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Hodge theorem for the logarithmic de Rham complex via derived intersections

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Abstract

In a beautiful paper, Deligne and Illusie (Invent Math 89(2):247–270, 1987) proved the degeneration of the Hodge-to-de Rham spectral sequence using positive characteristic methods. Kato (in: Igusa (ed) ALG analysis, geographic and numbers theory, Johns Hopkins University Press, Baltimore, 1989) generalized this result to logarithmic schemes. In this paper, we use the theory of twisted derived intersections developed in Arinkin et al. (Algebraic Geom 4:394–423, 2017) and the author of this paper to give a new, geometric interpretation of the Hodge theorem for the logarithmic de Rham complex.

1 Introduction

Let X be a smooth proper variety over an algebraically closed field k of characteristic 0. The algebraic de Rham complex is defined as the complex

$$\Omega^\cdot_X := 0 \to \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \dots$$

where d is the usual differential on the *algebraic* forms. The de Rham cohomology of X is defined as the hypercohomology of the de Rham complex,

$$H^*_{\mathrm{dR}}(X) = R^*\Gamma(X, \Omega^\cdot_X).$$

The Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{\mathrm{dR}}(X)$$

is given by the stupid filtration on the de Rham complex whose associated graded terms are the Ω^p_X.

In their celebrated paper [9], Deligne and Illusie proved the degeneration of the Hodge-to-de Rham spectral sequence holds in positive characteristics by showing the following result.

Theorem 1.1 [9] *Let X be a smooth proper scheme over a perfect field k of positive characteristic $p \geq \dim X$. Assume that X lifts to the ring $W_2(k)$ of second Witt vectors of k. Then the Hodge-to-de Rham spectral sequence for X degenerates at E_1.*

Then they showed that the corresponding result in characteristic 0 follows from a standard reduction argument.

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Similar results can be obtained in the logarithmic setting. Consider a smooth proper scheme X over a perfect field k of characteristic $p > \dim X$ and a reduced normal crossing divisor D on X. Around each point of the divisor, there exist local coordinates $x_1, \ldots, x_n$ so that in an étale neighborhood of that point the divisor is cut out by $x_1 \cdot \ldots \cdot x_k$ for some $k \leq n$. The logarithmic 1-forms are generated locally by the symbols

$$\frac{\mathrm{d}x_1}{x_1}, \ldots, \frac{\mathrm{d}x_k}{x_k}, \mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_n.$$

The logarithmic 1-forms form a locally free sheaf which we denote by $\Omega^1_X(\log D)$. We define the sheaf $\Omega^q_X(\log D)$ of logarithmic q-forms as the q-th wedge power of the sheaf of logarithmic 1-forms, $\wedge^q \Omega^1_X(\log D)$. The differential d of the *meromorphic* de Rham complex maps logarithmic forms to logarithmic forms, and hence, we define the logarithmic de Rham complex $\Omega^{\cdot}_X(\log D)$ as the subcomplex of the meromorphic de Rham complex consisting of logarithmic forms. The stupid filtration of this complex gives rise to a Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(Y, \Omega^p_X(\log D)) \Rightarrow R^{p+q}\Gamma(X, \Omega^{\cdot}_X(\log D)).$$

In [12], Kato generalized the result of Deligne–Illusie obtaining the following result.

Theorem 1.2 [12] *Assume that the pair (X, D) lifts to the ring $W_2(k)$. Then the Hodge-to-de Rham spectral sequence of the logarithmic de Rham complex degenerates at E_1.*

The corresponding result in characteristic 0 follows from a standard reduction argument similarly to the case of the theorem of Deligne and Illusie. (For a detailed treatment, and for applications to vanishing theorems see [10].)

In a recent paper [3], Arinkin, Căldăraru and the author of the present paper recast the problem of the degeneration of the Hodge-to-de Rham spectral sequence as a derived self-intersection problem. They considered the embedding of the Frobenius twist X' (see Sect. 3 for the definition) of X into its cotangent bundle T^*X' as the zero section. The sheaf of crystalline differential operators $\mathcal{D}_X$ on X can be regarded as a sheaf $\mathcal{D}$ of Azumaya algebras on T^*X'. This Azumaya algebra splits on the zero section $X' \to T^*X'$ giving rise to an embedding of Azumaya spaces $X' \to (T^*X', \mathcal{D})$[1]. We follow [3] and call this intersection problem as a twisted intersection problem. The main geometric observation in [3] is that there exists a line bundle on the ordinary derived self-intersection of Y inside S measuring the difference between the twisted and the ordinary derived self-intersections. In the case of the embedding $X' \to (T^*X', \mathcal{D})$, this line bundle is given by the dual of $F_*\Omega^{\cdot}_X$, the Frobenius pushforward of the de Rham complex of X.

Using the theory of twisted derived self-intersections, the authors of [3] obtain the following result giving a geometric interpretation of the result of Deligne–Illusie (see also [15] for the relation between the statements (1) and (2)).

Theorem 1.3 [3] *Let X be a smooth scheme over a perfect field k of characteristic $p >$ $\dim X$. Then the following statements are equivalent.*

1. *X lifts to $W_2(k)$.*
2. *D splits on the first infinitesimal neighborhood of X' in T^*X'.*
3. *The associated line bundle is trivial.*

[1] We emphasize that $\mathcal{D}$ is a sheaf of algebras on X, not a divisor.

4. *$F_*\Omega^\cdot_X$ is formal in $D(X')$ (meaning that it is quasi-isomorphic to the direct sum of its cohomology sheaves).*

In the present paper, we generalize the result of [3] to the logarithmic setting. We consider the logarithmic scheme (X, D) where X is a smooth proper scheme over a perfect field k of characteristic $p > \dim X$ and D is a reduced normal crossing divisor on X. As the category of quasi-coherent sheaves on (X, D), we consider the quasi-coherent parabolic sheaves on (X, D) defined by Yokogawa (see [21]).

These sheaves are generalizations of parabolic sheaves introduced by Mehta and Seshadri [14,19], Mehta and Seshadri define parabolic sheaves on a projective curve with finitely many marked points as locally free sheaves E with filtrations

$$0 = F_k(E) \hookrightarrow F_{k-1}(E) \hookrightarrow \cdots \hookrightarrow F_0(E) = E$$

at every marked point. They use these sheaves to generalize the correspondence between stable sheaves of rank 0 and irreducible unitary representations of the topological fundamental group to non-projective curves. One major advantage of Yokogawa's version of parabolic sheaves is that those sheaves form an Abelian category.

The sheaf of crystalline logarithmic differential operators $\mathcal{D}_X(\log D)$—defined as the universal enveloping Lie algebroid of the logarithmic tangent bundle—is not an Azumaya algebra over its center. On the other hand, one can equip $\mathcal{D}_X(\log D)$ with a filtration so that the corresponding parabolic sheaf becomes a parabolic Azumaya algebra over its center. (For a treatment of parabolic Azumaya algebras see [13].) The center of the parabolic Azumaya algebra can be identified with the structure sheaf of the logarithmic cotangent bundle $T^*X'(\log D')$ of the Frobenius twist (X', D') of (X, D) equipped with the divisor π^*D' where $\pi : T^*X'(\log D') \to X'$ is the bundle map. Therefore the parabolic sheaf of crystalline logarithmic differential operators can be regarded as a parabolic sheaf of algebras $\mathcal{D}(\log D)_*$ over $(T^*X'(\log D'), \pi^*D')$. Moreover, the parabolic Azumaya algebra $\mathcal{D}(\log D)_*$ splits on the zero section $(X', D') \to (T^*X'(\log D'), \pi^*D')$. We consider the two embeddings

- $(X', D') \to (T^*X'(\log D'), \pi^*D')$ and
- $(X', D') \to (T^*X'(\log D'), \pi^*D', \mathcal{D}(\log D)_*)$.

where the second embedding is an embedding of logarithmic Azumaya schemes. (For more details, see Sect. 5.)

The difference between the derived self-intersections of the embeddings is measured by a parabolic line bundle, which is the dual of the Frobenius pushforward of the logarithmic de Rham complex $F_*\Omega^\cdot_X(\log D)$ equipped with the filtration given by D.

We generalize the theory of twisted derived self-intersections to the logarithmic setting. Our main result is a generalization of Theorem 1.3.

Theorem 1.4 *Let X be a smooth variety over a perfect field of characteristic $p > \dim X$, with a reduced normal crossing divisor D. Then, the following statements are equivalent.*

1. *The logarithmic scheme (X, D) lifts to $W_2(k)$.*
2. *The associated line bundle is trivial.*
3. *The parabolic sheaf of algebras $\mathcal{D}(\log D)_*$ splits on the first infinitesimal neighborhood of (X', D') inside $(T^*X'(\log D'), \pi^*D')$.*

4. *The complex $F_*\Omega^{\cdot}_X(\log D)_*$ is quasi-isomorphic to a formal parabolic sheaf equipped with the trivial parabolic structure.*

As an easy corollary, we obtain Kato's theorem, Theorem 1.2.

