Logarithmic approach to the double ramification cycle

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Chapter 3

Logarithmic intersections of the double ramification cycle

This chapter contains the article written with David Holmes that appeared in Algebraic Geometry, [HS22]. We describe a theory of logarithmic Chow rings and tautological subrings for logarithmically smooth algebraic stacks, via a generalisation of the notion of piecewise-polynomial functions. Using this machinery we prove that the double-double ramification cycle lies in the tautological subring of the (classical) Chow ring of the moduli space of curves, and that the logarithmic double ramification cycle is divisorial (as conjectured by Molcho, Pandharipande, and Schmitt).

The article has been included as published. Therefore one should take care that the following notation has changed:

1. The Chow groups are denoted by CH instead of CH.
2. The distinction between $\mathsf{CH}_{\mathsf{op}}$ and CH is no longer made specific.
3. The operational DR cycle is no longer distinguished as $\mathsf{DR}^{\mathsf{op}}$, but now just denoted as DR.
4. The map AJ has lost its capital letters and is denoted aj.

3.1 Introduction

If C/S is a family of smooth algebraic curves and $\mathcal{L}$ on C a line bundle, the double ramification cycle measures the locus of points $s \in S$ where the line bundle $\mathcal{L}$ becomes trivial upon restriction to the fibre C_s. More formally, $\mathsf{DR}(\mathcal{L})$ is a virtual fundamental class of this locus, living in the Chow group of codimension g cycles on S. Extending this class in a natural way to families of (pre)stable curves, and giving a tautological formula, has been the subject of much recent research, including [FP05, Hai13, GZ14, Dud18, FP16, Sch18a, JPPZ17, MW20, JPPZ20, HKP18, Hol19, HS21, HPS19]. In particular, [BHP$^+$23] gives a definition of a double ramification cycle $\mathsf{DR}(\mathcal{L})$ for any line bundle $\mathcal{L}$ on any family C/S of prestable curves, and proves a tautological formula for this cycle.

3.1.1 Double-double ramification cycles are tautological

Suppose now we have two line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on a smooth curve C/S. Then the *double-double ramification cycle* $\mathsf{DR}(\mathcal{L}_1, \mathcal{L}_2)$ measures the locus of $s \in S$ such that both $\mathcal{L}_1$ and $\mathcal{L}_2$ become trivial on the fibre C_s – of course, this is just the intersection of the corresponding cycles $\mathsf{DR}(\mathcal{L}_1)$ and $\mathsf{DR}(\mathcal{L}_2)$. The key insight of [HPS19] was that this naive intersection is the 'wrong' way to extend this class to a family of (pre)stable curves. Instead, one should construct a new virtual class $\mathsf{DR}(\mathcal{L}_1, \mathcal{L}_2)$ for the product (see Section 3.1.2 below), and in general it will not equal the product of the virtual classes of the two factors:

$$\mathsf{DR}(\mathcal{L}_1, \mathcal{L}_2) \neq \mathsf{DR}(\mathcal{L}_1) \cdot \mathsf{DR}(\mathcal{L}_2). \tag{3.1.1.1}$$

Why is this new construction better than simply taking the intersection of the classes? One way to see this is to consider what happens when one tensors the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ together. For a family of smooth curves one sees easily the formula

$$\mathsf{DR}(\mathcal{L}_1)\mathsf{DR}(\mathcal{L}_2) = \mathsf{DR}(\mathcal{L}_1)\mathsf{DR}(\mathcal{L}_1 \otimes \mathcal{L}_2); \tag{3.1.1.2}$$

this also holds in compact type, which plays a key role in the construction of quadratic double ramification integrals and the noncommutative KdV hierarchy in [BR21]. However, (3.1.1.2) *fails* for general families of (pre)stable curves, obstructing the extension of quadratic double ramification integrals beyond the compact-type case (see [HPS19, §8] for an explicit example of this failure). On the other hand, the formula

$$\mathsf{DR}(\mathcal{L}_1, \mathcal{L}_2) = \mathsf{DR}(\mathcal{L}_1, \mathcal{L}_1 \otimes \mathcal{L}_2) \tag{3.1.1.3}$$

does hold for arbitrary families, giving hope of extending the results of [BR21] beyond compact-type. This is a particular instance of a $\mathrm{GL}_2(\mathbb{Z})$-invariance property for the double-double ramification cycles, which we generalise in Theorem 3.5.7 to $\mathrm{GL}_r(\mathbb{Z})$-invariance for r-fold products.

While the cycle $\mathsf{DR}(\mathcal{L}_1, \mathcal{L}_2)$ is in some ways better behaved than the product $\mathsf{DR}(\mathcal{L}_1)\mathsf{DR}(\mathcal{L}_2)$, until now the question of whether it is a tautological cycle has remained open, and is important to address if we hope to study quadratic double ramification integrals. Our first main theorem resolves this question:

Theorem 3.1.1. *Let g, n be non-negative integers, r a positive integer, and $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be line bundles on the universal curve over $\overline{\mathcal{M}}_{g,n}$. Then the r-fold double ramification cycle*

$$\mathsf{DR}(\mathcal{L}_1, \ldots, \mathcal{L}_r) \tag{3.1.1.4}$$

lies in the tautological subring of the Chow ring of $\overline{\mathcal{M}}_{g,n}$.

This theorem opens up the possibility of giving an explicit formula for the class $\mathsf{DR}(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ in terms of the standard generators of the tautological ring, as was done in [JPPZ17] for the case $r = 1$ (that $\mathsf{DR}(\mathcal{L})$ lies in the tautological ring was proven earlier by Faber and Pandharipande [FP05], but no formula was given at that time).

Remark 3.1.2. Ranganathan and Molcho have an independent approach to Theorem 3.1.1, by studying the virtual strict transforms of the DR cycle; see [MR21]. ♦

3.1.2 Logarithmic Chow rings

The fundamental reason for the failure of the product formula (3.1.1.2) for stable curves is that $\mathsf{DR}(\mathcal{L})$ should not really be viewed as a cycle on $\overline{\mathcal{M}}_{g,n}$, but rather it lives naturally on a *log blowup* of $\overline{\mathcal{M}}_{g,n}$ — essentially an iterated blowup in boundary strata. To avoid having to make a choice of blowup we work in $\mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$, which is defined to be the colimit of the Chow rings of all log blowups of $\overline{\mathcal{M}}_{g,n}$. This logarithmic Chow ring comes with a proper pushforward $\nu_*\colon \mathrm{LogCH}(\overline{\mathcal{M}}_{g,n}) \to \mathrm{CH}(\overline{\mathcal{M}}_{g,n})$ to the usual Chow ring, which is a group homomorphism but *not* a ring homomorphism. The construction of $\mathsf{DR}(\mathcal{L})$ can be upgraded (see Definition 3.4.4) to give a cycle $\mathsf{LogDR}(\mathcal{L}) \in \mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$, whose pushforward to $\mathrm{CH}(\overline{\mathcal{M}}_{g,n})$ is $\mathsf{DR}(\mathcal{L})$. The formula

$$\mathsf{LogDR}(\mathcal{L}_1)\mathsf{LogDR}(\mathcal{L}_2) = \mathsf{LogDR}(\mathcal{L}_1)\mathsf{LogDR}(\mathcal{L}_1 \otimes \mathcal{L}_2) \tag{3.1.2.1}$$

is not hard to prove in $\mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$ (see Theorem 3.5.7). We then define

$$\mathsf{DR}(\mathcal{L}_1, \ldots, \mathcal{L}_r) = \nu_* \left(\mathsf{LogDR}(\mathcal{L}_1) \cdots \mathsf{LogDR}(\mathcal{L}_r)\right), \tag{3.1.2.2}$$

from which the product formula (3.1.1.3) is immediate. The fact that equation (3.1.1.2) fails is then just a symptom of the fact that proper pushforward $\nu_*\colon \mathrm{LogCH}(\overline{\mathcal{M}}_{g,n}) \to \mathrm{CH}(\overline{\mathcal{M}}_{g,n})$ is not a ring homomorphism.

3.1.3 Logarithmic tautological rings

Our proof of Theorem 3.1.1 (that double-double ramification cycles are tautological) will run via showing that $\mathsf{LogDR}(\mathcal{L})$ is tautological; but first we have to decide what it means for a cycle in $\mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$ to be tautological.

In fact, we need to do something slightly more general. Our proof that $\mathsf{LogDR}(\mathcal{L})$ is tautological for a line bundle $\mathcal{L}$ on the universal curve over $\overline{\mathcal{M}}_{g,n}$ proceeds by reduction to the fact that the usual double ramification cycle is tautological. However, for this reduction step it will be necessary to modify the universal curve (so that it is no longer stable, only prestable), and also to modify the line bundle $\mathcal{L}$. This leads us to study double ramification cycles on the total-degree-zero[1] Picard stack $\mathfrak{Jac}$ of the universal curve over the stack $\mathfrak{M}$ of all prestable marked curves, exactly the setting considered in [BHP+23].

It is then necessary to define a tautological subring of $\mathrm{LogCH}(\mathfrak{Jac})$, which is slightly delicate as this smooth algebraic stack is neither Deligne-Mumford nor quasi-compact. For this we develop a theory of piecewise-polynomial functions on any log algebraic stack, and for log smooth stacks over a field or dedekind scheme we construct a map from the space of piecewise-polynomial functions to the log Chow ring. We then define the tautological subring as the ring generated by image of this map together with pullbacks of classes in the usual tautological ring on $\mathfrak{Jac}$ (as described in [BHP+23, definition 4]). This leads to our main technical result, from which Theorem 3.1.1 follows easily:

Theorem 3.1.3. LogDR *lies in the tautological subring of* $\mathrm{LogCH}(\mathfrak{Jac})$*.*

In fact we prove a stronger result (Corollary 3.4.20); if $\mathcal{L}$ is the tautological line bundle on the universal prestable curve $\pi\colon C \to \mathfrak{Jac}$, we define the class

$$\eta = \pi_*(c_1(\mathcal{L})^2) \in \mathrm{CH}(\mathfrak{Jac}), \tag{3.1.3.1}$$

and prove that LogDR lies in the subring of $\mathrm{LogCH}(\mathfrak{Jac})$ generated by boundary divisors and the class η.

[1] In [BHP+23] we do not assume total degree zero, but the double ramification cycle is supported in total degree zero, so this is only a superficial change.

3.1.4 LogDR is divisorial

Double ramification cycles in logarithmic Chow rings are also studied in the recent paper [MPS23], with a particular emphasis on the case where $\mathcal{L}$ is the trivial bundle (corresponding to the top Chern class of the Hodge bundle on the moduli space of curves). The objective there is to understand the complexity of $\mathsf{DR}(\mathcal{O}_C)$ in the Chow ring, in particular to understand when it can be written as a polynomial in divisor classes. It is shown that $\mathsf{DR}(\mathcal{O}_C)$ cannot be written as polynomial in divisor classes, and conjectured that $\mathsf{LogDR}(\mathcal{L})$ can be written as a polynomial in divisors for all $\mathcal{L}$. As a byproduct of the proof of Theorem 3.1.3 we obtain something a little more general. The ring $\mathrm{LogCH}(\mathfrak{Jac})$ is graded by codimension, and we write $\mathrm{divLogCH}(\mathfrak{Jac})$ for the subring generated in degree 1. Since LogDR lies in the ring generated by η and boundary divisors, we immediately obtain

Theorem 3.1.4.

$$\mathsf{LogDR} \in \mathrm{divLogCH}(\mathfrak{Jac}).$$

By pulling back to $\overline{\mathcal{M}}_{g,n}$ this proves [MPS23, Conjecture C].

3.1.5 Strategy of proof

As with many things in life, our strategy is best illustrated by carrying it out over $\overline{\mathcal{M}}_{1,2}$. We write C for the universal curve with markings p_1, p_2, and we let $\mathcal{L} = \mathcal{O}(2p_1 - 2p_2)$. Then $\mathsf{DR}(\mathcal{L})$ is invariant under various changes to $\mathcal{L}$; these are listed quite exhaustively in [BHP+23, §0.6]. In particular, let D be the prime divisor on C given by the rational tails (via the isomorphism $C = \overline{\mathcal{M}}_{1,3}$ this is the closure of the locus of curves with a single rational tail and all markings on the tail). Then Invariance V of [BHP+23, §0.6] states that

$$\mathsf{DR}(\mathcal{L}) = \mathsf{DR}(\mathcal{L}(D)). \tag{3.1.5.1}$$

Our toehold on LogDR is obtained by realising that it should satisfy a stronger invariance property, corresponding to twisting by vertical divisors which only exist after blowing up $\overline{\mathcal{M}}_{1,2}$. Let $x \in \overline{\mathcal{M}}_{1,2}$ be the 2-marked 2-gon (Figure 3.1), and let $\widetilde{\mathcal{M}}_{1,2}$ be the blowup of $\overline{\mathcal{M}}_{1,2}$ in x (Figure 3.2), with $\tilde{C}$ the pullback of C. The curve C_x has two irreducible components Y_1, Y_2 (say Y_1 carries p_1), and the pullbacks of these to $\tilde{C}$ are prime divisors supported over the exceptional locus of the blowup, which we denote $\tilde{Y}_i$. We would like to say that LogDR satisfies the invariance

$$\mathsf{LogDR}(\mathcal{L}) = \mathsf{LogDR}(\mathcal{L}(\tilde{Y}_1)), \tag{3.1.5.2}$$

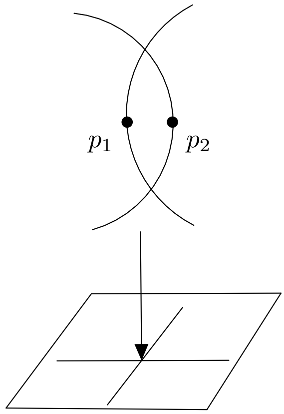

Figure 3.1: Curve over $\overline{\mathcal{M}}_{1,2}$

but this makes no sense because $\tilde{Y}_1$ is only a Weil divisor, not a Cartier divisor over the 'danger' points marked in Figure 3.2. To rectify this we blow up $\tilde{C}$ quite carefully so that the result $\tilde{\tilde{C}}$ is still a prestable curve, but now $\tilde{Y}_1$ is a Cartier divisor on $\tilde{\tilde{C}}$. Then the invariance

$$\mathsf{LogDR}(\mathcal{L}) = \mathsf{LogDR}(\mathcal{L}(\tilde{Y}_1)) \tag{3.1.5.3}$$

makes sense on $\tilde{\tilde{C}}$, and is moreover true.

