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Models of Jacobians of curves

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Abstract. We show that the Jacobians of prestable curves over toroidal varieties always admit Néron models. These models are rarely quasi-compact or separated, but we also give a complete classification of quasi-compact separated group-models of such Jacobians. In particular, we show the existence of a maximal quasi-compact separated group model, which we call the saturated model, and has the extension property for all torsion sections. The Néron model and the saturated model coincide over a Dedekind base, so the saturated model gives an alternative generalization of the classical notion of Néron models to higher-dimensional bases; in the general case we give necessary and sufficient conditions for the Néron model and saturated model to coincide. The key result, from which most others descend, is that the logarithmic Jacobian of [*S. Molcho* and *J. Wise*, The logarithmic Picard group and its tropicalization, preprint 2018] is a log Néron model of the Jacobian.

1. Introduction

1.1. Néron models. Let $X \to S$ be a prestable curve (this means that X/S is proper, flat, finitely presented, and the geometric fibers are reduced and connected of pure dimension 1 and with at worst ordinary double point singularities; for example, a stable curve) over a scheme S, smooth over a schematically dense open subscheme $U \subset S$. The Jacobian $\mathrm{Pic}^0_{X_U}$ is then an abelian scheme over U which (in general) admits no extension to an abelian scheme over all of S. Néron suggested that one should look instead for a *Néron model* of $\mathrm{Pic}^0_{X_U}$; a smooth algebraic space N over S, such that

$$N \times_S U = \mathrm{Pic}^0_{X_U},$$

satisfying the *Néron mapping property.*

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Néron mapping property. For any smooth map $T \to S$ of schemes, the natural restriction map $N(T) \to \mathrm{Pic}^0(T \times_S U)$ is a bijection.[1)]

Néron models are unique when they exist, and inherit a group structure extending that of the Jacobian. When S is Dedekind, the existence of a Néron model was proven by Néron and Raynaud [27, 34]; in this case, a Néron model is automatically separated and quasi-compact.

When the base S is higher dimensional, a separated, quasi-compact Néron model rarely exists. In [12], Holmes showed that over a regular base S, existence of such a model implies a delicate relation between the smoothing parameters of the nodes of X, which he called alignment. He also proved that alignment is sufficient if the total space X is also regular. In [31], Orecchia refines this into a necessary and sufficient condition when X is smooth over the complement of a normal crossings divisor $D \to S$, but not necessarily regular. In [33], Poiret constructs the Néron model assuming any smooth S-scheme is locally factorial (e.g., S is regular), and shows that a more restrictive version of alignment, *strict alignment*, is necessary and sufficient for it to be quasi-compact and separated.

The main result of this paper is that without the extra conditions of separatedness and quasi-compactness, a Néron model of the Jacobian exists whenever the base is a toroidal variety (or more generally a log regular scheme, see Section 6.2).

Theorem 1.1 (Corollary 6.13). *Let S be a toroidal variety with $U \subset S$ the open complement of the boundary divisor, and let X/S be prestable curve, smooth over U. Then a Néron model N/S for $\mathrm{Pic}^0_{X_U}$ exists and is quasi-separated.*

The Néron model admits a modular interpretation coming from logarithmic geometry: if one endows X and S with suitable logarithmic structures, one obtains the notion of *logarithmic line bundle* on X in the sense of [26]. The Néron model is then the algebraic space $\mathrm{sLPic}^0_{X/S}$ (on the category of schemes) representing the functor of log line bundles of degree zero; we call it the *strict logarithmic Jacobian*. We postpone further discussion of the log geometric side of the story to Section 1.5 of this introduction. If X is regular, then a more concrete description of the Néron model can be given as a quotient of the relative Picard functor of X/S, generalizing the approach of Raynaud when $\dim S = 1$; see Section 9.

1.2. Separated, quasi-compact models of the Jacobian. We have seen that Néron models always exist, but they are rarely separated or quasi-compact. A *model* of $\mathrm{Pic}^0_{X_U/U}$ is an algebraic space G/S which restricts to $\mathrm{Pic}^0_{X_U/U}$ over U. Our next result gives a complete classification of separated, quasi-compact group models of the Jacobian $\mathrm{Pic}^0_{X_U/U}$ in terms of subgroups of the tropical Jacobian. In the setting of Theorem 1.1 we define the *strict tropical Jacobian* $\mathrm{sTPic}^0_{X/S}$ to be the quotient of the Néron model by its fiberwise-connected component of identity $\mathrm{Pic}^0_{X_U/U}$. The quotient $\mathrm{sTPic}^0_{X/S}$ is an étale group algebraic space over S, and is trivial over U. If S is local Dedekind, then the group $\mathrm{sTPic}^0_{X/S}(S)$ is exactly the classical component group of the Néron model.

Suppose we are given Ψ/S an étale group space, and $\Psi \to \mathrm{sTPic}^0_{X/S}$ a group homomorphism. One obtains by fiber product a smooth group space $\mathscr{G}(\Psi) := \Psi \times_{\mathrm{sTPic}^0_{X/S}} \mathrm{sLPic}^0_{X/S}$. If $\Psi_U = 0$, then the restriction $\mathscr{G}(\Psi)_U$ is identified with the Jacobian $\mathrm{Pic}^0_{X_U/U}$.

[1)] The classical definition over Dedekind schemes and in [12] requires Néron models to be separated. When the base is a Dedekind scheme and the generic fiber is a group scheme, this is automatic by [5, Theorem 7.1.1]

Theorem 1.2 (Proposition 7.3, Corollary 7.10). *The map $\Psi \mapsto \mathcal{G}(\Psi)$ induces a bijection of partially ordered sets from the set of quasi-finite open subgroups of* $\mathrm{sTPic}^0_{X/S}$ *to the set of smooth, separated, quasi-compact S-group models of the Jacobian* $\mathrm{Pic}^0_{X_U/U}$.

As an application, we consider the case $S = \overline{\mathcal{M}}_{g,n}$ the moduli stack of stable curves, and we take X/S the universal curve. One shows that the strict tropical Jacobian is torsion-free in this case; an immediate consequence of Theorem 1.2 is:

Corollary 1.3 (Lemma 5.5). *The universal Jacobian* $\mathrm{Pic}^0_{X/\mathcal{M}_{g,n}}$ *admits a unique smooth, separated group model over $\overline{\mathcal{M}}_{g,n}$, namely the generalized Jacobian* $\mathrm{Pic}^0_{X/\overline{\mathcal{M}}_{g,n}}$ *parametrizing line bundles of multidegree* $(0, 0, \ldots, 0)$.

In particular, this shows the non-existence of a smooth separated group scheme (or space) $\mathcal{G}$ over $\overline{\mathcal{M}}_{g,n}$ whose every fiber $\mathcal{G}_s$ is isomorphic to the special fiber of the Néron model of the Jacobian of a regular 1-parameter smoothing of X_s,[2)] as such a space would be a model of $\mathrm{Pic}^0_{X/\mathcal{M}_{g,n}}$. A question about the existence of such "universal Néron models of the Jacobian" over a compactification of $\mathcal{M}_g$ was asked by Chiodo in the introduction of [8]. Caporaso investigates in [6] the analogous problem where, instead of Jacobians, one tries to fit into a universal family the special fibers of the Néron models of the $\mathrm{Pic}^d_{Y^s/T^s}$, where Y^s/T^s is a regular 1-parameter smoothing of X_s.

1.3. The saturated model. We define the *saturated model* of the Jacobian $\mathrm{Pic}^0_{X_U/U}$ to be the smooth separated quasi-compact group model which is maximal for the relation of inclusion.

Corollary 1.4 (Theorem 7.11). *The saturated model exists, and can be constructed as $\mathcal{G}(\Psi)$ for Ψ the torsion subgroup of* $\mathrm{sTPic}^0_{X/S}$.

When $\dim S = 1$, the Néron model and the saturated model coincide, so that the saturated model may be viewed as an alternative generalization of the Néron model to base schemes of higher dimension. The saturated model has the advantage of being separated and quasi-compact, but has a weaker extension property:

Corollary 1.5. *Let $x : U \to \mathrm{Pic}^0_{X_U/U}$ be a torsion section. Then x extends uniquely to a section of the saturated model.*

The saturated model admits a modular interpretation: it represents the "saturated Jacobian" functor $\mathrm{sPic}^{\mathrm{sat}}$ of log line bundles of degree zero on X which become line bundles after taking a suitable integer power.

1.4. The saturated model and the Néron model. The papers [12, 31, 33] present criteria for the existence of quasi-compact and separated models of $\mathrm{Pic}^0_{X_U/U}$ when the base is regular. When put together, our main results yield the following criterion, valid even over some non-regular bases.

[2)] This means a flat morphism of schemes $Y \to T$ where T is a trait and Y/T has smooth generic fiber; together with an isomorphism between X_s/s and the special fiber of Y/T.

Theorem 1.6 (Theorem 8.5). *Let S be a toroidal variety with $U \subset S$ the open complement of the boundary divisor, and let X/S be prestable curve, smooth over U (or more generally, a log smooth curve over a log regular base). Then the following are equivalent:*

(i) *The strict tropical Jacobian* $\mathrm{sTPic}^0_{X/S}$ *is quasi-finite over* S.

(ii) *The Néron model of* $\mathrm{Pic}^0_{X_U}$ *is separated over* S.

(iii) *The Néron model of* $\mathrm{Pic}^0_{X_U}$ *is quasi-compact and separated over* S.

(iv) *The saturated model and the Néron model of* $\mathrm{Pic}^0_{X_U}$ *are equal.*

1.5. Log geometric interpretation. Although many of our results concern classical algebraic geometry, they become more natural in the context of logarithmic geometry. To explain the connection, suppose that $X \to S$ is a family of logarithmic curves (see Definition 2.1). In [26], following ideas of Illusie and Kato, the authors constructed the analogue of the Picard scheme in the category of logarithmic schemes, the logarithmic Picard group $\mathrm{LogPic}_{X/S}$. This is the sheaf of isomorphism classes of the stack which parametrizes the logarithmic line bundles alluded to above, that is, certain[3] torsors under the associated group M_X^{gp} of the log structure.

The logarithmic Picard group is a group, is log smooth and proper over S, and on the locus U of S where the log structure is trivial it coincides with the ordinary Picard group Pic_{X_U}. Furthermore, logarithmic line bundles have a natural notion of degree, extending the notion of degree for ordinary line bundles, and $\mathrm{LogPic}_{X/S}$ splits into connected components according to degree. Thus, the *logarithmic Jacobian* $\mathrm{LogPic}^0_{X/S}$ provides a "best possible" extension of the Jacobian $\mathrm{Pic}^0_{X_U/U}$. The caveat is that the logarithmic Jacobian is a sheaf on the category of log schemes, not schemes, and it is in general not algebraic – i.e., it is not representable by an algebraic space with a log structure. In fact, it is "log algebraic", that is, it satisfies the analogous properties that algebraic spaces enjoy, but only in the category of log schemes; for instance, it has a logarithmically étale cover by a log scheme. See Example 3.60 for the case of the Tate curve.

Nevertheless, properness of $\mathrm{LogPic}^0_{X/S}$ suggests that it is close to a Néron model for $\mathrm{Pic}^0_{X_U/U}$. For example, in the simplest case when S is a trait[4], the valuative criterion tells us that every line bundle L on X_U extends uniquely to a log line bundle on X. In fact, this "limit bundle" is simply the pushforward $j_*L^\times$ of the $\mathcal{O}^\times$-torsor associated to L along the inclusion $j : X_U \hookrightarrow X$. Remarkably, the description of the limit goes through whenever S is a log regular scheme, showing that the logarithmic Jacobian satisfies the logarithmic version of the Néron mapping property:

Theorem 1.7 (Theorem 6.11). *Let S be a log regular scheme. Then* $\mathrm{LogPic}^0_{X/S}$ *satisfies the Néron mapping property for log smooth morphisms.*

In the case where S is Dedekind this answers positively for Jacobians of curves a question of Eriksson, Halle, and Nicaise in [9], who asked for the existence of a log Néron model.

To connect Theorem 1.7 with classical algebraic geometry, we have to bring the problem back from the category of log schemes to the category of schemes. There is a standard proce-

[3] The torsors must satisfy a condition called bounded monodromy – see Section 3 for details.

[4] The spectrum of a discrete valuation ring.

dure to do so: the category $\mathbf{Sch}/S$ embeds into $\mathbf{LSch}/S$ by giving $T \to S$ its pullback ("strict") log structure, and we may thus restrict the functor $\mathrm{LogPic}^0_{X/S}$ to $\mathbf{Sch}/S$. We denote the resulting functor, the "strict" log Jacobian, by $\mathrm{sLPic}^0_{X/S}$. It is an immediate consequence that the strict logarithmic Jacobian $\mathrm{sLPic}^0_{X/S}$ satisfies the classical Néron mapping property. The functors $\mathrm{LogPic}^0_{X/S}$ and $\mathrm{sLPic}^0_{X/S}$ are very different in nature: good properties of $\mathrm{LogPic}^0_{X/S}$ such as properness, or even quasi-compactness, are generally lost in passing to $\mathrm{sLPic}^0_{X/S}$. This is however compensated by the following positive result:

Theorem 1.8 (Theorem 4.4). *Let X/S be a vertical log curve. The functor $\mathrm{sLPic}^0_{X/S}$ is representable by a quasi-separated, smooth algebraic space over $\underline{S}$.*

If S is log regular (e.g., a toroidal variety with divisorial log structure), then $\mathrm{sLPic}^0_{X/S}$ is the Néron model of $\mathrm{Pic}^0_{X_U/U}$.

The strict tropical Jacobian $\mathrm{sTPic}^0_{X/S}$, defined above as a quotient, also has a natural log geometric interpretation. The log Picard group $\mathrm{LogPic}_{X/S}$ has a "tropicalization" $\mathrm{TroPic}_{X/S}$, an essentially combinatorial object which determines the features of $\mathrm{LogPic}_{X/S}$ which are not present in the Jacobian $\mathrm{Pic}^0_{X/S}$ of X/S. Restricting the tropical Jacobian $\mathrm{TroPic}^0_{X/S}$ – that is, the degree 0 part of $\mathrm{TroPic}_{X/S}$ – to schemes by giving a scheme its pullback log structure, as before, produces $\mathrm{sTPic}^0_{X/S}$. The tropical Jacobian plays an important role in the theory of compactifications of the universal Jacobian. It was essentially shown in [16, 17] that subdivisions of $\mathrm{TroPic}^0_{X/S}$ correspond to toroidal compactifications of $\mathrm{Pic}^0_{X/S}$. Theorem 1.2 provides a complementary view of the role of $\mathrm{TroPic}_{X/S}$: the quasi-finite open subgroups of its strict locus determine the quasi-compact, smooth, separated group models of $\mathrm{Pic}^0_{X/S}$.

2. Background

Here we collect for the convenience of the reader the necessary facts that we will use, especially from the paper [26].

2.1. Log schemes. All our log schemes are fine and saturated. For a log scheme S we denote by $\underline{S}$ the underlying scheme. We denote by $\mathbf{LSch}/S$ the category of log schemes over S, and by $(\mathbf{LSch}/S)_{\text{ét}}$ the (big) strict étale site over S; the small strict étale site is denoted $S_{\text{ét}}$.

We write M_S for the log structure of a log scheme S, $\overline{M}_S$ for its characteristic monoid $M_S/\mathcal{O}_S^*$ (these are sheaves on $S_{\text{ét}}$). For a map of log schemes $f : X \to S$ we denote by $\overline{M}_{X/S}$ the relative characteristic monoid $M_X/f^*M_S = \overline{M}_X/f^{-1}\overline{M}_S$.

Let S be a log scheme; we define the logarithmic and tropical multiplicative groups on S to be the sheaves of abelian groups on $(\mathbf{LSch}/S)_{\text{ét}}$ given by

$$\mathbb{G}_{m,S}^{\log} : (\underline{T}, M_T) \to M_T^{\mathrm{gp}}(\underline{T}),$$
$$\mathbb{G}_{m,S}^{\mathrm{trop}} : (\underline{T}, M_T) \to \overline{M}_T^{\mathrm{gp}}(\underline{T}).$$

2.2. Log curves.

Definition 2.1. A log curve $X \to S$ is a proper, vertical, integral, log smooth morphism of log schemes with connected and reduced geometric fibers of pure dimension 1.

The underlying morphism of schemes $\underline{X} \to \underline{S}$ is a prestable curve as in [37, Tag 0E6T]. Our definition is the same as that of [20] except that we have added the assumption that the morphism be vertical; this means that the characteristic sheaf $\overline{M}_{X/S}$ is a sheaf of groups, or equivalently that it is supported exactly on the non-smooth locus X^{nsm} of X over S.

2.3. Sites, constructibility and representability. We leave for a moment the category of log schemes. For a scheme X, we write $X_{\text{ét}}$ for the small étale site and $(\mathbf{Sch}/X)_{\text{ét}}$ for the big étale site.

There is a morphisms of sites $\mathrm{i} : (\mathbf{Sch}/X)_{\text{ét}} \to X_{\text{ét}}$ given by the inclusion of categories $X_{\text{ét}} \to (\mathbf{Sch}/X)_{\text{ét}}$. We have functors between the categories of sheaves

$$\mathrm{i}_* : \mathrm{Sh}(\mathbf{Sch}/X)_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}}); \mathrm{i}_*\mathcal{F}(U/X) = \mathcal{F}(U), \tag{2.3.1}$$

$$\mathrm{i}^* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(\mathbf{Sch}/X)_{\text{ét}} \tag{2.3.2}$$

defined by setting $\mathrm{i}^*\mathcal{F}(j : T \to X)$ to be $j^*\mathcal{F}(T)$.

Definition 2.2. A sheaf $\mathcal{F} \in \mathrm{Sh}(\mathbf{Sch}/X)_{\text{ét}}$ is *locally constructible* if the natural map $\mathrm{i}^*\mathrm{i}_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Representability reduces to local constructibility via the following well-known lemma:

Lemma 2.3 ([3, VII, 1.8]). *Let S be a Noetherian scheme. A sheaf $\mathcal{F}$ on $(\mathbf{Sch}/S)_{\text{ét}}$ is locally constructible if and only if it is representable by a (quasi-separated) étale algebraic space over S.*

Remark 2.4. The original statement of Lemma 2.3 in [3] states the equivalence for étale locally separated algebraic spaces; however an étale algebraic space is automatically locally separated as its diagonal is an open immersion ([37, Tag 05W1]). Moreover, an étale algebraic space over a locally noetherian base is automatically quasi-separated, as it is easily seen by combining [37, Tag 03KG] and [37, Tag 01OX].

2.4. Functors between the categories of log schemes and schemes. Fix a log scheme $S = (\underline{S}, M_S)$ and consider the functors

$$\begin{aligned} \mathfrak{f} &: \mathbf{LSch}/S \to \mathbf{Sch}/\underline{S}, & T &\mapsto \underline{T}, \\ \mathfrak{s} &: \mathbf{Sch}/\underline{S} \to \mathbf{LSch}/S, & (g : T \to \underline{S}) &\mapsto (T, g^*M_S). \end{aligned}$$

The first forgets the log structure; the second endows an $\underline{S}$-scheme with the strict (pullback) log structure from S.

The functor $\mathfrak{f}$ is the left adjoint of $\mathfrak{s}$: for $X \in \mathbf{LSch}/S$ and $Y \in \mathbf{Sch}/\underline{S}$, we have

$$\mathrm{Hom}(\underline{X}, Y) = \mathrm{Hom}(X, (Y, g^*M_S)).$$

We write $\mathrm{Sh}(\mathbf{LSch}/S)_{\text{ét}}$ (resp. $\mathrm{Sh}(\mathbf{Sch}/\underline{S})_{\text{ét}}$) for the category of sheaves on the strict étale site on $\mathbf{LSch}/S$ (resp. the étale site on $\mathbf{Sch}/\underline{S}$). The functors $\mathfrak{f}$ and $\mathfrak{s}$ give rise to pushforward functors $\mathfrak{f}_*$ and $\mathfrak{s}_*$ on the categories of sheaves:

$$\mathrm{Sh}(\mathbf{Sch}/\underline{S})_{\text{ét}} \xrightarrow{\mathfrak{f}_*} \mathrm{Sh}(\mathbf{LSch}/S)_{\text{ét}},$$

$$\mathrm{Sh}(\mathbf{LSch}/S)_{\text{ét}} \xrightarrow{\mathfrak{s}_*} \mathrm{Sh}(\mathbf{Sch}/\underline{S})_{\text{ét}}.$$

The functor $\mathfrak{s}$ takes étale coverings to strict étale coverings, and commutes with fibered products. Therefore, $\mathfrak{s}_*$ admits a left adjoint

$$\mathfrak{s}^* : \mathrm{Sh}(\mathbf{Sch}/\underline{S})_{\text{ét}} \to \mathrm{Sh}(\mathbf{LSch}/S)_{\text{ét}}$$

by [37, Tag 00WX].

Since the counit $\mathfrak{f} \circ \mathfrak{s} \to 1_{\mathbf{Sch}}$ is an isomorphism, $\mathfrak{s}_*\mathfrak{f}_*$ is isomorphic to the identity. One then obtains by adjunction a morphism of functors $\mathfrak{s}^* \to \mathfrak{f}_*$.

