# Logarithmic approach to the double ramification cycle

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# Chapter 2

# Piecewise-polynomial functions and divisors

This chapter is based on the presentation 'There and back again: an example session on translating between piecewise-polynomial functions and divisors' given on the 23rd of March 2022 at the conference 'Recent advances in moduli of curves' in Leysin. It is meant as a more detailed explanation with extra examples for the sections concerning piecewise-polynomial functions and the map to Chow group from the article [HS22], which is included in the next chapter of this thesis.

## 2.1 Notation and conventions

This chapter is meant to be independently understandable yet based on the article [HS22] (which is included in this thesis as chapter 3), and therefore all definitions and theorems that are necessary will be included here, along with the reference to the original in [HS22].

In the article [HS22], we work with log structures in the sense of Fontaine–Illusie–Kato, in particular working with the small étale site. That is, a *log scheme* $(X, \mathsf{M}_X, \alpha)$ is a scheme $X$ with sheaf of monoids with respect to the étale topology denoted by $\mathsf{M}_X$ and a morphism $\alpha\colon \mathsf{M}_X \to \mathcal{O}_X$ (where $\mathcal{O}_X$ is seen as a sheaf of monoids with the multiplication of functions) such that $\alpha$ identifies the units, i.e. $\alpha\colon \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism.

The *characteristic monoid sheaf* on a log scheme $(X, \mathsf{M}_X)$ is denoted $\bar{\mathsf{M}}_X$, which is sometimes referred to as the ghost sheaf and is obtained as the quotient $\mathsf{M}_X/\mathcal{O}_X^*$. The groupifications of these sheaves are denoted $\mathsf{M}_X^{\mathsf{gp}}$ and $\bar{\mathsf{M}}_X^{\mathsf{gp}}$ respectively. The first two examples arise simply from toric varieties: the notation for these toric varieties is the same as in [Ful93]. Each toric variety naturally gives rise to a log structure, as illustrated in the examples. Note that for toric varieties, a translation between piecewise-polynomial functions and divisors has been known for a longer period of time, for example by work of Payne [Pay06].

Most examples in this chapter will be log schemes, until Section 2.5, in which we explicitly describe a DM-stack and its log structure. A *log algebraic stack* is an algebraic stack equipped with an (fs) log structure. In [HS22], we mostly work with regular log regular log algebraic stacks. The stacks that we are interested in, including $\overline{\mathcal{M}}_{g,n}, \mathfrak{M}_{g,n}, \mathfrak{Pic}_{g,n}$, are of this form. This assumption allows us to use that, by [Niz06, 5.2], for a regular log regular log algebraic stack $S$ there exists a unique normal crossings divisor $Z$ on $S$ (the *boundary divisor* of $S$) with complement $i\colon U \to S$ and $\mathsf{M}_S = i_*\mathcal{O}_U$. Therefore, we may consider the log structure as the log structure associated to a certain divisor, making our discussion more explicit. These divisors shall be clearly described in all examples. Also, as the examples discussed in this chapter range over some possible boundary phenomena, these examples give a reasonable intuition for all possible log structures one may encounter while studying regular log regular log algebraic stacks.

In [HS22], we work over a field or Dedekind scheme $k$ equipped with trivial log structure. In the examples, we work over a field $k$ of characteristic zero, mostly for simplicity but also so that we can apply the results of [BHP$^+$23] and the previous chapter directly. The theory is developed in the *(2-)category of fine saturated (fs) log schemes (and stacks)* over $k$ with trivial log structure; in each of the examples the log structure is made explicit.

In [HS22], we work almost exclusively with *operational Chow groups* with rational coefficients, as defined in [BHP$^+$23, §2], denoted $\mathsf{CH}_{\mathsf{op}}$. All examples given here satisfy the properties of Lemma 1.2.6, and therefore, as well as for illustrational purposes, we use usual Chow groups in these examples.

## 2.2 Strict piecewise-polynomial functions

To start with, we consider the strict piecewise-polynomial functions, as discussed in [HS22, section 3.1-3.3] also found in this thesis at Section 3.3.1. Then, Section 2.3 discusses the maps to the Chow ring, after which we mention general piecewise-polynomial functions when assembling the defined maps to Chow. As we will see in example 3, to sensibly consider all piecewise-polynomial functions one may need to allow for strict piecewise-polynomial functions of a log blowup or subdivision.

Firstly, the definition contains a sheaf of symmetric algebras, which we will briefly introduce, for more detail see eg. [Sta13, 17.21]. In general, let $(X, \mathcal{O}_X)$ be a ringed site and $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules. We write $\operatorname{Sym}\mathcal{M}$ for the sheafification of the presheaf $U \mapsto \operatorname{Sym}(\mathcal{M}(U))$; it is a sheaf of $\mathcal{O}_X$-algebras. Note that this involves a sheafification and this may change possible global sections, e.g. as shown in Example 2.3.7.

If $X$ is any site and $\mathcal{A}$ a sheaf of abelian groups, then we view $\mathcal{A}$ as a sheaf of modules for the constant sheaf of rings $\mathbb{Z}$, yielding a sheaf $\operatorname{Sym}\mathcal{A}$ of graded $\mathbb{Z}$-algebras. In our case, we will apply the Sym construction to the sheaf of groupified characteristic monoids $\bar{\mathsf{M}}^{\mathsf{gp}}$.

**Example 2.2.1.** Consider the log scheme $\mathbb{A}^n = \operatorname{Spec}(k[\mathbb{N}^n])$ with the log structure induced by this monoid ring structure. This example will be important in Section 2.3 and will be explained more thoroughly in Example 2.3.1. For now, we just describe what happens to the groupified characteristic monoid sheaf when we apply the symmetric algebra construction. Writing $x_1, ..., x_n$ for each of the generators of $k[\mathbb{N}^n]$, the log structure map

$$\alpha\colon \mathsf{M}_{\mathbb{A}^n} \to \mathcal{O}_{\mathbb{A}^n}$$

is given on global sections by $k^* \oplus \mathbb{N}^n \to \mathsf{M}_{\mathbb{A}^n}(\mathbb{A}^n), (u, \underline{v}) \mapsto ux^{\underline{v}}$. The global sections of the sheaf of characteristic monoids, i.e. quotienting the units, is given by $\bar{\mathsf{M}}_{\mathbb{A}^n}(\mathbb{A}^n) = \mathbb{N}^n$. Applying the groupification then yields the scheme $\mathbb{A}^n$ with global sections $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{A}^n}(\mathbb{A}^n) = \mathbb{Z}^n$ as abelian groups and then $\operatorname{Sym}(\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{A}^n}(\mathbb{A}^n)) = \mathbb{Z}[X_1, \ldots, X_n]$. ♦

**Definition 2.2.2.** We define the *sheaf of strict piecewise-polynomial functions* on a log algebraic stack $S$ as

$$\mathrm{sPP}_S := \operatorname{Sym}\bar{\mathsf{M}}^{\mathsf{gp}}_S.$$

We write

$$\mathrm{sPP}^n_S = \operatorname{Sym}^n\bar{\mathsf{M}}^{\mathsf{gp}}_S,$$

for the graded pieces, and *strict piecewise-linear functions* are

$$\mathrm{sPP}^1_S = \mathrm{Sym}^1 \bar{\mathsf{M}}^{\mathsf{gp}}_S = \bar{\mathsf{M}}^{\mathsf{gp}}_S.$$

### 2.2.1 Example 1: Projective plane with toric boundary log structure

To study strict piecewise-polynomial functions, we firstly need to be clear on what the sheaf $\bar{\mathsf{M}}^{\mathsf{gp}}_S$ is. There are two ways to describe the log structure on a projective plane $\mathbb{P}^2_k$ that we refer to as the toric boundary log structure; either from the construction as toric variety, or via the log structure given by a certain boundary divisor.

#### Toric log structure

For the first perspective, construct $\mathbb{P}^2_k$ from the fan in $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^2$ as illustrated in Figure 2.1; that is, consider the fan containing the trivial cone $\tau = \{0\}$, the rays $\tau_0 = \langle -e_1 - e_2\rangle, \tau_1 = \langle e_1\rangle, \tau_2 = \langle e_2\rangle$, and finally the cones $\sigma_0 = \langle e_1, e_2\rangle, \sigma_1 = \langle -e_1 - e_2, e_2\rangle, \sigma_2 = \langle e_1, -e_1 - e_2\rangle$ (taking the convex polyhedral cone formed by these generators in the vector space, see [Ful93] for the notation).