It is at this point perhaps not clear what we have gained; we have replaced the rather simple bundle $\mathcal{L}$ on C by the rather complicated $\mathcal{L}(\tilde{Y}_1)$ on $\tilde{\tilde{C}}$. The magic is that $\mathcal{L}(\tilde{Y}_1)$ has multidegree $\underline{0}$ — that is, it has degree zero on every irreducible component of every fibre of $\tilde{\tilde{C}}$ (with the exception of the danger points, which we will sweep under the carpet for now). Now, for a line bundle of multidegree $\underline{0}$ the cycle LogDR is just the pullback of the corresponding DR (Lemma 3.4.7), and we know the latter to be tautological by Pixton's formula [BHP+23].

3.1.6 Interpretation as a new invariance of LogDR

Dimitri Zvonkine asked us whether the six invariance properties listed in [BHP+23, §0.6], together with knowledge of DR for families C/S, $\mathcal{L}$ of multidegree zero, would be enough to determine DR completely. The answer is no,

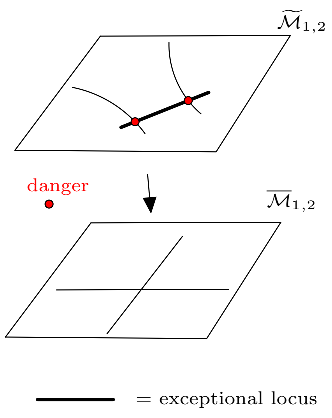

Figure 3.2: $\widetilde{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,2}$

essentially because the invariances in [BHP+23] do not allow us to twist by vertical divisors on C coming from non-separating edges. We saw above how to rectify this in the case of $\overline{\mathcal{M}}_{1,2}$; here we give a more general statement of the new invariance satisfied by LogDR.

Let C/S be a log curve with S log regular, and $\mathcal{L}$ on C a line bundle. We say C/S is *twistable*[2] if there exists a Cartier divisor D on C supported over the boundary of S (i.e. the points of S where the log structure is non-trivial) and such that $\mathcal{L}(D)$ has multidegree $\underline{0}$. We write $\mathsf{LogDR}(\mathcal{L})$ for the pullback of LogDR from $\mathfrak{Jac}$ along the map $S \to \mathfrak{Jac}$ induced by $\mathcal{L}$, and we write $\mathsf{DR}(\mathcal{L}(D))$ for the pullback of DR from $\mathfrak{Jac}$ along the map $S \to \mathfrak{Jac}$ induced by $\mathcal{L}(D)$. Viewing $\mathsf{DR}(\mathcal{L}(D))$ as an element of $\mathrm{LogCH}(S)$ by pullback, our new invariance states

$$\mathsf{LogDR}(\mathcal{L}) = \mathsf{DR}(\mathcal{L}(D)). \tag{3.1.6.1}$$

That this invariance holds is quite straightforward once the definitions are set up correctly, see Lemma 3.4.7. However, there are not enough twistable families that LogDR is determined by DR and the invariance (3.1.6.1); requiring multidegree $\underline{0}$ over *every* point in S is too restrictive a condition (e.g. it fails over the 'danger' points in $\widetilde{\mathcal{M}}_{1,2}$ mentioned above). Because of this we introduce in Definition 3.4.10 a notion of *almost twistable* families. In Lemma 3.4.13 we show the analogue of (3.1.6.1) for almost twistable families,

[2] We thank Rahul Pandharipande for suggesting this terminology

and in Lemma 3.4.17 we show that there are enough almost-twistable families to completely determine LogDR from DR.

3.1.7 Notation and conventions

We work with algebraic stacks in the sense of [Sta13], and with log structures in the sense of Fontaine-Illusie-Kato, for which we find [Ogu18] and [Kat89b] particularly useful general references. The sheaf of monoids on a log scheme (or stack) X is denoted M_X, and the characteristic (or ghost) sheaf is denoted $\bar{\mathsf{M}}_X$, with groupifications $\mathsf{M}_X^{\mathsf{gp}}$ and $\bar{\mathsf{M}}_X^{\mathsf{gp}}$. Occasionally we write $\underline{X}$ for the underlying scheme (or algebraic stack) of X.

We work over a field or Dedekind scheme k equipped with trivial log structure. We work in the category of fine saturated (fs) log schemes (and stacks) over k. In Theorem 3.5.6 and sections 3.4.6 and 3.4.7 we assume that k has characteristic zero, so that we can apply the results of [BHP+23]; it is plausible that the results would become false were this assumption omitted.

A *log algebraic stack* is an algebraic stack equipped with an (fs) log structure.

We work almost exclusively with operational Chow groups with rational coefficients, as defined in [BHP+23, §2], denoted CH.

3.1.8 Acknowledgements

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The first-named author is very grateful to Alessandro Chiodo for many extensive discussions on computing the double ramification cycle on blowups of $\overline{\mathcal{M}}_{g,n}$, which provided key motivation and examples.

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3.2 Logarithmic Chow rings

3.2.1 Logarithmic Chow rings of algebraic stacks

In this section we make a slight generalisation of some of the ideas from [HPS19], see also [MPS23]. We work extensively with log schemes (and stacks) which are both regular and log regular; equivalently, with log structures that are induced by normal crossings divisors (see [Niz06]). We make quite some effort in this and other sections to avoid unnecessary separatedness or quasi-compactness assumptions, and to work with algebraic stacks in place of (for example) schemes. This is not (primarily?) due to a particular personal preference, but rather because the objects we consider (such as the stack of log curves, or its universal Picard space) make this essential.

Definition 3.2.1 ([ALT18, Example 4.3])**.** A morphism $f\colon X \to Y$ of log algebraic stacks is a *monoidal alteration* if it is proper, log étale, and is an isomorphism over the locus in Y where the log structure is trivial.

Examples of monoidal alterations are log blowups and root stacks. We expect that every monoidal alteration can be dominated by a composition of log blowups and root stacks, but have not written down a proof.

Definition 3.2.2. Let X be an algebraic stack locally of finite type over k. We define $\mathrm{CH}(X)$ to be the operational Chow group of X with rational coefficients, using finite-type algebraic spaces as test objects, see [BHP$^+$23, §2] for details.

Remark 3.2.3. If in addition X is smooth and Deligne-Mumford then the intersection pairing induces an isomorphism from the usual Chow ring of X (as defined by Vistoli [Vis89]) to the operational Chow ring $\mathrm{CH}(X)$. ♦

Definition 3.2.4. Let X be a log smooth stack of finite type over k. We define the (operational) log Chow ring of X to be

$$\mathrm{LogCH}(X) = \mathrm{colim}_{\tilde{X}}\, \mathrm{CH}(\tilde{X}), \tag{3.2.1.1}$$

where the colimit runs over monoidal alterations $\tilde{X} \to X$ with $\tilde{X}$ smooth over k.

Remark 3.2.5. A fibre product of monoidal alterations is again a monoidal alteration, hence the above colimit is filtered. This implies that the colimit commutes with the forgetful functor to the category of sets. More concretely, the colimit can be realised as the disjoint union of the chow rings of monoidal alterations of X, modulo setting two elements of different chow rings to be equivalent if they become equal in some common refinement of their home monoidal alterations. ⧫

Definition 3.2.6. Let $z \in \mathrm{LogCH}(X)$ and let $U \hookrightarrow X$ be a quasi-compact open. We say the restriction of z to U is *determined* on a monoidal alteration $\tilde{U} \to U$ if there exists a cycle $z' \in \mathrm{CH}(\tilde{U})$ in the equivalence class of z as defined in the above remark; we then call z' the *determination* of z on $\tilde{U}$.

Remark 3.2.7. Because we work with rational coefficients, taking the colimit over log blowups would yield the same operational Chow ring; in particular, our Log Chow ring is the same as that in [HPS19, §9]. Throughout the paper we use the possibility of determining a cycle on a (smooth) log blowup without further comment. ⧫

Remark 3.2.8. The idea of allowing monoidal alterations rather than just log blowups was suggested to the authors by Leo Herr. It will play little role in most of the paper, but is absolutely essential in Section 3.4.6, where it allows us to apply ideas of [ALT18] on canonical resolution of singularities. ⧫

Remark 3.2.9. The ring $\mathrm{colim}_{\tilde{X}} \mathrm{CH}(\tilde{X})$ can be realised concretely as the disjoint union of the rings $\mathrm{CH}(\tilde{X})$, modulo the equivalence relation where we set cycles z_1 on $\tilde{X}_1$ and z_2 on $\tilde{X}_2$ to be equivalent if there exists a monoidal alteration $\tilde{X}$ dominating both $\tilde{X}_1$ and $\tilde{X}_2$ and on which the pullbacks of z_1 and z_2 coincide. ⧫

3.2.2 Operations on the logarithmic Chow ring

Throughout this subsection X and Y are log smooth stacks of finite type over k.

Definition 3.2.10 (LogCH is a CH-algebra)**.** Viewing X as a trivial log blowup of itself induces a ring homomorphism $\mathrm{CH}(X) \to \mathrm{LogCH}(X)$.

Definition 3.2.11 (Pullback for LogCH)**.** Let $f\colon X \to Y$ be a morphism and let $z \in \mathrm{LogCH}(Y)$. Let $\tilde{Y} \to Y$ be a log blowup on which z is determined. Then $X \times_Y \tilde{Y} \to X$ is a log blowup, and we write $\tilde{f}\colon X \times_Y \tilde{Y} \to \tilde{Y}$. We have a pullback $\tilde{f}^* z \in \mathrm{CH}(\tilde{X})$, which is independent of all choices, and the construction yields a ring homomorphism

$$f^*\colon \mathrm{LogCH}(Y) \to \mathrm{LogCH}(X). \qquad (3.2.2.1)$$

Lemma 3.2.12. *Let $f\colon X \to Y$ be a morphism, then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{CH}(Y) & \longrightarrow & \mathrm{LogCH}(Y) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{CH}(X) & \longrightarrow & \mathrm{LogCH}(X) \end{array} \tag{3.2.2.2}$$

Definition 3.2.13 (Pushforward from LogCH to CH)**.** Suppose X is smooth, and let $z \in \mathrm{LogCH}(X)$. Let $\tilde{X} \to X$ be a log blowup on which z is determined, with $\tilde{X}$ smooth. Then $\pi\colon \tilde{X} \to X$ is proper and lci, so we have a proper pushforward on operational Chow $\pi\colon \mathrm{CH}(\tilde{X}) \to \mathrm{CH}(X)$. These assemble into a pushforward

$$\mathrm{LogCH}(X) \to \mathrm{CH}(X). \tag{3.2.2.3}$$

3.2.3 Extension to the non-quasi-compact case

Definition 3.2.14. Let X be a log smooth log algebraic stack over k (we no longer assume X to be quasi-compact). Let $\mathrm{qOp}(X)$ denote the category of open substacks of X which are quasi-compact. We define the (operational) log Chow ring of X to be

$$\mathrm{LogCH}(X) = \lim_{U \in \mathrm{qOp}(X)} \mathrm{LogCH}(U). \tag{3.2.3.1}$$

Remark 3.2.15*.* Morally, we can think of an element of $\mathrm{LogCH}(X)$ as an (operational) cycle on the valuativisation[3] of X, which can be everywhere-locally represented on some finite log blowup of X. In the absence of a good theory of Chow groups of valuativisations of algebraic stacks, we make the above definition. ♦

Remark 3.2.16*.* All of the constructions of Section 3.2.2 carry through to this setting by restricting to suitable quasi-compact opens. We will make use of these extensions without further comment. ♦

3.3 Tautological subrings of logarithmic Chow rings

In this section we develop a fairly general theory of piecewise-polynomial functions on log algebraic stacks, generalising the theory for toric varieties

[3] See for example [Kat89a].

(for which see [Pay06] and the references therein). We use these piecewise-polynomial functions to build tautological subrings of the log Chow ring. Once again we need only log blowups in this section, root stacks are unnecessary.

In the toric case one can hope to realise every element of the Chow ring in terms of piecewise-polynomial functions, which is far from the case in the our context; for example, all piecewise-polynomial functions are zero on a scheme equipped with the trivial log structure, but the Chow ring can be large and interesting. However, in the presence of a non-trivial log structure the piecewise-polynomial functions can still generate many interesting Chow elements.

The theory in this section was largely developed before we became aware of the related work of Molcho, Pandharipande and Schmitt [MPS23], where 'normally decorated strata classes' approximately correspond to classes coming from our piecewise-polynomial functions. Their approach is probably better for writing formulae for (log) tautological classes, and ours has the advantage that piecewise-polynomials on opens can be glued (which is very useful when working on large algebraic stacks as we do in this paper; as far as we are aware the theory in [MPS23] has so far only been developed for finite-type Deligne-Mumford stacks).

3.3.1 Strict piecewise polynomial functions

Let $(X, \mathcal{O}_X)$ be a ringed site and $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules. We write $\operatorname{Sym}\mathcal{M}$ for the sheafification of the presheaf $U \mapsto \operatorname{Sym}(M(U))$; it is a sheaf of $\mathcal{O}_X$-algebras. If X is any site and $\mathcal{A}$ a sheaf of abelian groups, then we view $\mathcal{A}$ as a sheaf of modules for the constant sheaf of rings $\mathbb{Z}$, yielding a sheaf $\operatorname{Sym}\mathcal{A}$ of graded $\mathbb{Z}$-algebras.

Example 3.3.1. If X is a scheme and $\mathcal{A}$ is the constant sheaf $\mathbb{Z}^n$ of abelian groups, then $\operatorname{Sym}\mathcal{A}$ is the constant sheaf $\mathbb{Z}[x_1, \ldots, x_n]$. ⧫

Definition 3.3.2. We define the *sheaf of strict piecewise-polynomial functions* on a log algebraic stack S as

$$\mathrm{sPP}_S := \operatorname{Sym} \bar{\mathsf{M}}_S^{\mathsf{gp}}. \tag{3.3.1.1}$$

we write

$$\mathrm{sPP}_S^n = \operatorname{Sym}^n \bar{\mathsf{M}}_S^{\mathsf{gp}}, \tag{3.3.1.2}$$

for the graded pieces, and *strict piecewise-linear* functions are

$$\mathrm{sPP}_S^1 = \operatorname{Sym}^1 \bar{\mathsf{M}}_S^{\mathsf{gp}} = \bar{\mathsf{M}}_S^{\mathsf{gp}}. \tag{3.3.1.3}$$

Remark 3.3.3.

1. If S is a toric variety, then strict piecewise-linear (piecewise-polynomial) functions on S correspond exactly to piecewise linear (or piecewise-polynomial) functions on the fan of S which are linear (or polynomial) on each cone. Later (Definition 3.3.15) we will generalise this to allow functions which are linear (polynomial) on some subdivision of the fan.

2. The sheaf $\bar{\mathsf{M}}_S^{\mathsf{gp}}$ makes sense on the big strict étale site of S, so the same holds for the sheaf sPP_S.

3. There is natural map $\mathrm{Sym}^n(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) \to \mathrm{sPP}_S^n(S)$, but is in general not surjective unless $n = 1$, see Example 3.3.4; this will play a prominent role in what follows.