Lemma 2.5. *The map of functors $\mathfrak{s}^* \to \mathfrak{f}_*$ is an isomorphism. In particular, $\mathfrak{f}_*$ is exact.*

Proof. Let $\mathcal{F}$ be a presheaf on $\mathbf{Sch}/\underline{S}$, and $\mathcal{G}$ a presheaf on $\mathbf{LSch}/\mathbf{S}$. For any log scheme $X \to S$ with underlying scheme $\underline{X}$ we have a canonical factorization $X \to \mathfrak{s}\underline{X} \to S$. Thus, for any map of presheaves $\mathfrak{f}_*\mathcal{F} \to \mathcal{G}$ and $X \in \mathbf{LSch}/\mathbf{S}$, the map $\mathfrak{f}_*\mathcal{F}(X) := \mathcal{F}(\underline{X}) \to \mathcal{G}(X)$ factors as $\mathcal{F}(\underline{X}) \to \mathcal{G}(\mathfrak{s}\underline{X}) := \mathfrak{s}_*\mathcal{G}(X) \to \mathcal{G}(X)$, i.e., $\mathfrak{f}_*$ is left adjoint to $\mathfrak{s}_*$. □

Remark 2.6. We write $\mathrm{Ab}(-)$ for the category of sheaves of abelian groups on a site. Both $\mathfrak{s}$ and $\mathfrak{f}$ are continuous and cocontinuous functors (see [37, Tag 00XJ], [37, Tag 00WV]); it follows that the pushforwards

$$\mathfrak{s}_* : \mathrm{Ab}(\mathbf{LSch}/S)_{\text{ét}} \to \mathrm{Ab}(\mathbf{Sch}/\underline{S})_{\text{ét}}, \quad \mathfrak{f}_* : \mathrm{Ab}(\mathbf{Sch}/\underline{S})_{\text{ét}} \to \mathrm{Ab}(\mathbf{LSch}/S)_{\text{ét}}$$

are exact, by [37, Tag 04BD].

The tropical multiplicative group satisfies the following useful property (not shared by its logarithmic counterpart $\mathbb{G}_{m,S}^{\log}$):

Lemma 2.7. *Consider the sheaf $\mathbb{G}_{m,S}^{\mathrm{trop}}$ on $(\mathbf{LSch}/S)_{\text{ét}}$. We have a canonical isomorphism $\mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}} = \mathrm{i}^*\overline{M}_{\underline{S}}^{gp}$ of sheaves on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$, where i^* is the functor from* (2.3.1)*. In particular, $\mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}}$ is locally constructible.*

Proof. Let $g : T \to \underline{S}$ be a morphism of schemes. Then

$$\mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}}(T) = \mathbb{G}_{m,S}^{\mathrm{trop}}(\mathfrak{s}T) = g^{-1}\overline{M}_S^{\mathrm{gp}}(T) = \mathrm{i}^*\overline{M}_S^{\mathrm{gp}}(T).$$ □

Corollary 2.8. *The sheaves $\mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}}$ and $\mathfrak{s}_*\mathbb{G}_{m,S}^{\log}$ are representable by quasi-separated group algebraic spaces, respectively étale and smooth.*

Proof. The statement for $\mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}}$ follows by Lemma 2.3. Then we conclude by exactness of the sequence

$$0 \to \mathbb{G}_{m,S} \to \mathfrak{s}_*\mathbb{G}_{m,S}^{\log} \to \mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}} \to 0.$$ □

3. Tropical notions on the big site

In this section we develop the necessary tools to introduce the tropical and logarithmic Jacobian of [26]. We carefully define the sheaf of lattices $\mathcal{H}_{1,X/S}$ of first Betti homologies of the dual graphs of the fibers. Then we construct the tropical Jacobian as a global quotient of a tropical torus $\mathcal{H}om(\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger$ by the sheaf of lattices $\mathcal{H}_{1,X/S}$; this will facilitate the

proof of the representability of the strict tropical Jacobian in Section 4. The original definition of Molcho and Wise is slightly different but we show it to be equivalent to ours.

3.1. Tropicalization. A *graph* consists of finite sets V of vertices and H of half-edges, with an 'attachment' map $r : H \to V$ from the half-edges to the vertices, and an 'opposite end' involution $\iota : H \to H$ on the half-edges. To be consistent with our convention that log curves are vertical, we require this involution ι to have no fixed points; because our curves have connected geometric fibers we also require our graphs to be connected. An edge is an unordered pair of half-edges interchanged by the involution, and we denote the set of them by E.

Let $\overline{M}$ be a sharp monoid. A *tropical curve metrized by* $\overline{M}$ is a graph (V, H, r, i) together with a function $\ell : E \to \overline{M} \setminus \{0\}$.

Let S be a geometric logarithmic point[5)], and let X/S be a log curve. The associated tropical curve (*tropicalization*) $\mathfrak{X}$ of X has as underlying graph the usual dual graph of X (with a vertex for each irreducible component and a half-edge for each branch at each singular point). To define the labelling $\ell : H \to \overline{M}_S(S)$ we recall from [20] that the stalk of the log structure at a singular point x has characteristic monoid $\overline{M}_{X,x} \cong \overline{M}_S(S) \oplus_{\mathbb{N}} \mathbb{N}^2$, where the coproduct is over the diagonal map $\mathbb{N} \to \mathbb{N}^2$, and a map $\mathbb{N} \to \overline{M}_S(S)$. If e is the edge corresponding to x then we set $\ell(e)$ to be the image of 1 in $\overline{M}_S(S)$; this is independent of the choice of presentation.

A map of monoids $\varphi : \overline{M} \to \overline{N}$ determines a "contracted" tropical curve $\mathfrak{X}_\varphi$, whose graph is obtained from the graph underlying $\mathfrak{X}$ by contracting all edges e whose length $\ell(e)$ maps to 0 via φ, and with length of the remaining edges induced by φ. Molcho and Wise define a tropical curve over an arbitrary log scheme S as the data of a tropical curve for each geometric point of S, together with contraction maps between them compatible with geometric specializations, but we will not use this notion.

If S is a log scheme (but not necessarily a log point) and X/S a log curve, the edge-labellings of the tropicalization of X at various geometric points of S vary nicely in families, as a consequence of the following proposition:

Proposition 3.1. *Let $\pi : X \to S$ be a log curve and $\alpha : X^{\mathrm{nsm}} \to S$ the structure morphism of the non-smooth locus of π. The category of cartesian squares*

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X' & \longrightarrow & S', \end{array}$$

where $X' \to S'$ is a log curve and $S \to S'$ induces the identity on underlying schemes (with obvious morphisms) has a terminal object

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X^{\#} & \longrightarrow & S^{\#}, \end{array}$$

where the log structure $M_S^{\#}$ on $S^{\#}$ satisfies $\overline{M}_S^{\#} = \alpha_* \mathbb{N}$ (and $\mathbb{N}$ denotes the constant sheaf with value $\mathbb{N}$ on X^{nsm}).

[5)] A log scheme whose underlying scheme is the spectrum of an algebraically closed field.

Proof. This is [29, Theorem 2.7], combined with the observation that X/S is special in the sense of [29, Definition 2.6] if and only if $\overline{M}_S = \alpha_*\mathbb{N}$. □

3.2. Subdivisions. Given a tropical curve $\mathcal{X} = (V, H, r, i, \ell)$ with edges marked by a monoid $\overline{M}$, and an edge $e = \{h_1, h_2\}$, one can define a new tropical curve

$$\mathcal{X}' = (V', H', r', i', \ell')$$

as follows: V' is obtained from V by adjoining a new vertex v, H' is obtained by adding two half-edges γ_1, γ_2 with $r(\gamma_1) = r(\gamma_2) = v$. Then we set $i'(h_1) = \gamma_1$ and $i'(h_2) = \gamma_2$. Finally, we choose lengths $\ell'(\gamma_1)$ and $\ell'(\gamma_2)$ so that their sum is $\ell(h_1)$. On the remaining edges and vertices r', i', ℓ' are set to agree with r, i, ℓ.

Definition 3.2. A tropical curve $\mathcal{X}'$ constructed from $\mathcal{X}$ as in the paragraph above is called a *basic subdivision* of $\mathcal{X}$. A *subdivision* of $\mathcal{X}$ is a tropical curve obtained by composing a finite number of basic subdivisions.

We state a few simple facts regarding the behavior of subdivisions with respect to contractions.

Fact 3.3. Let $\mathcal{X}$ be a tropical curve metrized by a monoid $\overline{M}$, let $\varphi : M \to N$ be a map of monoids, and let $\mathcal{X}^\varphi$ be the contraction of $\mathcal{X}$ induced by φ as in Section 3.1. Let $\mathcal{Y}$ be a subdivision of $\mathcal{X}$. Then the contraction $\mathcal{Y}^\varphi$ is a subdivision of $\mathcal{X}^\varphi$.

Fact 3.4. Let S be a log scheme with log structure M_S and X/S a log curve. Let $(\bar\eta) \to (\bar s)$ be an étale specialization of geometric points, with induced morphism of characteristic monoids $\varphi : \overline{M}_{\bar s} \to \overline{M}_{\bar\eta}$ (see [7, Appendix A] for the notion of étale specialization). Let $\mathcal{X}_{\bar s}, \mathcal{X}_{\bar\eta}$ be the tropicalizations of $X_{\bar s}, X_{\bar\eta}$. Then $\mathcal{X}_{\bar\eta}$ is the contraction $\mathcal{X}_{\bar s}^\varphi$ of $\mathcal{X}_{\bar s}$ induced by φ.

Fact 3.5. Let $X \to S$ be a log curve, and $Y \to X$ a logarithmic modification such that $Y \to S$ is a log curve. For every log geometric point $t \to S$, the tropicalization of Y_t is a subdivision of the tropicalization of X_t. Moreover, for an étale specialization $(\bar\eta) \to (\bar s)$ as above, the subdivision of $\mathcal{X}_{\bar\eta}$ is $\mathcal{Y}^\varphi$, where $\mathcal{Y}$ is the subdivision of $\mathcal{X}_{\bar s}$ over s.

Lemma 3.6 ([26, 2.4.3]). *Let $\bar s \to S$ be a geometric point and $\mathcal{Y}$ a subdivision of $\mathcal{X}_{\bar s}$. Then there exist an étale neighborhood V of $\bar s$, a log curve Y/V and a logarithmic modification $Y \to X \times_S V$ inducing the subdivision $\mathcal{Y} \to \mathcal{X}_{\bar s}$.*

3.3. The tropical Jacobian over a point. Let $\mathcal{X}$ be a tropical curve over a monoid $\overline{M}$. After choosing an orientation on the edges we have a boundary map $\mathbb{Z}^E \to \mathbb{Z}^V$, whose kernel is the first homology group $H_1(\mathcal{X}, \mathbb{Z})$. Molcho and Wise define an intersection pairing

$$H_1(\mathcal{X}, \mathbb{Z}) \times H_1(\mathcal{X}, \mathbb{Z}) \to \overline{M}^{\mathrm{gp}}, \tag{3.3.1}$$

and then define the Jacobian of $\mathcal{X}$ to be

$$\mathrm{Hom}(H_1(\mathcal{X}, \mathbb{Z}), \overline{M}^{\mathrm{gp}})^\dagger / H_1(\mathcal{X}, \mathbb{Z}),$$

where the symbol † denotes the subgroup of elements of *bounded monodromy*, see [26, Definition 3.5.5], or Section 3.8.

For a log curve X/S, Molcho and Wise then define the tropical Jacobian to be the data of the Jacobians of the tropicalizations of X_s for every geometric point of s, together with their étale specialization maps (see [7, Appendix A]). However, for our purposes it is important to upgrade the groups $\mathbb{Z}^E$ and $\mathbb{Z}^V$, and thereby $H_1(\mathcal{X}, \mathbb{Z})$ and the tropical Jacobian, to sheaves of abelian groups on the big étale site of S. The definitions become slightly intricate, because a choice of orientation on the tropicalization does not exist in families, and because locally constructible *sheaves* of abelian groups are not isomorphic to their duals.

3.4. The sheaf $\mathcal{Z}^{\mathcal{E}}$. Let $\underline{S}$ be a scheme and $\pi : \underline{X} \to \underline{S}$ a prestable curve. Here we define a sheaf $\mathcal{Z}^{\mathcal{E}}$ on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$ which on each geometric point is isomorphic to the sheaf $\mathbb{Z}^E$ from above. Our sheaf is not of the form $\mathcal{H}om(\mathcal{E}, \mathbb{Z})$ for some sheaf $\mathcal{E}$ on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$, though it is locally of that form. Recall that $\mathrm{i} : (\mathbf{Sch}/\underline{X})_{\text{ét}} \to \underline{X}_{\text{ét}}$ is the morphism of sites induced by the inclusion of categories $\underline{X}_{\text{ét}} \to (\mathbf{Sch}/\underline{X})_{\text{ét}}$.

Definition 3.7. Equipping X/S with the minimal log structure, we define the sheaves

$$\mathbb{G}_{m,X/S}^{\text{trop}} = \mathrm{i}^* \overline{M}_{X/S}^{\text{gp}} \quad \text{and} \quad \mathcal{Z}^{\mathcal{E}} = \pi_* \mathbb{G}_{m,X/S}^{\text{trop}}. \tag{3.4.1}$$

3.4.1. Branches, half-edges and orientations.

Definition 3.8. We define $\mathcal{E}$ (the *sheaf of edges*) to be the sheaf on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$ represented by the non-smooth locus $\alpha : X^{\text{nsm}} \to \underline{S}$. We will use the notations $\mathcal{E}$ and X^{nsm} interchangeably, depending on context.

If we denote by $\{\star\}$ the final object in the category of sheaves of sets on $(\mathbf{Sch}/X^{\text{nsm}})_{\text{ét}}$, then $\mathcal{E}$ is $\alpha_!\{\star\}$. Then

$$\mathbb{Z}^{\mathcal{E}} := \mathcal{H}om_{\underline{S}}(\mathcal{E}, \mathbb{Z}) = \mathcal{H}om_{\underline{S}}(\alpha_!\{\star\}, \mathbb{Z}) = \alpha_* \mathcal{H}om_{X^{\text{nsm}}}(\{\star\}, \alpha^*\mathbb{Z}) = \alpha_* \mathbb{Z}_{X^{\text{nsm}}} \tag{3.4.2}$$

but in general this is *not* isomorphic to $\mathcal{Z}^{\mathcal{E}}$ due to the non-existence of global choices of orientations on the tropicalizations of the fibers; we expand on this in what follows.

Definition 3.9. The base change $X' := X \times_S X^{\text{nsm}} \to X^{\text{nsm}}$ along the finite unramified morphism $X^{\text{nsm}} \to S$ admits a natural section $a : X^{\text{nsm}} \to X \times_S X^{\text{nsm}}$ (a closed immersion). Let Y denote the blowup of X' along a, and define the *scheme of branches*

$$X^{\text{br}} := X^{\text{nsm}} \times_{X'} Y. \tag{3.4.3}$$

Remark 3.10. (i) The projection $X^{\text{br}} \to X^{\text{nsm}}$ is finite étale of degree 2; in fact, it is a $\mathbb{Z}/2\mathbb{Z}$-torsor.

(ii) The projection $X^{\text{br}} \to S$ is finite unramified; in particular, after replacing S by an étale cover, both $X^{\text{nsm}} \to S$ and $X^{\text{br}} \to S$ are disjoint unions of closed immersions, so that locally on S we have $X^{\text{br}} \cong X^{\text{nsm}} \sqcup X^{\text{nsm}}$ as schemes over X.

Definition 3.11. We say that $X \to S$ has *split branches* if

- $X^{\text{nsm}} \to S$ is a disjoint union of closed immersions, and
- $X^{\text{br}} \cong X^{\text{nsm}} \sqcup X^{\text{nsm}}$ over X.

Remark 3.12. A choice of section of the torsor $X^{\mathrm{br}} \to X^{\mathrm{nsm}}$ is equivalent to a choice of compatible orientations of the tropicalizations of the fibers of $X \to S$.

Example 3.13. Let k be a field and let X be the prestable irreducible curve

$$X := \operatorname{Proj} k[x, y, z]/(y^2 z - x^3 - x^2 z),$$

with the involution $\sigma : X \to X$ given by $y \mapsto -y$. Consider the prestable curve

$$Y := X \times_k X \to X.$$

Let P be a $\mathbb{Z}/2\mathbb{Z}$-torsor on X, and consider the twisted form of Y given by

$$Y^P := Y \otimes_{\mathbb{Z}/2\mathbb{Z}} P \to X,$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on Y/X by σ. Then the scheme of branches of Y^P/X is canonically isomorphic to the torsor $P \to X$. In particular, for P whose class in $H^1(X_{\text{ét}}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is non-zero, the curve Y^P does not have split branches.

Definition 3.14. We define the *sheaf of half-edges* $\mathcal{H} \in \mathrm{Sh}((\mathbf{Sch}/X^{\mathrm{nsm}})_{\text{ét}})$ to be X^{br}. We will use the notations $\mathcal{H}$ and X^{br} interchangeably. We will use the same notation for the pushforward (left adjoint to pullback) to $\mathrm{Sh}((\mathbf{Sch}/S)_{\text{ét}})$.

The sheaf $\mathcal{H}$ comes with a natural involution ι over X^{nsm} and hence over S. Composition induces an involution ι on $\mathbb{Z}^{\mathcal{H}} := \mathcal{H}om_S(\mathcal{H}, \mathbb{Z})$, and we find that:

Lemma 3.15. *We have*

$$\mathbb{Z}^{\mathcal{E}} = (\mathbb{Z}^{\mathcal{H}})^{\iota} \quad \text{and} \quad \mathcal{Z}^{\mathcal{E}} = (\mathbb{Z}^{\mathcal{H}})^{-\iota} \tag{3.4.4}$$

(respectively the invariants for ι and for $-\iota$).

Proof. The equality $\mathbb{Z}^{\mathcal{E}} = (\mathbb{Z}^{\mathcal{H}})^{\iota}$ is clear. Writing $\beta : X^{\mathrm{br}} \to X$, the locally-free rank 1 $\mathbb{Z}$-module $\beta^* \overline{M}^{\mathrm{gp}}_{X/S}$ has a *canonical* generating section m_0 coming from generating sections of the ideal sheaves of the preimages of the node on the two branches. The equality $\beta = \beta \circ \iota$ induces an isomorphism $\iota^* \beta^* \overline{M}^{\mathrm{gp}}_{X/S} \xrightarrow{\sim} \beta^* \overline{M}^{\mathrm{gp}}_{X/S}$, which takes $\iota^* m_0$ to $-m_0$.

There is a natural isomorphism

$$\tau : \beta^* \overline{M}^{\mathrm{gp}}_{X/S} \xrightarrow{\sim} \mathbb{Z}_{X^{\mathrm{br}}}, \quad m_0 \mapsto 1. \tag{3.4.5}$$

The involution ι induces an automorphism $\iota : \beta_* \mathbb{Z}_{X^{\mathrm{br}}} \xrightarrow{\sim} \beta_* \mathbb{Z}_{X^{\mathrm{br}}}$. We then define a map

$$\overline{M}^{\mathrm{gp}}_{X/S} \to \beta_* \mathbb{Z}_{X^{\mathrm{br}}}$$

sending m to $\tau(\beta^* m)$. Now $\iota(\tau(\beta^* m)) = -\tau(\beta^* m)$, so this map factors via the inclusion $(\beta_* \mathbb{Z}_{X^{\mathrm{br}}})^{-\iota} \hookrightarrow \beta_* \mathbb{Z}_{X^{\mathrm{br}}}$. Locally on X^{nsm} the induced map $\overline{M}^{\mathrm{gp}}_{X/S} \to (\beta_* \mathbb{Z}_{X^{\mathrm{br}}})^{-\iota}$ is a map of free rank-1 $\mathbb{Z}$-modules, and is easily checked to be an isomorphism, from which the second equality is immediate. □

Lemma 3.16. *The sheaves $\mathcal{Z}^{\mathcal{E}}$ and $\mathbb{Z}^{\mathcal{E}}$ are isomorphic étale locally on S.*

Proof. Étale locally on S the curve X has split branches, so we conclude by (3.15). □

3.4.2. Representability.

Lemma 3.17. *The sheaf $\mathbb{Z}^{\mathcal{E}}$ on $(\mathbf{Sch}/S)_{\text{ét}}$ is representable by a quasi-separated étale group algebraic space on S.*

Proof. Working étale locally on S, we reduce to the case where $\mathcal{E} \to \underline{S}$ is a finite disjoint union of closed immersions. We may then assume $i : \mathcal{E} \to S$ is a closed immersion with $\mathcal{E}$ connected, and with complement $j : \underline{U} \to S$. The constant sheaf $\mathbb{Z}_S$ is representable by an étale algebraic space, and so is its open subgroup $j_!\mathbb{Z}_U$. So the quotient $\mathcal{H}om_S(\mathcal{E}, \mathbb{Z}) = i_*\mathbb{Z}_{\mathcal{E}}$ is representable by an étale algebraic space. For quasi-separatedness, we may reduce to S locally noetherian because $\mathcal{E}$ is finitely presented, and then use Remark 2.4. □

Combining Lemmas 3.16 and 3.17 yields:

Lemma 3.18. *The sheaf $\mathcal{Z}^{\mathcal{E}}$ is representable by a quasi-separated étale group algebraic space over S.*

3.5. The sheaf $\mathcal{Z}^{\mathcal{V}}$. For any site Σ we denote by $\mathbb{Z}[\ ] : \mathrm{Sh}(\Sigma) \to \mathrm{Ab}(\Sigma)$ the functor taking a sheaf of sets $\mathcal{F}$ to the sheafification of the presheaf $\mathbb{Z}_{\mathrm{PSh}}[\mathcal{F}]$ of free abelian groups with basis $\mathcal{F}$. Note that $\mathbb{Z}[\ ]$ is left adjoint to the forgetful functor $\mathrm{Ab}(\Sigma) \to \mathrm{Sh}(\Sigma)$; indeed, for any sheaves $\mathcal{F}$ of sets and $\mathcal{G}$ of abelian groups, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}(\Sigma)}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathrm{PSh}(\Sigma)}(\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Hom}_{\mathrm{PAb}(\Sigma)}(\mathbb{Z}_{\mathrm{PSh}}[\mathcal{F}], \mathcal{G}) \\ &= \mathrm{Hom}_{\mathrm{Ab}(\Sigma)}(\mathbb{Z}[\mathcal{F}], \mathcal{G}).\end{aligned}$$

Definition 3.19. Let X/S be a prestable curve. We define the *sheaf of relative irreducible components* $\mathrm{Irr}_{X/S} := \pi_0(X^{sm}/S)$ by sending $T \to S$ to the set of X_T-isomorphism classes of open immersions $U \subset X_T$ such that

(i) for every geometric point $\bar{t}$ of T, the fiber $U_{\bar{t}}$ is irreducible.