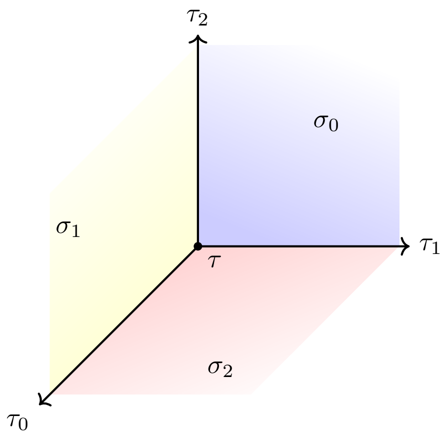



Figure 2.1: Fan for $\mathbb{P}^2_k$

This yields a toric variety as follows: firstly to each cone $\sigma$ one associates the dual cone $\sigma^\vee$ and then the dual lattice points in the dual cone form the monoids $\mathsf{S}_\sigma$. Then each cone specifies an affine variety $U_\sigma = \mathrm{Spec}(k[\mathsf{S}_\sigma])$. In this case, writing $e_i'$ for the dual generators, we obtain

$$\mathsf{S}_{\sigma_0} = \mathbb{N}\langle e_1', e_2'\rangle,$$
$$\mathsf{S}_{\sigma_1} = \mathbb{N}\langle -e_1', -e_1' + e_2'\rangle,$$
$$\mathsf{S}_{\sigma_2} = \mathbb{N}\langle e_1' - e_2', -e_2'\rangle,$$
$$\mathsf{S}_{\tau_0} = \mathbb{N}\langle e_1' - e_2', -e_1' + e_2', -e_1'\rangle,$$
$$\mathsf{S}_{\tau_1} = \mathbb{N}\langle e_1', e_2', -e_2'\rangle,$$
$$\mathsf{S}_{\tau_2} = \mathbb{N}\langle e_1', -e_1', e_2'\rangle,$$
$$\mathsf{S}_{\tau} = \mathbb{N}\langle e_1', -e_1', e_2', -e_2'\rangle.$$

Thus we can associate to $\sigma_0$ the variety $\mathrm{Spec}(k[X, Y])$, and similarly to $\sigma_1$ the variety $\mathrm{Spec}(k[X^{-1}, X^{-1}Y])$. As cones, $\sigma_0$ and $\sigma_1$ meet in the ray $\tau_2$ and thus glue as varieties over $\mathrm{Spec}(k[\mathsf{S}_{\tau_2}]) = \mathrm{Spec}(k[X, X^{-1}, Y])$. Rewriting to

$$\mathrm{Spec}(k[\mathsf{S}_{\sigma_0}]) = \mathrm{Spec}(k[t_1/t_0, t_2/t_0]), \text{ and}$$
$$\mathrm{Spec}(k[\mathsf{S}_{\sigma_1}]) = \mathrm{Spec}(k[t_0/t_1, t_2/t_1])$$

by strategically renaming the variables, we can observe this forms the projective plane with coordinates $(t_0 : t_1 : t_2)$; we may identify $\mathrm{Spec}(k[\mathsf{S}_{\sigma_0}])$ with the affine open where $t_0 \neq 0$, and $\mathrm{Spec}(k[\mathsf{S}_{\sigma_1}])$ with the affine open where $t_1 \neq 0$, which glue in the usual way over $\mathrm{Spec}(k[\mathsf{S}_{\tau_2}]) = \mathrm{Spec}(k[t_1/t_0, t_0/t_1, t_2/t_0])$.

More importantly, this procedure to construct the toric variety also specifies the log structure associated to the toric structure. The monoids $\mathsf{S}_\sigma$ on the affine open $U_\sigma$ give us the sheaf of characteristic monoids we are interested in. That is, in this case, the sheaf $\bar{\mathsf{M}} = \bar{\mathsf{M}}_{\mathbb{P}^2}$ is defined by the $\mathsf{S}_\sigma$ quotiented by the units, e.g.

$$\bar{\mathsf{M}}(U_{\sigma_1}) = \overline{\mathsf{S}_{\sigma_1}} = \overline{\mathbb{N}\langle -e_1', -e_1' + e_2'\rangle} = \mathbb{N}\langle -e_1', -e_1' + e_2'\rangle\,,$$
$$\bar{\mathsf{M}}(U_{\tau_1}) = \overline{\mathsf{S}_{\tau_1}} = \overline{\mathbb{N}\langle e_1, e_2, -e_2\rangle} = \mathbb{N}\langle e_1\rangle\,,$$

etc.

*Remark* 2.2.3. We are describing the sheaf of monoids as a sheaf on the Zariski site, while as convention we use log structures on the étale site. However, in the case of toric varieties, and so in our example 1 and 2, we have that the log structure is Zariski (terminology e.g. in [Niz06]). Therefore the log structure may be defined on the Zariski site, and for illustrational purposes we will do so. In example 3, we will work with an étale cover to compute the global sections instead. ♦

### Boundary log structure

Recall, the log structure associated to a normal crossings divisor $Z$ on $S$ with complement $i: U \to S$ is given by $\mathsf{M}_S = i_*\mathcal{O}_U$. We use the following more explicit description.

**Definition 2.2.4.** For a smooth scheme $X$ and $D$ a normal crossings divisor, the *log structure associated to the divisor* is given by

$$\mathsf{M}_D(U) = \{f \in \mathcal{O}_X(U) \mid f_{|U\setminus D} \in \mathcal{O}^*(U \setminus D)\} \subset \mathcal{O}_X^*(U),$$

which is a subsheaf of $\mathcal{O}_X^*$. The log structure map $\alpha_D : \mathsf{M}_D \to \mathcal{O}_X$ is the natural inclusion.

Define $D_0 := \{(0 : t_1 : t_2)\} \subset \mathbb{P}^2_k$, $D_1 := \{(t_0 : 0 : t_2)\} \subset \mathbb{P}^2_k$, and $D_2 := \{(t_0 : t_1 : 0)\} \subset \mathbb{P}^2_k$, then consider the log structure associated to $D_0, D_1, D_2$ as boundary divisors. Writing $D$ for the union of the divisors, these define the structure

$$\mathsf{M}_{\mathbb{P}^2_k, D}(U) = \{f \in \mathcal{O}_{\mathbb{P}^2_k}(U) \mid f \text{ invertible outside } D_0, D_1, D_2\}.$$

For example on the affine patch $U_{t_0 \neq 0}$, the monoid contains those functions in $k[t_1/t_0, t_2/t_0]$ that are invertible outside where $t_1 = 0$ or $t_2 = 0$. Thus, also the multiples of the functions $t_1/t_0$ or $t_2/t_0$ are examples of invertible functions in $\mathsf{M}_{\mathbb{P}^2_k, D}$.

**Lemma 2.2.5.** *The log structure associated to the boundary divisors $D_0, D_1, D_2$ is the same as (uniquely isomorphic to) the log structure described above coming from the toric fan.*

*Proof.* The first log structure map $\alpha_D \colon \mathsf{M}_D \to \mathcal{O}_{\mathbb{P}^2_k}$ is the natural inclusion, so the sheaf of monoids lies injectively in the structure sheaf. The toric log structure map $\alpha \colon \mathsf{M} \to \mathcal{O}_{\mathbb{P}^2_k}$ is induced by the maps $\mathsf{S}_\sigma \to k[\mathsf{S}_\sigma]$ (followed by the appropriate renaming of coordinates), again injectively in the structure sheaf.