4. Given a map of log algebraic stacks $f\colon S' \to S$ there is a natural map $f^*\bar{\mathsf{M}}_S \to \bar{\mathsf{M}}_{S'}$, inducing a natural map of sheaves of $\mathbb{Z}$-algebras $f^*\,\mathrm{sPP}_S \to \mathrm{sPP}_{S'}$.

♦

Example 3.3.4. Let $\underline{S} = \mathbb{P}_k^2$, and let E be an irreducible nodal cubic in $\underline{S}$, with complement $i\colon U \hookrightarrow \underline{S}$. We define $\mathsf{M}_S = i_*\mathcal{O}_U$, so that $\bar{\mathsf{M}}_S(S) = \mathbb{N}$, and $\mathrm{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) = \mathbb{Z}[e]$, where e corresponds to the divisor E. There is an étale chart for S at the singular point of E given by $k[\langle a, b\rangle]$ where a, b correspond to the two branches of E through the singular point. The image of $\mathrm{Sym}^2(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S))$ is the free module $\mathbb{Z}\left\langle (a+b)^2 \right\rangle$. However, there is another global section of sPP_S^2 given by ab, and in fact $\mathrm{sPP}_S^2(S) = \mathbb{Z}\left\langle (a+b)^2, ab \right\rangle = \mathbb{Z}\left\langle a^2 + b^2, ab \right\rangle$. ♦

3.3.2 Simple log algebraic stacks

Barycentric subdivision

If S is a regular log regular log algebraic stack then by [Niz06, 5.2] there exists a unique normal crossings divisor Z on S (the *boundary divisor* of S) with complement $i\colon U \to S$ and $\mathsf{M}_S = i_*\mathcal{O}_U$. We write the irreducible components of Z as $(D_i)_{i\in I}$.

If S is a regular log regular atomic[4] log scheme then we define the *barycentric ideal sheaf* to be the product

$$\prod_{J\subseteq I} \mathcal{I}\Big(\bigcap_{j\in J} D_j\Big)$$

[4] In the sense of [AW18]: S has a unique stratum that is closed and connected, and the restriction of the characteristic monoid to this stratum is a constant sheaf.

over *non-empty* subsets $J \subseteq I$, and the *barycentric subdivision of* S to be the blowup of S along the barycentric ideal sheaf. This blowup is stable under strict smooth pullback, defining a barycentric subdivision of any log regular log algebraic stack. A more explicit description can be found in [MPS23, §5.3]

Simple log algebraic stacks

Definition 3.3.5. If S is a regular log regular log algebraic stack with boundary divisor[5] $Z = \bigcup_{i\in I} D_i$, we say S is *simple* if for every $J \subseteq I$ the fibre product

$$D_J \coloneqq \bigtimes_{j\in J,S} D_j \tag{3.3.2.1}$$

is regular and in addition the natural map on sets of connected components $\pi_0(D_J) \to \pi_0(S)$ is injective. The closed connected substacks D_J are the *closed strata* of S.

This condition is more restrictive than requiring the boundary divisor to be a strict normal crossings divisor; consider the union of a line and a smooth conic in $\mathbb{P}^2$ meeting at two points, then the intersection is not connected.

Simplifying blowups

Lemma 3.3.6. *Let S be a log regular log algebraic stack. Then there exists a log blowup $\tilde{S} \to S$ such that $\tilde{S}$ is simple.*

The proof consists of three observations:

1. By [IT14] there exists a *functorial* resolution algorithm for log regular log schemes, hence there exists a log blowup of S which is regular and log regular;

2. If S is regular log regular then the barycentric subdivision has strict normal crossings boundary (i.e. the substacks D_J of (3.3.2.1) are regular);

3. If S is regular log regular with strict normal crossings boundary then the barycentric subdivision is simple.

[5] Here we implicitly mean that the D_i are reduced and irreducible substacks of pure codimension 1.

Piecewise polynomials as polynomials in boundary divisors

For an algebraic stack X, we write $\operatorname{Div}(X)$ for the monoid of isomorphism classes of pairs $(\mathcal{L}, \ell)$ where $\mathcal{L}$ is a line bundle on X and $\ell \in \mathcal{L}(X)$ a section, with monoid operation given by tensor product.

Let S be a log algebraic stack and $m \in \bar{\mathsf{M}}_S(S)$. The preimage $\mathcal{O}_S(-m)^\times$ of m in M_S is an $\mathcal{O}_S^\times$-torsor and the log structure equips it with a map to $\mathcal{O}_S(-m)^\times \to \mathcal{O}_S$. This map admits a unique $\mathcal{O}_S^\times$-equivariant extension to a map of line bundles $\mathcal{O}_S(-m) \to \mathcal{O}_S$, where we built $\mathcal{O}_S(-m)$ from $\mathcal{O}_S(-m)^\times$ by filling in the zero section. Finally, dualising this gives a map $\mathcal{O}_S \to \mathcal{O}_S(m) \coloneqq \mathcal{O}_S(-m)^\vee$, and the image of the section 1 of $\mathcal{O}_S$ gives a section of $\mathcal{O}_S(m)$. This defines a map

$$\mathcal{O}_S(-)\colon \bar{\mathsf{M}}_S(S) \to \operatorname{Div}(S). \tag{3.3.2.2}$$

This can be upgraded to a monoidal functor of fibred symmetric monoidal categories, see [BV12, §3.1].

If the log structure on S is trivial over a schematically-dense open $U \subseteq S$ (for example, this holds if S is log regular), then the given section of $\mathcal{O}_S(m)$ is non-vanishing over U, and so defines an effective Cartier divisor on S supported away from U, which we denote $\operatorname{div} \mathcal{O}_S(m)$.

Suppose now that S is a quasi-compact regular log regular log algebraic stack with boundary divisor $Z = \bigcup_{i\in I} D_i$, and write $\langle D_i : i \in I\rangle$ for the free commutative monoid generated by the D_i (so $\langle D_i : i \in I\rangle^{\mathsf{gp}}$ is the group of divisors on S with support contained in Z). Then the effective divisors $\operatorname{div} \mathcal{O}_S(m)$ are supported on Z, hence naturally lie in $\langle D_i : i \in I\rangle$. Using that the rank of $\bar{\mathsf{M}}_{S,s}$ is equal to the number of analytic branches of the D_i passing through s one deduces easily

Lemma 3.3.7. *Sending $m \mapsto \operatorname{div} \mathcal{O}_S(m)$ gives an isomorphism of monoids*

$$\bar{\mathsf{M}}_S(S) \to \langle D_i : i \in I\rangle\,. \tag{3.3.2.3}$$

Suppose now in addition that the D_i form a divisor with *strict* normal crossings (i.e. that all the fibre products in (3.3.2.1) are regular). Given $s \in S$, let $I_s = \{i \in I : s \in D_i\}$, so that $D_i : i \in I_s$ is the set of branches of Z through s. Then (for example by applying the above discussion to a small Zariski neighbourhood of s) we obtain an isomorphism

$$\bar{\mathsf{M}}_{S,s} \xrightarrow{\sim} \langle D_i : i \in I_s\rangle\,. \tag{3.3.2.4}$$

The restriction map simply corresponds to the map $\langle D_i : i \in I\rangle \to \langle D_i : i \in I_s\rangle$ sending those D_i with $i \notin I_s$ to zero; more precisely the square

$$\begin{array}{ccc} \bar{\mathsf{M}}_S(S) & \xrightarrow{=} & \langle D_i : i \in I\rangle \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow \\ \bar{\mathsf{M}}_{S,s} & \xrightarrow{=} & \langle D_i : i \in I_s\rangle \end{array} \tag{3.3.2.5}$$

commutes.

The isomorphism (3.3.2.3) and the definition of the symmetric algebra yields an isomorphism

$$\mathrm{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) \xrightarrow{\sim} \mathbb{Z}[D_i : i \in I] \tag{3.3.2.6}$$

to the free commutative ring on the D_i. Similarly one obtains an isomorphism

$$\mathrm{sPP}_{S,s} = \mathrm{Sym}(\bar{\mathsf{M}}_{S,s}^{\mathsf{gp}}) \xrightarrow{\sim} \mathbb{Z}[D_i : i \in I_s]. \tag{3.3.2.7}$$

We obtain corresponding commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) & \xrightarrow{=} & \mathbb{Z}[D_i : i \in I] \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow \\ \mathrm{Sym}(\bar{\mathsf{M}}_{S,s}^{\mathsf{gp}}) & \xrightarrow{=} & \mathbb{Z}[D_i : i \in I_s], \end{array} \tag{3.3.2.8}$$

where again the right vertical map sends any D_i with $i \notin I_s$ to zero.

Global generation on simple log schemes

In this section we prove the key technical result on piecewise-polynomial functions on log stacks.

Theorem 3.3.8. *Let S be a quasi-compact simple log algebraic stack. Then the natural map of $\mathbb{Z}$-algebras*

$$\mathrm{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) \to (\mathrm{Sym}\,\bar{\mathsf{M}}_S^{\mathsf{gp}})(S) \tag{3.3.2.9}$$

is surjective.

In other words, $\mathrm{PP}_S(S)$ is a quotient of the symmetric algebra on $\bar{\mathsf{M}}_S(S)$; every global piecewise-polynomial function can be written *globally* as a polynomial in piecewise-linear functions. We thank the referee for suggesting the following proof, which is much simpler than the one we started with.

Proof. Given $\underline{n} \in \mathbb{N}^I$ we write $D^{\underline{n}} \in \mathbb{Z}[D_i : i \in I_s]$ for the corresponding monomial in the D_i, and $Z(D^{\underline{n}})$ for the fibre product over S of those D_i with $\underline{n}_i \neq 0$; this only depends on whether each $\underline{n}_i$ is 0, and is always either empty or irreduclbe (by our simplicity assumption).

Now let $f \in (\operatorname{Sym} \bar{\mathsf{M}}_S^{\mathsf{gp}})(S)$; we will construct explicitly a preimage in $\operatorname{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S))$ by giving a coefficient for each monomial $D^{\underline{n}}$.

First suppose that $Z(D^{\underline{n}})$ is empty; then $D^{\underline{n}}$ maps to zero in $\operatorname{Sym}(\bar{\mathsf{M}}_{S,s}^{\mathsf{gp}})$ for every $s \in S$, hence maps to zero in $\operatorname{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}})$ by the sheaf property. We set $f_{\underline{n}} = 0$ for each such $\underline{n}$.

Now take an $\underline{n} \in \mathbb{Z}^I$ with that $Z(D^{\underline{n}})$ is non-empty, and let η be its generic point. Then $D^{\underline{n}}$ maps to a non-zero element of $\mathbb{Z}[D_i : i \in I_s] = \operatorname{Sym}(\bar{\mathsf{M}}_{S,\eta}^{\mathsf{gp}})$, and we write $f_{\underline{n}}$ for the coefficient of $D^{\underline{n}}$ in the restriction of f to $\operatorname{Sym}(\bar{\mathsf{M}}_{S,\eta}^{\mathsf{gp}})$.

It is easy to check that only finitely many $f_{\underline{n}}$ can be non-zero. We define

$$F = \sum_{\underline{n} \in \mathbb{N}^I} f_{\underline{n}} D^{\underline{n}} \in \mathbb{Z}[D_i : i \in I] = \operatorname{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)). \tag{3.3.2.10}$$

It is easy to see that the image of F equals f, because this can be checked one stratum at a time, and each stratum appears as some intersection of D_i. □

3.3.3 Map to the Chow group

Map on divisors

Composing the map $\bar{\mathsf{M}}_S(S) \to \operatorname{Div}(S)$ of (3.3.2.2) with the (operational) first Chern class yields a group homomorphism

$$\Phi^1 \colon \bar{\mathsf{M}}_S^{\mathsf{gp}}(S) \to \mathrm{CH}^1(S), \tag{3.3.3.1}$$

with image contained in the subgroup generated by Cartier divisors.

The case of simple finite-type stacks

Let S be a simple log algebraic stack, smooth[6] over k. The operational Chow group $\mathrm{CH}(S)$ has a commutative ring structure coming from composition of

[6]If k is a field of characteristic zero then being smooth is here equivalent to being locally of finite type (since simple implies regular).

operations. As such, the map

$$\Phi^1 \colon \bar{\mathsf{M}}_S(S) \to \mathrm{CH}^1(S) \tag{3.3.3.2}$$

of (3.3.3.1) extends uniquely to a ring homomorphism

$$\Phi' \colon \mathrm{Sym}(\bar{\mathsf{M}}_S(S)) \to \mathrm{CH}(S). \tag{3.3.3.3}$$

Lemma 3.3.9. *Any element of the kernel of the surjective morphism* $\mathrm{Sym}^n(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) \to (\mathrm{Sym}^n \bar{\mathsf{M}}_S^{\mathsf{gp}})(S)$ *maps to* 0 *in* $\mathrm{CH}(S)$.

Proof. Let P be a polynomial in boundary divisors on S which maps to zero in $(\mathrm{Sym}^n \bar{\mathsf{M}}_S^{\mathsf{gp}})(S)$, and let $s \in S$ be a point, then P maps to zero in $\mathrm{Sym}^n \bar{\mathsf{M}}_{S,s}$. Write $D_1, \ldots, D_n$ for the irreducible components of the boundary divisor of S, with $s \in D_i$ if and only if $i \leq r$ for some $1 \leq r \leq n$. Then (3.3.2.8) becomes the commutative diagram of rings

$$\begin{array}{ccc} \mathrm{Sym}^n(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) & \xrightarrow{=} & \mathbb{Z}[D_1, \ldots, D_n] \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n \bar{\mathsf{M}}_{S,s}^{\mathsf{gp}} & \xrightarrow{=} & \mathbb{Z}[D_1, \ldots, D_r] \end{array} \tag{3.3.3.4}$$

where the vertical arrow sends D_i to 0 for $r < i \leq n$. In particular we see that every monomial in the D_i with non-zero coefficient in P must contain some D_i for $r < i \leq n$ with non-zero exponent. This implies that the corresponding set-theoretic intersection of D_i does not contain s.

Since this argument holds for every $s \in S$ we see that each monomial in P has the set-theoretic intersection of its divisors being empty, hence it maps to 0 in $\mathrm{CH}(S)$. □

Hence this map Φ' descends to a unique ring homomorphism

$$\Phi \colon (\mathrm{Sym}\, \bar{\mathsf{M}}_S)(S) = \mathrm{PP}_S(S) \to \mathrm{CH}(S), \tag{3.3.3.5}$$

whose degree 1 part is Φ^1.