(ii) up to isomorphism, $U \to X_T$ is maximal among open immersions with property (i).

Remark 3.20. In [36], Romagny studies relative irreducible components in more detail and generality. He shows that $\mathrm{Irr}_{X/S}$ is representable by a finitely presented étale algebraic space over $\underline{S}$.

Definition 3.21. We define the sheaf $\mathcal{Z}^{\mathcal{V}} := \mathbb{Z}[\mathrm{Irr}_{X/S}]$ on $(\mathbf{Sch}/S)_{\text{ét}}$.

Remark 3.22. The sheaf $\mathcal{Z}^{\mathcal{V}}$ is the Néron–Severi group of $\underline{X}/\underline{S}$. If S is a geometric point, then there is a natural isomorphism $\mathcal{Z}^{\mathcal{V}} \xrightarrow{\sim} \mathbb{Z}^V$, motivating the notation.

Remark 3.23. We emphasize that the sheaf $\mathcal{Z}^{\mathcal{V}}$ should not be confused with the sheaf $\mathbb{Z}^{\mathrm{Irr}_{X/S}} = \mathrm{Hom}(\mathrm{Irr}_{X/S}, \mathbb{Z})$ on $(\mathbf{Sch}/S)_{\text{ét}}$. For example, if S is a strictly henselian discrete valuation ring and X/S a prestable curve, smooth over the generic point, and whose special fiber has two irreducible components, then the global sections of $\mathbb{Z}[\mathrm{Irr}_{X/S}]$ form a free abelian group of rank 2, and those of $\mathbb{Z}^{\mathrm{Irr}_{X/S}}$ a free abelian group of rank 1.

Lemma 3.24. *The sheaf $\mathcal{Z}^{\mathcal{V}}$ on $(\mathbf{Sch}/S)_{\text{ét}}$ is representable by a quasi-separated étale group algebraic space over S.*

Proof. As $\mathrm{Irr}_{X/S}$ is of finite presentation, we may assume S is locally Noetherian. By Lemma 2.3 it suffices to prove that $\mathcal{Z}^{\mathcal{V}}$ is locally constructible. From Remark 3.20 we know that $\mathrm{Irr}_{X/S}$ is locally constructible, in other words is of the form i^*J for some sheaf J on the small étale site of S. Then

$$\mathbb{Z}[\mathrm{Irr}_{X/S}] = \mathbb{Z}[\mathrm{i}^*J] = \mathrm{i}^*\mathbb{Z}[J],$$

where for the last equality we use that the functors i^* and $\mathbb{Z}[\ ]$ commute, which follows in turn from the easy observation that their right adjoints (i_* and the forgetful functor from sheaves of abelian groups to sheaves of sets) commute. □

Lemma 3.25. *Let $X \to S$ be a prestable curve. Then the sheaves $\mathrm{Irr} := \mathrm{Irr}_{X/S}$ and $\mathcal{Z}^{\mathcal{V}} = \mathbb{Z}[\mathrm{Irr}]$ on $(\mathbf{Sch}/S)_{\text{ét}}$ are generated by global sections étale-locally*[6] *on S.*

Proof. It suffices to prove Irr is generated by global sections étale-locally on S, which follows from [10, Lemma 18]. □

3.6. The tropical boundary map. If $\mathcal{X}$ is a tropical curve with a choice of orientation on the edges, there is a *boundary map*

$$\delta : \mathbb{Z}^E \to \mathbb{Z}^V \tag{3.6.1}$$

sending an edge E to its endpoint minus its startpoint. We will define an analogous map $\delta : \mathcal{Z}^{\mathcal{E}} \to \mathcal{Z}^{\mathcal{V}}$, independent of choices.

Remark 3.26. As pointed out in Section 3.4.1, choices of orientations compatible with specialization maps cannot in general be made in families, motivating our definition of $\mathcal{Z}^{\mathcal{E}}$ as $\pi_*\mathbb{G}_{m,X/S}^{\mathrm{trop}}$ instead of as $\mathbb{Z}^{\mathcal{E}}$.

Since both $\mathcal{Z}^{\mathcal{E}}$ and $\mathcal{Z}^{\mathcal{V}}$ are locally constructible, it suffices to define δ on the small étale site of S. In what follows, we work exclusively on small étale sites, implicitly applying the functor i_* wherever necessary. Working locally on $S_{\text{ét}}$, we may assume that X/S has split branches.

Write X^{nsm}/S as a disjoint union of closed immersions $\bigsqcup_i Z_i \xrightarrow{\alpha_i} S$, and for each i put

$$Z_i^{\mathrm{br}} := X^{\mathrm{br}} \times_{X^{\mathrm{nsm}}} Z_i\,.$$

Let $\mathcal{H}_i$ be Z_i^{br}, seen as a sheaf on $(Z_i)_{\text{ét}}$. We have

$$\mathcal{Z}^{\mathcal{E}} = \bigoplus_i \mathcal{Z}_i^{\mathcal{E}},$$

where $\mathcal{Z}_i^{\mathcal{E}}$ is the subgroup of $(-\iota)$-invariants of $(\alpha_i)_*\mathbb{Z}^{\mathcal{H}_i}$. We will construct natural maps $\mathcal{Z}_i^{\mathcal{E}} \to \mathcal{Z}^{\mathcal{V}}$, and sum them. Pick some i and put $Z := Z_i$, $\alpha := \alpha_i$. Here α is a closed immer-

[6] For a sheaf F on S, this means that every geometric point of S admits an étale neighborhood U such that the natural map from the constant sheaf on $F(U)$ to the restriction $F|U$ is surjective.

sion, and we use the notation from [24, Proposition II.3.14] for pushing and pulling along α of sheaves on the small étale site. Denote by j the open immersion $U := S \setminus Z \to S$. By [24, Proposition II.3.14], α_* is left adjoint to the functor $\alpha^! : \mathrm{Ab}(S_{\text{ét}}) \to \mathrm{Ab}(Z_{\text{ét}})$ taking a sheaf to (the pullback of) its subsheaf of sections supported on Z. Explicitly, for a sheaf $\mathcal{F}$ on S, $\alpha^!\mathcal{F}$ is the kernel of $\alpha^*\mathcal{F} \to \alpha^* j_* j^* \mathcal{F}$. We deduce

$$\mathrm{Hom}_{Z_{\text{ét}}}((\mathbb{Z}^{\mathcal{H}_i})^{(-\iota)}, \alpha^! \mathcal{Z}^{\mathcal{V}}) = \mathrm{Hom}_{S_{\text{ét}}}(\mathcal{Z}_i^{\mathcal{E}}, \mathcal{Z}^{\mathcal{V}}). \tag{3.6.2}$$

There is a natural map of S-algebraic spaces

$$b : X^{\mathrm{br}} \to \mathrm{Irr}_{X/S} \hookrightarrow \mathcal{Z}^{\mathcal{V}} \tag{3.6.3}$$

sending a point to the irreducible component of the normalization of its fiber on which it lies. Writing $\beta : Z^{\mathrm{br}} \to Z$, the map b induces a global section of $\beta^*\alpha^*\mathcal{Z}^{\mathcal{V}}$, i.e., a morphism $\mathbb{Z}_{Z^{\mathrm{br}}} \to \beta^*\alpha^*\mathcal{Z}^{\mathcal{V}}$. We have $\beta_*\mathbb{Z}_{Z^{\mathrm{br}}} = \mathbb{Z}^{\mathcal{H}_i}$. Since β is a disjoint union of isomorphisms, it follows that β_* is simultaneously the left and right adjoint of β^*. By adjunction and after restricting to the $(-\iota)$-invariants, we get a map $B \in \mathrm{Hom}_{Z_{\text{ét}}}((\mathbb{Z}^{\mathcal{H}_i})^{-\iota}, \alpha^*\mathcal{Z}^{\mathcal{V}})$. The image of B consists of sections supported on Z, so B factors through $\alpha^!\mathcal{Z}^{\mathcal{V}}$, and yields a map $\mathcal{Z}_i^{\mathcal{E}} \to \mathcal{Z}^{\mathcal{V}}$ by (3.6.2). Summing over i, we get the desired map $\delta : \mathcal{Z}^{\mathcal{E}} \to \mathcal{Z}^{\mathcal{V}}$. The fact that δ does not depend on the chosen expression of $X^{\mathrm{nsm}} \to S$ as a disjoint union of closed immersions is clear from the construction.

Definition 3.27. The *tropical boundary map* is the map

$$\delta : \mathcal{Z}^{\mathcal{E}} \to \mathcal{Z}^{\mathcal{V}}$$

constructed above.

Remark 3.28. If S is a log point we have a canonical isomorphism $\mathbb{Z}^V = \mathcal{Z}^{\mathcal{V}}$ (cf. Remark 3.22). A choice of orientations of the edges of the tropicalization of X/S provides an isomorphism $\mathbb{Z}^{\mathcal{E}} = \mathcal{Z}^{\mathcal{E}}$, which identifies δ with the boundary map of (3.6.1).

Definition 3.29. We denote by $\mathcal{H}_{1,X/S}$ the kernel of δ. It is a sheaf of abelian groups whose stalks are free and finitely generated. Its value at a geometric point s of S is isomorphic to the first homology group $H_1(\mathcal{X}_s, \mathbb{Z})$, cf. Remark 3.28.

Remark 3.30. Prompted by a request from an anonymous referee, we give an alternative interpretation of the map δ, via the interpretation of $\mathcal{Z}^{\mathcal{V}}$ as the Néron–Severi group of X/S (see Remark 3.22). For this remark, we work in the sites $(\mathbf{Sch}/\underline{X})_{\text{ét}}$ and $(\mathbf{Sch}/\underline{S})_{\text{ét}}$. Let X^{st} be X with the log structure pulled back from S. We restrict sheaves naturally defined on $(\mathbf{LSch}/X^{st})_{\text{ét}}$, $(\mathbf{LSch}/X)_{\text{ét}}$, $(\mathbf{LSch}/S)_{\text{ét}}$ to $(\mathbf{Sch}/\underline{X})_{\text{ét}}$, $(\mathbf{Sch}/\underline{S})_{\text{ét}}$ without additional decorations, to avoid overcrowding the notation. Recalling that $\mathcal{Z}^{\mathcal{E}} = \pi_*\mathbb{G}_{m,X/S}^{\mathrm{trop}}$, the cohomology of the short exact sequence

$$0 \to \mathbb{G}_{m,X^{\mathrm{st}}}^{\log} \to \mathbb{G}_{m,X}^{\log} \to \mathbb{G}_{m,X/S}^{\mathrm{trop}} \to 0 \tag{3.6.4}$$

gives a canonical map

$$\mathcal{Z}^{\mathcal{E}} = \pi_*\mathbb{G}_{m,X/S}^{\mathrm{trop}} \to R^1\pi_*\mathbb{G}_{m,X^{\mathrm{st}}}^{\log}. \tag{3.6.5}$$

We also have an exact sequence

$$0 \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,X^{\mathrm{st}}}^{\log} \to \pi^{-1}\mathbb{G}_{m,S}^{\mathrm{trop}} \to 0, \tag{3.6.6}$$

which splits after base changing to a strict étale cover $S' \to S$. Choosing a splitting gives a map $\mathbb{G}_{m,X_{S'}^{\mathrm{st}}}^{\log} \to \mathbb{G}_{m,X_{S'}}$, and taking cohomology gives a map

$$R^1\pi_*\mathbb{G}_{m,X_{S'}^{\mathrm{st}}}^{\log} \to R^1\pi_*\mathbb{G}_{m,X_{S'}} = \mathrm{Pic}_{X_{S'}/S'}. \tag{3.6.7}$$

This map depends on the choice of splitting, but we claim that the composite with the natural map to the Néron–Severi group $\mathrm{Pic}_{X_{S'}/S'} / \mathrm{Pic}^0_{X_{S'}/S'}$ does not depend on this choice. In particular, the composite descends to a canonical map

$$R^1\pi_*\mathbb{G}_{m,X^{\mathrm{st}}}^{\log} \to \mathrm{Pic}_{X/S} / \mathrm{Pic}^0_{X/S}. \tag{3.6.8}$$

The claim may be proven assuming $S' = S$. The difference between two choices of splittings is measured by a map

$$F : \pi^{-1}\mathbb{G}_{m,S}^{\mathrm{trop}} \to \mathbb{G}_{m,X}, \tag{3.6.9}$$

which by adjunction is the same as a map $G : \mathbb{G}_{m,S}^{\mathrm{trop}} \to \pi_*\mathbb{G}_{m,X} = \mathbb{G}_{m,S}$ (the latter equality since X/S is proper with reduced and connected geometric fibers). Then we have a map

$$R^1\pi_*\pi^{-1}G : R^1\pi_*\pi^{-1}\mathbb{G}_{m,S}^{\mathrm{trop}} \to R^1\pi_*\pi^{-1}\mathbb{G}_{m,S}, \tag{3.6.10}$$

and composing with the canonical map $\pi^{-1}\mathbb{G}_{m,S} \to \mathbb{G}_{m,X}$ yields a map

$$R^1\pi_*\pi^{-1}\mathbb{G}_{m,S}^{\mathrm{trop}} \to R^1\pi_*\mathbb{G}_{m,X} = \mathrm{Pic}_{X/S}; \tag{3.6.11}$$

we must show that this map factors via $\mathrm{Pic}^0_{X/S}$. Writing $\beta : \pi^{-1}\mathbb{G}_{m,S} \to \mathbb{G}_{m,X}$ for the natural inclusion, we find that $F = \beta \circ \pi^{-1}\pi_* F$, and so

$$R^1\pi_* F : R^1\pi_*\pi^{-1}\mathbb{G}_{m,S}^{\mathrm{trop}} \to R^1\pi_*\mathbb{G}_{m,X} = \mathrm{Pic}_{X/S} \tag{3.6.12}$$

factors as

$$R^1\pi_*\pi^{-1}\mathbb{G}_{m,S}^{\mathrm{trop}} \to R^1\pi_*\pi^{-1}\mathbb{G}_{m,S} \to \mathrm{Pic}_{X/S}, \tag{3.6.13}$$

and

$$R^1\pi_*\pi^{-1}\mathbb{G}_{m,S} \to \mathrm{Pic}_{X/S}$$

factors through $\mathrm{Pic}^0_{X/S}$. Indeed, the fiber of $T := R^1\pi_*\pi^{-1}\mathbb{G}_{m,S}$ over a geometric point $s \to S$ is the torus part of $\mathrm{Pic}^0_{X_s/s}$.

This shows the existence of a slight refinement of δ; we defined it as a map from $\mathcal{Z}^{\mathcal{E}}$ to the Néron–Severi group $\mathcal{Z}^{\mathcal{V}} = \mathrm{Pic}_{X/S} / \mathrm{Pic}^0_{X/S}$, but it comes naturally from a map to the quotient $\mathrm{Pic}_{X/S} / T$ by the "torus part of $\mathrm{Pic}_{X/S}$". We do not know if this lift has any application.

3.7. The monodromy pairing. If $\mathcal{X}$ is a tropical curve over a monoid $\overline{M}$, there is a map $E \times E \to \overline{M}^{\mathrm{gp}}$ sending (e, f) to $\ell(e)$ if $e = f$ and 0 otherwise. This induces a map $\mathbb{Z}^E \times \mathbb{Z}^E \to \overline{M}^{\mathrm{gp}}$, which, after choice of an orientation on E, restricts to a pairing

$$H_1(\mathcal{X}, \mathbb{Z}) \times H_1(\mathcal{X}, \mathbb{Z}) \to \overline{M}^{\mathrm{gp}} \tag{3.7.1}$$

called the *monodromy pairing*. We will define an analogous map

$$\mathfrak{f}_*\mathcal{H}_{1,X/S} \times \mathfrak{f}_*\mathcal{H}_{1,X/S} \to \mathbb{G}_{m,S}^{\mathrm{trop}}, \tag{3.7.2}$$

where $\mathfrak{f}_*$ is the functor $\mathrm{Ab}(\mathbf{LSch}/S)_{\text{ét}} \to \mathrm{Ab}(\mathbf{Sch}/\underline{S})_{\text{ét}}$ introduced in Section 2.4.

Let X/S be a log curve. We write $\alpha : X^{\mathrm{nsm}} \to \underline{S}$ and $\varphi : X^{\mathrm{br}} \to X^{\mathrm{nsm}}$. On $(\mathbf{Sch}/X^{\mathrm{nsm}})_{\text{ét}}$, we have a sheaf of abelian groups $\varphi_*\mathbb{Z}_{X^{\mathrm{br}}} = \mathrm{Hom}_{X^{\mathrm{nsm}}}(X^{\mathrm{br}}, \mathbb{Z})$ which is locally free of rank 2 and endowed with the involution ι. We write $(\varphi_*\mathbb{Z}_{X^{\mathrm{br}}})^{-\iota}$ for the invariants under $-\iota$; this is a locally free rank 1 sheaf of abelian groups, hence self-dual. Pushing forward the natural pairing $(\varphi_*\mathbb{Z}_{X^{\mathrm{br}}})^{-\iota} \times (\varphi_*\mathbb{Z}_{X^{\mathrm{br}}})^{-\iota} \to \mathbb{Z}$ via α, we get a pairing

$$\mathcal{Z}^{\mathcal{E}} \times \mathcal{Z}^{\mathcal{E}} \to \alpha_*\mathbb{Z}. \tag{3.7.3}$$

Next, we compose (3.7.3) with the natural map $\alpha_*\mathbb{Z} \to \mathfrak{i}^*\overline{M}_S^{\mathrm{gp}}$ of Proposition 3.1, to obtain a pairing

$$\mathcal{Z}^{\mathcal{E}} \times \mathcal{Z}^{\mathcal{E}} \to \mathfrak{i}^*\overline{M}_S^{\mathrm{gp}} = \mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}}, \tag{3.7.4}$$

where the last equality is Lemma 2.7.

Definition 3.31. The *monodromy pairing* is the pairing

$$\mathfrak{f}_*\mathcal{H}_{1,X/S} \times \mathfrak{f}_*\mathcal{H}_{1,X/S} \to \mathbb{G}_{m,S}^{\mathrm{trop}}$$

obtained by adjunction from (3.7.4) and restriction to $\mathcal{H}_{1,X/S}$, and the fact that $\mathfrak{s}^* = \mathfrak{f}_*$ (Lemma 2.5)

3.8. The condition of bounded monodromy. In order to define a tropical Jacobian for X/S, we would like to take the quotient of the tropical torus $\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})$ by the group of periods $\mathfrak{f}_*\mathcal{H}_{1,X/S} = \mathfrak{s}^*\mathcal{H}_{1,X/S}$, where the action is given by the monodromy pairing. However, the sheaf $\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})$ does not behave as we would like with respect to generization. Indeed, if s, η are two geometric points of S such that η is a generization of s, a homomorphism $H_1(\mathcal{X}_s) \to \overline{M}_{S,s}^{\mathrm{gp}}$ does not induce a homomorphism $H_1(\mathcal{X}_\eta) \to \overline{M}_{S,\eta}^{\mathrm{gp}}$. As a consequence, formal elements of the sheaf do not integrate to elements over complete noetherian algebras. This has the consequence that the sheaf

$$\mathfrak{s}_*\mathcal{H}om(\mathfrak{s}^*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}}) = \mathcal{H}om(\mathcal{H}_{1,X/S}, \mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}})$$

is not representable by an $\underline{S}$-algebraic space, since it fails to satisfy one of Artin's axioms ([2, Theorem 5.3, 2']). The authors in [26] introduce the condition of *bounded monodromy* which fixes this issue. We recall it now.

Definition 3.32. Let $\overline{M}$ be a sharp fs monoid, and a, b elements of $\overline{M}^{\mathrm{gp}}$. We say a is *bounded by* b if there exist integers n, m such that $nb \leq a \leq mb$ (for the partial order induced by $\overline{M}$). The elements bounded by b form a subgroup of $\overline{M}^{\mathrm{gp}}$.

Definition 3.33. Let $\mathcal{X}$ be a tropical curve marked by a monoid $\overline{M}$. A homomorphism $\varphi : H_1(\mathcal{X}) \to \overline{M}^{\mathrm{gp}}$ is of *bounded monodromy* if for every $\gamma \in H_1(\mathcal{X})$, $\varphi(\gamma)$ is bounded by the length of γ, i.e., the self-intersection of γ under the monodromy pairing.

Remark 3.34. The length ℓ is a map $H_1(\mathfrak{X}) \to \overline{M}^{\mathrm{gp}}$ which factors through $\overline{M}$. If two elements γ, γ' of $H_1(\mathfrak{X})$ have disjoint support, then we have $\ell(\gamma + \gamma') = \ell(\gamma) + \ell(\gamma')$.

This notion extends naturally to the relative setting:

Definition 3.35. Let $X \to S$ be a log curve. We define the morphism

$$\ell : \mathfrak{f}_* \mathcal{H}_{1,X/S} \to \mathbb{G}_{m,S}^{\mathrm{trop}}$$

as the composition of the monodromy pairing with the diagonal

$$\mathfrak{f}_* \mathcal{H}_{1,X/S} \to \mathfrak{f}_* \mathcal{H}_{1,X/S} \times \mathfrak{f}_* \mathcal{H}_{1,X/S}.$$

Definition 3.36. Let $X \to S$ be a log curve. We will use the name *bounded monodromy subsheaf*, and we write $\mathcal{H}om(\mathfrak{f}_* \mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger$, for the open subgroup sheaf of $\mathcal{H}om(\mathfrak{f}_* \mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})$ obtained by sheafifying the sub-presheaf of morphisms

$$\varphi : \mathfrak{f}_* \mathcal{H}_{1,X/S}(T) \to \overline{M}_T^{\mathrm{gp}}$$

such that for every $\gamma \in \mathfrak{f}_* \mathcal{H}_{1,X/S}(T)$, $\varphi(\gamma)$ is bounded by $\ell(\gamma)$.