We sketch the proof on the level of characteristic monoids sheaves on $U_{\sigma_0}$. The first log structure

$$\begin{aligned}\mathsf{M}_D(U_{\sigma_0}) &= \{f \in \mathcal{O}_{\mathbb{P}^2}(U_{\sigma_0}) \mid f \text{ invertible outside } D_0, D_1, D_2\} \\ &= \{f \in \mathcal{O}_{\mathbb{P}^2}(U_{t_0 \neq 0}) \mid f \text{ invertible outside } D_0, D_1, D_2\} \\ &= \{f \in k[t_1/t_0, t_2/t_0] \mid f \text{ invertible outside } t_0 = 0, t_1 = 0, t_2 = 0\},\end{aligned}$$

and after quotienting by the invertible functions, we obtain for the characteristic monoid $\bar{\mathsf{M}}_D(U_{\sigma_0}) = \mathbb{N}\langle t_1/t_0, t_2/t_0\rangle$ with the natural inclusion map $\mathbb{N}\langle t_1/t_0, t_2/t_0\rangle \hookrightarrow k[t_1/t_0, t_2/t_0]$. Regarding the toric description, the monoid is given by

$$\bar{\mathsf{M}}(U_{\sigma_0}) = \mathsf{S}_{\sigma_0} = \mathbb{N}\langle e_1', e_2'\rangle$$

and the log structure map $\alpha$ induces

$$\mathbb{N}\langle e_1', e_2'\rangle \to k[X, Y] \cong k[t_1/t_0, t_2/t_0]$$

simply sending to each generator to the associated variable. Hence the log structures are the same on the level of characteristic monoids. $\square$

### Drawing stalks of the log structure

By computing the stalks of the characteristic monoid, we may obtain the usual sketch of the log structure. Consider a point $p = (p_0 : p_1 : 1)$ in $U_{\sigma_2}$, then if both $p_0 \neq 0$ and $p_1 \neq 0$, all coordinates are invertible and we have $p \in U_{\{0\}}$. Therefore the stalk of the characteristic monoid is given by

$$\bar{\mathsf{M}}_p = \bar{\mathsf{M}}(U_{\{0\}}) = \overline{\mathbb{N}\langle e_1, e_2, -e_1, -e_2\rangle} = 0.$$

If however $p_0 \neq 0$ but $p_1 = 0$, then $p \in U_{\tau_1}$ but $p \notin U_{\{0\}}$, yielding

$$\bar{\mathsf{M}}_p = \bar{\mathsf{M}}(U_{\tau_1}) = \overline{\mathbb{N}\langle e_1, e_2, -e_2\rangle} = \mathbb{N}\langle e_1\rangle\,.$$

Finally, if both $p_0 = 0$ and $p_1 = 0$, then $p \notin U_{\tau_0}$ and $p \notin U_{\tau_1}$ so

$$\bar{\mathsf{M}}_p = \bar{\mathsf{M}}(U_{\sigma_2}) = \overline{\mathbb{N}\langle e_1 - e_2, -e_2\rangle} = \mathbb{N}\langle a, b\rangle\,.$$

Therefore, we may draw an overview of the log structure via the stalks of the characteristic monoid sheaf as in Figure 2.2.

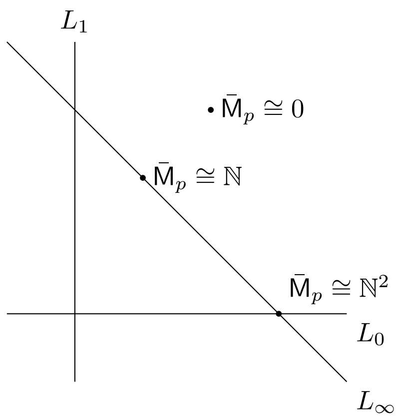



Figure 2.2: Stalks of the characteristic monoids for $\mathbb{P}^2_k$

*Remark* 2.2.6. It is also not a coincidence that the log structure associated to divisors $D_0, D_1, D_2$ yields Figure 2.2. When a normal crossings divisor $D$ locally at a $p \in D$ with a set of local coordinates $g_i$, is given by the vanishing of $g_1 \cdots g_e$ (that is, locally the intersection of $e$ coordinate hyperplanes), then we have an étale chart $\mathbb{N}^e \to \bar{\mathsf{M}}, e_i \mapsto g_i$ giving $\bar{\mathsf{M}}_p \cong \mathbb{N}^e$. Hence, we indeed see a $\mathbb{N}$ generator at the stalk of $\bar{\mathsf{M}}$ for each branch of the boundary divisor through that point. See [Kat96] for the formal explicit description of such log structures. ♦

### Gluing maps

To describe the strict piecewise-polynomial functions

$$\mathrm{sPP}_{\mathbb{P}^2_k} = \mathrm{Sym}\,\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k},$$

we view the global sections as functions on the stalks, or on small enough opens, that glue appropriately. Therefore, we will make the gluing maps for the sheaf of characteristic monoids more explicit, describing them from explicit restriction maps for the open cover and the toric log structure perspective. Rename the generators of $\mathsf{S}_{\sigma_0} = \mathbb{N}\langle a, b\rangle$, $\mathsf{S}_{\sigma_1} = \mathbb{N}\langle x, y\rangle$, $\mathsf{S}_{\sigma_2} = \mathbb{N}\langle u, v\rangle$; that is $a = e_1', b = e_2', x = -e_1', y = e_2' - e_1', u = e_1' - e_2', v = -e_2'$. These integer generators in the dual space are drawn in Figure 2.3, and this figure will also help to see what the specific restriction maps do.

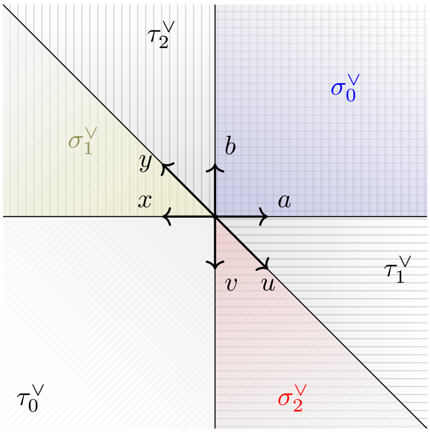



Figure 2.3: Monoids $\sigma_i^\vee$ in relation to the $\tau_i^\vee$.

Explicitly, to deduce what the restriction maps induce on the level of monoids, firstly observe that for an inclusion $\tau_2 \subset \sigma_0$, the natural inclusion $\sigma_0^\vee \hookrightarrow \tau_2^\vee$ gives the map

$$\mathsf{S}_{\sigma_0} \cong \mathbb{N}\langle a, b\rangle \hookrightarrow \mathbb{N}\langle a, -a, b\rangle \cong \mathsf{S}_{\tau_2},$$

which after quotienting by units yields $\bar{\mathsf{M}}(U_{\sigma_0}) \to \bar{\mathsf{M}}(U_{\tau_2})$ explicitly given by $\mathbb{N}\langle a, b\rangle \to \mathbb{N}\langle t\rangle, a \mapsto 0, b \mapsto t$.

By similar reasoning, we obtain all other restriction maps:

$$\begin{aligned}
\mathsf{S}_{\sigma_0} &\cong \mathbb{N}\langle a, b\rangle \hookrightarrow \mathbb{N}\langle a, b, -b\rangle \cong \mathsf{S}_{\tau_1} \\
\mathsf{S}_{\sigma_1} &\cong \mathbb{N}\langle x, y\rangle \hookrightarrow \mathbb{N}\langle x, -x, y\rangle \cong \mathsf{S}_{\tau_2} \\
\mathsf{S}_{\sigma_1} &\cong \mathbb{N}\langle x, y\rangle \hookrightarrow \mathbb{N}\langle x, y, -y\rangle \cong \mathsf{S}_{\tau_0} \\
\mathsf{S}_{\sigma_2} &\cong \mathbb{N}\langle u, v\rangle \hookrightarrow \mathbb{N}\langle u, -u, v\rangle \cong \mathsf{S}_{\tau_0} \\
\mathsf{S}_{\sigma_2} &\cong \mathbb{N}\langle u, v\rangle \hookrightarrow \mathbb{N}\langle u, v, -v\rangle \cong \mathsf{S}_{\tau_1}
\end{aligned}$$

Then quotienting by the units, and subsequently taking the groupification, yields the following diagram of restriction maps for the groupified characteristic monoid sheaf.

$$\begin{tikzcd}
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\sigma_0}) \cong \mathbb{Z}\langle a, b\rangle \arrow[r, "a\mapsto 0"] \arrow[dr, "b\mapsto 0"'] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_2}) \cong \mathbb{Z}\langle t\rangle \\
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\sigma_1}) \cong \mathbb{Z}\langle x, y\rangle \arrow[ur, "x\mapsto 0"] \arrow[dr, "y\mapsto 0"'] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_1}) \cong \mathbb{Z}\langle p\rangle \\
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\sigma_2}) \cong \mathbb{Z}\langle u, v\rangle \arrow[ur, "v\mapsto 0"] \arrow[r, "u\mapsto 0"'] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_0}) \cong \mathbb{Z}\langle q\rangle
\end{tikzcd}$$

Note that in order to visualise the interactions between the characteristic monoids, it is easier to consider the geometry of the cones in the fan than the dual cones. An attempt to draw the gluing rules is shown in Figure 2.4; note that here the $a = 0$ side of the triangle automatically coincides with the line corresponding to $\tau_2$, and indeed $a \mapsto 0$ maps to $\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_2})$ generated by one element.