We remark that there is another approach in the literature to deal with the sheaf of differential operators in the logarithmic setting. In [18], Schepler extends the theory of Ogus and Vologodsky [15] to the case of logarithmic schemes. He uses the theory of indexed modules and algebras, and he shows the sheaf of differential operators form an indexed Azumaya algebra. These results are generalized by Ohkawa [16,17] to the ring of differential operators for higher level. We choose not to work with indexed modules and algebras because of lack of functoriality. It would be interesting to compare the results of [16–18] with our approach.

The paper is organized as follows. In Sect. 2, we collect some basic facts about logarithmic schemes and parabolic sheaves. In Sect. 3, we introduce the parabolic sheaf of crystalline differential operators and the parabolic logarithmic de Rham complex. We show the Azumaya property of the parabolic sheaf of crystalline differential operators, and we show that there exists a Koszul duality between the parabolic sheaf of crystalline differential operators and the parabolic logarithmic de Rham complex. In Sect. 4, we summarize the theory of twisted derived intersections. We expand the theory to the case of logarithmic schemes in Sect. 5. We conclude the paper with Sect. 6 where we prove our main theorems Theorem 1.4 and Theorem 1.2.

2 Background on logarithmic schemes

In this section, we introduce the notion of the category of (quasi-)coherent sheaves on logarithmic schemes (X, D). We show how to apply pushforward and pullback functors to the objects, and we also provide some details about the monoidal structure of this category. We remark that we could have taken an alternative path, looking at the category of (quasi-)coherent sheaves on the infinite root stack (see [4,20]). In our case, the two categories are equivalent and we decided to follow the paper [21].

We regard $\mathbf{R}$ as the category whose objects are real numbers and whose morphism spaces between two objects $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}$ are defined as

$$Mor(\alpha, \beta) = \begin{cases} \{i^{\alpha,\beta}\} & \text{if } \alpha \geq \beta, \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, if $\alpha \geq \beta$, then there exists a unique morphism from α to β; otherwise, the morphism space is empty.

Definition 2.1 An $\mathbf{R}$-filtered $\mathscr{O}_X$-module is a covariant functor from the category $\mathbf{R}$ to the category of $\mathscr{O}_X$-modules. For an $\mathbf{R}$-filtered $\mathscr{O}_X$-module E we denote the $\mathscr{O}_X$-module $E(\alpha)$ by E_α, and the $\mathscr{O}_X$-linear homomorphisms $E(i^{\alpha,\beta})$ by $i_E^{\alpha,\beta}$.

Remark 2.2 The morphisms $i_E^{\alpha,\beta}$ are not necessarily injections.

We can shift $\mathbf{R}$-filtered sheaves.

Definition 2.3 For an $\mathbf{R}$-filtered $\mathscr{O}_X$-module E, we define the $\mathbf{R}$-filtered $\mathscr{O}_X$-module $E[\alpha]$ as $E[\alpha]_\beta = E_{\alpha+\beta}$ with morphisms $i_{E[\alpha]}^{\beta,\gamma} = i_E^{\beta+\alpha,\gamma+\alpha}$. In the sequel, we denote $\mathbf{R}$-filtered $\mathscr{O}_X$-modules by E_*.

Moreover, the $\mathbf{R}$-filtered sheaves can be equipped with a module structure over the monoid of $\mathcal{O}_X$-modules:

Definition 2.4 For an $\mathbf{R}$-filtered $\mathcal{O}_X$-module E_* and for an "ordinary" $\mathcal{O}_X$-module F, we define their tensor product as $(E_* \otimes F)_\alpha = E_\alpha \otimes F$ with homomorphisms $i^{\alpha,\beta}_{E_*\otimes F} = i^{\alpha,\beta}_{E_*} \otimes \mathrm{id}_F$.

We are ready to define the category of (quasi-)coherent parabolic sheaves with respect to an effective Cartier divisor D.

Definition 2.5 A (quasi-)coherent *parabolic* sheaf is an $\mathbf{R}$-filtered $\mathcal{O}_X$-module E_* together with an isomorphism of $\mathbf{R}$-filtered $\mathcal{O}_X$-modules

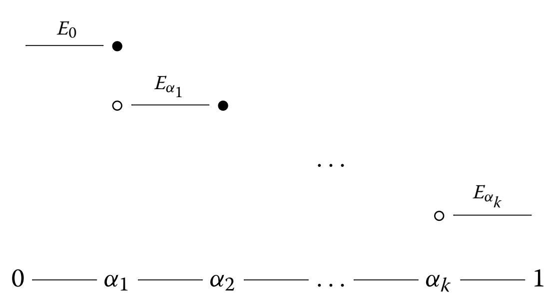
$$E_* \otimes \mathcal{O}_X(-D) \cong E_*[1].$$

Parabolic morphisms between parabolic sheaves E_* and E'_* are natural transformations $E_* \to E'_*$. We denote the Abelian group of parabolic morphisms by $Hom(E_*, E'_*)$.

Remark 2.6 It is enough to know E_α and $i^{\alpha,\beta}_E$ for $\alpha, \beta \in [0, 1]$ in order to determine the parabolic sheaf E_*. We say that a parabolic sheaf has weights $\alpha = (\alpha_1, \ldots \alpha_k)$

$$0 \le \alpha_1 < \alpha_2 \ldots < \alpha_k < 1$$

if $E_\beta = E\gamma$ and $i^{\gamma,\beta}_E = \mathrm{id}$ for all $\beta, \gamma \in [0, 1]$ satisfying $\alpha_i < \beta, \gamma \le \alpha_{i+1}$.
Visually a parabolic sheaf can be thought of as a finite collection of sheaves $E_0, E_{\alpha_1}, \ldots, E_{\alpha_k}$ on the interval $[0, 1]$.



Now, we can define the category of (quasi-)coherent sheaves on (X, D).

Definition 2.7 The category of (quasi-)coherent sheaves on (X, D, α) is the category whose objects are parabolic sheaves with weights α, and the Hom-spaces are the sets of parabolic morphisms. These categories will be denoted by $Coh(X, D, \alpha)$ or $QCoh(X, D, \alpha)$ respectively.

Throughout the paper, we only use two kinds of weights. Either the parabolic sheaves have trivial parabolic structure meaning that $\alpha = (0)$ or the parabolic sheaves have weights $\alpha = \left(0, \frac{1}{p}, \frac{2}{p}, \ldots \frac{p-1}{p}\right)$. The choice of the weights will be clear from the context, and hence, by abuse of notation we write $Coh(X, D)$ (or $QCoh(X, D)$) instead of $Coh(X, D, \alpha)$ (or $QCoh(X, D, \alpha)$).
The category $QCoh(X, D)$ is Abelian, the kernel and cokernel of a morphism can be defined pointwise. It has enough injectives (see [21]); we denote the corresponding derived category by $\mathbf{D}(X, D)$.

2.1 Pushforward and pullback functors

The pushforward and pullback functors can be quite complicated for a general map of logarithmic schemes $(X, D_1) \to (Y, D_2)$. On the other hand, these functors are simple in the following very special case.

Consider a morphism $f : X \to Y$ and an effective Cartier divisor D on Y such that f^*D is an effective divisor on X. These data give rise to a morphism of logarithmic schemes $(X, f^*D) \to (Y, D)$. We abuse notation and denote the induced map by f as well. We define the pushforward and pullback along f as follows. For any parabolic sheaf E_* on X its pushforward f_*E_* is defined as the parabolic sheaf where $(f_*E)_\alpha$ are the pushforward of the sheaves E_α along f and the morphisms $i^{\alpha,\beta}_{f_*E}$ are the morphisms $f_*i^{\alpha,\beta}_E$. Indeed, we obtain a parabolic sheaf on (Y, D), by the projection formula, we have

$$f_*(E_*[1]) = f_*(E_* \otimes \mathscr{O}_X(-f^*D)) = f_*E_* \otimes \mathscr{O}_Y(-D) = f_*E_*[1].$$

Similarly, for any parabolic sheaf E'_* on Y its pullback $f^*E'_*$, is defined as the parabolic sheaf where $(f^*E')_\alpha$ is the pullback of E'_α along f and the morphisms $i^{\alpha,\beta}_{f^*E'}$ are the morphisms $f^*i^{\alpha,\beta}_{E'}$. Again, we obtain a parabolic sheaf on (X, f^*D), we have

$$f^*(E'_*[1]) = f^*(E'_* \otimes \mathscr{O}_Y(-D)) = f^*E'_* \otimes \mathscr{O}_X(-f^*D) = f^*E'_*[1].$$

The pushforward and pullback functors descend to the derived categories and by abuse of notation, we denote the corresponding maps by f_* and f^* as well.

