ordered) four-point function (1.39) can also be determined.

$$\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4)=\frac{1}{N^2}\sum_{ij}^{N}\langle\mathcal{T}\psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\rangle-G(\tau_1,\tau_2)G(\tau_3,\tau_4) \tag{1.39}$$

We won't go into the details of this computation but encourage the reader to study this beautiful derivation in [55]. However, we note that the OTOC of the SYK can be computed from the four-point function (1.39), by analytically continuing to Lorentzian time, and behaves as

$$\begin{aligned}\mathrm{OTOC}(t)\approx&G(\beta/2)G(\beta/2)+\mathcal{F}(\frac{\beta}{4}+it,-\frac{\beta}{4}+it,0,-\frac{\beta}{2})\\=&\frac{\sqrt{\pi}}{2\beta J}\left(1-C\frac{\beta J}{N}e^{\kappa t}\right),\quad \beta\ll t\ll\beta\log\frac{N}{\beta J}\end{aligned} \tag{1.40}$$

This expression is valid in the range of small but non-zero temperatures $\frac{N}{J}\gg\beta\gg\frac{1}{J}$ ($\frac{J}{N}\ll T\ll J$) where $\mathcal{F}$ has two leading contributions, one coming from excitations of the soft-mode, as described by the Schwarzian action (1.38), and the other is related to fluctuations around the conformal action. One not only sees the exponential growth characterized by the Lyapunov coefficient:

$$\kappa\approx\frac{2\pi}{\beta}(1-\#\frac{1}{\beta J}), \tag{1.41}$$

it is also immediately obvious that in the IR ($\beta J\to\infty$) the SYK saturates the maximal chaos bound (1.14) and is a "fast scrambler".

1.4.2. Transverse Field Ising Chain

The one-dimensional transverse field Ising model (TI) consists of N spins acted on by Pauli operators $\hat{\sigma}_i^x\equiv\hat{X}_i$ and $\hat{\sigma}_i^z\equiv\hat{Z}_i$ residing on each site i; see Appendix 4.C.3.[5] The dynamics of the model is given by the TI Hamiltonian:

$$H_{\mathrm{TI}}=-J\sum_{i=1}^{N}\hat{Z}_i\hat{Z}_{i+1}-h_x\sum_{i=1}^{N}\hat{X}_i, \tag{1.42}$$

where $J>0$ is a dimensionful nearest-neighbor interaction constant, and h_x is the strength of the transverse field. In this thesis, the microscopic scale, set by the interaction constant, is fixed $J=1$, Consequently, the ground state of the TI Hamiltonian depends only on h_x, which is used to tune the system across the quantum critical point at $h_x=1$.

[5] There are many reviews; we shall use [50].

Let us first consider the phase with dominant nearest-neighbor interaction $h_x < 1$. At the extreme case of $h_x = 0$, the Hamiltonian is diagonalized by the eigenstates of $\hat{Z}_i$ ($\hat{Z}_i |\uparrow\rangle_i = |\uparrow\rangle_i$ and $\hat{Z}_i |\downarrow\rangle_i = -|\downarrow\rangle_i$). Due to the exact Z_2 symmetry of H_{TI} ($Z_i \to -Z_i, X_i \to X_i$) the ground state is degenerate with all spins either up or down:

$$|\uparrow\rangle = \prod_i^N |\uparrow\rangle_i \quad ; \quad |\downarrow\rangle = \prod_i^N |\downarrow\rangle_i \equiv \prod_i^N \hat{X}_i |\uparrow\rangle . \tag{1.43}$$

Systems with small fields $h_x \ll 1$ have a ground-state (labeled here as $|0_f\rangle$) with a small fraction of spins aligned in the opposite direction, but the global Z_2 symmetry preserves the degeneracy for any small h_x.[6] This symmetry is thus spontaneously broken and the normalized order parameter $\hat{m} = \frac{1}{N}\sum_i^N \hat{Z}_i$ is non-vanishing: $\langle 0_f | \hat{m} | 0_f \rangle \sim \pm 1$. Hence this phase is known as the ordered (ferromagnetic) phase.