Remark 3.37. Suppose S is a log point, and fix an orientation of the edges of the tropicalization $\mathfrak{X}$ of X at s (i.e., an isomorphism $\mathcal{Z}^{\mathcal{E}} = \mathbb{Z}^{\mathcal{E}}$). Then we recover the usual notions of length and bounded monodromy on the tropical curve $\mathfrak{X}$.

Remark 3.38. It will follow from Section 3.10 that $\mathcal{H}om(\mathfrak{f}_* \mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger$ is precisely the open subgroup sheaf of $\mathcal{H}om(\mathfrak{f}_* \mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})$ consisting of morphisms that have bounded monodromy at every strict geometric log point s of S.

One immediately checks that the morphism

$$\mathfrak{f}_* \mathcal{H}_{1,X/S} \to \mathcal{H}om(\mathfrak{f}_* \mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})$$

induced by the monodromy pairing factors via the bounded monodromy subgroup.

3.9. The tropical Jacobian.

Definition 3.39. The tropical Jacobian of X/S is the sheaf on $(\mathbf{LSch}/S)_{\text{ét}}$

$$\mathrm{TroPic}^0_{X/S} = \mathcal{H}om(\mathfrak{f}_* \mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger / \mathfrak{f}_* \mathcal{H}_{1,X/S}.$$

For s a geometric point of S, a choice of an orientation of the graph $\mathfrak{X}_s$ induces an isomorphism

$$\mathrm{TroPic}^0_{X/S}(s) \cong \mathrm{Hom}(H_1(\mathfrak{X}_s), \overline{M}_{S,s}^{\mathrm{gp}})^\dagger / H_1(\mathfrak{X}_s).$$

If $\mathfrak{X}/\overline{M}$ is a tropical curve, we write

$$\mathrm{TroPic}^0(\mathfrak{X}/\overline{M}) = \mathrm{Hom}(H_1(\mathfrak{X}_s), \overline{M}_{S,s}^{\mathrm{gp}})^\dagger / H_1(\mathfrak{X}_s).$$

As explained in [26], for η a generization of s, any homomorphism of bounded monodromy

$$H_1(\mathfrak{X}_s) \to \overline{M}_{S,s}^{\mathrm{gp}}$$

induces a unique homomorphism (of bounded monodromy)

$$H_1(\mathcal{X}_\eta) \to \overline{M}_{S,\eta}^{\mathrm{gp}};$$

moreover, an orientation of $\mathcal{X}_s$ induces a unique orientation of its contraction $\mathcal{X}_\eta$. There is therefore an induced generization map

$$\mathrm{TroPic}^0_{X/S}(s) \to \mathrm{TroPic}^0_{X/S}(\eta). \tag{3.9.1}$$

The original definition of $\mathrm{TroPic}^0_{X/S}$ in [26] is slightly different than the one we gave and relies on the generization maps defined above. The rest of this section is devoted to verifying that the two definitions are actually equivalent.

3.10. Equivalence with the definition of Molcho and Wise. All our log schemes are fine and saturated, so in particular they have étale-local charts by finitely presented monoids. This allows for the following convenient definitions of finiteness conditions for log schemes.

Definition 3.40. We say that a log scheme is *of finite type*, resp. *locally of finite type*, resp. *Noetherian*, resp. *locally Noetherian* if its underlying scheme is.

We say that a morphism of log schemes is *locally of finite type*, resp. *of finite type*, resp. *locally of finite presentation*, resp. *of finite presentation* if the underlying scheme map is.

Remark 3.41. Any log curve is of finite presentation. In particular, any log curve is the base change of a log curve over a log scheme locally of finite type.

Definition 3.42. Let $\pi : X \to S$ be a log curve. We say that S is *nuclear* (with respect to π) if:

(1) the stratification of S induced by the log structure $\overline{M}_S$ has only one closed stratum Z, Z is connected, and every connected component of every stratum specializes to Z,

(2) $\overline{M}_S$ is generated by global sections (in particular, there exists a surjection $\mathbb{Z}^J \to \mathrm{i}^*\overline{M}_S^{\mathrm{gp}}$ where J is a set),

(3) X/S has split branches,

(4) the sheaf $\mathcal{Z}^{\mathcal{V}}$ is generated by global sections.

We say that S is *pre-nuclear* if it satisfies (2), (3) and (4). We say that S is a *nuclear neighborhood* of a geometric point t of S if S is nuclear and t maps to the closed stratum.

Remark 3.43. Conditions (2), (3) and (4) are stable under strict étale base change. In particular, so is pre-nuclearity.

Remark 3.44. Suppose S is locally Noetherian and satisfies condition (2), and let Z be a stratum of S for the stratification induced by $\overline{M}_S$. Then the sheaves $\overline{M}_S$ and $\overline{M}_S^{\mathrm{gp}}$ are étale-locally constant on Z by definition of the stratification, and they are generated by global sections, so they are constant on each connected component of Z.

Lemma 3.45. *If X/S is a log curve, then S admits a strict étale cover by pre-nuclear schemes. If in addition S is locally Noetherian, then any geometric point has a nuclear strict étale neighborhood.*

Proof. First we show that S has a cover by pre-nuclear schemes. By Remark 3.43, it suffices to show that conditions (2), (3) and (4) individually hold locally on S for the strict étale topology. We can assume that S meets condition (3) by Remark 3.10; condition (4) by Lemma 3.25; and condition (2) by the existence of étale local charts for log schemes.

Now, assume S is locally Noetherian and let t be a geometric point of S. We will show that t has a nuclear strict étale neighborhood. Shrinking, we may assume that S is pre-nuclear and Noetherian. In particular, the connected components of strata (of the stratification induced by $\overline{M}_S$) form a partition of S into finitely many locally closed subschemes. If the closure $\overline{Y}$ of a piece Y does not meet t, take out $\overline{Y}$ from S. The resulting S is a nuclear neighborhood of t. □

Lemma 3.46. *Let X/S be a log curve with S locally Noetherian and pre-nuclear. Let Z be a connected component of a stratum of S (for the stratification induced by $\overline{M}_S$). Then for any two geometric points t, t' in Z, there is a canonical isomorphism $\mathfrak{X}(t) = \mathfrak{X}(t')$ of tropical curves over $\overline{M}_S(Z)$.*

Proof. The sheaf $\overline{M}_S$ is constant on Z by Remark 3.44. Since Z is locally Noetherian, t and t' can be joined by a sequence of geometric points

$$t_0 = t \leftsquigarrow \eta_0 \rightsquigarrow t_1 \leftsquigarrow \eta_2 \rightsquigarrow \cdots \rightsquigarrow t_n = t',$$

where the squiggly arrows denote étale specialization, and all points land in Z. Every specialization $\eta_i \rightsquigarrow t_j$ induces an edge contraction $\mathfrak{X}_{t_j} \to \mathfrak{X}_{\eta_i}$, which is an isomorphism of tropical curves over $\overline{M}_S(Z)$. The automorphism of $\mathfrak{X}_t$ induced by any such étale path from t to itself must be trivial: it sends every vertex to itself since $\mathrm{Irr}_{X/S}$ is generated by global sections, and every half-edge to itself since X/S has split branches. Therefore, the isomorphism $\mathfrak{X}_t \to \mathfrak{X}_{t'}$ is independent of the choice of étale path. □

Corollary 3.47. *If we assume the hypotheses and notations of Lemma* 3.46*, then the sheaf $\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X_Z/Z}, \mathbb{G}_{m,Z}^{\mathrm{trop}})^\dagger$ is constant. In particular,* $\mathrm{TroPic}^0_{X/S}$ *is constant on Z as well.*

Proof. By base-change, we can reduce to the following claim: if S is connected and $\overline{M}_S$ constant, then for any strict geometric point $s \to S$, the restriction map

$$\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger(S) \to \mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger(s)$$

is an isomorphism. Under the hypotheses of the claim, the sheaves $\mathcal{Z}^{\mathcal{E}}$, $\mathcal{Z}^{\mathcal{V}}$ and $\mathfrak{s}_*\mathbb{G}_{m,S}^{\mathrm{trop}}$ are constant on $\underline{S}$, so $\mathcal{H}_{1,X_S/S}$ is constant as well. In particular, we have

$$\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})(S) = \mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})(s). \tag{3.10.1}$$

Let φ be a global section of $\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})$. If φ has bounded monodromy, then its image φ_s in $\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})(s)$ also does. Conversely, suppose φ_s has bounded monodromy. Since

$$\mathfrak{f}_*\mathcal{H}_{1,X_S/S}(S) = \mathfrak{f}_*\mathcal{H}_{1,X_S/S}(s) \xrightarrow{\sim} H_1(\mathfrak{X}_s),$$

for every $\gamma \in \mathfrak{f}_*\mathcal{H}_{1,X/S}(S)$, the image $\varphi(\gamma)$ is bounded by the length of γ in $\overline{M}_S(S) = \overline{M}_S(s)$. Hence, φ has bounded monodromy. Thus, the isomorphism (3.10.1) respects the bounded monodromy subgroups, and the claim follows. □

Lemma 3.48. *Let X/S be a log curve with S locally Noetherian and nuclear. Let s be a geometric point of the closed stratum of S. Then the restriction map*

$$\mathrm{TroPic}^0_{X/S}(S) \to \mathrm{TroPic}^0_{X/S}(s)$$

is an isomorphism.

Proof. Since $\mathcal{E} = X^{\mathrm{nsm}} \to S$ is a disjoint union of closed immersions, and s lies in the unique closed stratum, the map $\mathbb{Z}^{\mathcal{E}}(S) \to \mathbb{Z}^{\mathcal{E}}(s)$ is an isomorphism. Because X/S has split branches, we can make a choice of isomorphism $\sigma : \mathbb{Z}^{\mathcal{E}} \cong \mathcal{Z}^{\mathcal{E}}$, so the restriction map $\mathcal{Z}^{\mathcal{E}}(S) \to \mathcal{Z}^{\mathcal{E}}(s)$ is an isomorphism. Since additionally $\mathcal{Z}^{\mathcal{V}}(S) = \mathcal{Z}^{\mathcal{V}}(s)$, it follows that the restriction $\mathcal{H}_{1,X/S}(S) = \mathcal{H}_{1,X_s/s}(s)$ is an isomorphism.

Next, we show that the restriction map

$$\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}^{\mathrm{trop}}_{m,S})^\dagger(S) \to \mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X_s/s}, \mathbb{G}^{\mathrm{trop}}_{m,s})^\dagger(s)$$

is an isomorphism. By adjunction, the map above is the map

$$\mathrm{Hom}(\mathcal{H}_{1,X/S}, \mathfrak{s}_*\mathbb{G}^{\mathrm{trop}}_{m,S})^\dagger \to \mathrm{Hom}(\mathcal{H}_{1,X_s/s}, \mathfrak{s}_*\mathbb{G}^{\mathrm{trop}}_{m,s})^\dagger$$

of Hom groups between sheaves on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$. The two sheaves $\mathcal{H}_{1,X/S}$ and $\mathfrak{s}_*\mathbb{G}^{\mathrm{trop}}_{m,S} = \mathrm{i}^*\overline{M}^{\mathrm{gp}}_S$ are both locally constructible. To give a morphism $\mathcal{H}_{1,X/S} \to \mathfrak{s}_*\mathbb{G}^{\mathrm{trop}}_{m,S}$ it suffices therefore to give a morphism $\mathrm{i}_*\mathcal{H}_{1,X/S} \to \overline{M}^{\mathrm{gp}}_S$ of sheaves on the small étale site. We are reduced to proving that

$$\mathrm{Hom}(\mathrm{i}_*\mathcal{H}_{1,X/S}, \overline{M}^{\mathrm{gp}}_S)^\dagger \to \mathrm{Hom}(H_1(\mathfrak{X}_s), \overline{M}^{\mathrm{gp}}_s)^\dagger \tag{3.10.2}$$

is an isomorphism.

To give a map $\varphi : \mathrm{i}_*\mathcal{H}_{1,X/S} \to \overline{M}^{\mathrm{gp}}_S$, it suffices to give, for every stratum $W \subset S$, a map φ_W between the restrictions to W, subject to the following compatibility condition: for every two strata W, W' such that W' is contained in the closure of W, let $U = W \cup W'$ (a locally closed in S) and $h : W' \hookrightarrow U$, $j : W \hookrightarrow U$, respectively a closed and open immersion. Then the diagram

$$\begin{array}{ccc} h^*\mathcal{H}_{1,X_U/U} & \xrightarrow{\varphi_{W'}} & h^*\overline{M}^{\mathrm{gp}}_U \\ \downarrow & & \downarrow \\ h^*j_*j^*\mathcal{H}_{1,X_U/U} & \xrightarrow{h^*j_*\varphi_W} & h^*j_*j^*\overline{M}^{\mathrm{gp}}_U \end{array}$$

should commute.

For every connected component W of a stratum, we choose a point η_W. Remember that we have made a global choice of isomorphism $\mathcal{Z}^{\mathcal{E}} = \pi_*\mathbb{G}^{\mathrm{trop}}_{m,X/S} \cong \mathbb{Z}^{\mathcal{E}}$ and that $\mathcal{H}_{1,X/S}$ is constant on connected components of strata. Thus $\mathcal{H}_{1,X_W/W}$ is identified with the constant sheaf with value $H_1(\mathfrak{X}_{\eta_W})$. Similarly, $\overline{M}^{\mathrm{gp}}_W$ is the constant sheaf with value $\overline{M}^{\mathrm{gp}}_{\eta_W}$, this time by condition (2) of Definition 3.42.

Let then t be a point of S and let η be a generization of t. We show that for every map of bounded monodromy $\alpha_t : H_1(\mathfrak{X}_t) \to \overline{M}^{\mathrm{gp}}_t$, the solid diagram

$$\begin{array}{ccc} H_1(\mathfrak{X}_t) & \xrightarrow{\alpha} & \overline{M}^{\mathrm{gp}}_t \\ \downarrow & & \downarrow \\ H_1(\mathfrak{X}_\eta) & \dashrightarrow & \overline{M}^{\mathrm{gp}}_\eta \end{array}$$

admits a unique dashed arrow α_η making the diagram commute. The uniqueness is due to the surjectivity of the left vertical map. For the existence part, a dashed arrow exists if α sends the kernel K of the left vertical map into the kernel K' of the right vertical map, i.e., if any cycle $\gamma \in H_1(\mathcal{X}_t)$ with length $\ell(\gamma)$ vanishing in $\overline{M}_\eta^{\mathrm{gp}}$ maps to zero in $\overline{M}_\eta^{\mathrm{gp}}$. This holds by the bounded monodromy condition, which completes the proof of existence. We write $(\eta \rightsquigarrow s)^*\alpha_t$ for the unique map α_η just constructed.

We omit the easy check that for every diagram of generizations

$$\begin{array}{ccc} \eta & \rightsquigarrow & \eta_1 \\ \downarrow & & \downarrow \\ \eta_2 & \rightsquigarrow & t \end{array}$$

and map of bounded monodromy $\alpha_t : H_1(\mathcal{X}_t) \to \overline{M}_t^{\mathrm{gp}}$, the maps $(\eta \rightsquigarrow \eta_1)^*(\eta_1 \rightsquigarrow t)^*\alpha_t$ and $(\eta \rightsquigarrow \eta_2)^*(\eta_2 \rightsquigarrow t)^*\alpha_t$ coincide.

Call Z the closed stratum of S. The above discussion, combined with condition (1) of the definition of nuclearity, shows that an arbitrary

$$\varphi \in \mathrm{Hom}(H_1(\mathcal{X}_s), \overline{M}_s^{\mathrm{gp}})^\dagger = \mathrm{Hom}(\mathrm{i}_*\mathcal{H}_{1,X_Z/Z}, \overline{M}_Z^{\mathrm{gp}})^\dagger$$

fits in a unique compatible system of bounded monodromy maps $\mathrm{i}_*\mathcal{H}_{1,X_W/W} \to \overline{M}_W^{\mathrm{gp}}$, where W ranges through the connected components of strata, i.e., φ comes uniquely from an element of $\mathrm{Hom}(\mathrm{i}_*\mathcal{H}_{1,X/S}, \overline{M}_S^{\mathrm{gp}})^\dagger$, which shows that

$$\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger(S) \to \mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X_s/s}, \mathbb{G}_{m,s}^{\mathrm{trop}})^\dagger(s)$$

is an isomorphism.

Summarizing, we have shown that the composition

$$\mathcal{H}om(\mathfrak{f}_*\mathcal{H}_{1,X/S}, \mathbb{G}_{m,S}^{\mathrm{trop}})^\dagger(S)/\mathfrak{f}_*\mathcal{H}_{1,X/S}(S) \to \mathrm{TroPic}^0_{X/S}(S) \to \mathrm{TroPic}^0_{X/S}(s)$$

is an isomorphism. In particular, the restriction map $\mathrm{TroPic}^0_{X/S}(S) \to \mathrm{TroPic}^0_{X/S}(s)$ is surjective. We will show that it is also injective. Let λ be an element of its kernel. It suffices to prove that λ vanishes in the étale stalk $\mathrm{TroPic}^0_{X/S}(\mathrm{Spec}\, \mathcal{O}^{et}_{S,\eta}) = \mathrm{TroPic}^0_{X/S}(\eta)$ at any geometric point η of S. Let W be the stratum containing η. Over some generic point of W, there is a geometric point η_W specializing to both η and a point of Z. By Corollary 3.47 we know $\mathrm{TroPic}^0_{X/S}$ is constant on strata, so the restriction map

$$\mathrm{TroPic}^0_{X/S}(S) \to \mathrm{TroPic}^0_{X/S}(\eta_W) = \mathrm{TroPic}^0_{X/S}(\eta)$$

factors via the "generization map"

$$\mathrm{TroPic}^0_{X/S}(s) \to \mathrm{TroPic}^0_{X/S}(\eta_W)$$

of (3.9.1). In particular, λ maps to 0 in $\mathrm{TroPic}^0_{X/S}(\eta)$, which concludes the proof. □

Definition 3.49. Let X/S be a log curve. We define a sheaf of sets $\mathcal{F}_{X/S}$ on $(\mathbf{LSch}/S)_{\text{ét}}$ as follows. If S is locally of finite type and T/S is of finite type, then $\mathcal{F}_{X/S}(T)$ is a system of elements $a \in \mathrm{TroPic}^0(t)$, one for each geometric point $t \to T$, and compatible with the generization maps (3.9.1). For general T, $\mathcal{F}(T)$ is the sheafification of the presheaf taking T to the colimit of the $\mathcal{F}_{X/S}(T_0)$, taken over all $T \to T_0 \to S$ with T_0 of finite type. For

general S, by Remark 3.41 we can pick a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & S_0 \end{array}$$

with S_0 locally of finite type and define $\mathcal{F}_{X/S}$ as the pullback of $\mathcal{F}_{X_0/S_0}$. This is independent from the choice of cartesian square, and the formation of $\mathcal{F}_{X/S}$ commutes with base change. When there is no ambiguity, we will write $\mathcal{F}$ instead of $\mathcal{F}_{X/S}$.

This is the way in which the tropical Jacobian is defined in [26].

Lemma 3.50. *The sheaves* $\mathrm{TroPic}^0_{X/S}$ *and* $\mathcal{F}$ *are canonically isomorphic.*

Proof. If S is locally of finite type and T is a S-log scheme of finite type, there is a natural map

$$\Phi(T) : \mathrm{TroPic}^0(T) \to \mathcal{F}(T)$$

taking $\alpha \in \mathrm{TroPic}^0(T)$ to the system $\{i^*\alpha \in \mathrm{TroPic}^0(t)\}_{i:t\to T}$, where $i : t \to T$ are all geometric points of T. Now, let T be an arbitrary S-log scheme. Since $\mathcal{H}_{1,X/S}$ is locally free and finite over $\underline{S}$, we find that locally on T, $\mathrm{TroPic}^0_{X/S}(T)$ is the colimit of the $\mathrm{TroPic}^0_{X/S}(T_0)$, taken over all $T \to T_0 \to S$ with T_0 of finite type. Combining this with the fact that the formation of $\mathrm{TroPic}^0_{X/S}$ commutes with strict base change, we get a morphism of abelian sheaves $\Phi : \mathrm{TroPic}^0_{X/S} \to \mathcal{F}$ for any log curve X/S.

We will show that Φ is an isomorphism. We may do so assuming that S is of finite type. Then, working locally, it suffices to show that $\Phi(T)$ is an isomorphism when T is nuclear.

Let x be a geometric point of T landing in the closed stratum Z. Consider the natural map

$$\psi : \mathcal{F}(T) \to \mathrm{TroPic}^0(x).$$

The composition $\psi \circ \Phi(T)$ is the restriction map $\mathrm{TroPic}^0(T) \to \mathrm{TroPic}^0(x)$, which is an isomorphism by Lemma 3.48. It suffices therefore to show that ψ is injective (in order to conclude that it is an isomorphism, and that therefore $\Phi(T)$ is an isomorphism as well).