We give an example of a more complicated pushforward. Consider the morphism $(X, D) \to (X, pD)$ which is the identity on X and the embedding of the divisor D into pD. The pushforward along this $(X, D) \to (X, pD)$ brings a parabolic sheaf E_* with weights $\alpha = (0)$ generated by

$$E \otimes \mathscr{O}(-D) \to E$$

to a parabolic sheaf generated by

$$E \otimes \mathscr{O}(-pD) \hookrightarrow E \otimes \mathscr{O}(-(p-1)D) \hookrightarrow \cdots \hookrightarrow E$$

with weights

$$\left(0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}\right).$$

2.2 Monoidal structure

In our cases (when $\alpha = (0)$ and $\alpha = (0, \frac{1}{p}, \ldots, \frac{p-1}{p})$) the categories $Coh(X, D)$ and $QCoh(X, D)$ are equipped with natural monoidal structures. We describe the monoidal structure in the case of $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$, the other case is very similar. In order to describe the monoidal structure, we take a quick detour.

An important subcategory of $QCoh(X, D)$ is the category of parabolic bundles (see [14, 19]).

Definition 2.8 A *parabolic bundle* is a triple (E, F_*, α_*) where E is a locally free sheaf on X, F_* is a filtration of E by locally free sheaves on X

$$F_k(E) = E \otimes \mathscr{O}_X(-D) \hookrightarrow F_{k-1}(E) \hookrightarrow F_{k-2}(E) \hookrightarrow \cdots \hookrightarrow F_0(E) = E$$

together with a sequence of weights α satisfying

$$0 \le \alpha_1 < \cdots < \alpha_k < 1.$$

The sequence of weights determines a family of coherent sheaves E_x for $0 \leq x \leq 1$ defined as

$$E_0 = E \quad \text{and} \quad E_x = F_i(E)$$

for $\alpha_i < x \leq \alpha_{i+1}$. A morphism between parabolic bundles (E, F_*, α_*) and (E', F'_*, α'_*) is a morphism of $\mathscr{O}_X$-modules $\varphi : E \to E'$ so that $\varphi(E_x) \subseteq E'_x$ for any $x \in [0, 1]$. By Remark 2.6, parabolic bundles give rise to parabolic sheaves and morphisms between parabolic bundles are exactly the parabolic morphisms between the corresponding parabolic sheaves.

Consider $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$ and the morphism

$$\psi : \operatorname{Pic} X \times \mathbf{Z}\left[\frac{1}{p}\right] \to Coh(X, D)$$

mapping the pair (L, a) to the parabolic bundle (L, F_*, α_*) where

$$L_x = \begin{cases} L & \text{if } x \leq a', \\ L \otimes \mathscr{O}_X(-D) & \text{if } a' < x \leq 1. \end{cases}$$

Here a' denotes the residue of a modulo 1. Parabolic bundles of this form are called the parabolic line bundles. The tensor product of parabolic sheaves is defined (for parabolic line bundles) to respect the group structure coming from the natural group structure on $\operatorname{Pic}(X) \times \mathbf{Z}\left[\frac{1}{p}\right]$ given by

$$(L_1, a_1) \cdot (L_2, a_2) = (L_1 \otimes L_2, a_1 + a_2).$$

The unit element of the tensor product is given by the parabolic sheaf $\psi(\mathscr{O}_X, 0)$, where

$$\psi(\mathscr{O}_X, 0)_x = \begin{cases} \mathscr{O}_X & \text{if } x = 0, \\ \mathscr{O}_X(-D) & \text{if } 0 < x \leq 1. \end{cases}$$

We define the *structure sheaf* of the logarithmic scheme (X, D) to be the parabolic sheaf $\psi(\mathscr{O}_X, 0)$ and in the sequel we denote it by $\mathscr{O}_{(X,D)}$.

Definition 2.9 For two parabolic sheaves E_*, F_*, we define the sheaf Hom as

$$\underline{\mathsf{Hom}}_x(E_*, F_*) := \mathsf{Hom}(E_*, F_*[x]).$$

Remark 2.10 If the parabolic sheaves E_* and F_* have weights $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$ (or $\alpha = (0)$), then the parabolic sheaf $\underline{\mathsf{Hom}}_*(E_*, F_*)$ has weights $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$ (or $\alpha = (0)$) as well.

For any parabolic line bundle L, its parabolic sheaf of endomorphisms is isomorphic to $\mathscr{O}_{(X,D)}$. The sheaf Hom functor and the tensor product satisfy the usual adjoint property (for more details, see [21]) giving rise to natural monoidal structures on $Coh(X, D)$ and $QCoh(X, D)$. We remark that the monoidal structures descend to the derived category $\mathbf{D}(X, D)$.

3 Background on schemes over fields of positive characteristics

In this section, we collect basic facts about schemes over fields of positive characteristics. We review the notion of logarithmic tangent sheaf, logarithmic q-forms and the crystalline sheaf of logarithmic differential operators. We show that the parabolic sheaf of crystalline logarithmic differential operators is an Azumaya algebra over its center. We conclude the section by showing that there exists a Koszul duality between the parabolic sheaf of crystalline logarithmic differential operators and the parabolic logarithmic de Rham complex.

We begin with the definition of the Frobenius twist.

Let X be a smooth scheme over a perfect field k of characteristic p. The absolute Frobenius map $\varphi : \operatorname{Spec} k \to \operatorname{Spec} k$ is given by the p-th power map $k \to k$. The Frobenius twist of X is defined as the base change of X along the absolute Frobenius morphism

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\varphi} & \operatorname{Spec} k. \end{array}$$

Since k is assumed to be perfect, the p-th power map $k \to k$ is an isomorphism; hence, X' is abstractly isomorphic to X (but not over $\operatorname{Spec} k$). The p-th power map $\mathscr{O}_X \to \mathscr{O}_X$ gives rise to a morphism $X \to X$ compatible with φ; thus, by the universal property of fiber product it factors through the Frobenius twist. The induced morphism $F : X \to X'$ is called the relative Frobenius morphism. For any effective Cartier divisor, D on X with ideal sheaf $\mathscr{I} = \mathscr{O}_X(-D)$; we obtain a corresponding Cartier divisor D', which is the pullback of D under the base change morphism $X' \to X$ and whose pullback F^*D' under the relative Frobenius morphism is the divisor pD. Thus, we have the following sequence of maps of logarithmic schemes

$$(X, D) \to (X, pD) \to (X', D') \to (X, D).$$

The morphism $(X, D) \to (X', D')$ is called the relative Frobenius morphism of logarithmic schemes; by abuse of notation, we denote it by F as well.

In this paragraph, we explain why we only consider the category of (quasi-)coherent sheaves on a logarithmic scheme as the category whose objects are parabolic sheaves with weights $\alpha = \left(0, \frac{1}{p}, \dots, \frac{p-1}{p}\right)$ and weights $\alpha = (0)$, and the Hom-spaces are the sets of parabolic morphisms (see Definition 2.7). Consider the category of parabolic sheaves on (X, D) with trivial weights $\alpha = (0)$. In other words, the objects of this category are just sheaves E on X equipped with the trivial parabolic structure

$$E \otimes \mathscr{O}(-D) \hookrightarrow E.$$

Consider the pushforward of such sheaves along the sequence of morphisms

$$(X, D) \to (X, pD) \to (X', D').$$

The pushforward along the first map $(X, D) \to (X, pD)$ brings E to the parabolic sheaf

$$E \otimes \mathscr{O}(-pD) \hookrightarrow E \otimes \mathscr{O}(-(p-1)D) \hookrightarrow \cdots \hookrightarrow E$$

with weights $\alpha = \left(0, \frac{1}{p}, \dots, \frac{p-1}{p}\right)$ (see Paragraph 2.11.2). The second map $(X, pD) \to (X', D')$ brings the above parabolic sheaf to the parabolic sheaf

$$F_*E \otimes \mathscr{O}(-D') \hookrightarrow F_*(E \otimes \mathscr{O}(-(p-1)D)) \hookrightarrow \cdots \hookrightarrow F_*E$$

with weights $\alpha = \left(0, \frac{1}{p}, \dots, \frac{p-1}{p}\right)$.

3.1 Logarithmic tangent and cotangent bundles

We say that a derivation $\delta \in T_X$ is *logarithmic* if for every open subset U we have $\delta(\mathscr{I}(U)) \subset \mathscr{I}(U)$. The logarithmic derivations form a subsheaf of the tangent bundle of

X which is called the logarithmic tangent sheaf, $T_X(\log D)$. The sheaf $T_X(\log D)$ is a Lie subalgebroid of T_X meaning that it is closed under the Lie-bracket on T_X. In characteristic $p > 0$, the p-th iteration $\delta^{[p]}$ of a derivation δ is again a derivation. Clearly, the sheaf $T_X(\log D)$ is closed under this operation as well; hence, $T_X(\log D)$ is a sub-p-restricted Lie algebroid of T_X.

In general $T_X(\log D)$ is not a subbundle of T_X, we say that a divisor is *free* if $T_X(\log D)$ is a locally free sheaf. For instance, reduced normal crossing divisors are free divisors. Indeed, we can find local coordinates $x_1, \dots, x_n$ around each point of the divisor so that in an étale neighborhood of that point the divisor is given by $x_1 \cdot \dots \cdot x_k$ for some $k \leq n$, and thus, the logarithmic tangent sheaf is generated by the logarithmic derivations $x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}, \dots, x_k \frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n}$.