The other limit of the model is at large fields $h_x \gg 1$ with ground state $|0_p\rangle$ that, to leading order in $1/h_x$, is given by the eigenstate of $\hat{X}_i$ with positive eigenvalue ($\hat{X}_i |\rightarrow\rangle_i = |\rightarrow\rangle_i$, and $\hat{X}_i |\rightarrow\rangle_i = -|\rightarrow\rangle_i$):

$$|0_p\rangle = \prod_i^N |\rightarrow\rangle_i , \tag{1.44}$$

$$|\rightarrow\rangle_i = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_i + |\downarrow\rangle_i \right) \quad ; \quad |\leftarrow\rangle_i = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_i - |\downarrow\rangle_i \right) \tag{1.45}$$

In this phase, the symmetry remains unbroken and the order parameter vanishes $\langle 0_p | \hat{m} | 0_p \rangle \sim 0$, so this is the disordered (paramagnetic) phase.

The usefulness of the TI model results from the observation that it can also be exactly solved by finding a basis in which the Hamiltonian is diagonal [50, 60]. Firstly, using the Jordan-Wigner transformation and the ladder operators ($\hat{\sigma}^{\pm}$), defined in Appendix 4.C.3, one can transform the spin-$\frac{1}{2}$ operators to spinless fermions:

$$c_i = \left(\prod_{j<i} \hat{Z}_j \right) \hat{\sigma}_i^+ \quad ; \quad c_i^\dagger = \left(\prod_{j<i} \hat{Z}_j \right) \hat{\sigma}_i^- \tag{1.46}$$

$$\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij}, \quad \{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0, \tag{1.47}$$

[6] The global Z_2 symmetry may be broken by boundary conditions, but we shall not consider that here.

which, upon rotation of the spin axis by 90°, constitute a basis in which the TI Hamiltonian (1.42) reads

$$H_{TI} = -J\sum_{i}^{N}\left(\hat{c}_i^\dagger\hat{c}_{i+1} + \hat{c}_{i+1}^\dagger\hat{c}_i + \hat{c}_{i+1}\hat{c}_i + \hat{c}_i^\dagger\hat{c}_{i+1}^\dagger - 2\frac{h_x}{J}\hat{c}_i^\dagger\hat{c}_i\right). \tag{1.48}$$

This form of the TI Hamiltonian does not have a conserved fermion number, due to the $\hat{c}_{i+1}\hat{c}_i$ and $\hat{c}_i^\dagger\hat{c}_{i+1}^\dagger$ terms. However, it is quadratic in operators suggesting it can be diagonalized. Indeed, the Fourier transformation:

$$\hat{c}_k = \frac{1}{\sqrt{N}}\sum_{i}^{N}\hat{c}_j e^{-ikaj}, \tag{1.49}$$

brings the Hamiltonian (1.48) to :

$$H_{TI} = J\sum_{k}^{N}\left(2\left(\frac{h_x}{J} - \cos(ka)\right)\hat{c}_k^\dagger\hat{c}_k + i\sin(ka)\left(\hat{c}_{-k}\hat{c}_k + \hat{c}_{-k}^\dagger\hat{c}_k^\dagger\right) - \frac{h_x}{J}\right). \tag{1.50}$$

From here, one can use the Bogoliubov transformation (1.51):

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger, \tag{1.51a}$$

$$u_k = \cos\left(\frac{\theta_k}{2}\right), \quad v_k = \sin\left(\frac{\theta_k}{2}\right) \quad ; \quad \tan\theta_k = \frac{\sin(ka)}{\cos(ka) - \frac{h_x}{J}}, \tag{1.51b}$$

$$u_k^2 + v_k^2 = 1, \quad u_{-k} = u_k, \quad v_{-k} = -v_k, \quad u_k, v_k \in \mathbb{R}, \tag{1.51c}$$

which map the $\hat{c}_i^{(\dagger)}$ operators to a new set of fermions (γ_k) that satisfy the cannonical fermionic relations $\{\hat{\gamma}_k, \hat{\gamma}_{k'}^\dagger\} = \delta_{kk'}, \{\hat{\gamma}_k, \hat{\gamma}_k'\} = \{\hat{\gamma}_k^\dagger, \hat{\gamma}_k'^\dagger\} = 0$. In this basis, the Hamiltonian has a conserved fermionic number and is manifestly solvable, since it represents a set of N free fermions with dispersion relation ε_k:

$$H_{\mathrm{TI}} = \sum_{k}^{N}\varepsilon_k\left(\hat{\gamma}_k\hat{\gamma}_k - \frac{1}{2}\right) \tag{1.52}$$

$$\varepsilon_k = 2J\sqrt{1 + \frac{h_x^2}{J^2} - 2\frac{h_x}{J}\cos(k)}\ . \tag{1.53}$$

