Logarithmic approach to the double ramification cycle

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Summary

The study of solutions of equations is very prominent in mathematics. An example of a polynomial equation that one might study is

$$x^2 + y^2 = 4$$

whose solutions are a circle in a x, y-plane over the real numbers $\mathbb{R}$.

Another example may be the equation

$$y^2 - x(x-1)(x-2) = 0$$

which is an equation for a so-called cubic curve: a *curve* is a one-dimensional object defined by a polynomial equation. Examples of curves are a line, a parabola or the above mentioned cubic. A curve is not necessarily one 'curvy line', but it may also consist of multiple parts, formally known as irreducible components. For example, when solving the equation $xy = 0$, we obtain both the line $x = 0$ as well as the line $y = 0$, so we may also consider the union of the x-axis and y-axis as a curve.

Suppose we are interested in solving these equations over the complex numbers $\mathbb{C}$, which extends the real numbers with an element i which functions as the square root of -1, so $i^2 = -1$. Then the study of those zero sets of such polynomial equations (up to adding in points at infinity) that are smooth (if drawn then somehow continuous without kinks) is actually the study of 'compact connected 1-dimensional complex manifolds', which is known as a Riemann surface. This string of words is best illustrated in the following simple example. Consider the Riemann sphere, $\mathbb{P}^1_{\mathbb{C}}$, which is obtained by adding a point at infinity to the complex numbers $\mathbb{C}$, so $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Although the complex numbers $\mathbb{C}$ are formally 1-dimensional over $\mathbb{C}$, it is easier to draw $\mathbb{C}$ viewed as a 2-dimensional plane over $\mathbb{R}$, adding a second axis for multiples of i. Adding one point at infinity to the plane of complex numbers, we obtain the picture of the Riemann sphere in Figure 1.

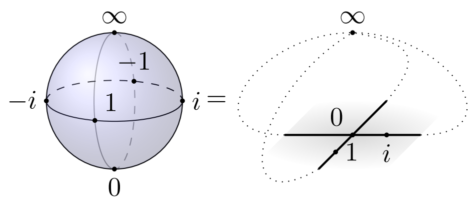

Figure 1: Riemann sphere $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Similarly to drawing the complex 1-dimensional complex numbers $\mathbb{C}$ as a 2-dimensional plane over $\mathbb{R}$, an algebraic curve (a 1-dimensional object over $\mathbb{C}$) is also a 2-dimensional surface over $\mathbb{R}$, explaining why we call it a 'surface' in Riemann surface and draw it as such. Now the word 'compact' means that we do not consider just the plane, where we may walk off into some direction forever, but we rather connect all these directions somehow (here to a point at infinity), intuitively remaining closer. The 'connected' means that the sphere is one entity and cannot be split into two separate parts, as would be the case for two distinct points in a plane. Finally, a 1-dimensional complex manifold means that if you take a small part of your sphere, like a patch of a football, you would on small scale have something that looks like a small part of our 1-dimensional $\mathbb{C}$, that is, a part of the plane of complex numbers.

Another example of such a Riemann surface would be a doughnut or a pretzel, as drawn in Figure 2.

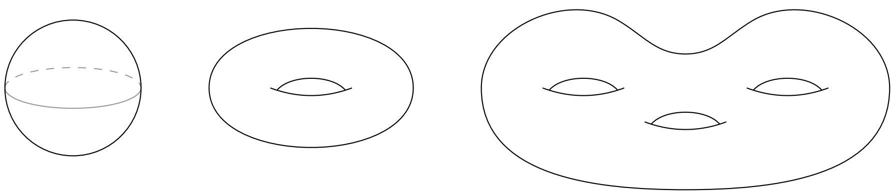

Figure 2: Examples of Riemann surfaces of genus 0,1, and 3

From this point of view, we can explain the genus g as the number of holes as a topological surface. For example, the doughnut has one hole and so has genus 0, but the Riemann sphere has no holes, meaning genus 0. The pretzel would have genus 3. The equation for the cubic above, $y^2 - x(x-1)(x-2) = 0$, would define a Riemann surface of genus 1. Now, as algebraic geometers, we are interested in what happens when you alter or deform these algebraic curves slightly and which properties remain invariant under such changes. This is called studying curves in families. For example, one may alter the defining polynomial for the cubic slightly by considering

$$y^2 - x(x-1)(x-\alpha) = 0$$

where the variable $\alpha \in \mathbb{C}$ may be any complex number. The parameter α means that we view this as a 1-parameter family. However, not all values of α give nice smooth curves and when we are studying this family, we are not only interested in Riemann surfaces of genus 1.