### Strict piecewise-polynomial functions

A strict piecewise-linear function in $\mathrm{sPP}^1_{\mathbb{P}^2}(\mathbb{P}^2_k) = \mathrm{Sym}^1 \bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(\mathbb{P}^2_k) = \bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(\mathbb{P}^2_k)$ can be computed by the equaliser diagram

$$\mathrm{sPP}_{\mathbb{P}^2_k}(\mathbb{P}^2_k) \to \mathrm{sPP}_{\mathbb{P}^2_k}(\mathcal{U}) \rightrightarrows \mathrm{sPP}_{\mathbb{P}^2_k}(\mathcal{U} \times_{\mathbb{P}^2_k} \mathcal{U})$$

for the cover $\mathcal{U}$ consisting of $U_{\sigma_0}, U_{\sigma_1}, U_{\sigma_2}$. Therefore a strict piecewise-linear function is given by

$$\begin{Bmatrix} \alpha a & + & \beta b \\ \gamma x & + & \delta y \\ \epsilon u & + & \zeta v \end{Bmatrix}$$

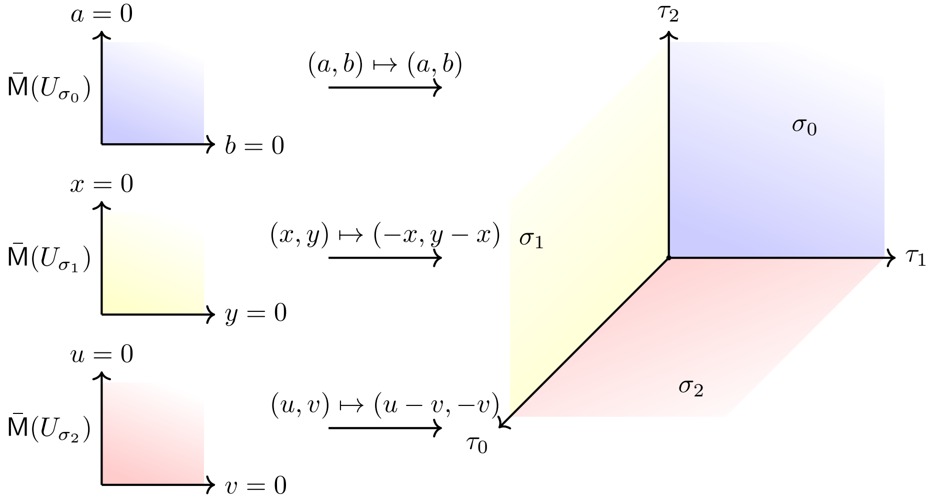



Figure 2.4: Visualisation of the interactions of the characteristic monoid sheaves in $\mathbb{P}^2_k$.

with $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{Z}$, which define local sections of the characteristic monoid sheaf, explicitly elements in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(U_{\sigma_0}), \bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(U_{\sigma_1})$ and $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(U_{\sigma_2})$. In order to glue to a global section, the images in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(U_{\tau_i})$ along the restriction maps must coincide, as for the Zariski cover we have $U_{\sigma_0} \times_{\mathbb{P}^2} U_{\sigma_1} = U_{\tau_2}$, etc. Hence the coefficients must satisfy $\beta = \delta, \alpha = \epsilon, \zeta = \gamma$, and so we have 3 coefficients generated by the rays through the $\tau_i$ that together define a global section of the sheaf of strict piecewise-linear functions.

A simple way to construct a strict piecewise-polynomial function of for example degree 2 is to take a product of piecewise-linear functions. For example one might consider

$$\begin{Bmatrix} a & + & b \\ & & y \\ u & & \end{Bmatrix} \begin{Bmatrix} & & b \\ x & + & y \\ & & v \end{Bmatrix} = \begin{Bmatrix} ab + b^2 \\ xy + y^2 \\ uv \end{Bmatrix}$$

as element in $\mathrm{Sym}^2(\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(\mathbb{P}^2_k))$. Essentially this approach is similar to simply taking elements of $\mathrm{Sym}^2(\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2_k}(U_{\sigma_i}))$ and ensuring the images along restriction maps for the sheaf of groupified characteristic monoids coincide, which defines a global section of the sheafification $\mathrm{sPP}_{\mathbb{P}^2_k}$.

### 2.2.2 Example 2: Blowup of a projective plane in a point with toric log structure

Similarly to the approach in the previous example, we start this section by explicitly describing the sheaf $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}$ for the blowup $\tilde{S} := \mathrm{Bl}_{(1:0:0)}(\mathbb{P}^2_k)$ of a projective plane $S := \mathbb{P}^2_k$ in a point, see Figure 2.5. The log structure can be realised from toric geometry with the subdivision of the fan from the previous example as shown in Figure 2.6, or taking the log structure associated to the boundary divisors $\bar{D}_0, \bar{D}_1, \bar{D}_2$ (the strict transforms of the boundary divisors $D_i$ in the previous example) and the exceptional divisor $E$. As discussed in the previous example, these yield the same log structure on $\tilde{S}$. Here we work out the toric perspective due to importance of the intuition attached to the concept of subdivision.

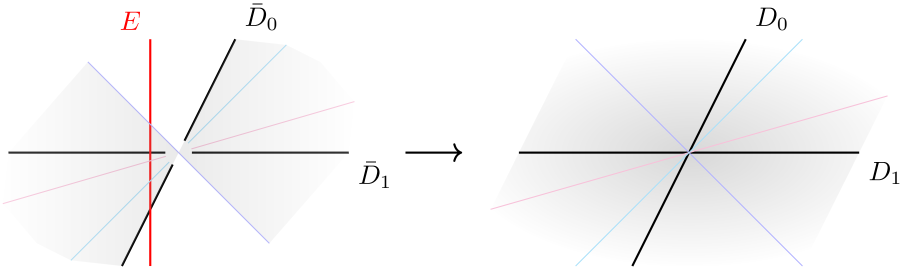



Figure 2.5: The blowup map $\mathrm{Bl}_{(1:0:0)}\,\mathbb{P}^2_k \to \mathbb{P}^2_k$ on affine patch $t_2 \neq 0$

This example will serve to illustrate that blowing up or subdividing yields more strict piecewise-polynomial (in particular strict piecewise-linear) functions, which we will use to define our map to the Chow group in the next section.

#### Toric log structure

Subdividing the correct cone in the fan for $\mathbb{P}^2_k$ yields the fan for the blowup $\tilde{S}$ as shown in Figure 2.6. Again, to each cone $\sigma$ in the fan, we associate the affine variety $U_\sigma$ and on these affine opens we define the sheaf of monoids $\bar{\mathsf{M}}_{\tilde{S}}(U_\sigma) = \overline{\mathsf{S}_\sigma}$. We make the gluing maps more explicit in order to describe the strict piecewise-polynomial functions. Rename the generators of $\mathsf{S}_{\bar{\sigma}} = \mathbb{N}\langle c, d\rangle$, $\mathsf{S}_{\sigma'} = \mathbb{N}\langle e, f\rangle$, $\mathsf{S}_{\sigma_1} = \mathbb{N}\langle x, y\rangle$, $\mathsf{S}_{\sigma_0} = \mathbb{N}\langle u, v\rangle$. Explicitly, the reader may draw all dual cones similarly as done in the previous example Figure 2.3, and take

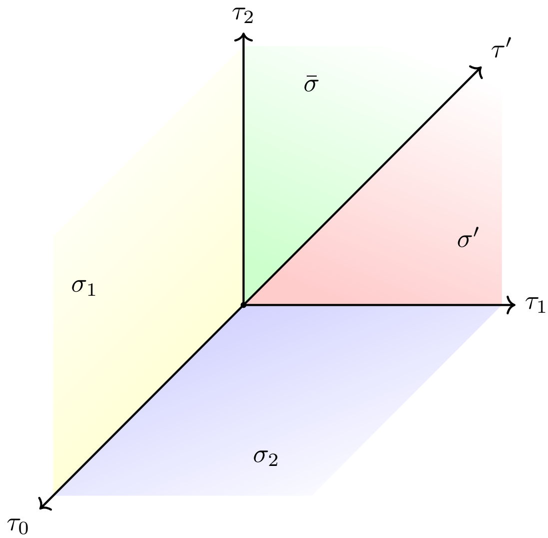



Figure 2.6: Fan for $\tilde{S} = \mathrm{Bl}_{(1:0:0)}(\mathbb{P}^2_k)$

our integer generators (expressed in standard dual basis vectors) to be $c = e_1'$, $d = e_2' - e_1', e = e_1' - e_2', f = e_2', x = -e_1', y = e_2' - e_1', u = e_1' - e_2', v = -e_2'$.