The argument is essentially the same as the end of the proof of Lemma 3.48. Let $y \to T$ be any geometric point, W_0 the stratum containing y, and W the connected component of W_0 containing y. By condition (1) in the definition of nuclearity there exists an étale specialization $\eta \rightsquigarrow \zeta$ with η in W and ξ in Z. By Corollary 3.47, we obtain a canonical map

$$\mathrm{TroPic}^0(x) = \mathrm{TroPic}^0(\zeta) \to \mathrm{TroPic}^0(\eta) = \mathrm{TroPic}^0(y).$$

The compatibility condition forces all elements of the system $\mathcal{F}(T)$ to be determined by the element belonging to $\mathrm{TroPic}^0(x)$. This proves the injectivity. □

3.11. The logarithmic Jacobian. Let $S = (\mathrm{Spec}\, k, M)$ be a logarithmic geometric point, with chart by a monoid $\overline{M}$, and $\pi : X \to S$ a log curve. We consider M_X^{gp}-torsors on X for the strict étale topology. Given such a torsor L, we denote by $\overline{L}$ the image of its isomorphism class via $H^1(X, M_X^{\mathrm{gp}}) \to H^1(X, \overline{M}_X^{\mathrm{gp}})$.

There is a natural surjective map

$$\mathrm{Hom}(H_1(\mathcal{X}), \overline{M}^{\mathrm{gp}}) = H^1(X, \pi^{-1}\overline{M}^{\mathrm{gp}}) \to H^1(X, \overline{M}_X^{\mathrm{gp}}).$$

The first equality is because $\overline{M}^{\mathrm{gp}}$ is torsion-free. Surjectivity is due to the fact that $\overline{M}^{\mathrm{gp}}_{X/S}$ is supported on X^{nsm} and therefore $H^1(X, \overline{M}^{\mathrm{gp}}_{X/S}) = 0$.

We say that L has *bounded monodromy* if some (equivalently, any) preimage of $\overline{L}$ in $\mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})$ has bounded monodromy as in Definition 3.33.

Let now $\pi : X \to S$ be a log curve over a general log base; we say that an M_X^{gp}-torsor L has bounded monodromy, if for every strict geometric point $s \to S$, the restriction L_s has bounded monodromy. Such a torsor is called a *logarithmic line bundle*.

Definition 3.51. The logarithmic Picard stack is the stack $\mathbf{LogPic}_{X/S}$ on $(\mathbf{LSch}/S)_{\text{ét}}$ with sections

$$\mathbf{LogPic}_{X/S}(T) = \{\text{log line bundles on } X_T = X \times_S T\}.$$

The logarithmic Picard sheaf $\mathrm{LogPic}_{X/S} \subset R^1\pi_*\mathbb{G}_{m,X}^{\log}$ is the sheafification of the functor of isomorphism classes of $\mathbf{LogPic}_{X/S}$.

Remark 3.52. To any line bundle L on a log curve $X \to S$ with associated $\mathcal{O}_X^\times$-torsor $L^\times$, one can associate the torsor $L^\times \otimes_{\mathcal{O}_X^\times} M_X^{\mathrm{gp}}$. Since its image in $H^1(X, \overline{M}_X^{\mathrm{gp}})$ vanishes, this has bounded monodromy and is therefore a log line bundle. There is therefore a natural map $\mathfrak{f}_*\mathbf{Pic}_{X/S} \to \mathbf{LogPic}_{X/S}$.

Consider the diagram

$$\begin{array}{ccc} \mathfrak{f}_*\mathbf{Pic}_{X/S} & \longrightarrow & \mathbf{LogPic}_{X/S} \\ \downarrow & & \vdots \\ \mathbb{Z}[\mathrm{Irr}_{X/S}] & \xrightarrow{\Sigma} & \mathbb{Z}, \end{array} \qquad \text{deg}$$

where $\mathbf{Pic}_{X/S}$ is the Picard stack, the left vertical map is the multidegree map, Σ is the sum, and the top horizontal map associates to a $\mathcal{O}_X^\times$-torsor L the log line bundle $L \otimes_{\mathcal{O}_X^\times} M_X^{\mathrm{gp}}$. It is shown in [26, 4.5] that there is a unique degree map making the diagram commute.

Definition 3.53. We define $\mathbf{LogPic}^0_{X/S}$ to be the substack of $\mathbf{LogPic}_{X/S}$ of logarithmic line bundles of degree zero, and similarly for the sheaf $\mathrm{LogPic}^0_{X/S} \subset \mathrm{LogPic}_{X/S}$, which is called *logarithmic Jacobian*.

For the convenience of the reader we recall from [26] a list of properties of the logarithmic Picard and Jacobian that we will use.

Property 3.54 ([26, Section 4.14]). There is a natural morphism

$$\mathrm{LogPic}^0_{X/S} \to \mathrm{TroPic}^0_{X/S},$$

the "tropicalization", which fits into a short exact sequence

$$0 \to \mathfrak{f}_* \mathrm{Pic}^0_{\underline{X}/\underline{S}} \to \mathrm{LogPic}^0_{X/S} \to \mathrm{TroPic}^0_{X/S} \to 0, \tag{3.11.1}$$

where $\mathrm{Pic}^0_{\underline{X}/\underline{S}}$ is the generalized Jacobian, i.e., the sheaf of line bundles of degree zero on every irreducible component, which is representable by a semiabelian scheme on $\underline{S}$. Moreover, $\mathfrak{f}_* \mathrm{Pic}^0_{\underline{X}/\underline{S}}$ is representable by a semiabelian scheme with pullback log structure from S.

Property 3.55 ([26, Proposition 4.3.2]). The bounded monodromy condition has the following concrete interpretation: an M_X^{gp}-torsor has bounded monodromy if and only if, étale-locally on S, there exist log modifications $S' \to S$ and $Y \to X \times_S S'$ such that $Y \to S'$ is a log curve and the induced M_Y^{gp}-torsor can be represented by a line bundle on Y.

Property 3.56 ([26, Corollary 4.4.14.1]). If $Y \to X$ is a log modification such that Y/S is a log curve, the induced maps $\mathrm{LogPic}^0_{X/S} \to \mathrm{LogPic}^0_{Y/S}$ and $\mathrm{TroPic}^0_{X/S} \to \mathrm{TroPic}^0_{Y/S}$ are isomorphisms. Although the statement for TroPic^0 is not explicitly stated in [26], this follows from [26, Corollary 4.4.14.1] together with the fact that $\mathrm{Pic}^0_{X/S} \to \mathrm{Pic}^0_{Y/S}$ is an isomorphism and the exact sequence (3.11.1).

Property 3.57 ([26, Section 4.4]). The stack $\mathbf{LogPic}_{X/S}$ is a stack in the (strict) étale topology. However, if the base S is logarithmically regular, $\mathbf{LogPic}_{X/S}$ is also a stack on the small *log* étale site of S, and in fact on the site generated by the small log étale site and arbitrary root stacks (perhaps of order not prime to the characteristic).

Property 3.58 ([26, Theorem 4.10.1]). The stack $\mathbf{LogPic}_{X/S}$ satisfies the valuative criterion for properness for log schemes: it has the unique lifting property with respect to valuation rings R whose log structure is the direct image of a valuative log structure on the fraction field.

Property 3.59 ([26, Theorem 4.13.1]). The stack $\mathbf{LogPic}_{X/S}$ is logarithmically smooth over S, meaning that it is locally of finite presentation and satisfies the infinitesimal lifting criterion for strict square 0 extensions, as in the original definition of [21].

In general, $\mathbf{LogPic}_{X/S}$ (resp. $\mathrm{LogPic}_{X/S}$) is not representable by a log algebraic stack (resp. log algebraic space), as the following example shows:

Example 3.60 ([17, Section I]). Let K be a complete discrete valuation field, with ring of integers $\mathcal{O}_K$, $\pi \in \mathfrak{m}_K$ a uniformizer, $q = \pi^2$ and E_q the corresponding Tate elliptic curve. Any proper log smooth model $\mathcal{E}_q$ of E_q over $\mathcal{O}_K$ gives the same logarithmic Jacobian $\mathrm{LogPic}^0_{\mathcal{E}_q}$, and comes by Theorem 1.7 with a unique (birational) morphism $\mathcal{E}_q \to \mathrm{LogPic}^0_{\mathcal{E}_q}$. The minimal regular model $\mathcal{E}_q^{min}$ has two irreducible components over the residue field; blowing down any of them to a point gives two log smooth models $\mathcal{E}_q^1, \mathcal{E}_q^2$. As both $\mathcal{E}_q^1$ and $\mathcal{E}_q^2$ map to $\mathrm{LogPic}^0_{X/S}$, we see that if the latter were a log algebraic space, its closed fiber would have to be a point, contradicting the log smoothness.

4. Strict log and tropical Jacobian and representability

Definition 4.1. Let $X \to S$ be a log curve. Recall the functor $\mathfrak{s}_*$ from Section 2.4. We define the *strict logarithmic Jacobian* $\mathrm{sLPic}^0_{X/S}$ to be $\mathfrak{s}_* \mathrm{LogPic}^0_{X/S}$, a sheaf on the big étale site $(\mathbf{Sch}/\underline{S})_{\text{ét}}$.

Similarly, we define the *strict tropical Jacobian* $\mathrm{sTPic}^0_{X/S}$ to be $\mathfrak{s}_* \mathrm{TroPic}^0_{X/S}$

Although these are sheaves on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$, they do not depend only on the prestable curve $\underline{X}/\underline{S}$ but also on the log structure M_S (see Example 4.7).

Remark 4.2. The article [26] is by no means the first time the term "logarithmic Jacobian" appears in the literature. To our knowledge the earliest appearances of the term are in [15, 16, 19]; a notion of logarithmic Jacobian was then studied, among others, by Olsson in [30] and Bellardini in [4]. Their notion is from the beginning a notion for schemes, and thus closer to our strict log Jacobian than to the log Jacobian of [26]. However, even so there are differences – for instance, in Olsson's and Bellardini's work the log structure on S is restricted, and the subset of M_X^{gp} torsors they consider is different from the subgroup of all bounded monodromy torsors. In particular, the objects constructed by Bellardini and Olsson will in general not satisfy the Néron mapping property over higher-dimensional bases.

There is an obvious modular interpretation for $\mathrm{sLPic}^0_{X/S}$, as the sheafification of the functor associating to a map of schemes $a : T \to \underline{S}$ the set

$$\{\text{log line bundles } L \text{ on } X \times_S \mathfrak{s}T \text{ of degree zero }\}/\cong.$$

By exactness of the functor $\mathfrak{s}_*$, the exact sequence (3.11.1) yields an exact sequence

$$0 \to \mathrm{Pic}^0_{X/S} \to \mathrm{sLPic}^0_{X/S} \to \mathrm{sTPic}^0_{X/S} \to 0. \tag{4.0.1}$$

Lemma 4.3. *Let X/S be a log curve and $f : T \to \underline{S}$ a morphism of schemes. Then*

$$\mathrm{sTPic}^0_{X/S} \times_{\underline{S}} T = \mathrm{sTPic}^0_{X_{\mathfrak{s}T}/\mathfrak{s}T}$$

and

$$\mathrm{sLPic}^0_{X/S} \times_{\underline{S}} T = \mathrm{sLPic}^0_{X_{\mathfrak{s}T}/\mathfrak{s}T}\,.$$

Proof. We have $\mathfrak{s}T = \mathfrak{f}_*T$, so $\mathfrak{s}_*\mathfrak{s}T = T$. The formation of the tropical Jacobian commutes with base change and therefore

$$\mathrm{TroPic}^0_{X/S} \times_S \mathfrak{s}T = \mathrm{TroPic}^0_{X_{\mathfrak{s}T}/\mathfrak{s}T}\,. \tag{4.0.2}$$

Now the right adjoint functor $\mathfrak{s}_*$ commutes with products, and applying it to (4.0.2) yields the result. The same proof works for $\mathrm{sLPic}^0_{X/S}$. □

Recall that the stack $\mathbf{LogPic}^0_{X/S}$ is proper and log smooth over S (Property 3.58 and Property 3.59), but in general not algebraic. The main purpose of this section is to prove the following:

Theorem 4.4. *Let X/S be a log curve. Then:*

(i) $\mathrm{sTPic}^0_{X/S}$ *is representable by a quasi-separated étale algebraic space over* $\underline{S}$.

(ii) $\mathrm{sLPic}^0_{X/S}$ *is representable by a quasi-separated smooth algebraic space over* $\underline{S}$.

Proof. Part (ii) of the theorem is immediate from part (i) and the exact sequence (4.0.1), as it realizes sLPic^0 as a Pic^0-torsor over sTPic^0, which makes it representable by a smooth separated algebraic space over sTPic^0. Let us prove part (i). Representability by a quasi-separated algebraic space is étale local on the target, and X/S is of finite presentation, so we reduce via Lemma 4.3 to the case where S is a log scheme of finite presentation (in particular Noetherian).

We reduce by virtue of Lemma 2.3 to checking that the sheaf $\mathcal{F} = \mathrm{TroPic}^0_{X/S}$ is locally constructible, i.e., that the canonical morphism $\alpha : \mathrm{i}^*\mathrm{i}_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

It suffices to show that for any morphism $T \to \underline{S}$ of schemes, the restriction of α to the small étale site over T is an isomorphism. That is, that for any $T \to \underline{S}$ and any geometric point t of T, the map

$$\operatorname*{colim}_{t \to V \to T} \mathrm{i}^* \mathrm{i}_* \mathcal{F}(V) \to \operatorname*{colim}_{t \to V \to T} \mathcal{F}(V)$$

is an isomorphism, where the colimits are over factorizations $t \to V \to T$ with $V \to T$ étale. The right-hand side is $\mathcal{F}(t)$, by Lemma 3.48 and the fact that factorizations such that V (with pullback log structure from S) is a nuclear neighborhood of t form a cofinal system. The left hand side becomes

$$\operatorname*{colim}_{t \to V \to T} \operatorname*{colim}_{V \to W \to S} \mathcal{F}(W),$$

where $V \to T$ and $W \to S$ are étale. This can in turn be replaced by the colimit of $\mathcal{F}(W)$ over the diagrams of the form

$$\begin{array}{ccccc} t & \longrightarrow & V & \longrightarrow & T \\ & & \downarrow & & \downarrow \\ & & W & \longrightarrow & S \end{array}$$

with $V \to T$ and $W \to S$ étale, where $t \to T$ and $T \to S$ are fixed and the remainder is allowed to vary. But this is simply the colimit over the factorizations $t \to W \to S$ with $W \to S$ étale. Since those factorizations with W a nuclear neighborhood of t in S form a cofinal system, by Lemma 3.48 the colimit is equal to $\mathcal{F}(t)$. □

4.1. Examples of strict logarithmic Jacobians.

Example 4.5. Let S be the spectrum of a discrete valuation ring with divisorial log structure, and X/S a log curve. Call η the generic point of S. The closed fiber of the strict tropical Jacobian is identified with the finite étale group scheme of components of the Néron model of $\mathrm{Pic}^0_{X_\eta/\eta}$, and $\mathrm{sLPic}^0_{X/S}$ is the Néron model itself. This will be shown in greater generality in Corollary 6.13. Another explicit description of the Néron model is known in this case: after a finite sequence of log blowups of X, we obtain a new log curve X'/S whose nodes all have length $1 \in \mathbb{N} = \overline{M}_S(S)$. By [5, 9.5, Theorem 4], the Néron model of $\mathrm{Pic}^0_{X_\eta/\eta}$ is the quotient of the Picard group of degree 0 line bundles $\mathrm{Pic}^{\mathrm{tot}0}_{X'/S}$ by the closure of its unit section. The relation between $\mathrm{sLPic}^0_{X/S}$ and quotients of Picard spaces is explored further in Section 9.

Example 4.6. Let $S = \operatorname{Spec} k[[u, v]]$, let $D \subset S$ be defined by $uv = 0$, and let

$$j : U = S \setminus D \hookrightarrow S.$$

Let E/S be the degenerate elliptic curve in $\mathbb{P}^2_S$ with equation

$$y^2 z = x^3 + x^2 z + uvz^3$$

We make $E \to S$ into a log smooth morphism by putting the log structures associated to the divisors D and $E \times_S D$ onto S and E respectively. Let s be a geometric point of S; there are essentially three possible structures for the tropical curve $\mathfrak{X}_s$ and for $\mathrm{TroPic}^0(\mathfrak{X}_s)$:

- if s lands in U, then $\mathfrak{X}_s$ consists of a single vertex and $\mathrm{TroPic}^0(\mathfrak{X}_s) = 0$,
- if s lands in $D \setminus \{u = 0, v = 0\}$, then $\overline{M}_{S,s} \cong \mathbb{N}$ and $\mathfrak{X}_s$ consists of a vertex and a loop labelled by $1 \in \mathbb{N}$; then $\mathrm{TroPic}^0(\mathfrak{X}_s) = 0$,

- if s maps to $(0,0)$, then $\overline{M}_{S,s} \cong \mathbb{N}^2$ and $\mathscr{X}_s$ consists of a vertex and a loop labelled by $(1,1) \in \mathbb{N}^2$; then $\mathrm{TroPic}^0(\mathscr{X}_s) \cong \mathbb{Z}$.

It follows that $\mathrm{sTPic}^0_{X/S}$ is an étale group space with fiber $\mathbb{Z}$ at the closed point of S and 0 everywhere else. In particular, $\mathrm{sLPic}^0_{X/S}$ is not quasi-compact. It will follow from Proposition 7.3 that it is not separated either.

Example 4.7. Note that the tropical Jacobian of a log curve depends on the log structures on C/S, and not only on the underlying scheme map $\underline{C}/\underline{S}$. Keeping the notations of Example 4.6, E/S comes via base-change from a log curve E_0/S_0, where S_0 is $\underline{S}$ with log structure given by

$$(k[[u,v]])^\times \oplus \mathbb{N}uv \to k[[u,v]].$$

The tropical Jacobian of E/S at the closed point is a free abelian group of rank 1, while that of E_0/S_0 is trivial. In this example, $E_0 \to S_0$ coincides with the log curve $E^\# \to S^\#$ provided by Proposition 3.1.

5. The saturation of $\mathbf{Pic}^0$ in $\mathbf{sLPic}^0$

We have seen in Example 4.6 that the étale algebraic space $\mathrm{sTPic}^0_{X/S}$ does not in general have finite fibers, and that consequentially $\mathrm{sLPic}^0_{X/S}$ is not in general quasi-compact. In this section we introduce a new quasi-compact quasi-separated (*qcqs*) smooth algebraic space naturally associated to the log curve X/S and sitting in between $\mathrm{Pic}^0_{X/S}$ and $\mathrm{sLPic}^0_{X/S}$.

Consider the subsheaf on $(\mathbf{LSch}/S)_{\text{ét}}$ of the tropical Jacobian

$$\mathrm{TroPic}^{\mathrm{tor}}_{X/S} \subset \mathrm{TroPic}^0_{X/S}$$

of torsion elements, and its strict version $\mathrm{sTPic}^{\mathrm{tor}}_{X/S} := \mathfrak{s}_* \,\mathrm{TroPic}^{\mathrm{tor}}_{X/S}$ of torsion elements.

Definition 5.1. Define the *saturated Jacobian* $\mathrm{Pic}^{\mathrm{sat}}_{X/S}$ to be the preimage of $\mathrm{TroPic}^{\mathrm{tor}}_{X/S}$ via the map $\mathrm{LogPic}^0_{X/S} \to \mathrm{TroPic}^0_{X/S}$.

Similarly, we define the *strict saturated Jacobian* $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$ to be the preimage of $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ via the map $\mathrm{sLPic}^0_{X/S} \to \mathrm{sTPic}^0_{X/S}$.

The exact sequence (3.11.1) restricts to an exact sequence

$$0 \to \mathfrak{f}_* \,\mathrm{Pic}^0_{X/S} \to \mathrm{Pic}^{\mathrm{sat}}_{X/S} \to \mathrm{TroPic}^{\mathrm{tor}}_{X/S} \to 0.$$

We apply the exact functor $\mathfrak{s}_*$ to find an exact sequence in $\mathrm{Sh}(\mathbf{Sch}/S)_{\text{ét}}$

$$0 \to \mathrm{Pic}^0_{X/S} \to \mathrm{sPic}^{\mathrm{sat}}_{X/S} \to \mathrm{sTPic}^{\mathrm{tor}}_{X/S} \to 0.$$

Remark 5.2. The strict saturated Jacobian has the following modular interpretation: it is the sheafification of the presheaf of abelian groups on $(\mathbf{Sch}/\underline{S})_{\text{ét}}$ whose T-sections are the isomorphism classes of log line bundles L of degree zero on $X \times_S sT$ such that some positive power of L is a line bundle. This line of thought is pursued further in [14].

Lemma 5.3. *The inclusion* $\mathrm{sTPic}^{\mathrm{tor}}_{X/S} \to \mathrm{sTPic}^0_{X/S}$ *is an open immersion; in particular* $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ *is representable by a quasi-separated* $\underline{S}$*-étale group algebraic space. Moreover,* $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ *is quasi-finite over* $\underline{S}$.

Proof. Denoting by $\mathrm{sTPic}[n]_{X/S}$ the subsheaf of n-torsion elements, for the first part of the statement it suffices to show that for every $n \in \mathbb{Z}$ the map $j : \mathrm{sTPic}[n]_{X/S} \to \mathrm{sTPic}^0_{X/S}$ is an open immersion. We have a pullback square

$$\begin{array}{ccc} \mathrm{sTPic}[n]_{X/S} & \longrightarrow & S \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 0} \\ \mathrm{sTPic}^0_{X/S} & \xrightarrow{n} & \mathrm{sTPic}^0_{X/S}\,. \end{array} \tag{5.0.1}$$

Since $\mathrm{sTPic}^0_{X/S}\,/\underline{S}$ is étale, the zero section $S \to \mathrm{sTPic}^0_{X/S}$ is an open immersion, hence so is j.

Now $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ is open in $\mathrm{sTPic}^0_{X/S}\,/\underline{S}$, and the latter is quasi-separated by Theorem 4.4, hence $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ is quasi-separated.