1.4.3. Mixed Field Ising Chain

The Mixed Field Ising model (MFI) is a simple generalization of the Transverse Field Ising model with dynamics is governed by the following Hamiltonian:

$$H_{\mathrm{MFI}} = -J\sum_{i=1}^{N}\hat{Z}_i\hat{Z}_{i+1} - h_x\sum_{i=1}^{N}\hat{X}_i - h_z\sum_{i=1}^{N}\hat{Z}_i. \tag{1.54}$$

This model has a few different manifestations, dependent on the point in the parameter space (h_x, h_z), which make it a rich playground for the theoretical testing of new hypotheses. For $h_z = 0$ this model reduces to the TI model (1.4.2), and for $h_x = 0$ reduces to the regular Ising model. We used the latter in Chapter 3 as a representative case of classical systems. The prominence of this model is that it is an archetype of quantum-chaotic behavior. At a particular point in the parameter space $h_x = -1.05$ and $h_z = 0.5$ it has the RMT-like spectral statistics [61]. Even though this point is not an isolated instance of chaotic behavior, sometimes it is referred to as "strong-chaoticity" [52], since, by varying (h_x, h_z), the spectral statistics of the model continuously transform between Wigner-Dyson distribution at $h_x = -1.05$ and $h_z = 0.5$ and Poisson distribution at an integrable point $h_x = 0$ or $h_z = 0$ in the parameter space. This smooth interpolation between chaotic and integrable regimes is what makes this model particularly convenient as a theoretical toy model.

1.5. Holographic duality

Though holographic duality itself is not part of the research presented in this thesis, the studies using holography to understand the dynamics of quantum black holes through its duality with non-gravitational quantum systems did provide a large motivating factor. For that reason we do briefly review it here.

The specific version of holography of relevance is the conjectured holographic duality of the SYK model with a 2D anti-de-Sitter (AdS) dilaton gravity [40, 55]. Below we outline how these pure two-dimensional dilaton gravities are described by the same Schwarzian action Eq.(1.38) in the IR limit and thus also exhibit maximal chaotic behavior with the Lyapunov coefficient saturating the bound Although it is much more difficult to find a gravitational theory and prove their duality in the full RG flow, the equivalence in the IR behavior of both theories has been an encouraging aspect in the quest for understanding their connection. There has been vast excitement around this duality because, for the first time, we have an exactly solvable field theory that can probe the realms of quantum gravity. Moreover, the far-reaching impact is far reaching since, as we'll demonstrate below, these 2D gravities appear as a universal behavior in the near-horizon limit of extremal black holes. Using an extension of this universal sector of the duality with the newly developed double-holographic paradigm [62, 63, 64, 65], that studies dualities between 1D field theories and 2D gravity with matter that's then dual to a 3D gravity, results from the SYK model have proven useful in understanding the process of black hole evaporation while preserving unitarity of the underlying physical process. In particular for the first time a tractable computation was able to reproduce the Page curve describing the evolution of the entanglement entropy between the black hole and its Hawking radiation. The

holographically dual description of this process in terms of the SYK physics is that of quenched cooling, and this is what motivated us to study the latter.[7]

First, we introduce pure AdS_2 in Euclidean signature and with radius set to unity. The particular choice of signature is useful for latter consideration of correlation functions in the dual boundary theory. In the Euclidean case, AdS_2 is simply a hyperbolic disk that is fully covered by both Poincaré:

$$ds^2 = \frac{dt^2 + dz^2}{z^2}, \tag{1.55}$$

and Rindler coordinates:

$$ds^2 = d\rho^2 + \sinh^2 \rho d\varphi^2. \tag{1.56}$$

In order to regularize the infinite volume of this space one cuts off the AdS_2 space along some curve $(t(\tau), z(\tau))$ close to the boundary, with a local affine parameter (the "time" on this boundary curve) in the interval $\tau \in [0, \beta)$. The physical length of this boundary curve goes to infinity in the limit $\epsilon \to 0$:

$$ds_b = \sqrt{\frac{ds^2}{d\tau^2}} d\tau = \sqrt{\frac{dt'^2 + dz'^2}{z^2}} = \frac{d\tau}{\epsilon}. \int ds_b \qquad = \int_0^\beta \frac{d\tau}{\epsilon} = \frac{d\beta}{\epsilon}, \tag{1.57}$$