The case of log smooth finite-type stacks

Let S be a quasi-compact log smooth log algebraic stack over k. By Lemma 3.3.6 there exists a log blowup $\pi \colon \tilde{S} \to S$ with $\tilde{S}$ simple. We define

$$\Phi_S \colon (\mathrm{Sym}\, \bar{\mathsf{M}}_S)(S) = \mathrm{sPP}_S(S) \to \mathrm{CH}(S) \tag{3.3.3.6}$$

as the composite

$$(\mathrm{Sym}\,\bar{\mathsf{M}}_S)(S) \to \mathrm{Sym}\,\bar{\mathsf{M}}_{\tilde{S}}(\tilde{S}) \xrightarrow{\Phi_{\tilde{S}}} \mathrm{CH}(\tilde{S}) \xrightarrow{\pi_*} \mathrm{CH}(S). \tag{3.3.3.7}$$

Lemma 3.3.6 actually yields a *canonical* choice of log blowup π, but we should still check that the map Φ_S is independent of the choice of π (for example, if S was already simple, we don't want to have changed the map by blowing up).

Lemma 3.3.10. *Let $\pi\colon \tilde{S} \to S$ be a log blowup with S and $\tilde{S}$ simple. The diagram*

$$\begin{array}{ccc} \mathrm{sPP}_{\tilde{S}}(\tilde{S}) & \xrightarrow{\Phi_{\tilde{S}}} & \mathrm{CH}(\tilde{S}) \\ {\scriptstyle\pi^*}\uparrow & & \downarrow{\scriptstyle\pi_*} \\ \mathrm{sPP}_S(S) & \xrightarrow{\Phi_S} & \mathrm{CH}(S) \end{array} \tag{3.3.3.8}$$

commutes.

Proof. Since S is simple it is enough to check this for a monomial in elements of $\bar{\mathsf{M}}_S(S)$ corresponding to prime boundary divisors on S, say $\prod_{a\in A} D_a$. Applying π^* corresponds to taking the total transforms of these divisors up to $\tilde{S}$. We then need to show that

$$\pi_*(\prod_a \pi^* D_a) = \prod_a D_a, \tag{3.3.3.9}$$

which follows from the projection formula and the fact that $\pi_*\pi^*$ is the identity. ☐

Lemma 3.3.11. *For any log regular S, the map Φ_S is independent of the choice of log blowup $\pi\colon \tilde{S} \to S$.*

Proof. Reduce to one blowup dominating another, then apply Lemma 3.3.10. ☐

Example 3.3.12. Resuming Example 3.3.4, recall $\mathrm{sPP}^2(S) = \mathbb{Z}\left\langle (a+b)^2, ab\right\rangle$. Then $(a+b)^2$ maps to $E^2 \in \mathrm{CH}^2(S)$, and ab maps to the class of the singular point of E in $\mathrm{CH}^2(S)$. ♦

Remark 3.3.13. If S is simple then we can choose π to be the identity, and it is clear that $\Phi_S\colon \mathrm{sPP}_S(S) \to \mathrm{CH}(S)$ is a ring homomorphism. If S is not simple then we suspect that Φ_S is still a ring homomorphism, but we have not managed to prove it. ♦

The case of log regular stacks locally of finite type

Let $\underline{S}$ be an algebraic stack locally of finite type over k, and write $\mathrm{qOp}(\underline{S})$ for the category of open substacks of $\underline{S}$ which are quasi-compact over k, with maps over $\underline{S}$. Then one sees easily that

$$\mathrm{CH}(\underline{S}) = \lim_{U \in \mathrm{qOp}(\underline{S})} \mathrm{CH}(U). \tag{3.3.3.10}$$

Lemma 3.3.14. *Let $i\colon S_1 \hookrightarrow S_2$ be a strict open immersion of quasi-compact log smooth log algebraic stacks over k. The diagram*

$$\begin{array}{ccc} \mathrm{sPP}_{S_2}(S_2) & \xrightarrow{\Phi_2} & \mathrm{CH}(S_2) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \mathrm{sPP}_{S_1}(S_1) & \xrightarrow{\Phi_1} & \mathrm{CH}(S_1) \end{array} \tag{3.3.3.11}$$

commutes.

Proof. A simplifying blowup for S_2 pulls back to one for S_1, so we may assume both S_i simple. Then it is enough to check the result for divisors, since both maps i^* are ring homomorphisms as are Φ_1 and Φ_2 (by our simplicity assumption), and $\mathrm{P}_{S_2}(S_2)$ is generated in degree 1. But the result for divisors is easy. □

Now let S be a log smooth log algebraic stack over k. Given $p \in \mathrm{PP}_S(S)$ and any $U \hookrightarrow S$ quasi-compact, we restrict p to $p_U \in \mathrm{sPP}_U(U)$, yielding an element $\Phi_U(p_U) \in \mathrm{CH}(U)$. By Lemma 3.3.14 these glue, yielding a map

$$\Phi_S\colon \mathrm{sPP}_S(S) \to \mathrm{CH}(S). \tag{3.3.3.12}$$

3.3.4 General piecewise-polynomials and the log-tautological ring

Definition 3.3.15. For a log algebraic stack S we define the group of *piecewise-polynomial functions* as

$$\mathrm{PP}'(S) = \mathrm{colim}_{\tilde{S} \to S} \mathrm{sPP}(\tilde{S}), \tag{3.3.4.1}$$

where $\tilde{S} \to S$ runs over all log blowups of S.

Lemma 3.3.16. *The pullback $\mathrm{sPP}(S) \to \mathrm{sPP}(\tilde{S})$ is injective for $\tilde{S} \to S$ any log blowup, so the natural maps to the colimit are injective.*

Proof. It suffices to show this locally, so we reduce to the atomic case. It is then enough to check that the natural map $\bar{\mathsf{M}}_S(S) \to \bar{\mathsf{M}}_{\tilde{S}}(\tilde{S})$ is injective. This is clear from the construction of the blowup in the toric case, but any log blowup is locally a strict base-change of a toric blowup. □

We define the sheaf of piecewise-polynomials PP on the small strict étale site of S as the sheafification of the presheaf of rings $\mathrm{PP}' \colon U \mapsto \mathrm{PP}'(U)$. This sheaf property then immediately yields

$$\mathrm{PP}(S) = \lim_{U \in \mathrm{qOp}(S)} \mathrm{PP}(U). \tag{3.3.4.2}$$

The natural maps

$$\Phi_i^{\mathsf{log}} \colon \mathrm{PP}(U_i) \to \mathrm{colim}_{\tilde{U}_i} \mathrm{CH}(\tilde{U}_i) \tag{3.3.4.3}$$

then assemble into a ring homomorphism

$$\Phi^{\mathsf{log}} \colon \mathrm{PP}(S) \to \mathrm{LogCH}(S). \tag{3.3.4.4}$$

Remark 3.3.17. The presheaf PP' is always separated, and is a sheaf if S is quasi-compact and quasi-separated; we make the above construction to avoid having to worry about finding common refinements of blowups of very large stacks. ♦

The log-tautological ring

Definition 3.3.18. Let S be a smooth log smooth log algebraic stack over k and let $T \subseteq \mathrm{CH}(S)$ be a subring. We define $T^{\mathsf{log}} \subseteq \mathrm{LogCH}(S)$ to be the sub-T-algebra of $\mathrm{LogCH}(S)$ generated by the image of

$$\Phi^{\mathsf{log}} \colon \mathrm{PP}(S) \to \mathrm{LogCH}(S). \tag{3.3.4.5}$$

A natural application is to take $S = \overline{\mathcal{M}}_{g,n}$ and T to be the usual tautological subring of the Chow ring. We want to ensure that after carrying out our logarithmic constructions and pushing back down to $\overline{\mathcal{M}}_{g,n}$ we still have tautological classes.

Definition 3.3.19. We say a subring $T \subseteq \mathrm{CH}(S)$ is *tectonic*[7] if the pushforward of T^{log} from $\mathrm{LogCH}(S)$ to $\mathrm{CH}(S)$ is equal to T.

[7] It contains many strata, which are formed by overlaps of other strata, perhaps after some things blow up...

Giving criteria for when a subring $T \subseteq \mathrm{CH}(S)$ is tectonic is somewhat subtle. Certainly if a subring is tectonic then it must contain all boundary strata, and the converse holds if S is simple, but not in general. Fortunately for us a precise criterion has been worked out in [MPS23] for the case where S is Deligne-Mumford and quasi-compact, which will be enough for our applications[8]. Their criterion goes by way of defining certain *normally decorated strata class* in $\mathrm{CH}(S)$; the definition is somewhat lengthy, and the details will not be so important for us. We need only the following lemma, and the fact that the usual tautological ring of $\overline{\mathcal{M}}_{g,n}$ contains these normally decorated strata classes.

Lemma 3.3.20. *Suppose S is Deligne-Mumford and quasi-compact. Then a subring $T \subseteq \mathrm{CH}(S)$ is tectonic if and only if it contains all normally decorated strata class.*

Proof. Let $\nu\colon \tilde{S} \to S$ be a log blowup. We denote by $R^\star(S)$ the ring of normally decorated strata classes on S, and similarly for $\tilde{S}$. By definition $R^\star(\tilde{S})$ contains[9] all boundary strata of $\tilde{S}$, and $\nu_*(R^\star(\tilde{S})) \subseteq R^\star(S)$ by [MPS23, Theorem 13]. This implies that every T containing $R^\star(S)$ is tectonic (which is the direction we will use). We deduce the converse from [MPS23, Theorem 11] and the fact that (on a simple log blowup) T^{log} contains all boundary divisors. □

3.3.5 Constructing a class

In this section we present a construction, formulated in a general setting, that will be used in Section 3.4.2 to define logarithmic double ramification cycles.

Suppose we are given the following data:

1. a quasi-separated log smooth log algebraic stack S/k which is stratified by global quotients[10];

2. $f\colon X \to S$ a birational representable log étale morphism;

3. J/k an algebraic stack and $i\colon e \rightarrowtail \mathsf{J}$ a regularly embedded closed substack (or more generally an lci morphism);

[8] It seems likely that their results (perhaps with slight modification) will also hold in the setting of smooth log smooth algebraic stacks, but we have not verified the details.

[9] If we take $\tilde{S}$ simple then $R^\star(\tilde{S})$ is in fact generated by strata, but this is not needed for our argument.

[10] In practise this last condition means we must be exclude $\mathfrak{M}_{1,0}$ from our results; but this is fairly harmless since we can just consider the corresponding cycle on $\mathfrak{M}_{1,1}$ with zero weighting on the new marking.

4. $\sigma\colon X \to \mathsf{J}$ a morphism over k.

Suppose also that the base-change $X \times_{\mathsf{J}} e$ is proper over S. Then we construct a class $[\sigma^* e]_{f,\mathsf{log}} \in \mathrm{LogCH}(S)$ — we often omit the f from the notation when it is clear from context.

The simplest case of our construction is when X is smooth and $f\colon X \to S$ is a log blowup (then $X \times_{\mathsf{J}} e \to S$ is automatically proper). Then $\sigma^! e$ is a well-defined class on X, and automatically gives an element of $\mathrm{LogCH}(S)$, which we denote $[\sigma^* e]_{f,\mathsf{log}}$.

In the general case a little more care is needed. Because $\mathrm{LogCH}(S)$ is defined as a limit over quasi-compact opens, we may assume that S is quasi-compact. Then by Lemma 3.3.25 there exist log blowups $\tilde{S} \to S$ and $\tilde{X} \to X$ and a strict open immersion $\tilde{f}\colon \tilde{X} \hookrightarrow \tilde{S}$ over f; after further log blowup we may also assume $\tilde{S}$ (and hence $\tilde{X}$) to be smooth.

Definition 3.3.21. We call such an $\tilde{S} \to S$ a *sufficiently fine* log blowup (for $f\colon X \to S$), and $\tilde{X} \hookrightarrow \tilde{S}$ the *lift* of X.

Set $Z = \tilde{X} \times_{\sigma,\mathsf{J},i} e$, and consider the diagram

$$\begin{array}{ccc} Z & \longrightarrow & e \\ \downarrow & & \downarrow{\scriptstyle i} \\ \tilde{X} & \xrightarrow{\ \sigma\ } & \mathsf{J} \\ \downarrow{\scriptstyle \tilde{f}} & & \\ \tilde{S} & & \end{array} \qquad (j\colon Z \to \tilde{S}) \tag{3.3.5.1}$$

We then define

$$[\sigma^* e]_{f,\mathsf{log}} = j_* i^! [\tilde{X}]. \tag{3.3.5.2}$$

To unravel this formula, recall that i is lci so we have a gysin morphism $i^!\colon A^*(\tilde{X}) \to A^* Z$. The composite $j\colon Z \to \tilde{S}$ is a closed immersion, in particular projective, so we have a pushforward $j_*\colon A^* Z \to A^* \tilde{S}$. Finally, $\tilde{S}$ is smooth, so the intersection product furnishes a map $A^*(S) \to \mathrm{CH}(\tilde{S})$, and we have a natural inclusion $\mathrm{CH}(\tilde{S}) \hookrightarrow \mathrm{LogCH}(S)$.

Remark 3.3.22. In place of the above construction it might be tempting to take the operational class on J defined by the regular embedding i, then pull it back to $\tilde{X}$. However, $\tilde{f}\colon \tilde{X} \to \tilde{S}$ is not necessarily proper, so we can't push the result forward to define a class on $\tilde{S}$. The essential feature of our construction is that we only assume that j is proper, not $\tilde{f}$. ♦

To see that the above construction is independent of the choice of $\tilde{S}$ we use that gysin pullbacks along lci maps commute with each other and with projective pushforward [Kre99, Theorem 2.1.12 (xi)]

Remark 3.3.23. Let $\tilde{S} \to S$ be a sufficiently fine log blowup. Then $[\sigma^* e]_{f,\mathsf{log}}$ is determined on $\tilde{S}$ (in the sense of Definition 3.2.6). ♦

Lemma 3.3.24. *Let $\varphi\colon X' \to X$ be another birational representable log étale morphism, and write $f'\colon X' \to S$, $\sigma'\colon X' \to \mathsf{J}$ for the obvious composites. Then*

$$[\sigma^* e]_{f,\mathsf{log}} = [\sigma'^* e]_{f',\mathsf{log}} \in \mathrm{LogCH}(S). \tag{3.3.5.3}$$

Proof. Let $T \to S$ be a blowup that is sufficiently fine for $X' \to S$, and which dominates a sufficiently fine blowup for $X \to S$. Unravelling the definitions one sees that the classes agree already in $\mathrm{CH}(T)$. □

The key to the above construction is the following lemma, whose proof is essentially the same as that of Lemma 6.1 of [Hol19].

Lemma 3.3.25. *Let S be a regular log regular qcqs stack and $f\colon X \to S$ birational separated log étale representable. Then there exist log blowups $\tilde{S} \to S$ and $\tilde{X} \to X$ and a strict open immersion $\tilde{X} \to \tilde{S}$ over S.*

Probably this lemma is false if one drops either the quasi-compactness or quasi-separatedness assumptions, but we have not managed to write down an example, and would be interested to see one.