We can write down the restriction maps on the monoids $\mathsf{S}_\sigma$ via the inclusions of cones, giving:

$$\begin{aligned}
\mathsf{S}_{\bar{\sigma}} &\cong \mathbb{N}\langle c, d\rangle \hookrightarrow \mathbb{N}\langle c, d, -d\rangle \cong \mathsf{S}_{\tau'}\\
\mathsf{S}_{\bar{\sigma}} &\cong \mathbb{N}\langle c, d\rangle \hookrightarrow \mathbb{N}\langle c, -c, d\rangle \cong \mathsf{S}_{\tau_2}\\
\mathsf{S}_{\sigma'} &\cong \mathbb{N}\langle e, f\rangle \hookrightarrow \mathbb{N}\langle e, -e, f\rangle \cong \mathsf{S}_{\tau'}\\
\mathsf{S}_{\sigma'} &\cong \mathbb{N}\langle e, f\rangle \hookrightarrow \mathbb{N}\langle e, f, -f\rangle \cong \mathsf{S}_{\tau_1}\\
\mathsf{S}_{\sigma_1} &\cong \mathbb{N}\langle x, y\rangle \hookrightarrow \mathbb{N}\langle x, -x, y\rangle \cong \mathsf{S}_{\tau_2}\\
\mathsf{S}_{\sigma_1} &\cong \mathbb{N}\langle x, y\rangle \hookrightarrow \mathbb{N}\langle x, y, -y\rangle \cong \mathsf{S}_{\tau_0}\\
\mathsf{S}_{\sigma_2} &\cong \mathbb{N}\langle u, v\rangle \hookrightarrow \mathbb{N}\langle u, -u, v\rangle \cong \mathsf{S}_{\tau_0}\\
\mathsf{S}_{\sigma_2} &\cong \mathbb{N}\langle u, v\rangle \hookrightarrow \mathbb{N}\langle u, v, -v\rangle \cong \mathsf{S}_{\tau_1}
\end{aligned}$$

Subsequently quotienting by the units and groupifying yields the following diagram of restriction maps for the characteristic monoid sheaf.

$$\begin{tikzcd}
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\bar\sigma}) \cong \mathbb{Z}\langle c,d\rangle \arrow[r, "d\mapsto 0"] \arrow[dr, "c\mapsto 0"] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau'}) \cong \mathbb{Z}\langle r\rangle \\
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\sigma'}) \cong \mathbb{Z}\langle e,f\rangle \arrow[ur, "e\mapsto 0"'] \arrow[dr, "f\mapsto 0"] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_2}) \cong \mathbb{Z}\langle t\rangle \\
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\sigma_1}) \cong \mathbb{Z}\langle x,y\rangle \arrow[ur, "x\mapsto 0"'] \arrow[dr, "y\mapsto 0"] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_1}) \cong \mathbb{Z}\langle p\rangle \\
\bar{\mathsf{M}}^{\mathsf{gp}}(U_{\sigma_2}) \cong \mathbb{Z}\langle u,v\rangle \arrow[ur, "v\mapsto 0"'] \arrow[r, "u\mapsto 0"'] & \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_0}) \cong \mathbb{Z}\langle q\rangle
\end{tikzcd}$$

Note that in order to visualise the interactions between the characteristic monoids, we can (similarly to previous example Figure 2.4) draw a visualisation as follows in Figure 2.7.

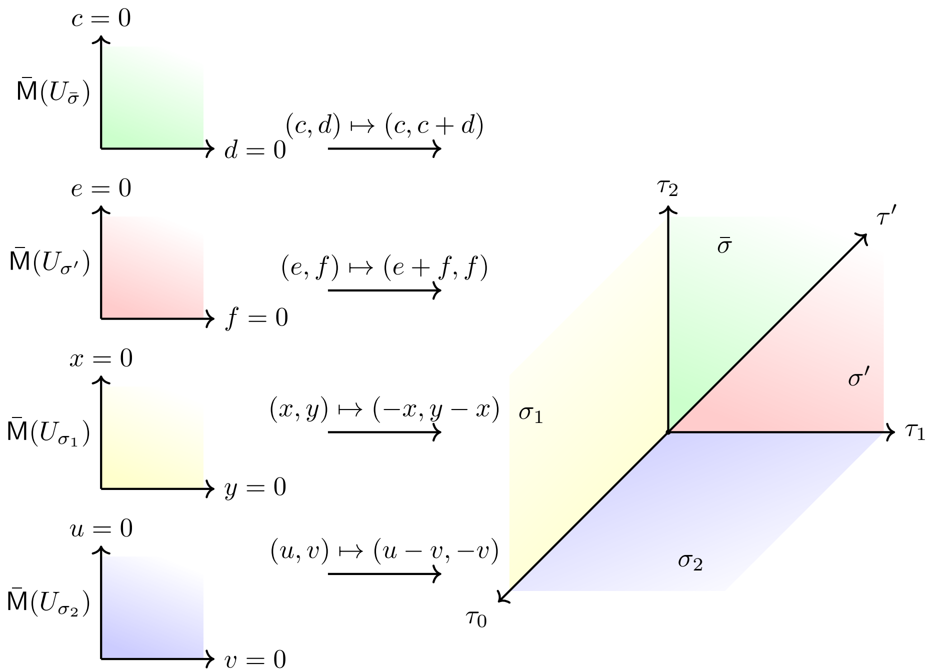



Figure 2.7: Visualisation of the interactions of the characteristic monoid sheaves in $\mathrm{Bl}_{(1:0:0)}\,\mathbb{P}^2_k$.

### Blowup morphism on log structures

Also useful for later discussion is the morphism $f^*\mathsf{M}_S \to \mathsf{M}_{\tilde{S}}$ that belongs to the blowup morphism $f : \tilde{S} \to S$. The map $f^*\bar{\mathsf{M}}_S \to \bar{\mathsf{M}}_{\tilde{S}}$ is defined via the inclusions $\tilde{\sigma} \to \sigma_0$ and $\sigma' \to \sigma_0$, which dually give $\sigma_0^\vee \to \bar{\sigma}^\vee$ and $\sigma_0^\vee \to \sigma'^\vee$, which yield $\bar{\mathsf{M}}_S(f(U_{\bar{\sigma}})) \to \bar{\mathsf{M}}_{\tilde{S}}(U_{\bar{\sigma}})$, and $\bar{\mathsf{M}}_S(f(U_{\sigma'})) \to \bar{\mathsf{M}}_{\tilde{S}}(U_{\sigma'})$.

The inclusions $\sigma' \hookrightarrow \sigma_0, \bar{\sigma} \hookrightarrow \sigma_0$ on the level of cones yield on the level of monoids $\mathsf{S}_\sigma$ the maps

$$\begin{aligned} \mathsf{S}_{\sigma_0} &\to \mathsf{S}_{\sigma'} \\ e_i' &\mapsto e_i'. \end{aligned}$$

Using the relations between the generators (recall that $a = e_1', b = e_2', c = e_1'$, $d = e_2' - e_1', e = e_1' - e_2', f = e_2', x = -e_1', y = e_2' - e_1', u = e_1' - e_2', v = -e_2'$), we obtain

$$\begin{aligned} \bar{\mathsf{M}}_S(U_{\sigma_0}) \cong \mathbb{N}\langle a, b\rangle &\to \mathbb{N}\langle c, d\rangle \cong \bar{\mathsf{M}}_{\tilde{S}}(U_{\sigma'}) \\ a &\mapsto c \\ b &\mapsto c + d, \text{ and} \end{aligned}$$

$$\begin{aligned} \bar{\mathsf{M}}_S(U_{\sigma_0}) \cong \mathbb{N}\langle a, b\rangle &\to \mathbb{N}\langle e, f\rangle \cong \bar{\mathsf{M}}_{\tilde{S}}(U_{\bar{\sigma}}) \\ a &\mapsto e + f \\ b &\mapsto f. \end{aligned}$$

Together with the identity maps on $U_{\sigma_2}$ and $U_{\sigma_1}$, these maps define the map $f^*\bar{\mathsf{M}}_S \to \bar{\mathsf{M}}_{\tilde{S}}$.