Due to the isometry of Euclidean AdS_2, curves with the following $SL(2,\mathbb{R})/\mathbb{Z}_2$ reparametrisation describe the same geometry.

$$\tilde{t}(\tau) = \frac{at(\tau) + b}{ct(\tau) + d} \quad ; \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{R} \tag{1.58}$$

Already here we notice the first possible hint of a duality with SYK as the dynamics in the AdS_2 space with cutoff is captured by a one-dimensional curve $t(\tau)$ with the same symmetry as the IR limit of the SYK model in Section 1.4.1.

In the same way that the full SYK model starts with weakly interacting fermions in the IR that flow to the strongly interacting theory in the IR, it is also useful to start from a full model defined in the UV on the holographic side. This UV will clearly *not* be the same UV as the SYK model, but it will serve as an anchor for studying the resulting AdS_2 IR. We will use electromagnetically charged AdS_4 Reissner-Nordström black holes to demonstrate how AdS_2 appears. These AdS_4 RN black holes are described by the metric

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r^2} dt^2 + \frac{r^2}{(r - r_+)(r - r_-)} dr^2 + r^2 d\Omega^2 \tag{1.59a}$$

$$r_\pm = l_p Q + l_p^2 E \pm \sqrt{2 l_p^3 Q E + l_p^4 E^2} \quad ; \quad F_{rt} = \frac{Q}{r^2}, \tag{1.59b}$$

[7] In earlier papers this was called evaporative cooling. However, the actual process is prompted by an instantaneous change and hence a quench in the language of non-equilibrium physics.

where $l_P = \sqrt{G}$ is the Planck length, M and Q are the mass and charge of the black hole and $r_\pm$ the two horizons that coincide at extremality $E = 0$. The near-horizon geometry is obtained by taking $r \to r_+$, $l_P \to 0$ while keeping the variable z constant:

$$z = \frac{Q^2 l_P^2}{r - r_+}. \tag{1.60}$$

Expressing the the RN metric (1.59) in the new coordinate z (1.60), the near-horizon geometry of extremal black holes factorizes into an AdS_2 and a 2D sphere:

$$ds^2 \approx Q^2 l_P^2 \left(\frac{-dt^2 + dz^2}{z^2} + d\Omega^2 \right). \tag{1.61}$$

While this is the ground state of the extremal RN black hole, excitations above it are described by two-dimensional dilaton gravities, Φ being the dilaton field. This follows from its derivation from the Einstein-Hilbert action:

$$I = -\frac{1}{16\pi l_P^2} \int dx^4 \sqrt{-g} \left(R_g - \frac{l_P^2}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{1.62}$$

and focusing only on static, spherically symmetric field configurations [42, 66, 56] of the form:

$$ds^2 = h_{ij}(r,t) dx^i dx^j + \Phi^2(r,t) d\Omega^2. \tag{1.63}$$

One obtains, after reduction over the two-sphere, an effective 2D theory:

$$I = -\frac{1}{4l_P^2} \int dx^2 \sqrt{-h} \left(\Phi^2 R_h + 2(\boldsymbol{\nabla}\Phi)^2 + 2 - \frac{2Q^2 l_P^2}{\Phi^2} \right). \tag{1.64}$$

This is a special case of a class of 2D dilaton gravity models with an arbitrary scalar potential U, studied extensively in [42].

$$I = \frac{1}{16\pi G_N} \int dx^2 \sqrt{-h} \left(\Phi^2 R_h + \lambda(\boldsymbol{\nabla}\Phi)^2 - U\left(\Phi^2\right) \right). \tag{1.65}$$