Remark 3.1 It is easy to see that the logarithmic derivations are just derivations of the parabolic bundle $\mathcal{O}_{(X,D)}$.

We say that a meromorphic q-form ω is logarithmic, if ω and $d(\omega)$ have a pole of order at most one along the divisor. The logarithmic q-forms form a subsheaf $\Omega_X^q(\log D)$ of the sheaf of *meromorphic* q-forms. The differential d of the meromorphic de Rham complex maps logarithmic forms to logarithmic forms. We define the logarithmic de Rham complex $\Omega_X^{\cdot}(\log D)$ as the subcomplex of the de Rham complex consisting of the logarithmic forms. This complex $\Omega_X^{\cdot}(\log D)$ is not a complex of $\mathcal{O}_X$-modules, the differential d is not linear in $\mathcal{O}_X$. On the other hand, we have

$$d(s^p\omega) = ps^{p-1}\mathrm{d}s \wedge \omega + s^p d\omega = s^p d\omega$$

for every $s \in \mathcal{O}_X$ and $\omega \in \Omega_X^q(\log D)$. This implies that $F_*\Omega_X^{\cdot}(\log D)$ is a complex of $\mathcal{O}_{X'}$-modules.

If the divisor is free, then the sheaves $\Omega_X^q(\log D)$ are locally free sheaves, and moreover, we have that the sheaf of logarithmic q-forms is the q-th wedge power of the sheaf of logarithmic 1-forms: $\Omega_X^q(\log D) = \wedge^q \Omega_X^1(\log D)$. In the case of a reduced normal crossing divisor, locally the logarithmic 1-forms are generated by

$$\mathrm{d}(\log x_1) = \frac{\mathrm{d}x_1}{x_1}, \dots, \mathrm{d}(\log x_k) = \frac{\mathrm{d}x_k}{x_k}, \mathrm{d}x_{k+1}, \dots, \mathrm{d}x_n.$$

Similarly to the non-logarithmic case, there is a perfect duality between $T_X(\log D)$ and $\Omega_X^1(\log D)$ given by contracting with polyvector fields.

In the sequel, D denotes a reduced normal crossing divisor.

The sheaf of crystalline logarithmic differential operators $\mathcal{D}_X(\log D)$ is defined as the universal enveloping algebra of the Lie algebroid $T_X(\log D)$. Explicitly it is defined locally as the k-algebra generated by sections of $T_X(\log D)$ and $\mathcal{O}_X$ modulo the relations

- $s \cdot \delta = s\delta$ for every $s \in \mathcal{O}_X$ and $\delta \in T_X(\log D)$,
- $\delta_1 \cdot \delta_2 - \delta_2 \cdot \delta_1 = [\delta_1, \delta_2]$ for every $\delta_1, \delta_2 \in T_X(\log D)$ and
- $\delta \cdot s - s \cdot \delta = \delta(s)$ for every $s \in \mathcal{O}_X$ and $\delta \in T_X(\log D)$.

We emphasize that we do not work with the sheaf of PD differential operators, for our purposes we need an algebra which is of finite type over X. Since $T_X(\log D)$ is a Lie subalgebroid of T_X, we have an inclusion of $\mathcal{D}_X(\log D)$ into the sheaf of crystalline differential operators $\mathcal{D}_X$ (defined as the universal enveloping algebra of the Lie algebroid T_X).

The map

$$\psi : T_X(\log D) \to \mathcal{D}_X(\log D)$$

mapping

$$\delta \mapsto \delta^p - \delta^{[p]}$$

is $\mathscr{O}_{X'}$-linear and its image is in the center of $\mathcal{D}_X(\log D)$ (see [5] for a detailed treatment in the non-logarithmic case) implying that the center of $\mathcal{D}_X(\log D)$ can be identified with the structure sheaf $\mathscr{O}_{T^*X'(\log D')}$ of the logarithmic cotangent bundle $\pi : T^*X'(\log D') \to X'$ over the Frobenius twist. The zero section $i : X' \to T^*X'(\log D')$ of the bundle map π gives rise to a natural embedding of logarithmic schemes

$$i_D : (X', D') \to (T^*X'(\log D'), \pi^*D'),$$

since $i^*\pi^*D' = D'$.
We equip the sheaf of algebras $\mathcal{D}_X(\log D)$ with the trivial logarithmic structure on (X, D) (in other words $\alpha = (0)$), we denote the corresponding logarithmic sheaf by $\mathcal{D}_X(\log D)_*$:

$$\mathcal{D}_X(\log D)_x = \begin{cases} \mathcal{D}_X(\log D) & \text{if } x = 0, \\ \mathcal{D}_X(\log D) \otimes \mathscr{O}_X(-D) & \text{if } 0 < x \leq 1. \end{cases}$$

After pushing forward the parabolic sheaf $\mathcal{D}_X(\log D)_*$ along the relative Frobenius morphism $F : (X, D) \to (X', D')$, we obtain the parabolic sheaf $F_*\mathcal{D}_X(\log D)_*$, whose filtration is given by

$$F_*\mathcal{D}_X(\log D) \otimes \mathscr{O}_{X'}(-D') = F_*(\mathcal{D}_X(\log D) \otimes \mathscr{O}_X(-pD)) \hookrightarrow \cdots \hookrightarrow F_*(\mathcal{D}_X(\log D)).$$

This parabolic sheaf has weights $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$.
By the discussion in Paragraph 3.4, we can regard the parabolic sheaf of algebras $F_*\mathcal{D}_X(\log D)_*$ as a parabolic sheaf of algebras on $(T^*X'(\log D'), \pi^*D')$. We denote the corresponding parabolic sheaf of algebras by $\mathcal{D}(\log D)_*$. The bundle map π identifies $\pi_*\mathcal{D}(\log D)_*$ with $F_*\mathcal{D}_X(\log D)_*$.
The following lemma is a straightforward generalization of [5] to the case of logarithmic schemes.

Lemma 3.2 *Assume that D is a reduced normal crossing divisor. Then, the parabolic sheaf of algebras $\mathcal{D}(\log D)_*$ is a parabolic Azumaya algebra over the logarithmic space $(T^*X'(\log D'), \pi^*D')$. Moreover, the parabolic Azumaya algebra $\mathcal{D}(\log D)$ splits on the zero section*

$$(X', D') \to (T^*X'(\log D'), \pi^*D').$$

Proof We only highlight the key steps. We need to show that $\mathcal{D}(\log D)_*$ becomes a parabolic matrix algebra under a flat cover. Consider an affine open set U' of X', and the corresponding open set $U \subseteq X$. Pick local coordinates $x_1, \ldots, x_n$ for U so that the reduced normal crossing divisor, D is given by the equation $x_1 \cdot \ldots \cdot x_k$. Then, the (non-parabolic) sheaf of algebras $\mathcal{D}_X(\log D)$ is generated by $\Gamma(U, \mathscr{O}_X)$ and the derivations $x_i \frac{\mathrm{d}}{\mathrm{d}x_i}$ for $1 \leq i \leq k$, and the derivations $\frac{\mathrm{d}}{\mathrm{d}x_i}$ for $k < i \leq n$. Consider $F_*\mathcal{D}_X(\log D)$ and the centralizer A_X of $\Gamma(U', \mathscr{O}_{X'})$ inside $F_*\mathcal{D}_X(\log D)$. A straightforward calculation shows that the centralizer is the $R = \Gamma(U', \mathscr{O}_{X'})$-algebra generated by the logarithmic derivations δ

of X, in other words

$$A_X = R\left[x_1\frac{\mathrm{d}}{\mathrm{d}x_1},\ldots,x_k\frac{\mathrm{d}}{\mathrm{d}x_k},\frac{\mathrm{d}}{\mathrm{d}x_{k+1}},\ldots,\frac{\mathrm{d}}{\mathrm{d}x_n}\right]$$

There is a natural logarithmic structure on $V = \operatorname{Spec} A_X$ given by the restriction of the divisor π^*D' to V. The ideal sheaf corresponding to the divisor is generated by the element $x_1^p \cdot \ldots \cdot x_k^p$. We denote by $\mathscr{I}$ the ideal sheaf of $\mathscr{O}_X$ generated by the element $x_1 \cdot \ldots \cdot x_k$. The sheaf

$$\mathcal{D}(\log D)|_V := \mathcal{D}(\log D) \otimes_{\mathscr{O}_{T^*X'(\log D')(U)}} A_X$$

is generated by $\Gamma(U', \mathscr{O}_{X'})$ and two copies $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ corresponding to the logarithmic derivations

$$x_1\frac{\mathrm{d}}{\mathrm{d}x_1},\ldots,x_k\frac{\mathrm{d}}{\mathrm{d}x_k},\frac{\mathrm{d}}{\mathrm{d}x_{k+1}},\ldots,\frac{\mathrm{d}}{\mathrm{d}x_n}.$$