To accommodate the study of such families, we use the concept of a moduli space of curves: a certain geometric object itself, whose points somehow represent or parameterise the curves we study. Note that when we say parameterise, we also want to specify when we consider two such algebraic curves over $\mathbb{C}$ (Riemann surfaces) the same, namely when they are biholomorphic (there is an isomorphism respecting the 'patches of a football' complex manifold structure). Then we denote an element in the moduli space as a class $[C]$ up to biholomorphism. In this instance, we study the moduli space $\mathcal{M}_g$ parameterising smooth Riemann surfaces of genus g. Around 1857, Riemann already considered such a moduli space of variations of complex structures which he knew to be essentially a complex manifold itself of dimension $3g-3$.

Sometimes, when asking certain questions in enumerative geometry about curves, it may be useful to introduce curves with a number of fixed ordered points. For example, when asking about whether curves exists that pass though 5 given points in a plane, or that are tangent to a certain line. In order to study these curves together with their specified points, we use the moduli space $\mathcal{M}_{g,n}$ parameterising smooth curves C together with n distinct marked points on C (up to biholomorphisms that respect the ordered markings):

$$[C, p_1, ..., p_n] \in \mathcal{M}_{g,n}.$$

For example, in genus 0 (that is, Riemann surfaces without holes), it may be proven that the surface up to biholomorphism equals the Riemann sphere, $\mathbb{P}^1_{\mathbb{C}}$, which we saw before in Figure 1. As each curve then looks like $\mathbb{P}^1_{\mathbb{C}}$, the moduli space $\mathcal{M}_{0,n}$ parameterises n distinct points on $\mathbb{P}^1_{\mathbb{C}}$ (up to biholomorphism). Suppose we are considering curves in $\mathcal{M}_{0,3}$, so having three distinct points $p_1, p_2, p_3 \in \mathbb{P}^1_{\mathbb{C}}$. Then we may choose a (unique) 'linear fractional' transformation $g\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ where we move p_1 to 0 and p_3 to ∞ (by choosing the zero and pole of g appropriately) and we may scale with elements from $\mathbb{C}^*$ to ensure that $g(p_2) = 1$, see Figure 3. Therefore, each curve of genus 0 with

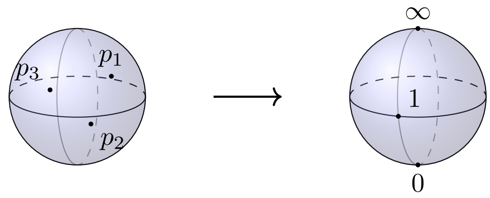

Figure 3: Biholomorphism on $\mathbb{P}^1_{\mathbb{C}}$ translating the markings to $0, 1, \infty$

three marked points we may consider is the 'same' as $\mathbb{P}^1_{\mathbb{C}}$ with points $0, 1, \infty$. So the moduli space $\mathcal{M}_{0,3}$ consists of one single class so one single point.

Now suppose we want to give a similar analysis for curves in $\mathcal{M}_{0,4}$. The genus 0 curve is still the 'same' as $\mathbb{P}^1_{\mathbb{C}}$, so we consider curves having four distinct points $p_1, p_2, p_3, p_4 \in \mathbb{P}^1_{\mathbb{C}}$. For the same reason as above, we can move the first three points to $0, 1, \infty$ respectively, but after choosing the zero, pole and scaling of our transformation, we have run out of freedoms to move our 4-th point. Therefore, the fourth marking will end somewhere in $\mathbb{P}^1_{\mathbb{C}}$, but because p_4 is distinct from p_1, p_2, p_3 it will not land on $0, 1$ or ∞. The moduli space $\mathcal{M}_{0,4}$ then just parameterises the choosing of the distinct fourth point, and is therefore itself isomorphic to $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$. So $\mathcal{M}_{0,4}$ is actually itself a Riemann sphere with certain points removed, and we clearly see that the moduli space is indeed a certain geometric object itself. Now imagine walking through your moduli space and correspondingly deforming your Riemann surface with 4 marked points. That is, while deforming, we let the fourth marking wander through $\mathcal{M}_{0,4} = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$, see Figure 4. Then while

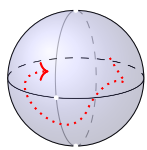

Figure 4: Deforming in $\mathcal{M}_{0,4} \cong \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$

approaching $0, 1$ or ∞, you can imagine that at these points you would fall into nothingness or drop into a sinkhole while walking. Therefore intuitively, we would much prefer to study $\mathbb{P}^1_{\mathbb{C}}$ itself, in order to do computations, such as study invariants, so-called cohomology classes or Chow classes. This enlarging of your moduli space so that it no longer has sinkholes is what is referred to as 'compactifying' the moduli space. In this case this was done by Deligne and Mumford in 1969 who constructed a bigger moduli space $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ by also parameterising certain nodal curves with only finitely many automorphisms (also called stable curves). Specifically, nodal means that we consider curves which may have singularities that locally look like the meeting of the x and y-axes in a complex 2-dimensional 'plane'.