### Strict piecewise-polynomial functions

A strict piecewise-linear function in $\mathrm{sPP}^1_{\tilde{S}}(\tilde{S}) = \mathrm{Sym}^1 \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S}) = \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S})$ is, by similar reasoning as in the last example, given by sections of each $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2}(U_\sigma)$ whose images in the $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2}(U_\tau)$ coincide, that is

$$\begin{Bmatrix} \alpha c & + & \beta d \\ \gamma e & + & \alpha f \\ \delta x & + & \beta y \\ \gamma u & + & \delta v \end{Bmatrix} \tag{2.2.2.1}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Hence we have 4 coefficients, one for each ray $\tau_0, \tau_1, \tau_2, \tau'$ that together define a global section of the sheaf of strict piecewise-linear functions. The subdivision has given us more rays, and thus more coefficients to work with.

### 2.2.3 Example 3: Nodal cubic in a plane with divisorial log structure

To give a non-toric example, we consider $D$ an irreducible nodal cubic in $S = \mathbb{P}^2_k$. Also important is that this example will illustrate that the strict piecewise-polynomial functions are not generated by the strict piecewise-linear functions. Let $X = \operatorname{Spec}(k[x,y]) = \mathbb{A}^2_k$ a standard affine open of $S$, then $D_{|X}$ is given by the equation $y^2 - x^3 - x^2$ in $X$. We equip $S$ with the log structure associated to the normal crossings boundary $D$, or equivalently $\iota_*\mathcal{O}$ for $\iota\colon S \setminus D \to S$. Note that this is not a toric example as before, so we will explicitly describe the log structure associated to the boundary divisor. To describe $\bar{\mathsf{M}}_S^{\mathsf{gp}}$ on the étale site, we firstly describe an étale cover $\mathcal{U}$, describe the characteristic monoid sheaves, after which we compute $\mathrm{sPP}_S(S)$ using the gluing maps and the equaliser diagram

$$\mathrm{sPP}_S(S) \to \mathrm{sPP}_S(\mathcal{U}) \rightrightarrows \mathrm{sPP}_S(\mathcal{U} \times_S \mathcal{U}).$$

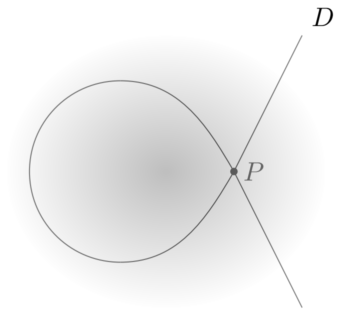



Figure 2.8: A nodal cubic in $\mathbb{P}^2_k$

#### Divisorial log structure on an étale cover

Let $P$ be the node of the cubic and let $V = S \setminus \{P\}$ with $i\colon V \to S$ a strict open immersion. Then the log structure on $V$ is given by the log structure associated to $i^{-1}D$. That means $\bar{\mathsf{M}}(V) \cong \mathbb{N}\langle r\rangle$ for we have one branch of a smooth boundary divisor remaining when disregarding $P$.

*Remark* 2.2.7. In general, we can deduce that the characterictic monoid sheaf associated to a smooth irreducible divisor $D$ can be given by $\mathbb{N}^n$ for $n$ the number of components.

In this case that would be argued as follows. Consider the map

$$\begin{aligned} \mathbb{N} &\to \bar{\mathsf{M}}_S(V) \\ 1 &\mapsto [y^2 - x^3 - x^2] = F \end{aligned}$$

where $[y^2 - x^3 - x^2]$ is the function in $\mathcal{O}(V)$ that on affine open $X$ is given by $y^2 - x^3 - x^2$. Consider also for $q \in V$ the composite

$$\mathbb{N} \to \bar{\mathsf{M}}_S(V) \to \bar{\mathsf{M}}_q.$$

For all $q \in V \setminus i^{-1}(D)$ this composite map is injective, and for all $q \in V$ (which does not include $P$) the composite map is surjective; for a small enough neighbourhood the order of sheafification and taking the quotient by units in $\mathsf{M}$ does not matter, so we may simply conclude all functions invertible outside $i^{-1}(D)$ are multiples of $F$ and these form the whole stalk. Then, writing $\varphi\colon i^{-1}D \hookrightarrow V$, we have that the sheaf $\bar{\mathsf{M}}_{|V}$ and the pushforward of the constant sheaf $\varphi_*\mathbb{N}$ are equal on the level of stalks. Therefore $\bar{\mathsf{M}}_{|V} = \varphi_*\mathbb{N}$ and this equality implies

$$\bar{\mathsf{M}}_{|V}(V) = \mathrm{H}^0(V, \bar{\mathsf{M}}_{|V}) = \mathrm{H}^0(i^{-1}(D), \mathbb{N}) = \mathbb{N}$$

as $i^{-1}(D)$ is connected. ♦

Secondly, we construct an étale open neighbourhood around $P$ denoted by $U$ in the cover $\mathcal{U}$ as follows. Write $R = k[x, y]$ so that $X = \operatorname{Spec} R$, and consider

$$A = \big(R[t]/(t^2 - (x+1))\big)_{(x+1)}$$

and

$$B = A_{(t-1)}$$

and the maps $R \to A \to B$ inclusion and localisation; that is, we are formally adjoining a square root of $x + 1$ which is now a unit, which will allow us to decompose $y^2 - x^3 - x^2 = y^2 - x^2(x+1) = (y - tx)(y + tx)$. Then consider the cover given by the composition of strict étale maps $j'\colon U = \operatorname{Spec} A \to X \to S$. By strictness, or by defining the log structure on $U$ as such, the log structure on $U$ is the log structure arising from the divisor $j'^{-1}D$. However, this boundary divisor has two nodes

$$j'^{-1}(P) = (y - tx, y + tx) = (y, x) = \{(x, y, t+1), (x, y, t-1)\}.$$

In order to simplify calculations, and in particular obtain an atomic cover[1], or even just and étale neighbourhood in which the divisor resembles the union of the coordinate axes in $\mathbb{A}^2$, we then localise at one of the two points, say $t-1$. We consider the cover given by the composition of strict étale maps $j\colon U = \operatorname{Spec} B \to X \to S$.

We may then visualise our étale cover as taking 2 copies of the normalisation, and drawing them as 2 parabolae intersecting in two points, and then localising to delete one of the intersection points. A sketch of $\operatorname{Spec} B \to \operatorname{Spec} R$ is given in Figure 2.9.

The log structure on $U$ defined by $j^{-1}D$ is

$$\begin{aligned}\mathsf{M}_S(U) &= \{f \in \mathcal{O}_S(U) \mid f \text{ invertible outside } j^{-1}(D)\}\\ &= \{f \in B \mid f \text{ invertible outside } Z(y^2 - x^3 - x^2)\}\\ &= \{f \in B \mid f \text{ invertible outside } Z((y+tx)(y-tx))\}\end{aligned}$$

and so for the characteristic monoids we obtain

$$\bar{\mathsf{M}}_S(U) \cong \mathbb{N}\left\langle \underline{y+tx}, \underline{y-tx} \right\rangle .$$

Indeed there is one generator for each branch through the boundary node in $U$, by similar reasoning as discussed in previous example. (Namely, we are in a similar situation as considering functions are invertible outside the union of the coordinate axes in the proof of Lemma 2.2.5).

The data of log schemes $U, V$ and strict étale maps $U \to S, V \to S$ specify the étale cover sketched in Figure 2.10.

## Gluing maps

The restriction maps in our étale covering may be described as follows. Firstly, the fibre product $V \times_S V$ simply equals $V$ and so both maps $V \times_S V \to V$ are the identity. Therefore the gluing maps induced on $\bar{\mathsf{M}}_S(V)$ are trivial.

Secondly, consider the pullback diagram

$$\begin{tikzcd} U \times_S V \arrow[r] \arrow[d, hook] & V \arrow[d, hook, "i"] \\ U \arrow[r, "j"] & S \end{tikzcd}$$

[1] For our purposes we may think of this as a small enough neighbourhood for which the characteristic monoid sheaf is uncomplicated enough. Formally this is meant in the sense of [AW18]: $S$ has a unique stratum that is closed and connected, and the restriction of the characteristic monoid to this stratum is a constant sheaf.