The action $u_i.x = u_ix$ and $v_i.x = xv_i$ for $x \in \mathcal{D}(\log D)$ gives rise to an action of $\mathcal{D}(\log D)|_V$ on the parabolic sheaf

$$\mathcal{D} := \mathcal{D}(\log D) \otimes \mathscr{I}^p \hookrightarrow \mathcal{D}(\log D) \otimes \mathscr{I}^{p-1} \hookrightarrow \cdots \hookrightarrow \mathcal{D}(\log D)$$

viewed as a parabolic sheaf over $\operatorname{Spec} A_X$. A local calculation similar to one in [5] shows that we have an isomorphism

$$\mathcal{D}(\log D)_x|_V = \underline{\mathrm{End}}_{(V,\pi^*D'|_V)}(\mathcal{D})_x$$

hence $\mathcal{D}(\log D)_*$ is an Azumaya algebra over $(T^*X'(\log D'), \pi^*D')$.
Next we show that the parabolic Azumaya algebra $\mathcal{D}(\log D)_*|_{X'}$ splits. More explicitly, we show that $\mathcal{D}(\log D)_*|_{X'} = \underline{\mathrm{End}}_{(X',D')}(F_*\mathscr{O}_{(X,D)})$. We remind the reader that the parabolic sheaf $F_*\mathscr{O}_{(X,D)}$ is defined as the parabolic sheaf given by the filtration

$$F_*\mathscr{O}_X \otimes \mathscr{O}_{X'}(-D') = F_*\mathscr{O}_X(-pD) \hookrightarrow \cdots \hookrightarrow F_*\mathscr{O}_X(-D) \hookrightarrow F_*\mathscr{O}_X.$$

The sheaf of crystalline logarithmic differential operators acts non-trivially on $\mathscr{O}_X(-D)$, $\mathscr{O}_X(-2D)$, ..., $\mathscr{O}_X(-(p-1)D)$ and moreover the elements which act trivially are generated by the symbols $\delta^p - \delta^{[p]}$ for $\delta \in T_X(\log D)$. These are exactly the elements which vanish under the pullback map $\mathscr{O}_{T^*X'(\log D')} \to \mathscr{O}_{X'}$, hence $\mathcal{D}(\log D)_*|_{X'}$ acts on $F_*\mathscr{O}_{(X,D)}$. As before, a local calculation similar to one in [5] shows that we have an isomorphism

$$\mathcal{D}(\log D)_*|_{X'} = \underline{\mathrm{End}}_{(X',D')}(F_*\mathscr{O}_{(X,D)}).$$

This concludes the proof. □

Remark 3.3 If D is a non-reduced normal crossing divisor, then the lemma above does not hold anymore. It would be interesting to generalize the above lemma to any normal crossing divisor. For instance, in the case of $p = 5$, $X = \operatorname{Spec} k[x]$, $\mathscr{I} = (x^3)$, the parabolic sheaf of algebras $\mathcal{D}(\log D)_*$ is not a parabolic Azumaya algebra over the logarithmic space $(T^*X'(\log D'), \pi^*D')$.
As a consequence of Lemma 3.2, we obtain a Morita equivalence between the category of coherent sheaves $Coh(X', D')$ on (X', D') and the category of coherent sheaves $Coh(\mathcal{D}(\log D)_*|_{(X',D')})$ on (X', D') with a left $\mathcal{D}(\log D)_*|_{X'}$-action: the functors

$$m_* : Coh(X', D') \to Coh(\mathcal{D}(\log D)_*|_{(X',D')}) :$$
$$E_* \mapsto E_* \otimes F_*\mathscr{O}_{(X,D)}$$

and

$$m^* : Coh(\mathcal{D}(\log D)_*|_{(X',D')}) \to Coh(X', D') :$$
$$E_* \mapsto \underline{\mathsf{Hom}}_{\mathcal{D}(\log D)_*|_{X'}}(F_*\mathcal{O}_{(X,D)}, E_*)$$

are inverses to each other. These functors give rise to an equivalence between the corresponding derived categories $\mathbf{D}(X', D')$ and $\mathbf{D}(X', D', \mathcal{D}(\log D)_*|_{(X',D')})$.

We conclude this section by showing that the sheaf of crystalline logarithmic differential operators and the logarithmic de Rham complex are Koszul dual (our reference is [7]). The sheaf $\mathcal{O}_X$ is naturally a left $\mathcal{D}_X(\log D) \subset \mathcal{D}_X$-module given by the action of the logarithmic derivations on $\mathcal{O}_X$. The logarithmic Spencer complex $Sp^\cdot(\mathcal{O}_X)$ defined as the complex of left $\mathcal{D}_X(\log D)$-modules

$$0 \to \mathcal{D}_X(\log D) \otimes \wedge^n T_X(\log D) \to \cdots \to \mathcal{D}_X(\log D) \otimes T_X(\log D) \to \mathcal{D}_X(\log D)$$

where the differentials

$$d_{Sp} : \mathcal{D}_X(\log D) \otimes \wedge^i T_X(\log D) \to \mathcal{D}_X(\log D) \otimes \wedge^{i-1} T_X(\log D)$$

are given by

$$\begin{aligned} d_{Sp}(T \otimes \delta_1 \wedge \cdots \wedge \delta_i) = & \sum_{l=1}^{i} (-1)^{i-1} T\delta_l \otimes \delta_1 \wedge \cdots \wedge \hat{\delta_l} \wedge \cdots \wedge \delta_i \\ & + \sum_{l<k=1}^{i} (-1)^{l+k} T \otimes [\delta_l, \delta_k] \wedge \delta_1 \wedge \cdots \wedge \hat{\delta_l} \wedge \cdots \wedge \hat{\delta_k} \wedge \cdots \wedge \delta_i \end{aligned}$$

is a locally free resolution of $\mathcal{O}_X$ by locally free left $\mathcal{D}_X(\log D)$-modules. As a consequence, for any left $\mathcal{D}_X(\log D)$ module F we have that the object $\mathbf{R}\underline{\mathsf{Hom}}_{(X,\mathcal{D}_X(\log D))}(\mathcal{O}_X, F)$ can be represented by the logarithmic de Rham complex of F

$$0 \to F \to F \otimes \Omega^1_X(\log D) \to \cdots \to F \otimes \Omega^n_X(\log D).$$

In the case of $F = \mathcal{O}_X$, we obtain an isomorphism

$$\Omega^\cdot_X(\log D) = \mathbf{R}\underline{\mathsf{Hom}}_{(X,\mathcal{D}_X(\log D))}(\mathcal{O}_X, \mathcal{O}_X).$$

Similarly, for any left $\mathcal{D}_X(\log D)$-module E we obtain a complex $Sp^\cdot(E)$ defined as

$$0 \to \mathcal{D}_X(\log D) \otimes \wedge^n T_X(\log D) \otimes E \to \cdots \to \mathcal{D}_X(\log D) \otimes E$$

where the differentials

$$d : \mathcal{D}_X(\log D) \otimes \wedge^i T_X(\log D) \otimes E \to \mathcal{D}_X(\log D) \otimes \wedge^{i-1} T_X(\log D) \otimes E$$

are given by

$$\begin{aligned} d(T \otimes \delta_1 \wedge \cdots \wedge \delta_i \otimes e) = & \, d_{Sp}(T \otimes \delta_1 \wedge \cdots \wedge \delta_i) \otimes e \\ & - \sum_{l=1}^{i} (-1)^{i-1} T \otimes \delta_1 \wedge \cdots \wedge \hat{\delta_l} \wedge \cdots \wedge \delta_i \otimes \delta_l(e). \end{aligned}$$

The Spencer complex of E gives rise to a locally free resolution of E by left $\mathcal{D}_X(\log D)$-modules. As a consequence, we can compute

$$\mathbf{R}\underline{\mathsf{Hom}}_{(X,\mathcal{D}_X(\log D))}(\mathcal{O}_X(lD), \mathcal{O}_X(mD))$$

for any $l, m \in \mathbf{Z}$ by replacing $\mathcal{O}_X(lD)$ by its Spencer complex, and we obtain isomorphisms

$$\mathbf{R}\underline{\mathsf{Hom}}_{(X,\mathcal{D}_X(\log D))}(\mathcal{O}_X(lD), \mathcal{O}_X(mD)) = \Omega^{\cdot}_X(\log D) \otimes \mathcal{O}_X((m-l)D). \tag{1}$$

The above discussion shows that for the parabolic sheaf $\mathcal{D}_X(\log D)$, we have

$$\mathbf{R}\underline{\mathsf{Hom}}_{(X,D,\mathcal{D}_X(\log D)_*)}(\mathcal{O}_{(X,D)}, \mathcal{O}_{(X,D)})_* = \Omega^{\cdot}_X(\log D)_*$$

where $\Omega^{\cdot}_X(\log D)_*$ is the parabolic sheaf given by the de Rham complex equipped with the trivial parabolic structure:

$$\Omega^{\cdot}_X(\log D)_x = \begin{cases} \Omega^{\cdot}_X(\log D) & \text{if } x = 0, \\ \Omega^{\cdot}_X(\log D) \otimes \mathcal{O}_X(-D) & \text{if } 0 < x \le 1. \end{cases}$$