To prove that $\mathrm{sTPic}^{\mathrm{tor}}_{X/S} \to \underline{S}$ is quasi-finite, we will show that it is quasi-compact with finite fibers. For $s \in \underline{S}$ a geometric point, the restriction of $\mathrm{sTPic}^0_{X/S}$ to s has, by Lemma 4.3, as group of s-points the quotient $\mathrm{Hom}(H_1(\mathcal{X}_s), \overline{M}_s^{\mathrm{gp}})^\dagger / H_1(\mathcal{X}_s)$. This is finitely generated since $\overline{M}_s^{\mathrm{gp}}$ is finitely generated, hence its torsion part is finite.

For quasi-compactness, the question being local on the base, we assume that $\underline{S}$ is affine and that $\overline{M}_S$ has a global chart from an fs monoid. Then the stratification of $\underline{S}$ induced by $\overline{M}_S^{\mathrm{gp}}$ has only finitely many strata Z_i, and these strata are affine. Therefore, it suffices to show that for each i, the preimage of Z_i in $\mathrm{sTPic}^{\mathrm{tor}}_{X_{Z_i}/Z_i}$ is quasi-compact.

On each stratum Z, the sheaf $\mathrm{Hom}(\mathcal{H}_{1,X_Z/Z}, \mathfrak{s}_*\mathbb{G}_{m,Z}^{\mathrm{trop}})^\dagger$ is locally constant by Corollary 3.47, so $\mathrm{sTPic}^0_{X_Z/Z}$ is locally constant as well. Its torsion part $\mathrm{sTPic}^{\mathrm{tor}}_{X_Z/Z}$ is then a locally constant sheaf of finite abelian groups and in particular a finite étale scheme over Z. It is therefore quasi-compact. □

Corollary 5.4. *The inclusion* $\mathrm{sPic}^{\mathrm{sat}}_{X/S} \to \mathrm{sLPic}^0_{X/S}$ *is an open immersion, and* $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$ *is representable by an* $\underline{S}$*-smooth group algebraic space of finite presentation.*

Proof. The first part of the statement follows from Lemma 5.3 and base change. The representability then follows from Theorem 4.4.We deduce that $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$ is qcqs because it is a torsor over the quasi-finite and quasi-separated $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$, under the quasi-compact separated group scheme $\mathrm{Pic}^0_{X/S}$. □

Lemma 5.5. *Let* $S = \overline{\mathcal{M}}_{g,n}$*, the moduli stack of genus* g *stable curves with* n *marked points. Let* $\mathcal{X}_{g,n}$ *be the universal curve. Endow both stacks with the divisorial log structure so as to make the universal curve into a log curve. Then* $\mathrm{sPic}^{\mathrm{sat}}_{\mathcal{X}_{g,n}} = \mathrm{Pic}^0_{\mathcal{X}_{g,n}}$.

Proof. For s a geometric point of $\overline{\mathcal{M}}_{g,n}$, the tropicalization $\mathcal{X}$ of the fiber has each edge labelled by a distinct base element of the free monoid $\overline{M}_{S,s}$. It follows that $\mathrm{TroPic}^0(\mathcal{X})$ is torsion-free. □

6. The Néron mapping property of LogPic

In this section we prove the Néron mapping property for the logarithmic Jacobian, under the assumption that the base is log regular.

6.1. Classical and log Néron models. For comparison, we briefly recall the definitions of classical, non-logarithmic Néron models.

Definition 6.1. Let S be a scheme, $U \subset S$ a schematically dense open, and $\mathcal{N}/S$ a category fibered in groupoids over $\mathbf{Sch}/S$. We say that $\mathcal{N}/S$ has the *Néron mapping property* with respect to $U \subset S$ if for every smooth morphism of schemes $T \to S$, the restriction map $\mathcal{N}(T) \to \mathcal{N}(T \times_S U)$ is an equivalence.

If in addition $\mathcal{N}$ is an algebraic stack and is smooth over S, we say $\mathcal{N}$ is a *Néron model* of its restriction $\mathcal{N}_U/U$.

Remark 6.2. Note that the definition of Néron models in [12] and the classical one over Dedekind schemes require them to be separated. When the base is a Dedekind scheme and the generic fiber is a group scheme this is automatic by [5, Theorem 7.1.1]. However, over higher-dimensional bases this is not true, and non-separated Néron models exist in much greater generality than separated ones (the author of [12] did not realize this at the time). From this perspective, [12] should be seen as investigating the existence of *separated* Néron models.

This definition extends naturally into the logarithmic setting:

Definition 6.3. Let S be a log scheme, $U \subset S$ strict schematically dense open, and $\mathcal{N}/S$ a category fibered in groupoids over $\mathbf{LSch}/S$. We say that $\mathcal{N}/S$ *has the Néron mapping property* with respect to $U \subset S$ if for every log smooth morphism $T \to S$, the map $\mathcal{N}(T) \to \mathcal{N}(T \times_S U)$ is an equivalence.

Remark 6.4. In order to qualify as a Néron model, the functor $\mathcal{N}$ should not only satisfy the Néron mapping property, but also be a sheaf for a suitable topology, and be representable in some suitable sense. Requiring representability as an algebraic space or stack with log structure is too restrictive; the log Jacobian does not satisfy these criteria, and in general the log Néron model will not exist as an algebraic space or stack with log structure. The most appropriate notions of representability are perhaps:

(i) for the functor, that it is a sheaf for the strict log étale topology, has diagonal representable by log schemes, and admits a log étale cover by a log scheme,

(ii) for the stack, that it is a sheaf for the strict log smooth topology, it has diagonal representable in the sense of (i), and admits a log smooth cover by a log scheme.

The Log Picard space and stack do satisfy these additional conditions. This is shown in [26], except for verifying that LogPic is a stack for the log smooth topology ([26] only verify it for the log étale topology), but this will be proven in the forthcoming [25]).

One consequence of imposing the above assumptions is that Néron models are unique,[7] by analogous descent arguments to those for the classical case. For example, suppose that $\mathcal{N}$ and $\mathcal{N}'$ are log Néron models, and let $V \to \mathcal{N}$ be a log étale cover by a log scheme; then the map $V_U \to \mathcal{N}_U \xrightarrow{\sim} \mathcal{N}'_U$ extends uniquely to a map $V \to \mathcal{N}'$ by the universal property of the latter, and a further application of uniqueness and the sheaf property shows that this descends to a map $\mathcal{N} \to \mathcal{N}'$.

[7] Either up to unique isomorphism, for the functor, or up to 1-isomorphism which is itself unique up to a unique 2-isomorphism, in the stack case.

Remark 6.5. In classical algebraic geometry the distinction between the smooth and étale topologies is often not so important since every smooth cover can be refined to an étale cover (see [37, Tag 055V]). In the logarithmic setting this is no longer true; for example the map

$$\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1, \quad (x,t) \mapsto xt^p, \tag{6.1.1}$$

where $\mathbb{G}_m, \mathbb{A}^1$ have their toric log structure is a log smooth cover, but does not admit a refinement by a log étale cover in characteristic p. It does admit a refinement by a Kummer cover, and this is in fact a general phenomenon; one can show using the results of [1] that every log smooth cover can be refined by a cover that is a composite of log étale covers and Kummer maps.

Remark 6.6. Suppose $\mathcal{N}/S$ is a functor on $(\mathbf{LSch}/S)$ with the log Néron mapping property with respect to $U \subset S$. Then $\mathfrak{s}_*\mathcal{N}$ has the Néron mapping property with respect to $\underline{U} \subset \underline{S}$ since, for every smooth morphism of schemes $g : T \to \underline{S}$, the morphism

$$(T, g^* M_S) \to S$$

is log smooth.

6.2. Log regularity. We will show that the logarithmic Jacobian of X/S is a log Néron model when S is log regular, a notion which we now recall.

Definition 6.7 ([22, 28]). Let S be a locally Noetherian fs log scheme. For a geometric point s of S, we denote the ideal generated by the image of $M_{S,s} \setminus \mathcal{O}^*_{S,s} \to \mathcal{O}_{S,s}$ by $I_{S,s}$. We say that S is *log regular* at s if $\mathcal{O}_{S,s}/I_{S,s}$ is regular and $\dim \mathcal{O}_{S,s} = \dim \mathcal{O}_{S,s}/I_{S,s} + \mathrm{rk}\overline{M}^{\mathrm{gp}}_{S,s}$. We say that S is *log regular* if it is log regular at all geometric points.

Remark 6.8. Any toric or toroidal variety with its natural log structure is log regular. In particular, a regular scheme equipped with the log structure from a normal crossings divisor is log regular. If S is log regular then the locus on which the log structure is trivial is schematically[8)] dense open.

Lemma 6.9 ([28, Lemma 5.2]). *Let S be a log regular scheme. Then the underlying scheme of S is regular if and only if the characteristic sheaf $\overline{M}_S$ is locally free. In this case, the log structure on S is the log structure associated to a normal crossings divisor on S.*

Lemma 6.10 ([22, Theorem 11.6]). *Let S be a log regular scheme, and $U \hookrightarrow S$ the dense open locus where the log structure is trivial. Then*

$$M_S = j_*\mathcal{O}^*_U \times_{j_*\mathcal{O}_U} \mathcal{O}_S$$

and

$$M_S^{gp} = j_*\mathcal{O}^*_U,$$

where $j : U \to X$ is the inclusion.

[8)] Log regular schemes are reduced, so density is equivalent to schematic density.

6.3. The log Jacobian is a log Néron model.

Theorem 6.11. *Let X/S be a log curve over a log regular scheme, and $U \hookrightarrow S$ the dense open locus where the log structure is trivial. The stacks* $\mathbf{LogPic}_{X/S}$ *and* $\mathbf{LogPic}^0_{X/S}$ *on* $(\mathbf{LSch}/S)_{\text{ét}}$ *are log smooth over S and have the Néron mapping property with respect to U (hence likewise for their sheafifications* $\mathrm{LogPic}_{X/S}$, $\mathrm{LogPic}^0_{X/S}$*).*

Proof. The log smoothness is Property 3.59. We check the Néron mapping property. For any log smooth morphism $T \to S$, the restriction map $\mathrm{LogPic}_{X/S}(T) \to \mathrm{LogPic}_{X/S}(T \times_S U)$ is obtained, locally on T, from $\mathbf{LogPic}_{X/S}(T) \to \mathbf{LogPic}_{X/S}(T \times_S U)$ by taking isomorphism classes on both sides. Hence, if the stack $\mathbf{LogPic}_{X/S}$ has the mapping property, then so does the sheaf $\mathrm{LogPic}_{X/S}$. Likewise, $\mathrm{LogPic}^0_{X/S}$ has the Néron mapping property if $\mathbf{LogPic}^0_{X/S}$ does. Since the constant sheaf $\mathbb{Z}$ on $\mathbf{LSch}/S$ has the Néron mapping property, the same property for $\mathbf{LogPic}_{X/S}$ implies it for the kernel $\mathbf{LogPic}^0{}_{X/S}$ of the degree map.

It remains to prove the property for $\mathbf{LogPic}_{X/S}$. Let $T \to S$ be a log smooth map, $V = T \times_S U$. Let $i : V \to T$, $j : X_V \to X_T$ denote the inclusions. We want to prove that the map

$$\mathbf{LogPic}_{X/S}(T) \to \mathbf{LogPic}_{X_U/U}(V) = \mathbf{Pic}_{X_U/U}(V) \tag{6.3.1}$$

given by $P \mapsto j^*P$ is an equivalence. We start with full faithfulness: let P, Q be log line bundles on $X \times_S T$. It suffices to show that the map $\mathcal{I}som(P, Q) \to j_*\mathcal{I}som(j^*P, j^*Q)$ of isomorphism sheaves is itself an isomorphism. The natural map $M^{\mathrm{gp}}_{X_T} \to j_*\mathcal{O}^\times_{X_V}$ is an isomorphism by log regularity of X_T and Lemma 6.10. Hence the natural map $P \to j_*j^*P$ is a map of $M^{\mathrm{gp}}_{X_T}$-torsors, hence is an isomorphism (and similarly for Q). The map

$$\mathcal{A}ut(P) \to j_*\mathcal{A}ut(j^*P)$$

is simply the natural map $M^{\mathrm{gp}}_{X_T} \to j_*\mathcal{O}^\times_{X_V}$, hence is also an isomorphism. The map

$$\mathcal{I}som(P, Q) \to j_*\mathcal{I}som(j^*P, j^*Q)$$

is then a map of $M^{\mathrm{gp}}_{X_T}$-pseudotorsors, so it suffices to show that $\mathcal{I}som(P, Q)$ is non-empty whenever $j_*\mathcal{I}som(j^*P, j^*Q)$ is. But given an isomorphism $j^*P \xrightarrow{\sim} j^*Q$, taking j_* gives an isomorphism

$$P = j_*j^*P \xrightarrow{\sim} j_*j^*Q = Q$$

(the above argument can be summarized by saying that j_* is a quasi-inverse to j^*).

We now prove essential surjectivity. Letting L be a, $\mathcal{O}^*_{X_V}$-torsor on X_V, we will exhibit an $M^{\mathrm{gp}}_{X_T}$-torsor P of bounded monodromy such that $j^*P = L$. The torsor P will simply be j_*L; as a priori this is only a pseudotorsor under $j_*\mathcal{O}^*_{X_V} = M^{\mathrm{gp}}_{X_T}$, and, if it is a torsor, it is not obvious that it has bounded monodromy, we will give a geometric construction of P, using the invariance of **LogPic** under log blowups and taking root stacks. Let D be a Cartier divisor on X_V representing L. By [1, Theorem 4.5], there is a cover $u : T' \to T$, which is a composition of a log blowup and a root stack, such that u restricts to an isomorphism over V, and a log modification $p : X' \to X_{T'} = X \times_T T'$ with $X' \to T'$ semistable ([1, Definition 2.2]). In particular, $X' \to T'$ is a log curve, and the total space of X' is regular. The schematic closure $\overline{D}$ of D in X' is therefore a Cartier divisor, and $\mathcal{O}(\overline{D})$ is a line bundle whose associated $\mathcal{O}^*_{X'}$-torsor extends L. By Property 3.56, it follows that there is a $M^{\mathrm{gp}}_{X_{T'}}$-torsor P' of bounded monodromy on $X_{T'}$ such that $j^*P' = L$. Then, by the full faithfulness of (6.3.1) and the fact $\mathbf{LogPic}_{X/S}$

is a stack for the log étale topology (Property 3.57), this P' descends to an $M_{X_T}^{\mathrm{gp}}$-torsor P with bounded monodromy. □

Remark 6.12. It will be shown in the work [25] that, when S is logarithmically smooth, $\mathbf{LogPic}_{X/S}$ is a stack, not only for the log étale topology, but for the topology generated by all logarithmically flat morphisms, from which it follows that $\mathbf{LogPic}_{X/S}$ and $\mathbf{LogPic}^0_{X/S}$ are in fact the Néron models of their restrictions to U.

Corollary 6.13. *Let X/S be a log curve over a log regular scheme. The smooth algebraic spaces* $\mathrm{sLPic}_{X/S}$ *and* $\mathrm{sLPic}^0_{X/S}$ *are Néron models of* $\mathrm{Pic}_{X/S}$ *and* $\mathrm{Pic}^0_{X/S}$ *respectively.*

Proof. This follows from Theorem 6.11 together with Remark 6.6. □

Corollary 6.14. *The strict saturated Jacobian* $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$ *satisfies the Néron mapping property for torsion sections. That is, for every $T \to S$ smooth and $L : T_U \to \mathrm{Pic}^0_{X/S}$ of finite order, there exists a unique extension to a map $T \to \mathrm{sPic}^{\mathrm{sat}}_{X/S}$. In particular, for every prime l*

$$T_l\,\mathrm{sPic}^{\mathrm{sat}}_{X/S} = j_* T_l\,\mathrm{Pic}^0_{X_U/U}$$

as sheaves on the étale site over $\underline{S}$, where $j : U \to S$ is the inclusion and T_l indicates the l-adic Tate module.

Proof. By Theorem 6.11, L extends uniquely to $\mathcal{L}$ in $\mathrm{sLPic}^0_{X/S}(T)$. As the image of $\mathcal{L}$ in $\mathrm{sTPic}^0_{X/S}(T)$ is torsion, $\mathcal{L}$ actually lies in $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$. □

7. Models of Pic^0

We saw in Section 6 that the logarithmic Jacobian and its strict version are Néron models. However, while $\mathrm{LogPic}^0_{X/S}$ is proper (in a suitable sense), its strict version $\mathrm{sLPic}^0_{X/S}$ is in general neither quasi-compact, separated, nor universally closed.

In this section we undertake the study of smooth *separated* group-models of $\mathrm{Pic}^0_{X_U}$. We establish a precise correspondence between such models and subgroups of the strict tropical Jacobian $\mathrm{sTPic}^0_{X/S}$.

7.1. A tropical criterion for separatedness. We start by considering a tropical curve $\mathcal{X} = (V, H, r, i, \ell)$ metrized by a sharp monoid $\overline{M}$. For a given a monoid homomorphism $\varphi : \overline{M} \to \overline{N}$ we denote by $\mathcal{X}_\varphi$ the induced tropical curve metrized by $\overline{N}$. We recall that by assumption all monoids we work with are fine and saturated.

Lemma 7.1. *Let $\mathcal{X}$ be a tropical curve metrized by a sharp monoid $\overline{M}$ and $\varphi : \overline{M} \to \overline{N}$ a map of monoids not contracting any edge of $\mathcal{X}$. The induced homomorphism of tropical Picard groups* $\mathrm{TroPic}^0(\mathcal{X}) \to \mathrm{TroPic}^0(\mathcal{X}_\varphi)$ *has free kernel.*

Proof. The map φ induces an isomorphism $H_1(\mathcal{X}) \to H_1(\mathcal{X}_\varphi)$. By the snake lemma, the kernel of the map of tropical Jacobians is equal to the kernel of the map

$$\mathrm{Hom}(H_1(\mathcal{X}), \overline{M}^{\mathrm{gp}})^\dagger \to \mathrm{Hom}(H_1(\mathcal{X}), \overline{N}^{\mathrm{gp}})^\dagger$$

induced by $\varphi^{\mathrm{gp}} : \overline{M}^{\mathrm{gp}} \to \overline{N}^{\mathrm{gp}}$. Because $\overline{M}$ is fine and saturated, $\overline{M}^{\mathrm{gp}}$ is free, hence so is $\mathrm{Hom}(H_1(\mathcal{X}), \overline{M}^{\mathrm{gp}})$ and any subgroup of it. □

Lemma 7.2. *Let $\mathcal{X}$ be a tropical curve metrized by a sharp monoid $\overline{M}$, let Ψ be an abelian group, and let $\alpha : \Psi \to \mathrm{TroPic}^0(\mathcal{X})$ be a homomorphism. The following are equivalent:*

(i) *α is injective and Ψ is finite.*

(ii) *For every monoid P and homomorphism $\varphi : \overline{M} \to P$ not contracting any edge of $\mathcal{X}$, the composition $\Psi \to \mathrm{TroPic}^0(\mathcal{X}) \to \mathrm{TroPic}^0(\mathcal{X}_\varphi)$ is injective.*

(iii) *For every monoid homomorphism $\varphi : \overline{M} \to \mathbb{N}$ not contracting any edge of $\mathcal{X}$, the composition $\Psi \to \mathrm{TroPic}^0(\mathcal{X}) \to \mathrm{TroPic}^0(\mathcal{X}_\varphi)$ is injective.*

Proof. (i) ⇒ (ii) The kernel of $\mathrm{TroPic}^0(\mathcal{X}) \to \mathrm{TroPic}^0(\mathcal{X}_\varphi)$ is free by Lemma 7.1. Since Ψ is finite, the intersection of the kernel with Ψ is zero.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i) We first establish the existence of a monoid homomorphism $\varphi : \overline{M} \to \mathbb{N}$ not contracting any edge of $\mathcal{X}$: since $\overline{M}$ is fine and saturated, it is the intersection of a cone $\sigma \subset P^{\mathrm{gp}} \otimes \mathbb{R}$ with P^{gp} for a finitely generated lattice P^{gp}. Then any integral element in the interior of the dual cone $\sigma^\vee \subset \mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{R})$ will not contract any non-zero element of M and thus gives rise to such a map. Injectivity of α thus follows. Now $\mathrm{TroPic}^0(\mathcal{X}_\varphi)$ is finite since it is the cokernel of the homomorphism $H_1(\mathcal{X}_\varphi) \to \mathrm{Hom}(H_1(\mathcal{X}_\varphi), \mathbb{Z})$ induced by the monodromy pairing, which is injective by [26, Corollary 3.4.8]. Hence, Ψ is finite as well. □

We fix a log curve X/S with $\underline{S}$ locally noetherian. We introduce the category **Et** whose objects are pairs (Ψ, α) of an étale group algebraic space Ψ/S and a homomorphism

$$\alpha : \Psi \to \mathrm{sTPic}^0_{X/S}.$$

Similarly, we let **Sm** be the category of pairs $(\mathcal{G}, \alpha)$ of a smooth group algebraic space $\mathcal{G}/S$ and a homomorphism $\alpha : \mathcal{G} \to \mathrm{sLPic}^0_{X/S}$.

There is an obvious base change functor

$$F : \mathbf{Et} \to \mathbf{Sm}, \quad (\Psi \to \mathrm{sTPic}^0_{X/S}) \mapsto (\Psi \underset{\mathrm{sTPic}^0_{X/S}}{\times} \mathrm{sLPic}^0_{X/S} \to \mathrm{sLPic}^0_{X/S}). \tag{7.1.1}$$

Recall the functor $\mathfrak{f}_*$ introduced in Section 2.4. Notice that for an object (Ψ, α) of **Et**, we have a natural map $\mathfrak{f}_*\Psi \to \mathfrak{f}_* \mathrm{sTPic}^0 \to \mathrm{TroPic}^0$ and similarly for an object of **Sm**.