Proof. Consider first the case where S is an affine toric variety, given by some cone $c \xrightarrow{\sim} \mathbb{N}^r$. Then X is given by a fan F consisting of a collection of cones contained in c. Let $\bar{F}$ be a complete fan in c such that every cone in F is a union of cones in c; after further refinement of $\bar{F}$ we can assume that it corresponds to a log blowup $\tilde{S} \to S$. The restriction of $\bar{F}$ to F gives a log blowup $\tilde{X} \to X$, and a strict open immersion $\tilde{X} \to \tilde{S}$.

In the case where S is an atomic log scheme we can follow essentially the same procedure, where the cone $c \xrightarrow{\sim} \mathbb{N}^r$ is replaced by the stalk of the ghost sheaf over the closed stratum of S.

In the general case we can find a smooth cover S by finitely many atomic patches (by quasi-compactness), and each intersection can be covered by finitely many atomic patches (by quasi-separatedness). A strict map of atomic patches corresponds to some inclusion of a cone as a face of another cone: $\mathbb{N}^r \hookrightarrow \mathbb{N}^s$.

Given a face inclusion $\mathbb{N}^r \hookrightarrow \mathbb{N}^s$ and a subdivision of $\mathbb{N}^s$, we can pull the subdivision back to a (unique) subdivision of $\mathbb{N}^r$. But also, given a subdivision of $\mathbb{N}^r$ we can turn it into a subdivision of $\mathbb{N}^s$ in a *canonical* way, by taking the product. Hence if $\mathbb{N}^r \hookrightarrow \mathbb{N}^s$ is a face map (where we allow $r = s$) and we have subdivisions of $\mathbb{N}^r$ and $\mathbb{N}^s$, then we can find 'common refinements' to subdivisions of $\mathbb{N}^r$ and $\mathbb{N}^s$ which agree along the face map. Moreover if both starting subdivisions were log blowups then so are these common refinements.

To conclude the proof, we just need to extend this 'common refinement' procedure from a single face map to any diagram D of face maps with finitely many objects. Such a diagram necessarily also has finitely many morphisms (since there are only finitely many face maps between any two cones), hence the same is true for the category D' obtained by formally inverting all the maps in D. By the discussion in the previous paragraph we can pull back a log blowup along any map in D'.

We are given a log blowup of each cone in D. For a fixed cone c there are only finitely many pairs d, f where d is another cone and $f\colon d \to c$ is a morphism in D'. We then give c the log blowup which is the superposition over all these pairs (f, d) of the pullback along f of the given log blowup of d. In this way we equip every object of D with a log blowup, in such a way that these are compatible along all the face maps in D. These then glue to a global log blowup of S, which pulls back to a global log blowup of X. □

3.4 Logarithmic double ramification cycles

3.4.1 Notation

Here we introduce notation needed for applying the machinery developed in the previous two sections to moduli of curves and to double ramification cycles.

1. $\mathfrak{M}_{g,n}$ denotes the (smooth, algebraic) stack of prestable curves of genus g with n ordered disjoint smooth markings. This has a normal crossings boundary, inducing a log smooth log structure. Equivalently, this is the stack of log curves of genus g and n markings, with a choice of total ordering on the markings (see [Kat00], [GS13, Appendix A]; the underlying algebraic stack is then given by the machinery of minimal log structures [Gil12]).

2. $\overline{\mathcal{M}}_{g,n}$ is the open substack of $\mathfrak{M}_{g,n}$ consisting of Deligne-Mumford-Knudsen stable curves.

3. $\mathfrak{M} = \bigsqcup_{g,n} \mathfrak{M}_{g,n}$ denotes the stack of all log curves with a choice of total ordering on their markings. Often the genus and markings will not be so important to us, so we can use this more compact notation.

4. C is the universal curve over $\mathfrak{M}$. We will abusively use the same notation for the tautological curve over any stack over $\mathfrak{M}$ (so for example, for the universal curve over $\overline{\mathcal{M}}_{g,n}$).

5. $\mathfrak{Pic}$ is the relative Picard stack of C over $\mathfrak{M}$; objects are pairs of a curve and a line bundle on the curve. This is smooth over $\mathfrak{M}$ with relative inertia $\mathbb{G}_m$; we equip it with the strict (pullback) log structure over $\mathfrak{M}$.

6. $\mathfrak{Jac}$ denotes the connected component of $\mathfrak{Pic}$ corresponding to line bundles of (total) degree 0 on every fibre.

7. J is the relative coarse moduli space over $\mathfrak{M}$ of the *fibrewise* connected component of identity in $\mathfrak{Jac}$ (or equivalently in $\mathfrak{Pic}$). Over the locus of smooth curves in $\mathfrak{M}$ this is an abelian variety, the classical jacobian. In general it is a semiabelian variety over $\mathfrak{M}$ which parametrises isomorphism classes of line bundles on $\mathfrak{C}/\mathfrak{M}$ which have degree 0 on every irreducible component of every geometric fibre (sometimes we refer to this condition as having *multidegree* $\underline{0}$). The morphism $\mathsf{J} \to \mathfrak{M}$ is separated, quasi-compact, and relatively representable by algebraic spaces (none of which hold for $\mathfrak{Pic}$ or $\mathfrak{Jac}$).

8. $\bar{\mathsf{J}}$ is the relative coarse moduli space of $\mathfrak{Jac}$ over $\mathfrak{M}$; it can be defined analogously to J except that we require total degree 0 instead of multidegree $\underline{0}$. In particular we have an open immersion $\mathsf{J} \hookrightarrow \bar{\mathsf{J}}$, which is an isomorphism over the locus of irreducible curves.

Lemma 3.4.1. $\mathfrak{Jac}$ *is quasi-separated.*

Proof. First we check that $\mathfrak{M}$ is quasi-separated; equivalently, that the diagonal is qcqs. In other words, if C/S is a prestable curve, then $\mathrm{Isom}_S(C)$ is qcqs over S - but this is well-known.

Now we show that $\mathfrak{Jac}$ is quasi-separated over $\mathfrak{M}$. In other words, we fix a prestable curve C/S and a line bundle $\mathcal{L}$ on C, and look at the automorphisms of $\mathcal{L}$ over C; but this is just $\mathbb{G}_m$. □

Piecewise linear functions

If C/S is a log curve and $\alpha \in \bar{\mathsf{M}}_C^{\mathsf{gp}}(C)$, the *outgoing slope* at a marked section c of C/S is the image of α in the stalk of the relative characteristic monoid $\bar{\mathsf{M}}_{C/S,s} = \mathbb{N}$.

Definition 3.4.2. A piecewise-linear (PL) function on a log curve C/S is an element $\alpha \in \bar{\mathsf{M}}_C^{\mathsf{gp}}(C)$ (cf. (3.3.1.3)) with all outgoing slopes vanishing[11].

The preimage of α in the exact sequence

$$1 \to \mathcal{O}_C^\times \to \mathsf{M}_C^{\mathsf{gp}} \to \bar{\mathsf{M}}_C^{\mathsf{gp}} \to 1 \tag{3.4.1.1}$$

defines an associated $\mathbb{G}_m$-torsor $\mathcal{O}^\times(\alpha)$, which we compactify to a line bundle $\mathcal{O}(\alpha)$ by glueing in the ∞ section (this is just a choice of sign; it corresponds to $\mathcal{O}(-p)$ being an ideal sheaf, rather than its dual).

The bundle $\mathcal{O}(\alpha)$ always has total degree zero, but rarely multidegree $\underline{0}$; more precisely, it has multidegree $\underline{0}$ if and only if $\mathcal{O}(\alpha)$ is a pullback from S, if and only if α is constant on geometric fibres.

3.4.2 Defining LogDR

Before defining the logarithmic double ramification cycle it seems useful to summarise the construction of the 'usual' double ramification cycle (in this thesis to be found in chapter 1) from [BHP+23]. Various constructions of double ramification cycles are given in various places in the literature in various levels of generality (e.g. [Hol19, MW20, HKP18, BHP+23]). They are mostly[12] equivalent, but descriptions of the relations between the constructions are scattered across various sketches in various papers at various levels of generality[13], making it troublesome to assemble a complete picture. Here we attempt to rectify this by giving a precise and general statement of the relation between the two most widely-used definitions, that of the first-named author by resolving rational maps, and that of Marcus and Wise via tropical divisors (in the form used in [BHP+23]).

Tropical divisors

If C/S is a log curve and α a PL function on C, then the line bundle $\mathcal{O}_C(\alpha)$ determines a map $S \to \mathfrak{Jac}$. In this way we have an Abel-Jacobi map from the stack of pairs $(C/S, \alpha)$ to $\mathfrak{Jac}$. We can see this Abel-Jacobi map as a first approximation of the double ramification cycle, but the map has relative

[11] It would be cleaner to work with vertical ('unmarked') log curves, but we will make use of smooth sections of C/S in other places in our arguments, so we do not wish to impose verticality.

[12] With the exception of classes given by the closure of the double ramification locus on the moduli space of smooth curves, as for example in the theory of admissible cover compactifications.

[13] This is in fairly large part the responsibility of the first-named author.

dimension 1 (a section α admits no non-trivial automorphisms, whereas the line bundle $\mathcal{O}_C(\alpha)$ has a $\mathbb{G}_m$ worth of automorphisms), and hence will not induce a good Chow class on $\mathfrak{Jac}$. To fix this we need a little more setup. Given a log scheme $S = (S, M_S)$, we write

$$\mathbb{G}_m^{\mathsf{trop}}(S) = \Gamma(S, \bar{\mathsf{M}}_S^{gp}),$$

which we call the tropical multiplicative group; it can naturally be extended to a presheaf on the category $\mathbf{LSch}_S$ of log schemes over S. A *tropical* line on S is a $\mathbb{G}_m^{\mathsf{trop}}$ torsor on S for the strict étale topology. Then a point of **Div** is a triple

$$(C/S, \alpha, \mathcal{M}) \tag{3.4.2.1}$$

where C/S is a log curve, $\alpha\colon C \to \mathbb{G}_M^{\mathsf{trop}}$ a morphism over S with zero outgoing slopes, and $\mathcal{M}$ is a line bundle on S. An isomorphism

$$(\pi\colon C \to S, \alpha, \mathcal{M}) \to (\pi\colon C \to S, \alpha', \mathcal{M}') \tag{3.4.2.2}$$

in **Div** is an isomorphism $\mathcal{M} \to \mathcal{M}'$, and the Abel-Jacobi map $\mathsf{aj}\colon \mathbf{Div} \to \mathfrak{Jac}$ sends $(\pi\colon C \to S, \alpha, \mathcal{M})$ to $\pi^*\mathcal{M}(\alpha)$.

In [BHP+23] we defined $\mathsf{DR} \in \mathrm{CH}(\mathfrak{Jac})$ to be the fundamental class of the proper log monomorphism $\mathbf{Div} \to \mathfrak{Jac}$ (we describe this in more detail in Definition 3.4.3.)

Universal σ-extending morphisms

Over the locus of irreducible curves in $\mathfrak{Jac}$ the notions of total degree and multidegree coincide, so that J comes with a tautological map from $\mathfrak{Jac}$. We can think of this as a rational map

$$\sigma\colon \mathfrak{Jac} \dashrightarrow \mathsf{J} \tag{3.4.2.3}$$

(rational because it is only defined on the open locus of irreducible curves).

Let $t\colon T \to \mathfrak{Jac}$ be a map of algebraic stacks over $\mathfrak{M}$. We say t is σ*-extending* if[14]

[14] The analogous definition in [Hol19] had the additional assumption that T be normal. At the time this was needed in order to be able to apply [BHdJ17, Theorem 4.1] at a certain critical step in the arguments, but since then Marcus and Wise have proven the analogue of [BHdJ17, Theorem 4.1] with no regularity assumptions, see [MW20, Corollary 3.6.3]. This can then be use to modify they theory of [Hol19] without a normality assumption. Alternatively one can reinstate the condition that T be normal, and all of the subsequent discussion will go through unchanged except that we will have to insert normalisations in various places. By Costello's Theorem [HW22] this will have no effect on the resulting cycles, but will make things much less readable, which is why we prefer to omit the condition.

1. The pullback along t of the locus of line bundles on smooth curves is schematically dense in T;

2. The rational map $T \dashrightarrow \mathsf{J}$ induced by σ extends to a morphism (necessarily unique if exists, by separatedness of J over $\mathfrak{M}$).

One can then show just as in [Hol19] that the category of σ-extending stacks over $\mathfrak{Jac}$ has a terminal object, which we denote $\mathfrak{Jac}^\Diamond$. The natural map

$$f\colon \mathfrak{Jac}^\Diamond \to \mathfrak{Jac} \tag{3.4.2.4}$$

is separated, relatively representable by algebraic spaces, of finite presentation, and an isomorphism over the locus of smooth (even treelike) curves, but it is not in general proper. The construction equips it with a map

$$\sigma\colon \mathfrak{Jac}^\Diamond \to \mathsf{J}. \tag{3.4.2.5}$$

The functor of points of $\mathfrak{Jac}^\Diamond$

One can describe the functor of points (on the category of log schemes) of $\mathfrak{Jac}^\Diamond$ in a manner very similar to the definition of **Div**. Namely, a point of $\mathfrak{Jac}^\Diamond$ is a triple

$$(C/S, \alpha, \mathcal{L}) \tag{3.4.2.6}$$

where C/S is a log curve, $\alpha\colon C \to \mathbb{G}_M^{\mathsf{trop}}$ a morphism over S with zero outgoing slopes, and $\mathcal{L}$ a line bundle on C such that the line bundle $\mathcal{L}(\alpha)$ has *multidegree* $\underline{0}$ on every fibre of C/S.

Given such $(C/S, \alpha, \mathcal{L})$, the map $S \to \mathsf{J}$ given by $\mathcal{L}(\alpha)$ is an extension of σ, so by the universal property of $\mathfrak{Jac}^\Diamond$ we obtain a map from the stack of such quadruples to $\mathfrak{Jac}^\Diamond$. To show this is an isomorphism, we may work locally (so assume C/S to be nuclear in the sense of [HMOP23], and smooth over a dense open of S), then it is enough to show that the extension of σ is given by a PL function; but this follows from [MW20, Corollary 3.6.3].