Similarly, the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points (where our markings are not allowed to be at the nodes that the curves may have) allows us to study invariants of marked Riemann surfaces. Then, while deforming our Riemann surface, instead of falling into depths of despair during our 'walk' through $\mathcal{M}_{0,4}$, we would, for example at 0, encounter a stable nodal curve as drawn in Figure 5: two copies of $\mathbb{P}^1_{\mathbb{C}}$ glued at a node, where one copy contains the 4-th and 0-th marking, and the other 1 and ∞.

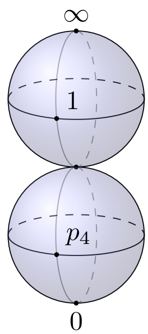

Figure 5: Example of stable curve in $\overline{\mathcal{M}}_{0,4}$

(Because both copies of $\mathbb{P}^1_{\mathbb{C}}$ have three points that an automorphism should respect, a node and two markings, the curve has only one and so finitely many automorphisms.) One may show that the 'compactified' version of $\mathcal{M}_{0,4} \cong \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ is actually isomorphic to the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ and so in itself a nice geometric space to study. In general, the moduli space $\overline{\mathcal{M}}_{g,n}$ has actually by itself also a good geometric structure; which allows us to use it to answer certain geometric questions.

In particular, this thesis is concerned with the *double ramification cycle* on $\overline{\mathcal{M}}_{g,n}$. One may ask the following question: given a certain curve C, is there a rational function f on C which has certain specified orders of zeroes and poles? For example, we may ask for a zero with multiplicity two and two poles with multiplicity one. These specified orders are given in a vector $A = (a_1, ..., a_n)$ of n integers satisfying $\sum_{i=1}^{n} a_i = 0$ where the positive a_i are orders of zeroes that must occur, and the negative a_i are orders of poles. Then the question translates to: is there a rational function $f: C \to \mathbb{P}^1_{\mathbb{C}}$ with zeroes of orders specified by the a_i satisfying $a_i > 0$, and poles of orders specified by the a_i satisfying $a_i < 0$? Because we specify the ramification profile (orders of zeroes and poles) A over two points, namely 0 and ∞, we use the term *double ramification*. The collection of all curves in $\mathcal{M}_{g,n}$ which allow a rational function with a zero or pole of order a_i at the i-th marked point forms the basis of the *double ramification cycle*. Only, as mentioned before, one generally prefers to work with $\overline{\mathcal{M}}_{g,n}$ and so this intuitive definition has to be extended to give what we really call the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$.

This thesis discusses several questions regarding the double ramification cycle using tools from so-called logarithmic geometry. One question is whether there is a more universal construction of the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$, that also includes the generalisations that are called twisted double ramification cycles. This question is answered in [BHP+23], which is the basis of chapter 1 in this thesis. Another commonly studied question with regard to cycles such as the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$ is whether or not a class is provably in the 'tautological' ring. That is, whether a class lies in a subring generated by 'computable and known' classes. The fact that the double ramification cycle is tautological is already known from [FP05], and there is a specific formula to compute it in [JPPZ17]. However, in the article [HS22] (which is included in chapter 3 of this thesis) we are able to show more classes of interest are tautological, via describing or deciding what tautological should be in logarithmic geometry. Key to describing 'logarithmically tautological' is a logarithmic geometry approach to describing classes, namely using piecewise-polynomial functions. The purpose of chapter two is to provide illustrations of piecewise-polynomial functions and their relation to classical algebraic geometry. It is a chapter explaining some concisely stated content of [HS22]. One other issue we can answer in chapter 3, is giving a good definition of a double-double ramification cycle, where we consider two ramification profiles simultaneously. That is, suppose we have two vectors $A, B \in \mathbb{Z}^n$, then the *double-double ramification cycle* $\mathsf{DR}(A, B)$ measures those marked stable curves where both a rational function f with ramification profile A and a rational function g with ramification profile B exist. We also show this double-double ramification cycle is tautological.