Returning to the specific model (1.64), one can remove the dilaton kinetic term with a Weyl transformation [42]:

$$I = -\frac{1}{4l_P^2} \int dx^2 \sqrt{-h} \left(\Phi^2 R_h + 2 - \frac{2Q^2 l_P^2}{\Phi^2} \right). \tag{1.66}$$

and readily obtain the saddle point solution,

$$\delta_\Phi I|_{\phi_0} = 0 \Rightarrow R_h = -2\frac{Q^2 l_P^2}{\phi_0^4} \tag{1.67}$$

which is an AdS_2-like space with radius $L = \phi_0^2/(|Q|l_P)$. The remaining variation $\delta_{h_{ij}} I_{h_{ij,0}} = 0$ has the AdS_4 RN black hole as solution but also just the near-horizon AdS_2 limit of the extremal RN black hole (1.61), in which case $\Phi(x, r) = \phi_0$.

Without proof we will state that the fluctuations around this solution with non-trivial dependency on the coordinates on the two-sphere are all energetically costly and the low lying excitations are given by small fluctuations of the dilaton field around its AdS_2 saddle point:

$$\Phi^2 = \phi_0^2 + \phi(x^0, x^1), \quad \phi(x^0, x^1) \ll \phi_0^2. \tag{1.68}$$

Substituting this ansatz in (1.66) and expanding to second order in ϕ/ϕ_0^2 the following action is obtained:

$$\begin{aligned} I \approx &- \frac{1}{2l_P^2}\int dx^2\sqrt{-h} - \frac{\Phi_0^2}{4l_P^2}\left[\int dx^2\sqrt{-h}\left(R_h + \frac{2}{L^2}\right) + 2\int_{\partial \mathrm{AdS}} \mathcal{K}\right] \\ &- \frac{1}{4l_P^2}\left[\int dx^2\sqrt{-h}\phi\left(R_h + \frac{2}{L^2}\right) + 2\int_{\partial \mathrm{AdS}} \phi_b \mathcal{K}\right], \end{aligned} \tag{1.69}$$

where we have restored the proper AdS_2 boundary terms proportional to trace of the extrinsic curvature $\mathcal{K}$ [66]. In this action, the first term is proportional to the volume of the AdS_2 space, the second represents the ordinary 2D Einstein gravity, and the last one (1.70), known as the Jackiw-Teitelboim (JT) gravity [67, 68], is the subject of our interest since it gives rise to a non-trivial dynamics.

$$I_{JT} = -\frac{1}{4l_P^2}\left[\int dx^2\sqrt{-h}\phi\left(R_h + \frac{2}{L^2}\right) + 2\int_{\partial \mathrm{AdS}} \phi_b \mathcal{K}\right], \tag{1.70}$$

If one derives the equation of motion for the dilaton ϕ, with a boundary condition given by ϕ_b, the bulk term in the JT gravity manifestly sets the spacetime to an (asymptotically) AdS_2 with constant scalar curvature $R_h = -2$. When we restrict the analysis to a nearly-AdS geometry the last term encodes the dynamics of the cutoff curve $(t(\tau), z(\tau))$, which after proper treatment of the arising infinities [69, 43, 55] is described by the Schwarzian action to leading order in ϵ:

$$I_b \approx -\frac{\phi_r}{8\pi G}\int_0^\beta d\tau \mathrm{Sch}[t(\tau), \tau], \tag{1.71}$$

where $\phi_r(\tau)$ is renormalized boundary dilaton ($\phi_r(\tau)/\epsilon \approx \phi_b$). The Schwarzian action is obtained under the assumption that the dilaton is constant at the boundary $\phi_r(\tau) = \phi_r$, a more detailed treatment, leading to the same result, can be found in [56, 66] and citations therein. With the knowledge that the dynamics of the cutoff modes in fact comprise the full low energy, this proves the duality with the IR action of the reparametrization modes in the SYK (1.38): they are both described by the Schwarzian action. It is a beautiful example of IR universality in that the two wildly differing UVs flow to the same fixed point in the IR.