Comparison

The key actor in Section 3.3.5 is the fibre product of the diagram

$$\begin{array}{ccc} & & \mathfrak{M} \\ & & \downarrow{\scriptstyle e} \\ \mathfrak{Jac}^\Diamond & \xrightarrow{\sigma} & \mathsf{J}. \end{array} \tag{3.4.2.7}$$

From the functor-of-points description of $\mathfrak{Jac}$ this fibre product is exactly given by tuples $(C/S, \alpha, \mathcal{L}) \in \mathfrak{Jac}^\diamondsuit$ such that $\mathcal{L}(-\alpha)$ is the pullback of some line bundle on S, say $\mathcal{L}(-\alpha) = \pi^*\mathcal{M}$. Giving the data of $\mathcal{L}$ or of $\mathcal{M}$ is exactly equivalent, and the tuple $(C/S, \alpha, \mathcal{M})$ is exactly a point of **Div**; in other words we have a pullback square

$$\begin{array}{ccc} \mathbf{Div} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow e \\ \mathfrak{Jac}^\diamondsuit & \xrightarrow{\sigma} & \mathsf{J}. \end{array} \tag{3.4.2.8}$$

Marcus and Wise show that the composite $\mathbf{Div} \to \mathfrak{Jac}$ is proper, and in [BHP+23] we define DR to be the associated cycle on $\mathfrak{Jac}$. The full construction of the operational class is a little subtle (see [BHP+23, §2] for details), but is easy to describe for a smooth stack S mapping to $\mathfrak{Jac}$.

Definition 3.4.3. Let S be a smooth stack and $\varphi\colon S \to \mathfrak{Jac}$ a morphism. Then φ is lci, so we have a gysin pullback $\varphi^!\colon A^*(\mathbf{Div}) \to A^*(\mathbf{Div} \times_{\mathfrak{Jac}} S)$, and following [Sko12] a proper pushforward $i_*\colon A^*(\mathbf{Div} \times_{\mathfrak{Jac}} S) \to A^*(S)$. Since S is smooth the intersection pairing furnishes a map $\cap\colon A^*(S) \to \mathrm{CH}(S)$, and we define

$$\varphi^*\mathsf{DR} = \cap(i_*\varphi^![\mathbf{Div}]) \in \mathrm{CH}(S), \tag{3.4.2.9}$$

where $[\mathbf{Div}]$ denotes the fundamental class of **Div** as a cycle on itself.

Defining LogDR

We construct the cycle LogDR in $\mathrm{LogCH}(\mathfrak{Jac})$ as hinted at in [BHP+23, §3.8]. We apply the construction of Section 3.3.5, taking $S = \mathfrak{Jac}$, $X = \mathfrak{Jac}^\diamondsuit$, $\mathsf{J} = \mathsf{J}$, and $\sigma = \sigma$. We need the natural map

$$\mathfrak{Jac}^\diamondsuit \times_{\mathsf{J}} \mathfrak{M} \to \mathfrak{Jac} \tag{3.4.2.10}$$

to be proper; this can be proven in the same way as in [Hol19, §5], or follows by the comparison to the construction of Marcus-Wise in Section 3.4.2.

Definition 3.4.4. The construction specified in Section 3.3.5 yields a class $\mathsf{LogDR} \coloneqq [\sigma^* e]_{f,\mathsf{log}} \in \mathrm{LogCH}(\mathfrak{Jac})$, the *log double ramification cycle.*

Comparing the constructions yields

Lemma 3.4.5. *Applying the pushforward $\nu_*\colon \mathrm{LogCH}(\mathfrak{Jac}) \to \mathrm{CH}(\mathfrak{Jac})$ of Definition 3.2.13 to $\mathsf{LogDR} \in \mathrm{LogCH}(\mathfrak{Jac})$ recovers the double ramification cycle $\mathsf{DR} \in \mathrm{CH}(\mathfrak{Jac})$ of [BHP+23].*

Proof. The class $\mathsf{DR} \in \mathrm{CH}(\mathfrak{Jac})$ can be obtained by applying Definition 3.4.3 to the smooth stack $S = \mathfrak{Jac}$ (with the map $\varphi\colon \mathfrak{Jac} \to \mathfrak{Jac}$ being the identity), so we need to compare this to the construction in Section 3.3.5. We will begin by making a slight simplifying assumption, namely that $X = \mathfrak{Jac}^{\Diamond}$ can be embedded in a smooth log blowup $\tilde{S} = \mathfrak{Jac}^{\blacklozenge}$ of S (in other words, that we can take $\tilde{X} = X$). Then we have a commutative diagram with cartesian square:

$$\begin{array}{ccc}
\mathbf{Div} & \longrightarrow & \mathfrak{M} \\
\downarrow & & \downarrow e \\
\mathfrak{Jac}^{\Diamond} & \xrightarrow{\sigma} & \mathsf{J} \\
\big\downarrow & & \\
\mathfrak{Jac}^{\blacklozenge} & & \\
\downarrow & & \\
\mathfrak{Jac}. & &
\end{array} \tag{3.4.2.11}$$

Now $\nu_* \mathsf{LogDR}$ is by definition the pushforward to $\mathfrak{Jac}$ of the class $e^![\mathfrak{Jac}^{\Diamond}]$ on $\mathbf{Div}$, so it suffices to show that $e^![\mathfrak{Jac}^{\Diamond}]$ is equal to the fundamental class of $\mathbf{Div}$. But $e^!$ takes the fundamental class to the fundamental class, so this is clear.

What if we cannot embed $\mathfrak{Jac}^{\Diamond}$ in a smooth log blowup $\mathfrak{Jac}^{\blacklozenge}$ of $\mathfrak{Jac}$? Then we must first replace $\mathfrak{Jac}^{\Diamond}$ by some log blowup of itself. The argument then proceeds as above, with the additional input that replacing $\mathbf{Div}$ by a log blowup does not change the class of the final pushforward to $\mathfrak{Jac}$. □

3.4.3 Invariance of LogDR in twistable families

Throughout this subsection, C/S is a log curve over a smooth log smooth base, and $\mathcal{L}$ is a line bundle on C.

Definition 3.4.6. We say the pair $(C/S, \mathcal{L})$ is *twistable* if there exists a PL function α on C such that $\mathcal{L}(\alpha)$ has multidegree $\underline{0}$; we call such an α a *twisting function.*

Being twistable is equivalent to the existence of a Cartier divisor on D on C supported over the boundary of S and such that $\mathcal{L}(D)$ has multidegree $\underline{0}$; see [BHdJ17, Theorem 4.1] or [MW20, Corollary 3.6.3].

Lemma 3.4.7. *Let $(C/S, \mathcal{L})$ be twistable with α a twisting function. Write $\varphi_{\mathcal{L}} \colon S \to \mathfrak{Jac}$ for the map induced by $\mathcal{L}$, and $\varphi_{\mathcal{L}(\alpha)} \colon S \to \mathfrak{Jac}$ for the map induced by $\mathcal{L}(\alpha)$. Then*

$$\varphi_{\mathcal{L}}^* \mathsf{LogDR} = \varphi_{\mathcal{L}(\alpha)}^* \mathsf{DR} \tag{3.4.3.1}$$

in LogCH(S) *(where we view $\varphi_{\mathcal{L}(\alpha)}^* \mathsf{DR}$ in* LogCH(S) *by pullback, cf. Section 3.2.2).*

Proof. Write $\sigma \colon S \dashrightarrow \mathsf{J}$ for the rational map induced by $\mathcal{L}$. Then the identity on S is the universal σ-extending morphism! More precisely, the extension is given by $\mathcal{L}(\alpha) \colon S \to \mathsf{J}$, and it is easily seen to be universal among extensions (see Remark 3.4.8). We have a pullback diagram

$$\begin{array}{ccc} S \times_J e & \longrightarrow & e \\ \big\downarrow j & & \big\downarrow i \\ S & \xrightarrow{\sigma} & \mathsf{J}, \end{array} \tag{3.4.3.2}$$

and the definitions of $\varphi_{\mathcal{L}}^* \mathsf{LogDR}$ and $\varphi_{\mathcal{L}(\alpha)}^* \mathsf{DR}$ simplify to

$$\varphi_{\mathcal{L}}^* \mathsf{LogDR} = j_* i^! [S] \quad \text{and} \quad \varphi_{\mathcal{L}(\alpha)}^* \mathsf{DR} = j_* \sigma^! [e], \tag{3.4.3.3}$$

which are equal since $\sigma^![e] = i^![s]$ (commutativity of the intersection pairing). □

Remark 3.4.8. If $(C/S, \mathcal{L})$ is twistable then α is not unique, but the line bundle $\mathcal{L}(\alpha)$ is uniquely determined up to pullback from S. Hence $\varphi_{\mathcal{L}(\alpha)}^* \mathsf{DR}$ does not depend on the choice of α. ♦

Remark 3.4.9. Unfortunately the notion of twistable families seems a little too restrictive; not enough of them seem to exist to determine LogDR from DR (though we have not written down a proof). Because of this we now introduce a weaker notion. ♦

Definition 3.4.10. We say $(C/S, \mathcal{L})$ is *almost twistable* if there exist a PL function α on C and a dense open $i \colon U \hookrightarrow S$ such that:

1. the restriction of α to U is a twisting function;

2. The map $U \xrightarrow{\mathrm{i} \times \varphi_{\mathcal{L}(\alpha)}} S \times_{\mathfrak{M}} \mathsf{J}$ is a closed immersion (equivalently, its image is closed).

We give some alternative formulations of the second condition in Definition 3.4.10, which will be used in the proof of Lemma 3.4.17.

Lemma 3.4.11. *Suppose we are given $(C/S, \mathcal{L})$, a PL function α on C and a dense open $i\colon U \hookrightarrow S$ such that the restriction of α to U is a twisting function. Then the following are equivalent:*

1. *The map $U \xrightarrow{\mathrm{i} \times \varphi_{\mathcal{L}(\alpha)}} S \times_{\mathfrak{M}} \mathsf{J}$ is a closed immersion (i.e. $(C/S, \mathcal{L})$ is almost twistable);*

2. *for any trait[15] T with generic point η and any map $T \to S$ sending η to a point in U, if the map $\eta \to S \times_{\mathfrak{M}} \mathsf{J}$ induced by $\mathcal{L}(\alpha)$ can be extended to a map $T \to S \times_{\mathfrak{M}} \mathsf{J}$ then the map $T \to S \times_{\mathfrak{M}} \mathsf{J}$ factors via $U \xrightarrow{\mathrm{i} \times \varphi_{\mathcal{L}(\alpha)}} S \times_{\mathfrak{M}} \mathsf{J}$*

3. *for any trait T with generic point η and any map $T \to S$ sending η to a point in U, if the map $\eta \to \mathsf{J}$ induced by $\mathcal{L}(\alpha)$ can be extended to a map $T \to \mathsf{J}$ then the map $T \to S$ factors via U.*

Proof. Condition (2) is exactly the valuative criterion for properness for the map $U \xrightarrow{\mathrm{i} \times \varphi_{\mathcal{L}(\alpha)}} S \times_{\mathfrak{M}} \mathsf{J}$, hence (1) and (2) are equivalent.

The difference between conditions (2) and (3) is about whether we allow the underlying curve to change when we construct an extension of the line bundle, but in fact this makes no difference, by Lemma 3.4.12. □

Lemma 3.4.12. *Let T be a trait with generic point η and let C, C' be prestable curves over T. Let $\varphi\colon C_\eta \to C'_\eta$ be an isomorphism over η. Write J for the multidegree-$\underline{0}$ jacobian of C/T (so J is the pullback of J along the classifying map $T \to \mathfrak{M}$ of C), and similarly define J' from C'. Then the isomorphism $J_\eta \xrightarrow{\sim} J'_\eta$ induced by φ extends uniquely to a T-isomorphism $J \xrightarrow{\sim} J'$.*

It is easy to see that the fibres of J and J' over the closed point t of T are isomorphic, since C_t and C'_t just differ by inserting some chains of rational curves at nodes, and adding some rational tails (see the explicit description of J_t in [BLR90, Example 9.2.8]. However, this is not enough; we need a (unique) global isomorphism compatible with φ.

Proof. This follows from the uniqueness of semiabelian prolongations over normal noetherian bases in [Ray70, XI, 1.15]. The result is stated there in the case where the generic fibre C_η is of compact type (so J_η is abelian, but the proof goes through unchanged if one allows arbitrary prestable curves and replaces Néron models of abelian schemes by Néron aft-models of semiabelian schemes. □

[15] A *trait* is the spectrum of a discrete valuation ring.

Lemma 3.4.13. *Let* $(C/S, \mathcal{L})$ *be* almost *twistable with* α *a twisting function. Write* $\varphi_{\mathcal{L}}\colon S \to \mathfrak{Jac}$ *for the map induced by* $\mathcal{L}$*, and* $\varphi_{\mathcal{L}(\alpha)}\colon S \to \mathfrak{Jac}$ *for the map induced by* $\mathcal{L}(\alpha)$*. Then*

$$\varphi_{\mathcal{L}}^{*}\mathsf{LogDR} = \varphi_{\mathcal{L}(\alpha)}^{*}\mathsf{DR} \tag{3.4.3.4}$$

in LogCH(S)*.*

Proof. Write $\sigma\colon S \dashrightarrow \mathsf{J}$ for the rational map induced by $\mathcal{L}$. Then the inclusion $U \hookrightarrow S$ is the universal σ-extending morphism. More precisely, the extension is given by $\mathcal{L}(\alpha)\colon U \to \mathsf{J}$, and the second property of Definition 3.4.10 shows it to be universal among extensions. Since $\mathcal{L}(\alpha)$ is of total degree 0 over the whole of S, it defines a map $\bar{\sigma}\colon S \to \bar{\mathsf{J}}$ over the whole of S. Consider the diagram

$$\begin{array}{ccc} U \times_{\mathsf{J}} e & \longrightarrow & e \\ \downarrow & & \downarrow i \\ U & \xrightarrow{\ \sigma\ } & \mathsf{J} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\ \bar{\sigma}\ } & \bar{J} \end{array} \qquad (j\colon U \times_{\mathsf{J}} e \to S) \tag{3.4.3.5}$$

where both squares are pullbacks (the top by construction, the bottom by the defining property of U), so that $U \times_{\mathsf{J}} e = S \times_{\bar{\mathsf{J}}} e$. In the notation of Section 3.3.5 we take $S = \tilde{S}$ and $X = \tilde{X} = U$. Then the definitions of $\varphi_{\mathcal{L}}^{*}\mathsf{LogDR}$ and $\varphi_{\mathcal{L}(\alpha)}^{*}\mathsf{DR}$ simplify to

$$\varphi_{\mathcal{L}}^{*}\mathsf{LogDR} = j_{*}i^{!}[U] \quad \text{and} \quad \varphi_{\mathcal{L}(\alpha)}^{*}\mathsf{DR} = j_{*}\bar{\sigma}^{!}[e] = j_{*}\sigma^{!}[e], \tag{3.4.3.6}$$

which are equal by the commutativity of the intersection pairing. □

The hard work remaining in this paper is to show that there are 'enough' almost-twistable families for Lemma 3.4.13 to determine LogDR from DR.