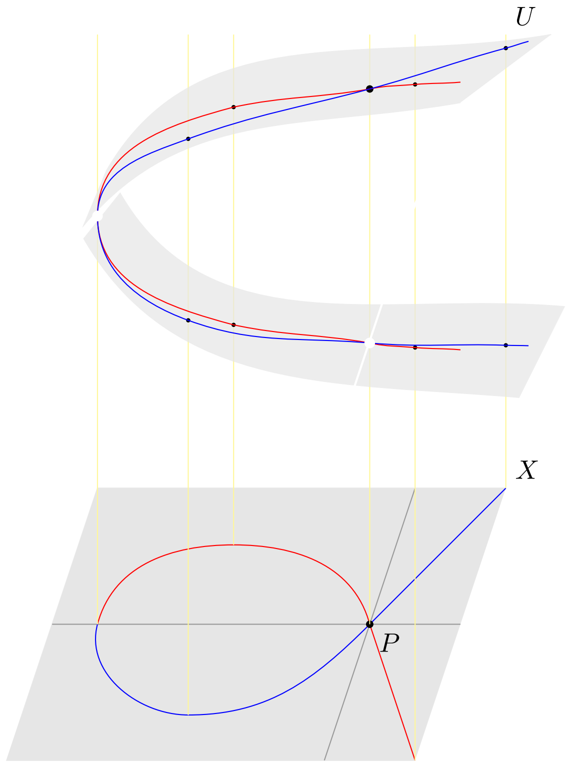



Figure 2.9: Étale neighbourhood $U$ of intersection point $P$ of the nodal cubic.

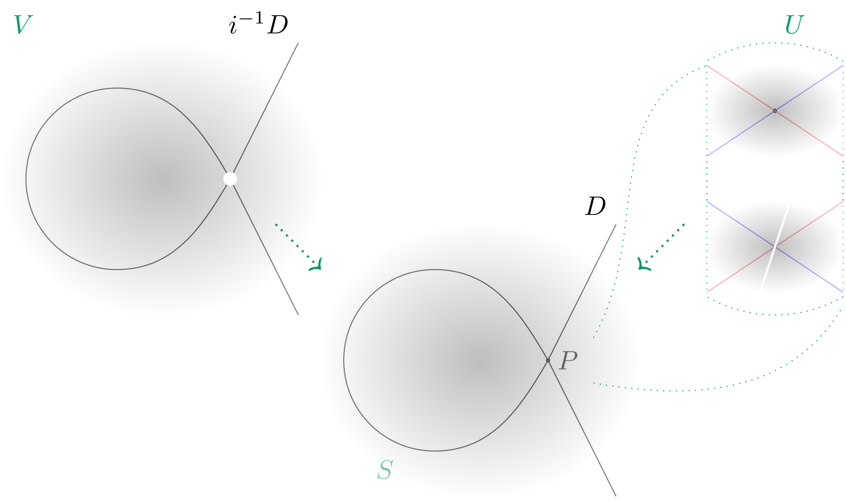



Figure 2.10: Cover of $S$ defined by $\{U, V\}$.

where all maps are strict. Then the fibre product equals $U$ minus the maximal ideal $j^{-1}(P) = (x, y, t+1)$, which is equipped with the divisorial log structure. Then we are again in the situation of Remark 2.2.7 where we consider, in this case two smooth irreducible divisors, and we obtain that the global sections in $\bar{\mathsf{M}}_S(U \times_S V)$ are given by $\mathbb{N}\langle m, n\rangle$, where generator $m$ corresponds to the branch of the divisor given by $y + tx$ and $n$ to the branch given by $y - tx$.

The gluing maps are then given by

$$\begin{aligned} \bar{\mathsf{M}}_S(V) &\to \bar{\mathsf{M}}_S(U \times_S V), \text{ i.e.} \\ \mathbb{N}\langle r\rangle &\to \mathbb{N}\langle m, n\rangle \\ r &\mapsto m + n \end{aligned}$$

for the morphism $U \times_S V \to V$ induced by $j\colon U \to S$, and

$$\begin{aligned} \bar{\mathsf{M}}_S(U) &\to \bar{\mathsf{M}}_S(U \times_S V), \text{ i.e.} \\ \mathbb{N}\langle \underline{y+tx}, \underline{y-tx}\rangle &\to \mathbb{N}\langle m, n\rangle, \\ \underline{y+tx} &\mapsto m, \\ \underline{y-tx} &\mapsto n, \end{aligned}$$

for the morphism $U \times_S V \to U$.

Finally, consider the pullback diagram

$$\begin{tikzcd} U \times_S U \arrow[r] \arrow[d] & U \arrow[d, "i"] \\ U \arrow[r, "j"] & S \end{tikzcd}$$

where all maps are strict. Note that because we have defined $U$ as composition $U \to X \to S$ over the standard affine open $X$, we may also compute the fibre product as $U \times_X U$. Therefore, note that the fibre product

$$\begin{aligned} B \times_R B &= \left(\frac{R[t]}{t^2 - x - 1}\right)_{(t),(t-1)} \times_R \left(\frac{R[t']}{t'^2 - x - 1}\right)_{(t'),(t'-1)} \\ &\cong \left(\frac{R[t,t']}{t^2 - x - 1, t'^2 - x - 1}\right)_{(t),(t'),(t-1),(t'-1)} \\ &= \left(\frac{R[t,t']}{t^2 - x - 1, t^2 - t'^2}\right)_{(t),(t'),(t-1),(t'-1)} \\ &= \left(\frac{R[t,t']}{t^2 - x - 1, (t-t')(t+t')}\right)_{(t),(t'),(t-1),(t'-1)} \\ &\cong \left(\frac{R[t]}{t^2 - x - 1}\right)_{(t),(t-1)} \times \left(\frac{R[t]}{t^2 - x - 1}\right)_{(t),(t-1),(t+1)} \\ &= B \times B_{(t+1)}, \end{aligned}$$

where we use the Chinese remainder theorem, setting $t = t'$ on one factor and $t = -t'$ on the other. The composition of these maps is given by $t \otimes 1 \mapsto (t, t)$ and $1 \otimes t' \mapsto (t, -t)$. The two restriction maps $U \times_S U \to U$ are then induced by $B \to B \times_R B \to B \times B_{(t+1)}, t \mapsto (t, t)$ and $t \mapsto (t, -t)$ respectively.

We may conclude that $U \times_X U \cong U \sqcup (U \setminus Z(t+1))$, and we know both $\bar{\mathsf{M}}_S(U) \cong \mathbb{N}\left\langle \underline{y+tx}, \underline{y-tx} \right\rangle$ and $\bar{\mathsf{M}}_S(U \setminus Z(t+1)) \cong \mathbb{N}\left\langle \underline{y+tx}, \underline{y-tx} \right\rangle$.

Then the induced gluing maps are given by

$$\begin{aligned} \bar{\mathsf{M}}_S(U) &\to \bar{\mathsf{M}}_S(U) \times \bar{\mathsf{M}}_S(U \setminus Z(t+1)), \\ \underline{y+tx} &\mapsto (\underline{y+tx}, \underline{y+tx}) \\ \underline{y-tx} &\mapsto (\underline{y-tx}, \underline{y-tx}), \end{aligned}$$

to which we may refer as $(\mathrm{id}, \mathrm{id})$ and

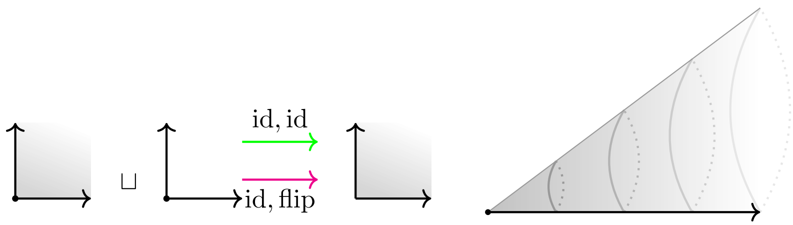
$$\begin{aligned} \bar{\mathsf{M}}_S(U) &\to \bar{\mathsf{M}}_S(U) \times \bar{\mathsf{M}}_S(U \setminus Z(t+1)), \\ \underline{y+tx} &\mapsto (\underline{y+tx}, \underline{y-tx}) \\ \underline{y-tx} &\mapsto (\underline{y-tx}, \underline{y+tx}), \end{aligned}$$

to which we may refer as $(\mathrm{id}, \mathrm{flip})$.