Consider the parabolic sheaf $F_*\mathcal{D}_X(\log D)_*$ on (X', D'). We remind the reader that this parabolic sheaf has weights $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$. As before, using the isomorphisms 1 we obtain an isomorphim

$$\mathbf{R}\underline{\mathsf{Hom}}_{(X',D',F_*\mathcal{D}_X(\log D)_*)}(F_*\mathcal{O}_{(X,D)}, F_*\mathcal{O}_{(X,D)})_* = F_*\Omega^{\cdot}_X(\log D)_*$$

where $F_*\Omega^{\cdot}_X(\log D)_*$ is the complex of parabolic bundles given by the pushforward de Rham complex with the filtration

$$F_*\Omega^{\cdot}_X(\log D) \otimes \mathcal{O}_{X'}(-D') \hookrightarrow \cdots \hookrightarrow F_*(\Omega^{\cdot}_X(\log D) \otimes \mathcal{O}_X(-D)) \hookrightarrow F_*\Omega^{\cdot}_X(\log D)$$

with weights $\alpha = \left(0, \frac{1}{p}, \ldots, \frac{p-1}{p}\right)$.

4 Derived self-intersection of (Azumaya) schemes

In this section, we summarize the theory of derived self-intersections of (Azumaya) schemes. Our references are [1,2,8,11].

Let S be a smooth variety and X a smooth subvariety of S of codimension n. Assume that the base field is either of characteristic 0 or of $p > n$. We denote the embedding of X inside S by i and the corresponding normal bundle by N. The derived self-intersection W of X inside S is a dg-scheme whose structure sheaf is constructed by taking the derived tensor product of the structure sheaf of X with itself over $\mathcal{O}_S$. The derived self-intersection is equipped with a map from the underived self-intersection, X, by the universal property of fiber product.

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

We say that a derived scheme W is *formal* over a scheme $\pi : W \to Z$ if $\pi_*\mathcal{O}_W$ is a *formal* complex of $\mathcal{O}_Z$-modules, meaning that there exists an isomorphism of commutative differential graded algebras

$$\pi_*\mathcal{O}_W = \bigoplus_k \mathcal{H}^k(\pi_*\mathcal{O}_W)[-k].$$

A local calculation [6] shows that the cohomology sheaves of the structure sheaf of the derived self-intersection W (over S) are given by

$$\mathscr{H}^{-*}(\mathscr{O}_W) = \mathrm{Tor}_*^S(\mathscr{O}_X, \mathscr{O}_X) = \wedge^* N^\vee.$$

Therefore, the formality of the derived self-intersection asserts that there is a quasi-isomorphism (of commutative dg-algebras)

$$\pi_* \mathscr{O}_W = \bigoplus_{i=0}^{n} \wedge^i N^\vee[i] =: \mathbb{S}(N^\vee[1]).$$

(We omit writing the pushforward of $N^\vee$ to Z along the map $X \to W \to Z$.) The main result of [1] is the following.

Theorem 4.1 [1] *The following statements are equivalent.*

1. *There exists an isomorphism of dg-autofunctors of $D(X)$*

   $$i^* i_*(-) = (-) \otimes \mathbb{S}(N^\vee[1]).$$

2. *W is formal over $X \times X$.*
3. *The natural map $X \to W$ is split over $X \times X$.*
4. *The short exact sequence*

   $$0 \to T_X \to T_S|_X \to N \to 0$$

   of vector bundles on X splits.

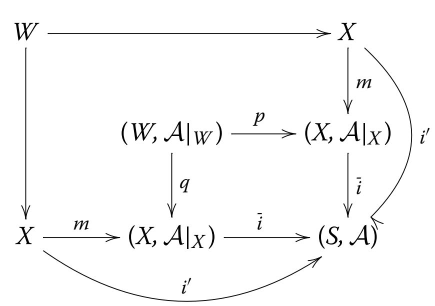
Remark 4.2 For instance, the derived self-intersection of X' inside $T^*X'(\log D')$ is formal: the bundle map $\pi : T^*X'(\log D') \to X'$ splits the embedding of the zero section, and hence, the injection

$$T_{X'} \hookrightarrow T_{T^*X'(\log D')}|_{X'}$$

is split.

An *Azumaya scheme* is a pair $(S, \mathcal{A})$ where S is a scheme or dg-scheme and $\mathcal{A}$ is an Azumaya algebra over S. We say that $(f, E) : (X, \mathcal{B}) \to (S, \mathcal{A})$ is a 1-morphism of Azumaya schemes if f is a morphism of schemes and E is an $f^*\mathcal{A}^{opp} \otimes \mathcal{B}$-module that provides a Morita equivalence between $f^*\mathcal{A}$ and $\mathcal{B}$. Given an embedding of Azumaya schemes $\bar{i} : (X, \mathcal{A}|_X) \to (S, \mathcal{A})$, the derived self-intersection is given by $\overline{W} = (W, \mathcal{A}|_W)$.

From now on, assume that $\mathcal{A}|_X$ is a split Azumaya algebra with splitting module E, in other words there exists a 1-isomorphism (id, E) of Azumaya schemes $m : X \to (X, \mathcal{A}|_X)$. We denote the induced map $X \to (S, \mathcal{A})$ by i'. We organize our spaces into the following diagram.



The Azumaya schemes W and $(W, \mathcal{A}|_W)$ are abstractly isomorphic, but in general the isomorphism is not over (X, X). (We remark that W can be thought of as a dg-scheme over $X \times X$; on the other hand, Azumaya spaces do not have absolute products, and thus, it is more natural to think of them as spaces equipped with two morphisms to X.) The structure sheaves of derived self-intersections W and $(W, \mathcal{A}|_W)$ regarded as dg-schemes endowed with a map to (X, X) are the kernels of the dg-autofunctors of $D(X)$ $i^*i_*(-)$ and $i'^*i'_*(-)$ respectively. In [2], the authors show that there exists an isomorphism of dg-autofunctors of $D(X)$

$$i'^*i'_*(-) = q_*p^*(- \otimes L)$$

for some line bundle L on the derived scheme W. This line bundle is called the associated line bundle of the derived self-intersection in [2].

Consider the object $\bar{i}^*\bar{i}_*E$ of $D(X, \mathcal{A}|_X)$. A local calculation similar to one in [6] shows that there exist isomorphisms

$$\mathscr{H}^k(\bar{i}^*\bar{i}_*E) = E \otimes \wedge^{-k} N^\vee[-k].$$

Therefore, we obtain a triangle in $D(X, \mathcal{A}|_X)$

$$E \otimes N^\vee[1] \to \tau^{\geq -1}\bar{i}^*\bar{i}_*E \to E \to E \otimes N^\vee[2].$$

The rightmost map of the triangle

$$\alpha_E \in H^2_{(X,\mathcal{A}|_X)}(E, E \otimes N^\vee) = H^2(X, N^\vee)$$

is called the HKR class of E.

The HKR class gives the obstruction of lifting E to the first infinitesimal neighborhood of $(X, \mathcal{A}|_X)$ inside $(S, \mathcal{A})$. Having a lifting of E is equivalent to having a splitting module F of the Azumaya algebra $\mathcal{A}|_{X^{(1)}}$ so that $F|_X = E$ where $X^{(1)}$ denotes the first infinitesimal neighborhood of X inside S. Such lifting exists, if i' splits to first-order meaning that there exists a map $\varphi : X^{(1)} \to X$ splitting the natural inclusion $X \to X^{(1)}$, so that φ^*E is a splitting module for $\mathcal{A}|_{X^{(1)}}$. We are ready to state the main result concerning about the triviality of the associated line bundle.

Theorem 4.3 [2] *Assume that W is formal over $X \times X$. Then, the following statements are equivalent.*

1. *The dg-schemes W and $(W, \mathcal{A}|_W)$ are isomorphic over $X \times X$.*
2. *There exists an isomorphism of dg-autofunctors of $D(X)$*

$$i^*i_*(-) \cong i'^*i'_*(-) = (-) \otimes \mathbb{S}(N^\vee[1]).$$

3. *The associated line bundle is trivial.*
4. *The morphism i' splits to first order.*
5. *The HKR class α_E vanishes.*

5 Derived self-intersection of logarithmic (Azumaya) schemes

In this section, we expand the theory of twisted derived intersections to the logarithmic setting.

Let X be a smooth subscheme of a smooth scheme S. We denote the embedding by i. Let us equip S with an effective Cartier divisor D so that i^*D is an effective divisor on X. Consider the induced embedding of logarithmic schemes $i_D : (X, D|_X) \to (S, D)$. We say that i_D splits to first order if there is a left inverse of the induced morphism

$$(X, D|_X) \to (X^{(1)}, D|_{X^{(1)}}),$$

where $X^{(1)}$ denotes the first infinitesimal neighborhood of X inside S. Equivalently, we say that i_D splits to first order if there is a splitting $\rho : X^{(1)} \to X$ of the embedding $X \to X^{(1)}$ so that $\rho^*D|_X = D|_{X^{(1)}}$.