3.4.4 Extending piecewise-linear functions

Let C/S be a log curve. The key to showing the existence of enough almost-twistable families will be to extend PL functions over open subsets of S to PL functions over the whole of S, perhaps after some monoidal alteration.

Lemma 3.4.14. *Let* C/S *be a log curve with* S *a smooth log smooth log algebraic stack. Then there exist*

1. *a monoidal alteration* $\tilde{S} \to S$;
2. *a subdivision* $\tilde{C} \to C \times_S \tilde{S}$

with $\tilde{C}/\tilde{S}$ *a log curve and* $\tilde{C}$ *regular.*

Proof. This follows from [ALT18]. More precisely, their Theorem 4.4 gives a canonical monoidal resolution over schemes, which therefore applies to stacks. The argument in the proof of their Theorem 4.5 then shows that this monoidal resolution has $\tilde{C}$ regular. □

After applying the above lemma we will show that PL functions always extend. We start by considering the case where the base S is very small (*nuclear* in the sense of [HMOP23]), after which we will glue to a global solution.

Lemma 3.4.15. *Let C/S be a regular log curve over a (regular) log regular base, with C/S nuclear. Let $U \hookrightarrow S$ be strict dense open and let α be a PL function on C_U/U. Then we construct an extension $\bar{\alpha}$ to a PL function on C/S, and this construction is compatible with strict open base-change.*

Proof. Let s be the generic point of the closed stratum of S. Let r be the rank of $\bar{\mathsf{M}}_{S,s}$, and let $D_1, \ldots, D_r$ be the divisorial strata of the boundary of S. Let Γ be the graph of C/S over the closed stratum, and let Γ_i be the graph over the generic point of D_i (obtained by contracting those edges of Γ whose lengths differ from D_i). Our assumption that C/S be nuclear implies that S is simple (equivalently every intersection of D_is is regular and connected), and moreover that the intersection of *all* of the D_is is non-empty.

On each D_i with non-empty intersection with U we equip Γ_i with the PL function from α, and for the other D_i we put the zero PL function.

Now let z be a stratum of S, with graph Γ_z, and let $N_z \subseteq \{1, \ldots, r\}$ be such that $\overline{\{z\}} = \bigcap_{i \in N_z} D_i$. Then by (3.3.2.4) we have $\bar{\mathsf{M}}_{S,z} = \bigoplus_{i \in N_z} \mathbb{N} \cdot D_i$; write $f_i \colon \mathbb{N} \cdot D_i \to \bar{\mathsf{M}}_{S,z}$ for the natural inclusion. Let v be a vertex of Γ_z, and for each $i \in N_z$ let v_i be its image in Γ_i under specialisation. Then we define

$$\bar{\alpha}(v) = \sum_{i \in N_z} f_i(\alpha(v_i)). \tag{3.4.4.1}$$

To check that $\bar{\alpha}$ is a PL function on Γ_z, suppose that e is an edge of Γ_z between vertices u and v. By regularity of C we know that the length of e is D_i for some $i \in N_z$; suppose it is D_1. Then $f_i(u) = f_i(v)$ for every $i \neq 1$, and $D_1 \mid \alpha(f_1(u)) - \alpha(f_1(v))$. It is easy to see that $\bar{\alpha}$ restricts to α over U.

Suppose that $S' \to S$ is a strict open map such that $C_{S'}/S'$ is also nuclear and $C_{S'}$ is regular. Let s' be the generic points of the closed stratum of S'; it is enough to check the result for the restriction of $\bar{\alpha}$ to $\Gamma_{s'}$. Let $N' \subseteq \{1, \dots, r\}$ be the set of those D_i meeting the image of s'. Then each of those D_i meet the image of S', and their pullbacks are exactly the divisorial strata on S' (so in particular the rank of $\bar{\mathsf{M}}_{S',s'}$ is $\#N'$). Then $\bar{\alpha}$ on $\Gamma_{s'}$ is constructed by interpolating the values of α on the D_i for $i \in N'$, regardless of whether we compute this on S or on S'; in particular, these give the same result. □

Lemma 3.4.16. *Let C/S be a log curve with C (and hence S) regular log regular, S a log algebraic stack. Let $U \hookrightarrow S$ be a strict dense open immersion and α a PL function on C_U. Then there exists a PL function $\bar{\alpha}$ on S restricting to α.*

Proof. By [HMOP23, lemma 3.40] we know that S admits a strict étale cover $\{S_i\}_i \to S$ with each $C \times_S S_i/S_i$ nuclear. In Lemma 3.4.15 we give a canonical choice of extension for each $C \times_S S_i/S_i$, and these are compatible with smooth base-change, so descend to algebraic stacks. □

3.4.5 LogDR from DR

We wish to compute LogDR in LogCH($\mathfrak{Jac}$). Let $i\colon S \hookrightarrow \mathfrak{Jac}$ be a strict open immersion with S quasi-compact, and write C/S for the universal curve and $\mathcal{L}$ on C for the universal line bundle.

Lemma 3.4.17. *There exist*

1. *a monoidal alteration $\psi\colon \tilde{S} \to S$;*
2. *a subdivision $\tilde{C}$ of $C \times_S \tilde{S}$;*

such that the pair $(\tilde{C}/\tilde{S}, \psi^*\mathcal{L})$ is almost twistable.

Proof. Write $\sigma\colon S \dashrightarrow \mathsf{J}$ for the rational map induced by the line bundle $\mathcal{L}$, and let $\psi_1\colon S^\diamond \to S$ be the universal σ-extending morphism.

Claim: there exists a representable monoidal alteration $S^{\diamond\diamond}$ of $S^\diamond$ over which the map $\sigma\colon S^\diamond \to \mathsf{J}$ can be represented as $\psi_1^*\mathcal{L}(\alpha)$ for some PL function α over $S^{\diamond\diamond}$.

The claim is clear from Section 3.4.2 locally on $S^\diamond$, but these PL functions are only unique up to addition of a PL function from the base, and so need not glue. We define $S^{\diamond\diamond}$ to be the subfunctor of $S^\diamond$ where the maps $\alpha\colon C \to P$ (in the notation of Section 3.4.2) can be chosen such that their set of values is totally ordered (in the ordering on P induced by the monoid structure on $\bar{\mathsf{M}}^{\mathsf{gp}}$). That this subfunctor is a representable monoidal alteration of $S^\diamond$ is proven exactly as in [MW20, Theorem 5.3.4] for the map **Rub** $\to$ **Div**, to which it is closely analogous.

Now we can construct these α locally on $S^{\diamond\diamond}$ just as before, but with $P = \mathbb{G}_M^{\mathsf{trop}}$ and the additional requirement that the smallest value taken by α on any vertex is zero. This makes the α unique, hence they glue to a global PL function, proving the claim.

Now let $S^{\blacklozenge} \to S$ be a sufficiently fine log blowup for $S^{\diamond\diamond} \to S$, and let $U \hookrightarrow S^{\blacklozenge}$ be the lift of $S^{\diamond\diamond}$.

Writing $C^{\blacklozenge}/S^{\blacklozenge}$ for the pullback of C/S, we apply Lemma 3.4.14 to the $C^{\blacklozenge}/S^{\blacklozenge}$ to construct a monoidal alteration $\tilde{S} \to S^{\blacklozenge}$ and a subdivision $\tilde{C}$ of $C^{\blacklozenge} \times_{S^{\blacklozenge}} \tilde{S} = C \times_S \tilde{S}$ with $\tilde{C}$ (and hence $\tilde{S}$) regular. Writing $\psi \colon \tilde{S} \to S$ for the composite, we claim that the pair $(\tilde{C}/\tilde{S}, \psi^*\mathcal{L})$ is almost twistable.

Let $\tilde{U}$ be the pullback of U to $\tilde{S}$ (a twistable open), and let α be a twisting function over $\tilde{U}$. Then Lemma 3.4.16 implies that this α can be extended to a PL function over the whole of $\tilde{S}$.

Now let $T \to S$ be a trait with generic point η landing in $\tilde{U}$. Suppose that the map $\eta \to \mathsf{J}$ given by $(\psi_1^*\mathcal{L})(\alpha)$, then it is proven in [Hol19, Lemma 4.3] that this cannot be extended to a map $T \to \mathsf{J}$ unless it can already be extended to a map $T \to U$. By Lemma 3.4.11 this shows that $(\tilde{C}/\tilde{S}, \psi^*\mathcal{L})$ is almost twistable, with $\tilde{U} \hookrightarrow \tilde{S}$ the largest twistable open. □

Let $(\tilde{C}/\tilde{S},\ \psi^*\mathcal{L})$ be as in the statement of Lemma 3.4.17, with twisting function α over $\tilde{S}$. Then we have maps

$$\varphi_{\mathcal{L}} \colon \tilde{S} \to \mathfrak{Jac} \quad \text{and} \quad \varphi_{\mathcal{L}(\alpha)} \colon \tilde{S} \to \mathfrak{Jac} \tag{3.4.5.1}$$

induced by $\psi^*\mathcal{L}$ and $\psi^*\mathcal{L}(\alpha)$ respectively. Then

Theorem 3.4.18. *We have an equality of cycles*

$$\varphi_{\mathcal{L}}^* \mathsf{LogDR} = \varphi_{\mathcal{L}(\alpha)}^* \mathsf{DR} \tag{3.4.5.2}$$

in $\mathrm{LogCH}(S)$.

Proof. Immediate from Lemmas 3.4.13 and 3.4.17. □

3.4.6 LogDR is tautological

If $\mathcal{L}$ is the universal line bundle on the universal curve $\pi \colon C \to \mathfrak{Jac}$, we define the class

$$\eta = \pi_*(c_1(\mathcal{L})^2) \in \mathrm{CH}(\mathfrak{Jac}). \tag{3.4.6.1}$$

In [BHP+23, Definition 4] we defined a tautological subring of $\mathrm{CH}(\mathfrak{Jac})$; it is the $\mathbb{Q}$-span of certain decorated prestable graphs of degree 0, as described in [BHP+23, Section 0.3.3]; in particular, it includes the class η from (3.4.6.1). Here we prove that LogDR is contained in the corresponding tautological subring of $\mathrm{LogCH}(\mathfrak{Jac})$. In fact, we can prove something stronger[16]. We write $\mathbb{Q}[\eta] \subseteq \mathrm{CH}(\mathfrak{Jac})$ for the sub-$\mathbb{Q}$-algebra of the Chow ring generated by the class η of (3.4.6.1), and recall from Definition 3.3.18 that $\mathbb{Q}[\eta]^{\mathsf{log}}$ denotes the corresponding subring of $\mathrm{LogCH}(\mathfrak{Jac})$; we show that LogDR lies in $\mathbb{Q}[\eta]^{\mathsf{log}}$.

Continuing in the notation of the previous subsection, we can pull back $\mathbb{Q}[\eta]^{\mathsf{log}}$ along $\varphi_{\mathcal{L}}\colon S \to \mathfrak{Jac}$ to give a subring of $\mathrm{LogCH}(S)$. Since $\varphi_{\mathcal{L}}$ is strict this is equivalent to pulling back $\mathbb{Q}[\eta] \subseteq \mathrm{CH}(\mathfrak{Jac})$ to $\mathrm{CH}(S)$, then taking the corresponding subring of $\mathrm{LogCH}(S)$. We denote the resulting subring $\mathbb{Q}[\eta]^{\mathsf{log}}_S \subseteq \mathrm{LogCH}(S)$.

Lemma 3.4.19. *Suppose k has characteristic zero. In the notation of Theorem 3.4.18, the cycle $\varphi^*_{\mathcal{L}(\alpha)}\mathsf{DR}$ lies in $\mathbb{Q}[\eta]^{\mathsf{log}}_S \subseteq \mathrm{LogCH}(S)$.*

We are grateful to Johannes Schmitt for pointing out an omission in an earlier version of the proof (as well as the strengthening mentioned above).

Proof. This is an easy consequence of Pixton's formula for DR on $\mathfrak{Jac}^\diamond$, as stated in equation (56) of [BHP+23, §0.7]. The formula expresses DR as a polynomial in the following classes:

1. The class $\pi_*(c_1(\psi^*\mathcal{L}(\tilde\alpha))^2)$;
2. Classes $\psi_h + \psi_{h'}$ where h, h' are the two half-edges forming an edge of a graph of $C/\tilde S$.

It hence suffices to show that the above classes lie in $\mathbb{Q}[\eta]^{\mathsf{log}}_S$; we treat them in order:

1. The class $\pi_*(c_1(\psi^*\mathcal{L}(\tilde\alpha))^2)$ can be expanded as a sum

$$\pi_*(c_1(\psi^*\mathcal{L})^2) + 2\pi_*(c_1(\psi^*\mathcal{L})c_1(\mathcal{O}_C(\tilde\alpha)) + \pi_*(c_1(\mathcal{O}_C(\tilde\alpha))^2).$$

   The first summand is the pullback of the tautological class $\eta = \pi_*(c_1(\mathcal{L})^2)$ from $\mathrm{CH}(\mathfrak{Jac})$, hence is in $\mathbb{Q}[\eta]^{\mathsf{log}}_S$. For the second summand, we can reduce to computing $\pi_*(c_1(\psi^*\mathcal{L})D)$ where D is some vertical prime divisor on $C/\tilde S$, say with image a prime divisor Z on $\tilde S$. Then for dimension

[16] This improvement was suggested to us by Johannes Schmitt, to whom we are very grateful for permission to include it.

reasons we see that $\pi_*(c_1(\psi^*\mathcal{L})D)$ is an integer multiple of the class of the boundary divisor Z, in particular is in $\mathbb{Q}[\eta]_S^{\mathsf{log}}$.

Finally, the class $c_1(\mathcal{O}_C(\tilde{\alpha}))$ can be written as a sum of vertical boundary divisors on $C/\tilde{S}$. If D and E are distinct vertical prime divisors then D and E meet properly, and their locus of intersection is a union of vertical codimension 2 loci in C (which push down to zero on $\tilde{S}$ for dimension reasons) and horizontal boundary strata which push forward to boundary classes on $\tilde{S}$.

It remains to show that $\pi_*(D^2)$ is tautological. For this, let Z be the prime divisor in $\tilde{S}$ which is the image of D, and let E be the vertical divisor lying over Z such that $\pi^*Z = D + E$. Then we reduce to the previous case by noting $\pi_*(D^2) = \pi_*(D \cdot (\pi^*Z - E)) = \pi_*(D \cdot E)$.