The next proposition is the key statement of the section.

Proposition 7.3. *Let X/S be a log curve with $\underline{S}$ locally noetherian. Let $(\Psi, \alpha) \in \mathbf{Et}$, with $\mathcal{G} \to \mathrm{sLPic}^0_{X/S}$ its image under the functor F. Consider the two conditions:*

(i) *$\alpha : \Psi \to \mathrm{sTPic}^0_{X/S}$ is an open immersion and Ψ/S is quasi-finite,*

(ii) *$\mathcal{G}/S$ is separated.*

Then (i) *implies* (ii). *Moreover, if (S, M_S) is log regular and Ψ has trivial restriction to the dense open $U \subset S$ where $\overline{M}_S$ vanishes, then* (ii) *implies* (i) *as well, and the two conditions are equivalent to*

(iii) *$\mathcal{G}/S$ is separated and quasi-compact.*

Remark 7.4. The restriction of $\mathrm{sLPic}^0_{X/S}$ to U is naturally identified with $\mathrm{Pic}^0_{X_U/U}$, and therefore $\mathrm{sTPic}^0_{X/S}$ restricts to $\{0\}$. The fact that $\Psi_U = \{0\}$ implies that $\mathcal{G}_U = \mathrm{Pic}^0_{X_U/U}$.

Proof. We prove that (i) implies (ii). Since $\underline{S}$ is locally noetherian, it follows that Ψ/S is quasi-separated, hence so is its base change $\mathcal{G}/S$. By [37, Tag 0ARI] we may check the valuative criterion for a strictly henselian discrete valuation ring V; we write η for its generic point and s for its closed point. Fix a map $a : \eta \to \mathcal{G}$ and two lifts to $b, b' : V \to \mathcal{G}$; using the group structure of $\mathcal{G}$, we may assume $b' = 0$ and therefore $a = 0$.

Denote by d the map from V to S; we endow η with the log structure M_η pulled back from S, and we write M_V for the maximal extension of M_η to a log structure on V. We obtain a commutative diagram

$$\begin{array}{ccccccc}
(\eta, M_\eta) & \longrightarrow & (\eta, M_\eta) & \xrightarrow{0} & \mathfrak{f}_*\mathcal{G} & \longrightarrow & \mathrm{LogPic}^0_{X/S} \\
\downarrow & & \downarrow & \overset{b}{\nearrow}\overset{0}{\searrow} & \downarrow & & \downarrow \\
(V, M_V) & \longrightarrow & (V, d^*M_S) & & S & \longrightarrow & S.
\end{array}$$

Let $\mathcal{X}_s$ (resp. $\mathcal{X}^V$) denote the tropicalization of X_s metrized by $\overline{M}_{S,s}$ (resp. $\overline{M}_{V,s}$). Notice that the map $\overline{M}_{S,s} \to \overline{M}_{V,s}$ of characteristic monoids does not contract any edge of $\mathcal{X}_s$, since $d^*M_S \to M_V$ is a map of log structures and therefore sends non-units to non-units. As V is strictly henselian, we have by Lemma 3.48 that $\mathrm{TroPic}^0(V, M_V) = \mathrm{TroPic}^0(\mathcal{X}^V)$ and $\mathrm{TroPic}^0(V, d^*M_S) = \mathrm{TroPic}^0(\mathcal{X}_s)$. Consider the commutative diagram whose rows are exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0_{X/S}(V) & \longrightarrow & \mathfrak{f}_*\mathcal{G}(V, d^*M_S) & \longrightarrow & \mathfrak{f}_*\Psi(V, d^*M_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0_{X/S}(V) & \longrightarrow & \mathrm{LogPic}^0_{X/S}(V, d^*M_S) & \longrightarrow & \mathrm{TroPic}^0(\mathcal{X}_s) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}^0_{X/S}(V) & \longrightarrow & \mathrm{LogPic}^0_{X/S}(V, M_V) & \longrightarrow & \mathrm{TroPic}^0(\mathcal{X}^V) & \longrightarrow & 0.
\end{array}$$

The element b lies in $\mathfrak{f}_*\mathcal{G}(V, d^*M_S)$. The following three facts imply that $b = 0$:

- By properness of $\mathrm{LogPic}^0_{X/S}$, the image of b in $\mathrm{LogPic}^0_{X/S}(V, M_V)$ is the trivial log line bundle, so b maps to 0 in $\mathrm{TroPic}^0(\mathcal{X}^V)$.
- Combining the assumption that (i) holds with Lemma 7.2 yields that the composition
$$\mathfrak{f}_*\Psi(V, d^*M_S) = \Psi(V) \to \mathrm{sTPic}^0(V) = \mathrm{TroPic}^0(V, d^*M_S) \to \mathrm{TroPic}^0(V, M_V)$$
is injective.
- If $b \in \mathfrak{f}_*\mathcal{G}(V, d^*M_S)$ lies in the image of $\mathrm{Pic}^0_{X/S}(V)$, then $b = 0$ since $\mathrm{Pic}^0_{X/S}$ is separated.

We move on to the next part of the statement, so from now on we suppose that S is log regular and that $\Psi_U = \{0\}$. This in particular implies that $\mathcal{G}_U = \mathrm{Pic}^0_{X_U/U}$. It is clear that (iii) implies (ii); if we show that (ii) implies (i), then we immediately obtain (ii) $\Rightarrow$ (iii). Indeed, quasi-finiteness of Ψ together with the fact that $\mathrm{Pic}^0_{X/S}$ is quasi-compact, implies that $\mathcal{G}/S$ is quasi-compact.

It remains to prove that (ii) implies (i). Write K for the kernel of $\Psi \to \mathrm{sTPic}^0_{X/S}$. It is étale, and since $\Psi_U = 0$, K also vanishes over the open dense $U \subset S$. Moreover, K is identified with the kernel of $\mathcal{G} \to \mathrm{sLPic}^0_{X/S}$; as $\mathcal{G}/S$ is separated, so is K, hence K is trivial. This shows that $\Psi \to \mathrm{sTPic}^0_{X/S}$ is an open immersion.

Now let $t \to S$ be a geometric point, with image $s \in S$. Because S is locally noetherian, by a special case of [11, 7.1.9] there exists a morphism $Z \to S$ from the spectrum of a discrete valuation ring such that the closed point is mapped to s and the generic point to U. Now consider the composition $\omega : Z^{\mathrm{sh}} \to Z \to S$ with the strict henselization induced by $t \to s$. $\mathcal{G}$ is a smooth, separated model of $\mathrm{Pic}^0_{X_U}$ and therefore $\omega^*\mathcal{G}$ is a Z^{sh}-smooth separated model of $\omega^* \mathrm{Pic}^0_{X_U}$. The restriction of the latter to the generic point of Z^{sh} is an abelian variety, which therefore admits a Néron model of finite type $\mathcal{N}/Z^{\mathrm{sh}}$ by [5, Corollary 1.3.2]. We claim that the natural map $\Phi : \omega^*\mathcal{G} \to \mathcal{N}$ is an open immersion. In the case of schemes this would follow immediately from [5, Proposition 7.4.3]. The same proof shows for algebraic spaces that the natural map on identity components $\omega^*\mathcal{G}^0 \to \mathcal{N}^0$ is an isomorphism. This implies that Φ is flat, hence is an open immersion by Lemma 7.5.

In particular, the map of fibers $\mathcal{G}_t \to \mathcal{N}_t$ is an open immersion. As $\mathcal{N}_t$ is of finite type (and S is locally noetherian), so is $\mathcal{G}_t$. Then by descent Ψ_t is of finite type as well; as it is moreover étale, it is finite over $k(t)$. In particular, the map $\Psi_t \to \mathrm{sTPic}^0_t$ factors via $\mathrm{sTPic}^{\mathrm{tor}}_t$ (which is quasi-finite by Lemma 5.3); it follows that the open immersion $\Psi \to \mathrm{sTPic}^0$ factors via the open immersion $\mathrm{sTPic}^{\mathrm{tor}} \to \mathrm{sTPic}^0$. The resulting open immersion $\Psi \to \mathrm{sTPic}^{\mathrm{tor}}$ is quasi-compact (indeed $\mathrm{sTPic}^{\mathrm{tor}}$ is locally noetherian since S is). This proves that Ψ/S is quasi-finite. □

Lemma 7.5. *Let $f : X \to Y$ be a flat, separated, locally finitely presented morphism of algebraic spaces. Let $U \subseteq Y$ be open such that f is an open immersion over U and such that $f^{-1}U$ is schematically dense in X. Then f is an open immersion.*

Proof. We follow [23, Lemma 2.0]. Replacing Y by the open image of f, we may assume f is faithfully flat. Base-changing along f, the assumptions are preserved and we may assume f has a section $e : Y \to X$. Since f is separated, the section e is a closed immersion. But the open immersion $j : f^{-1}U \to X$ factors via e, so schematic density of j implies that e is an isomorphism. □

The image of $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ under the functor F is $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$, yielding:

Corollary 7.6. *Let X/S be a log curve with $\underline{S}$ locally noetherian. Then* $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$ *is separated.*

7.2. Equivalence of categories. From this point until the end of the section we will assume that S is log regular and denote by $U \subset S$ the open dense where $\overline{M}_S$ is trivial. Recall that the étale algebraic space $\mathrm{sTPic}^0_{X/S}$ has trivial restriction to U while $\mathrm{sLPic}^0_{X/S}$ is a smooth group space whose restriction to U is naturally identified with $\mathrm{Pic}^0_{X_U/U}$.

We consider the full subcategory of **Et**

$$\begin{aligned}\mathbf{Et}^\circ := \{&(\Psi, \alpha) \text{ with } \Psi \to S \text{ an étale group-algebraic space}\\ &\text{such that } \Psi_U = \{0\},\ \alpha : \Psi \to \mathrm{sTPic}^0_{X/S} \text{ a homomorphism}\}.\end{aligned}$$

and the full subcategory of **Sm**

$$\begin{aligned}\mathbf{Sm}^\circ := \{&(\mathcal{G}, \beta) \text{ with } \mathcal{G} \to S \text{ a smooth quasi-separated group algebraic space,}\\ &\text{with fiberwise-connected component of identity } \mathcal{G}^0 \text{ separated over } S,\\ &\beta : \mathcal{G} \to \mathrm{sLPic}^0_{X/S} \text{ a homomorphism such that } \beta_U : \mathcal{G}_U \to \mathrm{Pic}^0_{X_U/U}\\ &\text{is an isomorphism}\}.\end{aligned}$$

The fiberwise connected-component of identity $\mathcal{G}^0$ is an open subgroup space of $\mathcal{G}$ containing the identity section and whose geometric fibers $\mathcal{G}^0_s$ are the connected component of identity of $\mathcal{G}_s$. See Appendix A for details on $\mathcal{G}^0$.

Lemma 7.7. *Let $(\mathcal{G}, \beta)$ be in* **Sm**$^\circ$. *Then $\mathcal{G}^0$ is naturally identified with* $\mathrm{Pic}^0_{X/S}$.

Proof. Over U there is already a natural identification $\mathcal{G}_U = \mathcal{G}^0_U = \mathrm{Pic}^0_{X_U}$. For every point s of S of codimension 1, the restriction of $\mathrm{Pic}^0_{X/S}$ to $\mathcal{O}_{S,s}$ is the identity component of its own Néron model. By [5, 7.4.3], the same holds for $\mathcal{G}^0$. Now by [35, XI, 1.15], the isomorphism $\mathcal{G}^0_U \to \mathrm{Pic}^0_{X_U}$ extends uniquely to an isomorphism $\mathcal{G}^0 \to \mathrm{Pic}^0_{X/S}$. □

Because of the Néron mapping property of $\mathrm{sLPic}^0_{X/S}$ (Corollary 6.13) there is a natural equivalence between **Sm**$^\circ$ and the category with objects

$$\begin{aligned}\{&(\mathcal{G}, \varphi) \text{ with } \mathcal{G} \to S \text{ a smooth quasi-separated group-algebraic space,}\\ &\text{with fiberwise-connected component of identity } \mathcal{G}^0 \text{ separated over } S,\\ &\varphi : \mathcal{G}_U \to \mathrm{Pic}^0_{X_U/U} \text{ an isomorphism}\},\end{aligned}$$

so we will not distinguish between the two.

We obtain by restriction of the functor F of (7.1.1) a functor

$$F^\circ : \mathbf{Et}^\circ \to \mathbf{Sm}^\circ.$$

We are going to construct a quasi-inverse to F°. We denote by $\pi_0(\mathcal{G})$ the étale algebraic space $\mathcal{G}/\mathcal{G}^0$ of Definition A.2. By the universal property of $\mathcal{G}/\mathcal{G}^0$ for maps to étale spaces (Lemma A.4), together with the fact that $\pi_0(\mathrm{sLPic}^0_{X/S}) = \mathrm{sTPic}^0_{X/S}$ (Lemma A.5), we obtain a functor

$$\Pi_0 : \mathbf{Sm}^\circ \to \mathbf{Et}^\circ, \quad (\mathcal{G}, \beta : \mathcal{G} \to \mathrm{sLPic}^0_{X/S}) \mapsto (\pi_0(\mathcal{G}), \pi_0(\beta) : \pi_0(\mathcal{G}) \to \mathrm{sTPic}^0_{X/S}).$$

Lemma 7.8. *The functor $F^\circ : \mathbf{Et}^\circ \to \mathbf{Sm}^\circ$ is an equivalence with Π_0 as a quasi-inverse.*

Proof. Let $\Psi \to \mathrm{sTPic}^0_{X/S}$ be in **Et**$^\circ$, with image $\mathcal{G} \to \mathrm{sLPic}^0_{X/S}$ via F°. The surjective map $\mathcal{G} \to \Psi$ factors by the universal property via an étale surjective map $\pi_0(\mathcal{G}) \to \Psi$. It remains to show that its kernel K vanishes. We obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \mathcal{G} & \longrightarrow & \pi_0(\mathcal{G}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & \mathcal{G} & \longrightarrow & \Psi & \longrightarrow & 0. \end{array}$$

By Lemma 7.7, the left vertical map is an isomorphism and we conclude.

Conversely, let $\mathcal{G} \in \mathbf{Sm}^\circ$. Because the functor F° is defined as a fiber product there is a natural map $f : \mathcal{G} \to F^\circ(\Pi_0(\mathcal{G})) = \pi_0(\mathcal{G}) \times_{\mathrm{sTPic}^0} \mathrm{sLPic}^0$, and we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \mathcal{G} & \xrightarrow{p\circ f} & \pi_0(\mathcal{G}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & \pi_0(\mathcal{G}) \times_{\mathrm{sTPic}^0} \mathrm{sLPic}^0 & \xrightarrow{p} & \pi_0(\mathcal{G}) & \longrightarrow & 0, \end{array}$$

where the rightmost vertical map is the identity and the leftmost vertical map h is the induced map $\ker(p \circ f) \to \ker(p)$. Since f restricts over U to the identity of $\mathrm{Pic}^0_{X_U}$, so does h. It follows that h is the isomorphism of Lemma 7.7, and that f is an isomorphism as well. □

As a corollary of Proposition 7.3 we refine the equivalence F°.

Definition 7.9. We let $\mathbf{Et}^{\mathbf{qf,mono}}$ to be the full subcategory of $\mathbf{Et}^\circ$ of those (Ψ, α) with α a monomorphism (i.e., an open immersion) and Ψ/S quasi-finite. We let $\mathbf{Sm}^{\mathbf{qc,sep}}$ be the full subcategory of $\mathbf{Sm}^\circ$ of those $(\mathcal{G}, \varphi)$ with $\mathcal{G} \to S$ separated and quasi-compact.

Both $\mathbf{Et}^{\mathbf{qf,mono}}$ and $\mathbf{Sm}^{\mathbf{qc,sep}}$ are equivalent to partially ordered sets. For $\mathbf{Et}^{\mathbf{qf,mono}}$ this is clear, and for $\mathbf{Sm}^{\mathbf{qc,sep}}$ we observe that, for an object $(\mathcal{G}, \varphi : \mathcal{G} \to \mathrm{sLPic}^0_{X/S})$ of $\mathbf{Sm}^{\mathbf{qc,sep}}$, φ is the base change of $\pi_0(\mathcal{G}) \to \mathrm{sTPic}^0_{X/S}$, by Lemma 7.8. The latter is an open immersion by Proposition 7.3, so φ is an open immersion.

The following corollary allows us to describe all possible smooth separated group models of $\mathrm{Pic}^0_{X_U}$ in terms of open subgroups of the strict tropical Jacobian.

Corollary 7.10. *The equivalence $F^\circ : \mathbf{Et}^\circ \to \mathbf{Sm}^\circ$ restricts to an order-preserving bijection $F^* : \mathbf{Et}^{\mathbf{qf,mono}} \to \mathbf{Sm}^{\mathbf{qc,sep}}$.*

Proof. This is immediate by Proposition 7.3. □

The partially ordered set $\mathbf{Et}^{\mathbf{qf,mono}}$ has a maximal element, namely the quasi-finite étale group space $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ representing the torsion part of the sheaf $\mathrm{sTPic}^0_{X/S}$. From Corollary 7.10 we deduce:

Theorem 7.11. *Let X/S be a log curve over a log regular base S, and $U \subset S$ the open where the log structure is trivial. The partially ordered set of smooth separated group-S-models of finite type of $\mathrm{Pic}^0_{X_U}$ has*

$$F^*(\mathrm{sTPic}^{\mathrm{tor}}_{X/S}) = \mathrm{sPic}^{\mathrm{sat}}_{X/S}$$

as maximum element. Namely, any other such model has a unique open immersion to $\mathrm{sPic}^{\mathrm{sat}}_{X/S}$.

7.3. Possible extensions to the case of log abelian varieties. It is natural to ask which of the results of this paper remain valid when the Jacobian of a curve is replaced by an arbitrary abelian variety. Suppose that we have a log regular log scheme S, and a log abelian variety $A^{\log}/S$ (which is necessarily an abelian variety A_U over the open locus $U \subseteq S$ on which the log structure is trivial). Now $A^{\log}$ is a sheaf on $(\mathbf{LSch}/S)_{\text{ét}}$ which has a tropicalization A^{trop} over S. We can restrict $A^{\log}$ and A^{trop} to sheaves on the strict étale site $(\mathbf{Sch}/\underline{S})_{\text{ét}}$ to obtain

algebraic spaces $sA^{\log}$ and sA^{trop}. The tropicalization sA^{trop} can equivalently be defined as the quotient

$$sA^{\mathrm{trop}} = sA^{\log}/sA^{\mathbf{0}}$$

of $sA^{\log}$ by the (semiabelian) fiberwise connected component of the identity $sA^{\mathbf{0}}$, and, as described in detail in [18, 4.1.2] also has an explicit combinatorial description étale locally in S as $\mathrm{Hom}(X, \overline{M}_S^{\mathrm{gp}})_{(Y)}/Y$ for lattices X and Y; the subscript (Y) indicates a subgroup of $\mathrm{Hom}(X, \overline{M}_S^{\mathrm{gp}})$ (see [18, 3.1]), analogous to the bounded monodromy subgroup of the Jacobian.

Conjecture 7.12. The log abelian variety $A^{\log}$ has the log Néron mapping property (Definition 6.3) with respect to U.

Conjecture 7.12 would in particular imply that $sA^{\log}$ is always a Néron model for A_U, but is rarely separated: it is separated if and only if sA^{trop} is finite. At the moment, we do not have a proof of Conjecture 7.12. Our proof for the Jacobian uses the geometry of the curve to produce the extension. On the other hand, the proof that abelian varieties are their own Néron models goes by extending line bundles on the dual abelian variety. This argument would extend to the case of logarithmic abelian varieties if we had a theory of log Picard functors for higher dimensional logarithmic schemes which satisfies analogues of Property 3.55, Property 3.56, Property 3.57, and the usual duality axioms.

Conditional on Conjecture 7.12, our proof of Theorem 1.2 goes through verbatim to show that there is a bijection between quasi-finite open subgroups of sA^{trop} and smooth, separated, quasi-compact S-group models of A_U.

8. Alignment and separatedness of strict log Pic

For X/S a log curve over a log regular base S with $U \subset S$ the largest open where the log structure is trivial, we have shown in Corollary 6.13 that $\mathrm{sLPic}^0_{X/S}$ is the Néron model of $\mathrm{Pic}^0_{X_U/U}$. It is worth stressing the fact that classically, the term Néron model is reserved for separated, quasi-compact models satisfying the Néron mapping property. The strict logarithmic Jacobian fails in general to satisfy these properties, as observed in Example 4.6.

In the papers [12, 31, 33], several criteria were introduced for the Jacobian of a prestable curve X/S (or for an abelian variety in [32]) to admit a separated, quasi-compact Néron model. They are all closely related to the general notion of *log alignment* that we introduce here:

Definition 8.1. Let $\overline{M}$ be a sharp fs monoid. The 1-dimensional faces of $\overline{M} \otimes_{\mathbb{Z}} \mathbb{R}_{\geq 0}$ are called the *extreme rays* of $\overline{M}$.

Definition 8.2. A *cycle* in a graph is a path that begins and ends at the same vertex, and which otherwise repeats no vertices. A subset S of the edges of a graph is called *cycle-connected* if for every pair $e, e' \in S$ of distinct edges there exists a cycle in S containing e and e'. It is shown in [13, Lemma 7.2] that the maximal cycle-connected subsets (which are there called circuit-connected) form a partition of the edges of the graph.

Definition 8.3. We say that a tropical curve $\mathfrak{X}$ metrized by a sharp fs monoid $\overline{M}$ is *log aligned* when for every cycle γ in $\mathfrak{X}$, all lengths of edges of γ lie on the same extreme ray

of $\overline{M}$. Let $X \to S$ be a log curve. We say that X/S is *log aligned* at a geometric point $\bar{s}$ of $\underline{S}$ when the tropicalization of X at $\bar{s}$ is log aligned. We say that X/S is *log aligned* if it is log aligned at every geometric point of $\underline{S}$.