At last, to establish the connection in the chaotic behavior, the introduction of matter fields $\xi(t, z)$ is required on top of the empty dilaton gravity. We consider a minimally coupled free theory in the cut-off AdS_2 background,

$$I_m = \frac{1}{2}\int dx^2\sqrt{-h}\left[\partial^a\xi\partial_a\xi + m^2\xi^2\right] . \tag{1.72}$$

Its saddle point solution is completely determined by its boundary value $\xi_r(t)$ defined as

$$\xi(t, z) = z^{1-\Delta}\xi_r(t) + \cdots \quad ; \quad \Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}. \tag{1.73}$$

The AdS/CFT dictionary tells us that, in the dual field theory, there is an operator $V(\tau)$ with conformal dimension Δ that is conjugate to $\xi_r(\tau)$. Introducing a second free field $\chi(t, z)$, with the same action (1.72), that is conjugate to another operator $W(\tau)$ with the same conformal dimension Δ, one can show [70, 43] that the leading behavior of the OTOC of those operators arises from the fluctuations of the boundary shape and is given by:

$$\text{OTOC}(t) \approx \left(\frac{\pi}{\beta}\right)^2\left(1 - 2\Delta^2\frac{\beta G}{\phi_r}e^{\frac{2\pi t}{\beta}}\right), \quad \beta \ll t \ll \beta\log\frac{\phi_r}{\beta G}. \tag{1.74}$$

From the exponential growth of the second term in (1.74) with a Lyapunov coefficient $\lambda = 2\pi/\beta$ that saturates the bound (1.14) we conclude that the 2D dilaton gravity exhibits a maximally chaotic behavior that coincides with the SYK (1.13) as it should.

1.6. Thesis outline

This thesis studies the non-equilibrium dynamics of composite quantum systems following the quench described in the introduction at early times specifically. The time scales of interest are well before the validity of ETH and even hydrodynamics, so the analysis is based on the evolution of energy, the von Neumann and relative entropies, as well as the information spread between different parts of

the system. This was initially motivated by an observation in numerical simulations of a peculiar early-time energy increase in the hotter of two quench-coupled systems. This thesis not only provides a detailed explanation of this quantum phenomenon but also explores how it can be used for experimentally measuring quantum correlations, both as a feature for detecting a lab realization of an SYK system as well as the implications it has on the formation of black holes. Those results have been published in three different papers [71, 72, 73] plus one yet unpublished manuscript. Before elaborating on them we provided a brief overview of each of the four chapters.

In the first Chapter 2, we introduce this paradoxical early time rise in energy in the hotter of the two baths and the quantum thermodynamic approach to studying post-quench dynamics. To understand the essence of this phenomenon, without the interference of model-dependent peculiarities, we use the 1D free fermion model as a case study. Conveniently, in the time regime of interest, we obtain analytical expressions for the energy and for the von Neumann entropies, which is an additional benefit of this model. In addition to the theoretical study, we have suggested an experimentally realizable quench protocol that can measure entanglement between two subsystems using the energy increase and its relation to von Neumann entropy. In this quench, one starts with two originally independent systems of free fermions A and B initially prepared in quantum thermal states at temperatures T_A and T_B. At low temperatures, when quenched, the increasing entanglement contribution to the von Neumann entropy is dominant over the decreasing thermal one. As a consequence the von Neumann entropy of each subsystem increases for a certain period after the subsystems are coupled. If in this period one decouples the subsystems there is an energy transfer to the system in the amount set by the von Neumann entropy accumulated during the joint evolution of A and B. This energy transfer appears as work produced by the quench to decouple the reservoirs. Once A and B are disconnected, the information about their mutual correlations – von Neumann entropy – is stored in the energy increment of each reservoir which allows a direct readout of quantum correlations by measuring the energy of the subsystems. While this thesis doesn't cover the details on the feasibility of the experimental realization and subtleties when the temperature of either subsystem approaches $T \to 0$, interested readers are advised to consult [71].