2. These are exactly the first Chern classes of conormal bundles to boundary divisors on $\tilde{S}$. As such they can be realised as self-intersections of these boundary divisors (perhaps after a harmless further blowup we may assume $\tilde{S}$ simple), hence are in $\mathbb{Q}[\eta]_S^{\mathsf{log}}$. □

Putting together Theorem 3.4.18 and Lemma 3.4.19 we obtain

Corollary 3.4.20. *Suppose k has characteristic zero. Write $T \subseteq \mathrm{CH}(\mathfrak{Jac})$ for the tautological ring as in [BHP+23, Definition 4], and $\mathbb{Q}[\eta] \subseteq T$ for the subring generated by the class η from (3.4.6.1). Then the log double ramification cycle $\mathsf{LogDR} \in \mathrm{LogCH}(\mathfrak{Jac})$ lies in $\mathbb{Q}[\eta]^{\mathsf{log}} \subseteq T^{\mathsf{log}}$.*

3.4.7 Conjecture C

In this section we prove Conjecture C of [MPS23]; we thank Johannes Schmitt and the anonymous referee for corrections and improvements to this argument. Given non-negative integers g, n satisfying $2g-2+n > 0$, we write $\overline{\mathcal{M}}_{g,n}$ for the corresponding stack of stable marked curves, with log structure given by the boundary divisor. We write $R_{g,n}$ for the subring of $\mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$ generated by the classes $\psi_1, \ldots, \psi_n$ and the boundary divisors of logarithmic blowups of $\overline{\mathcal{M}}_{g,n}$.

Let $a_1, \ldots, a_n$ be integers summing to 0, where $p_1, \ldots, p_n$ are the markings on the universal curve $C_{g,n}/\overline{\mathcal{M}}_{g,n}$. Write $\varphi_a \colon \overline{\mathcal{M}}_{g,n} \to \mathfrak{Jac}$ for the map induced by $\mathcal{O}_{C_{g,n}}(\sum_i a_i p_i)$.

Theorem 3.4.21 ([MPS23, Conjecture C]). *Suppose k is a field of characteristic zero. Then $\varphi_a^*\mathsf{LogDR}$ lies in $R_{g,n} \subseteq \mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$.*

Proof. We know $\mathsf{LogDR} \in \mathbb{Q}[\eta]^{\mathsf{log}}$, and one easily computes $\varphi_a^*\eta = \sum_i a_i^2\psi_i$. By definition there exists a simple log blowup $\widetilde{\mathcal{M}}_{g,n}$ on which $\varphi_a^*\mathsf{LogDR}$ is determined; write $t \in \mathrm{CH}(\widetilde{\mathcal{M}}_{g,n})$ for the determination. By Theorem 3.3.8 there exists a polynomial p in piecewise-linear functions on $\widetilde{\mathcal{M}}_{g,n}$ such that $\Phi_{\widetilde{\mathcal{M}}_{g,n}}(p) = t$, where $\Phi_{\widetilde{\mathcal{M}}_{g,n}}$ is the map as in (3.3.3.12). Now $\Phi_{\widetilde{\mathcal{M}}_{g,n}}$ is a ring homomorphism and piecewise-linear functions map to linear combinations of logarithmic boundary divisors, so the result follows. □

3.5 The double-double ramification cycle

3.5.1 Iterated double ramification cycles

Let r be a positive integer, and let $\mathfrak{Jac}^r$ be the fibre product of r copies of $\mathfrak{Jac}$ over $\mathfrak{M}$. This is smooth and log smooth, and comes with r projection maps to $\mathfrak{Jac}$. According to Definition 3.2.11 we can pull back LogDR along each of the projection maps, yielding r elements of $\mathrm{LogCH}(\mathfrak{Jac}^r)$. We define LogDR_r to be the product of these elements in the ring $\mathrm{LogCH}(\mathfrak{Jac}^r)$.

We can also give a more direct construction of LogDR_r. Write J^r for the r-fold fibre product of J with itself over $\mathfrak{M}$, with e_r the unit section. Then over the locus of smooth curves we have a tautological morphism $\sigma_r\colon \mathfrak{Jac}^r \to \mathsf{J}^r$, we view it as a rational map $\sigma_r\colon \mathfrak{Jac}^r \dashrightarrow \mathsf{J}^r$, and let $\mathfrak{Jac}^{r\lozenge}$ be the universal σ_r-extending stack over $\mathfrak{Jac}^r$. The pullback $\sigma_r^* e_r$ is proper over $\mathfrak{M}$, so we can apply the construction in Section 3.3.5 to obtain a class $[\sigma_r^* e_r]_{\mathsf{log}} \in \mathrm{LogCH}(\mathfrak{Jac}^r)$.

Lemma 3.5.1. *These two constructions of* LogDR_r *coincide, i.e.*

$$\mathsf{LogDR}_r = [\sigma_r^* e_r]_{\mathsf{log}}. \tag{3.5.1.1}$$

Proof. We begin by comparing $\mathfrak{Jac}^{r\lozenge}$ with $(\mathfrak{Jac}^{\lozenge})^r$, where the latter denotes the r-fold fibre product over $\mathfrak{M}$ in the category of fs log algebraic stacks. The composites $\mathfrak{Jac}^{r\lozenge} \to \mathsf{J}^r \to \mathsf{J}$ are σ-extending, hence the universal property furnishes r maps $\mathfrak{Jac}^{r\lozenge} \to \mathfrak{Jac}^{\lozenge}$, hence a map

$$\mathfrak{Jac}^{r\lozenge} \to (\mathfrak{Jac}^{\lozenge})^r \tag{3.5.1.2}$$

to the fibre product. On the other hand, the fibre product $(\mathfrak{Jac}^{\lozenge})^r$ is σ_r-extending, yielding an inverse to (3.5.1.2).

The claimed equality of cycles is then immediate from the construction in Section 3.3.5 and an application of [Ful84, example 6.5.2] (whose proof carries over to this setting essentially unchanged). □

Write $\mathcal{L}_1, \dots, \mathcal{L}_r$ for the tautological line bundles on the universal curve over $\mathfrak{Jac}^r$, with corresponding classes

$$\eta_i = \pi_*(c_1(\mathcal{L})^2) \in \mathrm{CH}(\mathfrak{Jac}), \tag{3.5.1.3}$$

and let $\mathbb{Q}[\eta^r]$ denote the sub-$\mathbb{Q}$-algebra of $\mathrm{CH}(\mathfrak{Jac}^r)$ generated by these classes. From Corollary 3.4.20 and the first construction of LogDR_r we obtain

Lemma 3.5.2.

$$\mathsf{LogDR}_r \in \mathbb{Q}[\eta^r]^{\mathsf{log}} \subseteq \mathrm{LogCH}(\mathfrak{Jac}^r). \tag{3.5.1.4}$$

3.5.2 $\mathrm{GL}_r(\mathbb{Z})$-invariance

Let G/S be a commutative group scheme and M an $r \times r$ matrix with integer coefficients. Writing $G^{\times_S^r}$ for the fibre product of G with itself r times over S, we write

$$[M]\colon G^{\times_S^r} \to G^{\times_S^r} \tag{3.5.2.1}$$

for the endomorphism induced by M. If $M \in \mathrm{GL}_r(\mathbb{Z})$ then this is an automorphism.

Applying this to $\mathfrak{Jac}$ over $\mathfrak{M}$ with $M \in \mathrm{GL}_r(\mathbb{Z})$ yields an automorphism

$$[M]\colon \mathfrak{Jac}^r \to \mathfrak{Jac}^r, \tag{3.5.2.2}$$

and pulling back along the map yields an automorphism

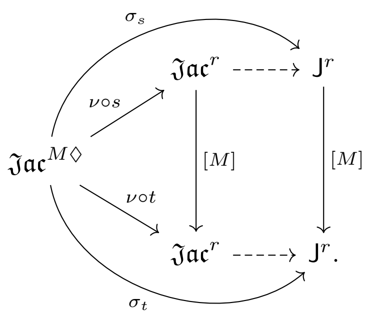
$$[M]^*\colon \mathrm{LogCH}(\mathfrak{Jac}^r) \to \mathrm{LogCH}(\mathfrak{Jac}^r). \tag{3.5.2.3}$$

Theorem 3.5.3. *The map $[M]^*$ of (3.5.2.3) takes LogDR_r to itself.*

Proof. For this we use the second construction of LogDR_r, going via $\mathfrak{Jac}^{r\diamondsuit}$. We write $\sigma_r\colon \mathfrak{Jac}^r \dashrightarrow \mathsf{J}^r$, and we write e for the unit section of J^r. We define $\mathfrak{Jac}^{M\diamondsuit}$ to be the limit (in the fs category) of the solid diagram

$$\begin{array}{ccccc} & & \mathfrak{Jac}^{r\diamondsuit} & \xrightarrow{\nu} & \mathfrak{Jac}^r \\ & \overset{s}{\nearrow} & & & \downarrow{\scriptstyle [M]} \\ \mathfrak{Jac}^{M\diamondsuit} & & & & \\ & \underset{t}{\searrow} & & & \\ & & \mathfrak{Jac}^{r\diamondsuit} & \xrightarrow{\nu} & \mathfrak{Jac}^r \end{array} \tag{3.5.2.4}$$

(we can think of $\mathfrak{Jac}^{M\diamondsuit}$ as the common refinement of $\mathfrak{Jac}^{r\diamondsuit}$ with its translation along $[M]$). Now the composite $\nu \circ s$ is σ_r-extending, as is the composite $\nu \circ t$, so we obtain a commutative diagram



(3.5.2.5)

Now $\sigma_s^* e$ is a cycle on $\mathfrak{Jac}^{M\diamondsuit}$, which can induce (following Section 3.3.5) a logarithmic cycle on $\mathfrak{Jac}^r$ in two ways; either via the map $\nu \circ s$ or via the map $\nu \circ t$. Our notation is $[\sigma_s^* e]_{\nu\circ s,\mathsf{log}}$ for the former and $[\sigma_s^* e]_{\nu\circ t,\mathsf{log}}$ for the latter, and we define analogously $[\sigma_t^* e]_{\nu\circ s,\mathsf{log}}$ and $[\sigma_t^* e]_{\nu\circ t,\mathsf{log}}$, all elements of $\mathrm{LogCH}(\mathfrak{Jac}_r)$. Applying lemma Lemma 3.5.1 and commutativity of the diagram yields the relations

$$\mathsf{LogDR}_r = [\sigma_s^* e]_{\nu\circ s,\mathsf{log}} = [\sigma_t^* e]_{\nu\circ t,\mathsf{log}}, \tag{3.5.2.6}$$

$$[M^{-1}]^* \mathsf{LogDR}_r = [\sigma_s^* e]_{\nu\circ t,\mathsf{log}} \quad \text{and} \quad [M]^* \mathsf{LogDR}_r = [\sigma_t^* e]_{\nu\circ s,\mathsf{log}}. \tag{3.5.2.7}$$

Finally, we note that $[M]^* e = e$ and $\sigma_t = [M] \circ \sigma_s$, so that

$$[M]^* \mathsf{LogDR}_r = [\sigma_t^* e]_{\nu\circ s,\mathsf{log}} = [\sigma_s^* M^* e]_{\nu\circ s,\mathsf{log}} = [\sigma_s^* e]_{\nu\circ s,\mathsf{log}} = \mathsf{LogDR}_r. \tag{3.5.2.8}$$

□

Remark 3.5.4. One can alternatively prove this theorem by appealing to the invariance of **Div** (see Section 3.4.2) under the action of M. ⧫

3.5.3 On the moduli space of curves

Here we translate the above results into the setting of [HPS19]. We fix non-negative integers g, n and a positive integer r. We choose r line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ of total degree zero on the universal curve $\pi\colon C \to \overline{\mathcal{M}}_{g,n}$, for example of the form

$$\mathcal{L}_i = \omega^{k_i}\Big(-\sum_{j=1}^{n} a_{i,j} x_j\Big) \tag{3.5.3.1}$$

where $a_{i,1}, \dots, a_{i,n}$ are integers summing to $k_i(2g-2)$. The tuple $\mathcal{L}_1, \dots, \mathcal{L}_r$ defines a morphism

$$\Psi\colon \overline{\mathcal{M}}_{g,n} \to \mathfrak{Jac}^r, \tag{3.5.3.2}$$

and we denote

$$\mathsf{LogDR}(\mathcal{L}_1, \dots, \mathcal{L}_r) = \Psi^*\mathsf{LogDR}_r \in \mathrm{LogCH}(\overline{\mathcal{M}}_{g,n}), \tag{3.5.3.3}$$

and

$$\mathsf{DR}(\mathcal{L}_1, \dots, \mathcal{L}_r) = \nu_*\mathsf{LogDR}(\mathcal{L}_1, \dots, \mathcal{L}_r) \in \mathrm{CH}(\overline{\mathcal{M}}_{g,n}). \tag{3.5.3.4}$$

Remark 3.5.5. Note that $\mathsf{LogDR}(\mathcal{L}_1, \dots, \mathcal{L}_r)$ is just the product of the classes $\mathsf{LogDR}(\mathcal{L}_i)$ for $1 \le i \le r$; this is *not* in general the case with DR in place of LogDR. ♦

Theorem 3.5.6 (DDR is tautological)**.** *Suppose k has characteristic zero. The cycle $\mathsf{DR}(\mathcal{L}_1, \dots, \mathcal{L}_r)$ lies in the tautological subring of $\mathrm{CH}(\overline{\mathcal{M}}_{g,n})$.*

Proof. For $1 \le i \le r$ define a codimension 1 class

$$\eta_i \coloneqq \pi_*(c_1(\mathcal{L}_i)^2) \in \mathrm{CH}(\overline{\mathcal{M}}_{g,n}); \tag{3.5.3.5}$$

these are of codimension 1 and hence tautological, so that $\mathbb{Q}[\eta_1, \dots, \eta_r]$ is a subring of the tautological ring T of $\overline{\mathcal{M}}_{g,n}$.

Pulling back Lemma 3.5.2 implies that the class $\mathsf{LogDR}(\mathcal{L}_1, \dots, \mathcal{L}_r)$ lies in $\mathbb{Q}[\eta_1, \dots, \eta_r]^{\mathsf{log}} \subseteq T^{\mathsf{log}}$. Now T is tectonic by Lemma 3.3.20, and therefore $\mathsf{DR}(\mathcal{L}_1, \dots, \mathcal{L}_r) \in T$ by Definition 3.3.19. □

Theorem 3.5.7 ($\mathrm{GL}(\mathbb{Z})$-invariance of DDR)**.** *If $M \in \mathrm{GL}_r(\mathbb{Z})$ and*

$$M[\mathcal{L}_1, \dots, \mathcal{L}_r] = [\mathcal{F}_1, \dots, \mathcal{F}_r],$$

then

$$\mathsf{DR}(\mathcal{L}_1, \dots, \mathcal{L}_r) = \mathsf{DR}(\mathcal{F}_1, \dots, \mathcal{F}_r). \tag{3.5.3.6}$$

Proof. Immediate from Theorem 3.5.3. □

In the case $r = 2$ this recovers [HPS19, Theorem 1.2].