Lemma 8.4. *Let $\overline{M}$ be a sharp fs monoid, $\mathfrak{X}$ a tropical curve metrized by $\overline{M}$ and $\mathfrak{Y}$ a subdivision of $\mathfrak{X}$. Then $\mathfrak{Y}$ is log aligned if and only if $\mathfrak{X}$ is.*

Proof. It suffices to treat the case where $\mathfrak{Y}$ is a basic subdivision of $\mathfrak{X}$. Suppose it is, and call e the subdivided edge: it is replaced in $\mathfrak{Y}$ by a chain of two edges e_1, e_2 of the same total length. There is a canonical bijection between cycles of $\mathfrak{X}$ and of $\mathfrak{Y}$. Let γ be a cycle of $\mathfrak{X}$. It suffices to show that the lengths of edges of γ in $\mathfrak{X}$ lie on the same extreme ray of $\overline{M}$ if and only if the lengths of edges of γ in $\mathfrak{Y}$ do. If γ does not contain e, this is clear. Otherwise, it follows from observing that the length of e is in an extreme ray R of $\overline{M}$ if and only if the lengths of e_1 and e_2 are both in R. □

Theorem 8.5. *Let X/S be a log curve. Consider the following conditions:*

(i) *X/S is log aligned,*

(ii) $\mathrm{sTPic}^0_{X/S}$ *is quasi-finite over S,*

(iii) $\mathrm{sLPic}^0_{X/S}$ *is separated over S.*

Then we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). *If S is log regular and $U \subset S$ is the largest open where the log structure is trivial, we additionally have* (iii) $\Rightarrow$ (ii)*, and the conditions above are equivalent to the following two:*

(iv) $\mathrm{sLPic}^0_{X/S}$ *is a separated Néron model of finite type for* $\mathrm{Pic}^0_{X_U}$.

(v) $\mathrm{Pic}^0_{X_U/U}$ *admits a separated Néron model of finite type over S.*

Proof. First, Proposition 7.3 for $(\Psi, \alpha) = (\mathrm{sTPic}^0_{X/S}, \mathrm{Id})$ gives (ii) $\Rightarrow$ (iii) and if S is log regular also (iii) $\Rightarrow$ (ii). If $\mathrm{sLPic}^0_{X/S}$ is separated then by Proposition 7.3 it is also quasi-compact, hence of finite type. The equivalence of (iii), (iv) and (v) in the log regular case then follows from Theorem 6.11 and the uniqueness of Néron models.

It remains to prove (i) $\Leftrightarrow$ (ii). By Theorem 7.11, $\mathrm{sTPic}^{\mathrm{tor}}_{X/S}$ is the maximum open quasi-finite subgroup of $\mathrm{sTPic}^0_{X/S}$. Condition (ii) is then equivalent to $\mathrm{sTPic}^{\mathrm{tor}}_{X/S} = \mathrm{sTPic}^0_{X/S}$, which in turn is equivalent to $\mathrm{sTPic}^0_{X/S}$ having finite fibers. We immediately reduce to the case where S is a geometric log point, and we write $M := M_S(S)$. Denote by $\mathfrak{X}$ the tropicalization of X/S, and by $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ the maximal cycle-connected components of $\mathfrak{X}$. We have a canonical isomorphism

$$H_1(\mathfrak{X}) = \bigoplus_{i=1}^{n} H_1(\mathfrak{X}_i).$$

Suppose first that $\mathfrak{X}/\overline{M}$ is log aligned, so that for every $0 \leq i \leq n$ there is an irreducible element ρ_i of an extremal ray of $\overline{M}$ such that all edges of $\mathfrak{X}_i$ have length in $\mathbb{N}\rho_i$. Thus $\mathfrak{X}_i$ can be seen as a tropical curve metrized by $\mathbb{N}\rho_i$, and any bounded monodromy map $H_1(\mathfrak{X}_i) \to \overline{M}^{\mathrm{gp}}$ factors uniquely through the inclusion $\mathbb{Z}\rho_i \to \overline{M}^{\mathrm{gp}}$. We get isomorphisms

$$\mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})^\dagger = \bigoplus_{i=1}^{n} \mathrm{Hom}(H_1(\mathfrak{X}_i), \overline{M}^{\mathrm{gp}})^\dagger = \bigoplus_{i=1}^{n} \mathrm{Hom}(H_1(\mathfrak{X}_i), \mathbb{Z}\rho_i)^\dagger,$$

where the first equality holds since bounded monodromy can be checked separately on each $\mathcal{X}_i$ by Remark 3.34. Quotienting by $H_1(\mathcal{X})$, we obtain

$$\mathrm{TroPic}^0(\mathcal{X}/\overline{M}) = \bigoplus_{i=1}^{n} \mathrm{TroPic}^0(\mathcal{X}_i/\overline{M}) = \bigoplus_{i=1}^{n} \mathrm{TroPic}^0(\mathcal{X}_i/\mathbb{N}\rho_i).$$

The right-hand side is finite, as the rank of $\mathrm{Hom}(H_1(\mathcal{X}_i), \mathbb{Z}\rho_i)$ is equal to the rank of $H_1(\mathcal{X}_i)$.

For the reverse implication, suppose X/S is not log aligned; we will show

$$\mathrm{TroPic}^0(\mathcal{X}/\overline{M}) = \mathrm{Hom}(H_1(\mathcal{X}), \overline{M}^{\mathrm{gp}})^\dagger/H_1(\mathcal{X})$$

is not finite, by showing its rank is at least 1. Note that if $\overline{M} \to \overline{M}'$ is a finite index homomorphism, a homomorphism $\varphi : H_1(\mathcal{X}) \to \overline{M}^{\mathrm{gp}}$ has bounded monodromy if and only if its composition with $\overline{M}^{\mathrm{gp}} \to \overline{M}'^{\mathrm{gp}}$ has bounded monodromy. Thus, the rank of $\mathrm{TroPic}^0(\mathcal{X}/\overline{M})$ is equal to the rank of $\mathrm{TroPic}^0(\mathcal{X}, \overline{M}')$ for any finite index inclusion $\overline{M} \to \overline{M}'$. Let $\ell(e) \in \overline{M}$ denote the length of the edge e in $\mathcal{X}$. As the extreme rays of $\overline{M}$ span $\overline{M}$ over $\mathbb{Q}$, and we are free to replace $\overline{M}$ by finite index extensions, we can assume that each length $\ell(e)$ can be written as a sum of elements in $\overline{M}$ that lie on the extreme rays of $\overline{M}$. We may then subdivide $\mathcal{X}$ so that each edge in the subdivision has length along the extreme rays of $\overline{M}$. Using Lemma 8.4 and the invariance of the tropical Jacobian under subdivisions of $\mathcal{X}$, we may then assume that each edge of $\mathcal{X}$ has length which lies along an extreme ray of $\overline{M}$. Pick a spanning tree T of $\mathcal{X}$. The edges $e_1, \ldots, e_r$ not in T correspond to cycles $\gamma_1, \ldots, \gamma_r$ forming a basis of $H_1(\mathcal{X})$. By hypothesis, $\mathcal{X}$ is not log aligned, so one of the γ_i, for example γ_1, has length not belonging to an extremal ray of $\overline{M}^{\mathrm{gp}}$. Therefore, there exists an edge e in γ_1 of length along an extreme ray of $\overline{M}$ different than the ray containing the length of e_1.

We claim that the intersection pairings of the family $(e, e_1, \ldots, e_r)$ are independent bounded monodromy maps $H_1(\mathcal{X}) \to \overline{M}^{\mathrm{gp}}$. The fact that intersection pairing with an edge has bounded monodromy is general: for any edge e, and any cycle $\gamma \in H_1(\mathcal{X})$, the intersection pairing $e.\gamma$ evidently has length bounded by the length of γ. To see that the pairings are independent, notice that $e_i.\gamma_j$ is $\delta_{ij}\ell(e_i)$ where δ_{ij} is the Kronecker delta. Consider a linear combination $b = ae + \sum a_i e_i$ with coefficients in $\mathbb{Z}$, and suppose the intersection pairing of b is trivial. Then $a\ell(e) + a_1\ell(e_1) = b.\gamma_1 = 0$, combined with the fact e and e_1 have independent lengths, yields $a = a_1 = 0$. Hence for $j > 1$ we have $a_j\ell(e_j) = b.\gamma_j = 0$, from which we deduce $a_j = 0$.

From this, we obtain

$$\mathrm{rank}(\mathrm{Hom}(H_1(\mathcal{X}), \overline{M}^{\mathrm{gp}})^\dagger) \geq r + 1 > r = \mathrm{rank}(H_1(\mathcal{X})).$$

Thus, $\mathrm{TroPic}^0(\mathcal{X}/\overline{M}) = \mathrm{Hom}(H_1(\mathcal{X}), \overline{M}^{\mathrm{gp}})^\dagger/H_1(\mathcal{X})$ is not finite. □

9. The strict logarithmic Jacobian and the Picard space

Over a Dedekind base S, Raynaud constructed the Néron model of the Jacobian of a curve X/S as the quotient of the relative Picard space by the closure $\bar{e}$ of the unit section (see [5, Section 9.5]). When $\dim S > 1$, the closure $\bar{e}$ is in general neither S-flat nor a subgroup, and so this quotient is not representable. In [12] and [31], necessary and sufficient conditions for the flatness of $\bar{e}$ are given, proving the existence of separated Néron models when these conditions hold. In this section we show that Raynaud's approach can be extended over higher-

dimensional bases even when $\bar{e}$ is not S-flat, simply by replacing $\bar{e}$ by its largest open subspace $\bar{e}^{\text{ét}}$ which is étale over S. This allows us to describe the Néron model (constructed above as the algebraic space sLPic^0) as the quotient of the Picard space by $\bar{e}^{\text{ét}}$, under somewhat more restrictive assumptions on X/S. The construction of the Néron model in [33] is done by gluing local models of this form.

Letting $\pi : X \to S$ be a log curve, we write $\mathrm{Pic}^{\mathrm{tot}0}$ for the kernel of the composition

$$\mathrm{Pic}_{X/S} \to \mathbb{Z}[\mathrm{Irr}_{X/S}] \xrightarrow{\Sigma} \mathbb{Z}.$$

There is a canonical map $\mathrm{Pic}^{\mathrm{tot}0}_{X/S} \to \mathrm{sLPic}^0_{X/S}$ taking a line bundle to the associated log line bundle.

Write $\bar{e}$ for the schematic closure of the unit section in $\mathrm{Pic}^{\mathrm{tot}0}_{X/S}$. If U, $V \hookrightarrow \bar{e}$ are open immersions étale over S, then the same is true of their union. Hence $\bar{e}$ has a largest open subscheme which is étale over S, which we denote by $\bar{e}^{\text{ét}}$, a locally closed subscheme of $\mathrm{Pic}^{\mathrm{tot}0}_{X/S}$.

Theorem 9.1. *Suppose that S is log regular. Then:*

(i) *The map $f : \mathrm{Pic}^{\mathrm{tot}0}_{X/S} \to \mathrm{sLPic}^0_{X/S}$ has kernel $\bar{e}^{\text{ét}}$.*

(ii) *If in addition $\underline{X}$ is regular, then f is surjective.*

Remark 9.2. By Property 3.56, a log modification $X' \to X$ induces an isomorphism $\mathrm{sLPic}^0_{X/S} \to \mathrm{sLPic}^0_{X'/S}$. On the other hand $\mathrm{Pic}^{\mathrm{tot}0}_{X/S} \to \mathrm{Pic}^{\mathrm{tot}0}_{X'/S}$ is an open immersion but in general not an isomorphism. When $\underline{S}$ is log regular and regular, we can always find, étale locally on S, a log modification of X/S with regular total space. To see this, note that since S is locally Noetherian, it has an étale cover by nuclear schemes by Lemma 3.45, so we can assume that S is nuclear. Let $\mathcal{X}$ denote the tropicalization of X over the closed stratum. As S is log regular and regular, every edge of $\mathcal{X}$ is marked by an element of a free monoid $\mathbb{N}^r$. Put $\mathcal{X}' = \mathcal{X}$. As long as $\mathcal{X}'$ has an edge whose length is not one of the generators of $\mathbb{N}^r$, replace $\mathcal{X}'$ by any basic subdivision at that edge. The process terminates, and provides a maximal subdivision of $\mathcal{X}$. This subdivision lifts to a log modification $X' \to X$ whose total space is regular: X' is evidently log regular, and the log structure around a node of length a generator of $\mathbb{N}^r$ is isomorphic to $\mathbb{N}^r \oplus_{\mathbb{N}} \mathbb{N}^2 \cong \mathbb{N}^{r+1}$. Combining this observation with Theorem 9.1, we obtain a local description of $\mathrm{sLPic}^0_{X/S}$ as a quotient of a Picard space.

Corollary 9.3. *Let X/S be a prestable curve over a toroidal variety, smooth exactly over the complement of the toroidal boundary, and assume X is regular. Then the quotient $\mathrm{Pic}^{\mathrm{tot}0}_{X/S}/\bar{e}^{\text{ét}}$ is the Néron model of the Jacobian of X.*

Proof of Theorem 9.1. (i) We write Ψ for the kernel of the summation map $\mathcal{Z}^{\mathcal{V}} \to \mathbb{Z}$; this is étale as it is the kernel of a map of étale group spaces. The multidegree map $\mathrm{Pic}^{\mathrm{tot}0}_{X/S} \to \Psi$ is the cokernel of the open immersion $\mathrm{Pic}^0_{X/S} \to \mathrm{Pic}^{\mathrm{tot}0}_{X/S}$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & \mathrm{Pic}^{\mathrm{tot}0}_{X/S} & \longrightarrow & \Psi & \longrightarrow & 0 \\
 & & \| & & \downarrow f & & \downarrow g & & \\
0 & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & \mathrm{sLPic}^0_{X/S} & \longrightarrow & \mathrm{sTPic}^0_{X/S} & \longrightarrow & 0,
\end{array}$$

where $K := \ker f$ is equal to $\ker g$ by the snake lemma. The map g is a map of étale group spaces (Theorem 4.4), hence its kernel is étale, i.e., K is étale.

Note that the locus $U \hookrightarrow S$ over which the morphism $X \to S$ is smooth is schematically dense by our log regularity assumption. One checks immediately that K is trivial over U, hence the map $K \to \mathrm{Pic}^{\mathrm{tot}\mathbf{0}}_{X/S}$ factors via the closure $\bar{e}$ of the unit section (since $K \to S$ is étale, so the pullback of $U \subseteq S$ is schematically dense in K). The map $K \to \bar{e}$ is a quasi-compact immersion. To show that it is open, we may pick a geometric point $p \in K$ and restrict to the strict henselization of S at p. There is a unique section $S \to K$ through p, which by Lemma 9.4 is open in $\bar{e}$. Thus $K \to \bar{e}$ is open, and factors through an open immersion $K \subseteq \bar{e}^{\text{ét}}$.

The reverse inclusion is easier: the map $\bar{e}^{\text{ét}} \to \mathrm{sLPic}^{\mathbf{0}}_{X/S}$ is zero when restricted to U, so by the Néron mapping property of $\mathrm{sLPic}^{\mathbf{0}}_{X/S}$ (Theorem 6.11), the map $\bar{e}^{\text{ét}} \to \mathrm{sLPic}^{\mathbf{0}}_{X/S}$ is zero and therefore $\bar{e}^{\text{ét}} \subset K$.

(ii) Note that since S is log regular, so is X; thus, if in addition $\underline{X}$ is regular, its log structure is locally free. Since the log structure of S is isomorphic to the log structure of X away from the singular points of the fibers, the log structure on S must therefore also be locally free, and thus $\underline{S}$ must also be regular. As the Néron model $N = \mathrm{sLPic}^{\mathbf{0}}_{X/S}$ of $\mathrm{Pic}^{\mathbf{0}}_{X_U/U}$ is smooth over S, it follows that X_N is regular as well. The canonical isomorphism

$$N_U = \mathrm{Pic}^{\mathbf{0}}_{X_U/U}$$

corresponds to a line bundle L on X_{N_U}, represented by a Cartier divisor D. The scheme-theoretical closure $\overline{D}$ of D in X_N is Cartier by regularity of the latter. Thus the line bundle $\mathcal{O}(\overline{D})$ provides a lift of L under the natural morphism $\mathrm{Pic}^{\mathrm{tot}\mathbf{0}}_{X/S} \to N$, which is therefore surjective. □

Lemma 9.4. *Let S be the spectrum of a local ring, let $U \hookrightarrow S$ be open, and let $X \to S$ be a morphism such that $X_U \to U$ is an isomorphism, and X_U is schematically dense in X. Let $\sigma : S \to X$ be a section. Then σ is open.*

Proof. Writing s for the closed point of S, let $\sigma(s) \in V \hookrightarrow X$ be an affine open neighborhood; then σ factors via V (the preimage of V via σ is open and contains s). We write $\sigma' : S \to V$ for the factored map.

Since $V \to S$ is separated, the map $\sigma' : S \to V$ is a closed immersion. On the other hand, its image contains the schematically dense $V_U \hookrightarrow V$, hence σ' is an isomorphism. □

A. The functor of connected components of a smooth quasi-separated group algebraic space

Throughout this appendix, S denotes a scheme and G/S a group algebraic space with unit section $e \in G(S)$. We extend some results of Romagny [36] to the case where G/S is smooth and quasi-separated, avoiding the quasi-compactness assumptions of [36].

Lemma A.1. *Suppose that G/S is quasi-separated, flat, locally of finite presentation, and has reduced geometric fibers. Then there is a unique open subspace G_0 of G such that each fiber $G_{s,0}$ of G_0/S is the connected component of G_s containing $e(s)$. Moreover, G_0 is a subgroup of G.*

Definition A.2. We call G_0 the *fiberwise-connected component of identity in* G. The sheaf quotient G/G_0 is a group algebraic space by [5, Proposition 8.3.9], which we call the *group of connected components* of G.

Proof of Lemma A.1. The assertion that G_0 is a subgroup of G is immediate from the continuity of the multiplication and inversion operations. The first part of the statement is also immediate, from [36, Proposition 2.2.1], if in addition G/S is quasi-compact.

We will prove the general case by reduction to the quasi-compact case. We define a sub*set* $|G_0|$ of the underlying topological space $|G|$ of G to be the union of the $G_{s,0}$ as s runs over $|S|$. It suffices to show that $|G_0|$ is open in $|G|$. Given $s \in S$ and $x \in G_{0,s}$, let $U \to G$ be an étale map from an affine scheme with x and $e(s)$ in its topological image $\mathcal{W} \subset |G|$; the latter is open and we let W be the corresponding open subspace of G. Base-changing to an open neighborhood of s in S, we can assume e factors through W. Then by [36, Proposition 2.2.1] we obtain an open subspace $W_0 \to W$ through which e factors and whose fibers over S are connected components of the fibers of W.

Since any connected group scheme over a field is irreducible, for any $s \in S$ the connected component of identity $G_{s,0}$ is irreducible. Then the intersection $G_{s,0} \cap W_s$ is connected and therefore coincides with $W_{0,s}$. This shows that $|W_0| = |W| \cap |G_0|$. In particular, we have $x \in |W_0| \subset |G_0|$ with $|W_0|$ open in $|G|$. □

Lemma A.3. *Suppose that G/S is smooth and quasi-separated. Then the structure morphism $G/G_0 \to S$ is étale.*

Proof. We prove this locally at $x \in G/G_0$. Since $G/G_0 \to S$ is smooth we can choose $S' \to S$ étale and a section $S' \to G/G_0$ through x. Translating by this section, we may assume x lies in the image of the unit section $u_0 : S \to G/G_0$. It then suffices to show that $G/G_0 \to S$ is étale in an open neighborhood of the unit section, but the unit section is itself open (as the image G_0/G_0 of the open $G_0 \hookrightarrow G$). □

Lemma A.4. *Suppose that G/S is smooth and quasi-separated, and let $T \to S$ be an étale algebraic space. Then any S-morphism $G \to T$ factors uniquely via $G \to G/G_0$.*

Proof. Fix an S-morphism $f : G_0 \to T$, and write $e : S \to T$ for the map induced by the unit section of G_0. By the universal property of the quotient G/G_0, it suffices to show that the following diagram commutes:

(A.0.1)
$$\begin{array}{ccc} G_0 & \xrightarrow{f} & T. \\ \downarrow & \nearrow_{e} & \\ S & & \end{array}$$

If S is the spectrum of a separably closed field then T is a scheme (e.g., by Lemma 2.3), e is an open and closed immersion, and G_0 is connected, so the result is clear. In the general case it follows that the diagram on k-points commutes for any separably closed field k. By descent, we may replace T by an étale cover and assume it is a scheme. Since the diagonal of T is an open immersion, the equalizer of f and $G_0 \to S \to T$ is open in G. Since this open subspace contains all geometric points, it is G_0. □

Lemma A.5. *Let X/S be a log curve and $\mathcal{G} = \mathrm{sLPic}^0_{X/S}$. Then $\mathcal{G}_0 = \mathrm{Pic}^0_{X/S}$.*

Proof. The natural map $\mathcal{G}/\mathcal{G}_0 \to \mathrm{sTPic}^0_{X/S}$ coming from the universal property (Lemma A.4) induces a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}/\mathcal{G}_0 & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0_{X/S} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{sTPic}^0_{X/S} & \longrightarrow & 0. \end{array}$$

By the snake lemma, the left vertical map is injective, and its cokernel is identified with the kernel K of the right vertical map. As the latter is étale, so is K. However, $\mathrm{Pic}^0_{X/S}$ has connected geometric fibers, hence the map $\mathrm{Pic}^0_{X/S} \to K$ is constantly zero. □

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