Next, in Chapter 3 we study the post-quench quantum dynamics of both strongly correlated SYK systems and weakly correlated mixed field Ising chains. As previously mentioned, the quantum thermodynamic relations require exact knowledge of the time evolution of energy and the von Neumann and relative entropies. For those systems these cannot be obtained analytically, therefore, we resort to exact diagonalization. While this approach is limited to finite sized systems, it allows us to study their evolution at an arbitrary time-scale allowing us to distinguish two qualitatively different behaviours. Namely, the early time polynomial increase of

energy in both sub-systems is followed by a conventional classical-like evaporation with the energy of the hotter/colder system exponentially decreasing/increasing. At the transition time t_m the energy of the hotter system peaks, a feature known as the energy bump. We show that even in the quantum regime the origin of this energy bump is not due to thermal flux from the hot to the cold, contrary to what has been reported before [13]. Instead the early-time energy increase of the hotter subsystem is not related to a temperature increase but results from the potential energy gained by coupling the two systems. The size of the energy increase is set by the entropy gain and lasts until correlations between the subsystems saturate. When the systems of interests are SYK dots we have numerically found that the energy bump appears regardless of the initial temperature difference, which isn't surprising, given the ultra quantum nature of this system.[8] To answer the question "Why we haven't seen this phenomenon in everyday life?", we use the MFI model which has classical, integrable, chaotic and critical regimes. At the critical point, same as with SYK, the energy bump always appears, but moving even slightly away from criticality there is a distinct temperature T_c such that, when the hotter system is initiated above $T_A > T_c$, its energy decreases from the very beginning in full agreement with our (classical) intuition. If MFI is tuned to the classical regime this distinct temperature vanishes $T_c = 0$ and classical dynamics is recovered at any initial configuration.

In order to better understand the numerical results for the SYK and MFI models we analyze in Chapter 4 their properties with a perturbative time expansion of each subsystem's energy. We derive the coefficients up to the third order and focus on situations where both subsystems are initialized in a thermal state. In this case the first and third coefficients vanish and the second coefficient captures the relation between the appearance of the early time energy increase in the hotter subsystem and before-quench thermodynamic states. More precisely, the bump appears whenever the second coefficient is larger than zero. Using this approach we have shown that, for two quench-coupled SYKs, the existence of the quantum regime isn't conditioned on the temperature difference. since the second coefficient of each subsystem depends only on its own parameters and they're always positive. Consequently, the early time energy increase will always appear, as we've suspected from the numerical results. However, if the systems under consideration are MFI models, the second coefficient has two competing contributions. One is always positive, similar to the SYK case, and depends only on the analyzed subsystem, whereas the other is negative and depends on the initial parameters of both, hence, the bump persists as long as the former term is dominant over the former. Both terms have the same magnitude at the critical temperature T_c, after which the second term dominates resulting in the bump disappearance, for any temperature $T \geq T_c$. Here, using the perturbative

[8] This is intrinsically linked to the ANEC inequality when modeling evaporative black hole formation with SYK dots [14, 15].

analytical expressions, we were able to compute the critical temperature as a function of quantities evaluated in the initial thermal state and prove it matches with results obtained from numerical time evolution.

In the last Chapter 5 we study the post-quench dynamics, using the same protocol of coupling two initially independent subsystems, but now for charged SYKs. Furthermore, the first hot system is a SYK_4 dot, but unlike before, the second is a much larger disordered Fermi liquid heat bath modeled by an SYK_2 at zero temperature and chemical potential. This project was motivated by efforts to engineer an experimental realization of the SYK model and the fragility upon measurement of the non-Fermi liquid SYK_4 phase. By studying the time evolution of the SYK's spectral properties we demonstrated that, while the non-Fermi liquid phase eventually breaks under the influence of the bath, it is preserved right after the quench and we were able to estimate the transition timescale, which matches with results obtained from extrapolation of equilibrium studies. In early post-quench times, before the non-FL/Fermi liquid transition, we discovered that the charged current undergoes a power-law increase and reaches maximum before exponentially relaxing, resembling the energy behavior analyzed in the three previous chapters. The half-life of the current is inversely proportional to the coupling between the system and the bath and it increases with temperature. Investigating this temperature dependence of the half-life we have found that at low temperatures its temperature dependence behaves as $t_{1/2} \sim T$ when the hot system is a non-Fermi liquid SYK_4 dot. However, when the hot system is a disordered regular Fermi liquid SYK_2 system, at the same chemical potential, the half-life scales as $t_{1/2} \sim T^2$. Based on this result, we proposed measuring the half-life of the discharging current in the quench-coupling protocol, as a feasible experimental setup for detecting the fragile non-Fermi liquid SYK phase and differentiating it from a disordered Fermi-liquid phase.