We define the derived-self intersection of $(X, D|_X)$ inside (S, D) as the logarithmic dg-scheme $(W, D|_W)$. We assume that the dg-scheme W is formal over $X \times X$. Notice that the divisor $D|_W$ can be thought of restricting $D|_{X\times X}$ to W. We generalize Theorem 4.1 to the logarithmic setting.

Proposition 5.1 *Assume further that i_D splits to first order. Then we have an isomorphism of dg-autofunctors of $D(X, i^*D)$*

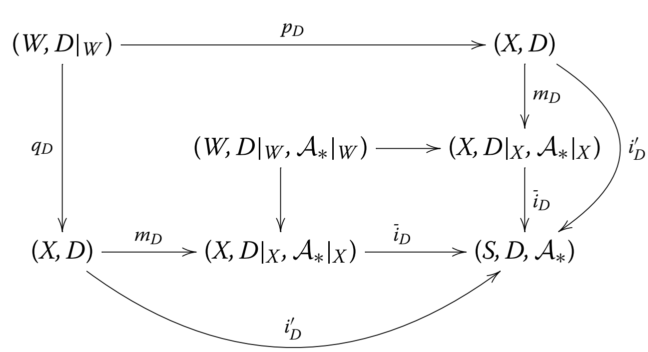
$$i_D^* i_{D,*}(-) = (-) \otimes \mathbb{S}(N^\vee[1]).$$

In other words, $(W, D|_W)$ is formal over $(X \times X, D|_{X\times X})$.

Proof The proof is entirely similar to the proof of Theorem 0.7 of [1]. □

Remark 5.2 All the assumptions of the Proposition above are satisfied in the case when S is a vector bundle $\pi : S \to X$ over X, $i : X \to S$ is the zero section and the divisor D is the pullback along π of a divisor on X.

We turn our attention to embeddings of logarithmic Azumaya schemes. Let us equip the logarithmic scheme (S, D) with a parabolic sheaf of Azumaya algebras $\mathcal{A}_*$ and assume that $\mathcal{A}_*$ splits over $(X, D|_X)$ with splitting module E_*. We remind the reader that E_* induces an isomorphism of spaces $m_D : (X, D) \to (X, D|_X, \mathcal{A}_*|_X)$. We denote the embedding of logarithmic Azumaya spaces $(X, D|_X, \mathcal{A}_*|_X) \to (S, D, \mathcal{A}_*)$ by $\bar{i}_D$, and the composite of embeddings $(X, D|_X) \to (X, D|_X, \mathcal{A}_*|_X) \to (S, D, \mathcal{A}_*)$ by i'_D. We organize our spaces as follows.



The spaces $(W, D|_W)$ and $(W, D|_W, \mathcal{A}_*|_W)$ are abstractly isomorphic, but in general not over the pair $((X, D), (X, D))$, since the splitting modules $p_D^*E_*$ and $q_D^*E_*$ may not be isomorphic. Two splitting modules differ by a parabolic line bundle; thus, the failure to have an isomorphism between $(W, D|_W)$ and $(W, D|_W, \mathcal{A}_*|_W)$ is measured by a parabolic line bundle on $(W, D|_W)$ which we call the associated parabolic line bundle $\mathcal{L}_*$. As a consequence, we obtain an isomorphism of dg-functors

$$i'^*_D i'_{D,*}(-) = q_{D,*}p^*_D(- \otimes \mathcal{L}_*).$$

In particular for the structure sheaf $\mathscr{O}_{(X,D|_X)}$, we have

$$i'^*_D i'_{D,*}\mathscr{O}_{(X,D|_X)} = q_{D,*}p^*_D\mathcal{L}_*.$$

Consider the object $\bar{i}^*_D \bar{i}_{D,*}E_*$. As before, a local calculation similar to one in [6] shows that there exist isomorphisms of parabolic sheaves

$$\mathscr{H}^k(\bar{i}^*_D \bar{i}_{D,*}E_*) = E_* \otimes \wedge^{-k}N^\vee[-k].$$

As above, we define the HKR class $\alpha^D_{E_*}$ as the rightmost map of the triangle

$$E_* \otimes N^\vee[1] \to \tau^{\geq -1}\bar{i}^*_D \bar{i}_{D,*}E_* \to E_* \to E_* \otimes N^\vee[2]. \tag{2}$$

A priori the HKR class is an element of

$$\mathrm{Ext}^2_{(X,D|_X,\mathcal{A}_*|_X)}(E_*, E_* \otimes N^\vee),$$

which is the obstruction of lifting the splitting module E_* to the first infinitesimal neighborhood of the embedding $\bar{i}_D$.

By Morita equivalence the extension group above is isomorphic to

$$Ext^2_{(X,D|_X)}(\mathscr{O}_{(X,D|_X)}, \mathscr{O}_{(X,D|_X)} \otimes N^\vee) = H^2((X, D|_X), \mathscr{O}_{(X,D|_X)} \otimes N^\vee).$$

It is easy to see that $\mathsf{Hom}_{(X,D|_X)}(\mathscr{O}_{(X,D|_X)}, E'_*) = \mathsf{Hom}_X(\mathscr{O}_X, E'_0)$ for any parabolic sheaf E'_*. Therefore, the HKR class $\alpha^D_{E_*}$ can be thought of as an element of $H^2(X, N^\vee)$. This element corresponds to the rightmost map of the triangle given by the triangle in 2 when $* = 0$, i.e., all objects are considered as ordinary sheaves. Summarizing the above discussion we obtain the following theorem which is a straightforward generalization of Theorem 4.3.

Theorem 5.3 *Assume that both maps i and i_D split to first order. Then, the following statements are equivalent.*

1. *The dg-schemes $(W, D|_W)$ and $(W, D|_W, \mathcal{A}|_W)$ are isomorphic over $(X \times X, D|_{X\times X})$.*
2. *There exists an isomorphism of dg-autofunctors of $D(X, D|_X)$*

$$i^*_D i_{D,*}(-) \cong i'^*_D i'_{D,*}(-) = (-) \otimes \mathbb{S}(N^\vee[1]).$$

3. *The associated parabolic line bundle is trivial.*
4. *The morphism i'_D splits to first order.*
5. *The HKR class $\alpha^D_{E_*}$ vanishes.*

6 Proof of the main theorems

In this section, we prove our main theorems, Theorems 6.1–6.3 and Corollary 6.5.

Let X be a smooth scheme over a perfect field k of characteristic $p > \dim X$, and D a reduced normal crossing divisor. We denote by D' the corresponding reduced normal crossing divisor on the Frobenius twist, X'. We consider the Frobenius twist, X' embedded into the vector bundle $T^*X'(\log D')$ as the zero section. Recall that the parabolic sheaf of crystalline logarithmic differential operators $\mathcal{D}(\log D)_*$ can be regarded as a parabolic sheaf of algebras over $T^*X'(\log D')$. Moreover, $\mathcal{D}(\log D)_*$ is a parabolic sheaf of Azumaya

algebras over the logarithmic scheme $(T^*X'(\log D'), \pi^*D')$, so that on the zero section $\mathcal{D}(\log D)_*|_{X'}$ is a split parabolic sheaf of Azumaya algebras.
We are in the context described in Paragraph 5.5. We compare the derived self-intersection $(W, D|_W)$ corresponding to the embedding

$$i : (X', D') \to (T^*X'(\log D'), \pi^*D')$$

and the derived self-intersection corresponding to

$$i' : (X', D') \to (T^*X'(\log D'), \pi^*D', \mathcal{D}(\log D)_*).$$

We denote the latter space by $(\overline{W}, D'|_W)$, and the map

$$(X', D', \mathcal{D}(\log D)_*|_{X'}) \to (T^*X'(\log D'), \pi^*D', \mathcal{D}(\log D)_*)$$

by $\bar{i}$. As an easy consequence of Proposition 5.1 we obtain that the structure sheaf of $(W, D|_W)$ is a formal parabolic sheaf.

Theorem 6.1 *The structure sheaf $\mathcal{O}_{(W,D|_W)}$ over X' is isomorphic to the dual of the formal complex*

$$\mathbb{S}(\Omega^1_{X'}(\log D')[-1]) := \mathcal{O}_X \xrightarrow{0} \Omega^1_{X'}(\log D') \xrightarrow{0} \Omega^2_{X'}(\log D') \xrightarrow{0} \dots$$

equipped with the trivial parabolic structure.

Proof The structure sheaf of $(W, D|_W)$ over X' is given by the object $i^*i_*\mathcal{O}_{(X',D')}$. By Remark 5.2, all the assumption of Proposition 5.1 are satisfied for the embedding i implying the statement above. □

Next we compute the associated parabolic line bundle $\mathcal{L}_*$ of the derived self-intersection corresponding to

$$i' : (X', D') \to (T^*X'(\log D'), \pi^*D', \mathcal{D}(\log D)_*).$$

Theorem 6.2 *The associated line bundle $\mathcal{L}$ is isomorphic to the dual of $F_*\Omega^\cdot_X(\log D)_*$.*

Proof We have the following sequence of maps

$$\begin{aligned} F_*\Omega^\cdot_X(\log D)_* &= \mathbf{R}\underline{\mathrm{Hom}}_{(X',D',F_*\mathcal{D}_X(\log D)_*)}(F_*\mathcal{O}_{(X,D)}, F_*\mathcal{O}_{(X,D)}) \\ &= \mathbf{R}\underline{\mathrm{Hom}}_{(X',D',\pi_*\mathcal{D}(\log D)_*)}(F_*\mathcal{O}_{(X,D)}, F_*\mathcal{O}_{(X,D)}) \\ &= \mathbf{R}\underline{\mathrm{Hom}}_{(X',D',\pi_*\mathcal{D}(\log D)_*)}(\pi_*\bar{i}_*F_*\mathcal{O}_{(X,D)}, \pi_*\bar{i}_*F_*\mathcal{O}_{(X,D)}) \\ &= \pi_*\mathbf{R}\underline{\mathrm{Hom}}_{(T^*X'(\log D'),\pi^*D',\mathcal{D}(\log D)_*)}(\bar{i}_*F_*\mathcal{O}_{(X,D)}, \bar{i}_*F_*\mathcal{O}_{(X,D)}) \end{aligned}$$

where the first isomorphism is the Koszul duality between $\Omega^\cdot_X(\log D)_*$ and $\mathcal{D}_X(\log D)_*$, the second is the isomorphism $F_*\mathcal{D}_X(\log D)_* = \pi_*\mathcal{D}(\log D)_*$ for the bundle map $\pi : T^*X'(\log D') \to X'$, the third is the identity $\pi \circ i = \mathrm{id}$ and the last one is the consequence of that π is affine. The map $\bar{i}_*$ has a right adjoint, which we denote by $\bar{i}^!$ (see [3,21]). We have

$$F_*\Omega^\cdot_X(\log D)_* = \pi_*i_*\mathbf{R}\underline{\mathrm{Hom}}_{(X',D',\mathcal{D}(\log D)_*|_{X'})}(F_*\mathcal{O}_{(X,D)}, \bar{i}^!\bar{i}_*F_*\mathcal{O}_{(X,D)}).$$

We use again that $\pi \circ i = \mathrm{id}$ to obtain

$$F_*\Omega^\cdot_X(\log D)_* = \mathbf{R}\underline{\mathrm{Hom}}_{(X',D',\mathcal{D}(\log D)_*|_{X'})}(F_*\mathcal{O}_{(X,D)}, \bar{i}^!\bar{i}_*F_*\mathcal{O}_{(X,D)}).$$

We remind the reader that there exists a Morita equivalence between (X', D') and $(X', D', \mathcal{D}(\log D)_*|_{X'})$ given by the functors

$$m_* : Coh(X', D') \to Coh(\mathcal{D}(\log D)_*|_{(X',D')}) :$$
$$M \mapsto M \otimes F_*\mathscr{O}_{(X,D)}$$

and

$$m^* : Coh(\mathcal{D}(\log D)_*|_{(X',D')}) \to Coh(X', D') :$$
$$M \mapsto \underline{\mathsf{Hom}}_{\mathcal{D}(\log D)|_{X'}}(F_*\mathscr{O}_{(X,D)}, M)$$

and therefore

$$F_*\Omega^{\cdot}_X(\log D)_* = m^*\bar{i}^!\bar{i}_*\mathscr{O}_{(X',D')}.$$

Moreover, the functor m^* is both left and right adjoint to m_* implying that

$$F_*\Omega^{\cdot}_X(\log D)_* = m^!\bar{i}^!\bar{i}_*\mathscr{O}_{(X',D')} = i'^!i'_*\mathscr{O}_{(X',D')}$$

completing the proof. □

We conclude the paper by applying Theorem 5.3 to our situation.

Theorem 6.3 *Let X be a smooth variety over a perfect field of characteristic $p > \dim X$, with a reduced normal crossing divisor D. Then, the following statements are equivalent.*

1. *The logarithmic scheme (X, D) lifts to $W_2(k)$.*
2. *The class of the extension*

$$0 \to \mathscr{O}_{X'} \to F_*\mathscr{O}_X \to F_*Z^1 \to \Omega^1_{X'}(\log D') \to 0$$

   vanishes. Here Z^1 denotes the image of d inside $F_*\Omega^1_X(\log D)$.
3. *The map i' splits to first order.*
4. *The associated line bundle is trivial.*
5. *The parabolic sheaf of algebras $\mathcal{D}(\log D)_*$ splits on the first infinitesimal neighborhood of (X', D') inside $(T^*X'(\log D'), \pi^*D')$.*
6. *The complex $F_*\Omega^{\cdot}_X(\log D)_*$ is a formal parabolic sheaf equipped with the trivial parabolic structure.*
7. *There exists an isomorphism*

$$F_*\Omega^{\cdot}_X(\log D)_* \cong \mathbb{S}(\Omega^1_{X'}(\log D')[-1])_*$$

   in $D(X', D')$ where the sheaf $\mathbb{S}(\Omega^1_{X'}(\log D')[-1])$ is equipped with the trivial parabolic structure.

Proof The HKR class corresponding to the embedding i' is the dual of the extension class in 2. Therefore, the equivalence

$$(2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7)$$

follows from Theorems 5.3, 6.1 and 6.2. The equivalence (1) $\Leftrightarrow$ (2) is proved in [10]. □

Remark 6.4 A particular interesting feature of the theorem above is part (7). The two parabolic complexes $F_*\Omega^{\cdot}_X(\log D)_*$ and $\mathbb{S}(\Omega^1_{X'}(\log D')[-1])_*$ have different filtration, the former one has weights $\alpha_i = \frac{i}{p+1}$, the latter has the trivial parabolic structure. On the other hand, under quasi-isomorphism the filtrations may change. We illustrate the phenomenon in the simplest case. Let $X = \operatorname{Spec} k[x]$ and $D = \{0\}$. Then the pushforward of the logarithmic de Rham complex is given by

$$\left(x^p k[x] \hookrightarrow x^{p-1}k[x] \hookrightarrow \cdots \hookrightarrow k[x]\right) \xrightarrow{d} \left(x^p \frac{k[x]dx}{x} \hookrightarrow \cdots \hookrightarrow \frac{k[x]dx}{x}\right).$$

The kernel of d is the parabolic sheaf

$$x^p k[x^p] \hookrightarrow x^p k[x^p] \hookrightarrow \cdots \hookrightarrow x^p k[x^p] \hookrightarrow k[x^p]$$

which is isomorphic $\mathscr{O}_{(X',D')}$. Similar calculations can be done for the cokernel.
As an immediate consequence, we obtain a Hodge decomposition for the logarithmic de Rham cohomology.

Theorem 6.5 *Let X be a smooth variety over a perfect field of characteristic $p > \dim X$, and D a reduced normal crossing divisor on X. Assume that (X, D) lifts to $W_2(k)$. Then,*

$$R^*\Gamma(X, \Omega^\cdot_X(\log D)) = \bigoplus_{p+q=*} H^q(X, \Omega^p_X(\log D)).$$

Proof By Theorem 6.3, there exists an isomorphism

$$F_*\Omega^\cdot_X(\log D)_* \cong \mathbb{S}(\Omega^1_{X'}(\log D')[-1])_*.$$

In particular setting $* = 0$, we obtain an isomorphism

$$F_*\Omega^\cdot_X(\log D) \cong \mathbb{S}(\Omega^1_{X'}(\log D')[-1])$$

of objects in $D(X')$. Thus,

$$R^*\Gamma(X', F_*\Omega^\cdot_X(\log D)) = R^*\Gamma(X', \mathbb{S}(\Omega^1_{X'}(\log D')[-1]) = \bigoplus_{p+q=*} H^q(X', \Omega^p_{X'}(\log D')).$$

On the one hand, since F is affine we have

$$R^*\Gamma(X', F_*\Omega^\cdot_X(\log D)) = R^*\Gamma(X, \Omega^\cdot_X(\log D)).$$

On the other hand, since X and X' are abstractly isomorphic, we have

$$\bigoplus_{p+q=*} H^q(X', \Omega^p_{X'}(\log D')) = \bigoplus_{p+q=*} H^q(X, \Omega^p_X(\log D))$$

completing the proof. □

The above theorem implies that the Hodge-to-de Rham spectral sequence degenerates by dimension counting. This gives a new, geometric proof of Kato's theorem.

Corollary 6.6 *Assume that the pair $(X; D)$ lifts to the ring $W_2(k)$. Then, the Hodge-to-de Rham spectral sequence of the logarithmic de Rham complex degenerates at the E_1-page.*

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