Thus we obtain the following diagram of gluing maps for the sheaf $\bar{\mathsf{M}}_S$:

$$\begin{tikzcd}
\mathbb{N}\left\langle \underline{y+tx}, \underline{y-tx} \right\rangle \arrow[r, equal] & \bar{\mathsf{M}}_S(U) \arrow[r, bend left, "\substack{\underline{y+tx}\mapsto(\underline{y+tx},\underline{y+tx})\\ \underline{y-tx}\mapsto(\underline{y-tx},\underline{y-tx})}"] \arrow[r, bend right, "\substack{\underline{y+tx}\mapsto(\underline{y+tx},\underline{y-tx})\\ \underline{y-tx}\mapsto(\underline{y-tx},\underline{y+tx})}"'] \arrow[dr, bend right, "\substack{\underline{y+tx}\mapsto m\\ \underline{y-tx}\mapsto n}"'] & \bar{\mathsf{M}}_S(U) \times \bar{\mathsf{M}}_S(U \setminus Z(t+1)) & \\
 & & \bar{\mathsf{M}}_S(U \times_S V) \arrow[r, equal] & \mathbb{N}\langle m, n \rangle \\
\mathbb{N}\langle r \rangle \arrow[r, equal] & \bar{\mathsf{M}}_S(V) \arrow[ur, bend left, "r\mapsto m+n"] \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{id}"'] & \bar{\mathsf{M}}_S(V \times_S V) \arrow[r, equal] & \bar{\mathsf{M}}_S(V).
\end{tikzcd}$$

If we strive to give a visualisation of the characteristic monoids, remember that the plane with log structure from the coordinate axes as divisors simply comes from the fan with $\{0\}$, 2 rays and the plane they span. Removing the intersection point of the coordinate axes from considerations, means we remove the plane from the fan. Hence we may try to visualise the characteristic monoids for $S$ now either as the colimit of the diagram on the left[2] or as the ice cream horn in Figure 2.11.




(a) Colimit diagram (b) Sketch of $\bar{\mathsf{M}}_S$

Figure 2.11: Visualisations of the characteristic monoids for $S = \mathbb{P}^2_k$ with respect to $D$.

[2]The colimit here is formally taken in the category of cone stacks, but may be thought of as the category of rational polyhedral cone complexes or sharp fine saturated monoids. See [CCUW20] for the details, however the figures here serve merely as a visualisation.

**Strict piecewise-polynomial functions**

Studying the diagram of restriction maps, the global sections of the sheaf of characteristic monoids are $\bar{\mathsf{M}}_S^{\mathsf{gp}}(U) = \mathbb{Z}\langle a, b\rangle = \mathbb{Z}\langle \underline{y+tx, y-tx}\rangle$, writing $a, b$ for its generators. Then those functions that are compatible with the maps to $\bar{\mathsf{M}}_S(U \times_S U)$, are invariant under the map that flips the coordinates, and so the strict piecewise-linear functions on $S$ are then of the form $\{\alpha a + \alpha b\}$ for $\alpha \in \mathbb{Z}$. Therefore all strict piecewise-linear functions on $S$ are symmetric in $a$ and $b$. However, there will be strict piecewise-polynomial functions that are not products of strictly piecewise-linear ones.

To study all strict piecewise-polynomial functions, firstly note that the symmetric algebra associated to the $\mathbb{Z}$-algebra $\mathbb{Z}\langle x_1, \ldots, x_n\rangle$ is equal to the polynomial ring $\mathrm{Sym}(\mathbb{Z}\langle x_1, \ldots, x_n\rangle) = \mathbb{Z}[x_1, \ldots, x_n]$ as graded $\mathbb{Z}$-algebra. (Here the inclusion map $\mathbb{Z}\langle x_1, ..., x_n\rangle \to \mathbb{Z}[x_1, \ldots, x_n], x_1 \cdot x_2 \mapsto x_1 + x_2$ into the 1-graded part (the linear polynomials), maps the unit in $\mathbb{Z}\langle x_1, \ldots, x_n\rangle$ as $\mathbb{Z}$-module to the additive unit $0 \in \mathbb{Z}[x_1, \ldots, x_n]$.)

Hence we know that $\mathrm{sPP}_S(U) = \mathbb{Z}[a, b]$. For $U \setminus Z(t+1)$, we may either guess the appropriate $\mathbb{Z}$-algebra from Figure 2.11, or more formally take the cover $V_+, V_-$ of $U \setminus Z(t+1)$ where we remove the branch $y + tx$ or the branch $y - tx$ respectively. Then we obtain the equalizer diagram

$$\mathrm{sPP}(U \setminus Z(t+1)) \to \mathrm{sPP}(V_+) \times \mathrm{sPP}(V_-) \rightrightarrows \mathrm{sPP}_S(V_+ \cap V_-)$$

which, as $V_+ \cap V_-$ is empty and for both $V_+$ and $V_-$ we are in the case of one smooth irreducible boundary divisor with one component, is equal to

$$\mathrm{sPP}(U \setminus Z(t+1)) \to \mathbb{Z}[A] \times \mathbb{Z}[B] \rightrightarrows \mathbb{Z}$$

where both maps are given by $A, B \mapsto 0$. Then we may conclude that

$$\mathrm{sPP}(U \setminus Z(t+1)) = \{(f, g) \in \mathbb{Z}[A] \times \mathbb{Z}[B] \mid f(0) = g(0)\} \cong \mathbb{Z}[A, B]/(AB).$$

To compute the strict piecewise-polynomial functions, we consider the diagram

$$\begin{tikzcd}
\mathbb{Z}[a,b] \arrow[r, bend left, "{\substack{a\mapsto(a,A)\\ b\mapsto(b,B)}}"] \arrow[r, bend right, "{\substack{a\mapsto(a,B)\\ b\mapsto(b,A)}}"'] \arrow[dr, bend right, "{\substack{a\mapsto m\\ b\mapsto n}}"] & \mathbb{Z}[e,f] \times \mathbb{Z}[A,B]/(AB) \\
\mathbb{Z}[r] \arrow[r, bend right, "r\mapsto m+n"'] & \mathbb{Z}[m,n].
\end{tikzcd}$$

Then the degree 2 function $ab \in \mathbb{Z}[a,b]$ maps to

$$(a,A)(b,B) = (ab, AB) = (ab, 0)$$

via one map and

$$(a,B)(b,A) = (ab, AB) = (ab, 0)$$

via the other. Hence this is an allowed strict piecewise-polynomial function, but it is not a product of strict piecewise-linear functions, as the product $\{\alpha(a+b)\}\{\beta(a+b)\}$ for $\alpha, \beta \in \mathbb{Z}$ will never yield $\{ab\}$. Therefore, in the next section, we will not be able to define a map to Chow by solely defining the image on piecewise-linear functions.

Note that higher degree strict piecewise-polynomial functions need not be symmetric: the degree 3 function $a^2b \in \mathbb{Z}[a,b]$ maps to

$$(a,A)^2(b,B) = (a^2b, A^2B) = (a^2b, 0)$$

via one map and

$$(a,B)^2(b,A) = (a^2b, AB^2) = (a^2b, 0)$$

via the other which is equal, so indeed gives an element of $\mathrm{sPP}_S(S)$.

## 2.3 Map to the Chow group

### 2.3.1 Map from characteristic monoid sheaf to divisor classes

The formal procedure to construct a Chow class from a given strict piecewise polynomial function is given in Section 3.3.3 or [HS22, Section 3.3]. The first step to construct this map is via a map $\bar{\mathsf{M}}_S(S) \to \mathrm{Div}(S)$ to line bundles and sections up to isomorphism; we will give a detailed description of this map to Div as described in Section 3.3.2 or [HS22, Section 3.2]. Then we describe a practical procedure associating a Chow class to a global section of the characteristic monoid, i.e. for strict piecewise-*linear* functions. For log stacks where we may write all strict piecewise-polynomial functions as products of strict piecewise-linear ones (that is, for simple log algebraic stacks, see Definition 2.3.5), we then have defined the map from the ring of strict piecewise-polynomial functions to the Chow group.

For a log algebraic stack $S$, we write $\mathrm{Div}(S)$ for the monoid of isomorphism classes of pairs $(\mathcal{L}, \ell)$ where $\mathcal{L}$ is a line bundle on $S$ and $\ell \in \mathcal{L}(S)$ a section, with monoid operation given by tensor product. Then the map

$$\mathcal{O}_S(-)\colon \bar{\mathsf{M}}_S(S) \to \mathrm{Div}(S). \tag{2.3.1.1}$$

is defined via the following three steps.

**Step 1: Building an $\mathcal{O}^*$-torsor**

Consider the exact sequence of monoid sheaves

$$1 \to \mathcal{O}_S^\times \to \mathsf{M}_S \to \bar{\mathsf{M}}_S \to 1. \tag{2.3.1.2}$$

Let $m \in \bar{\mathsf{M}}_S(S)$, then the preimage $\mathcal{O}_S(-m)^\times$ of $m$ in $\mathsf{M}_S$ is an $\mathcal{O}_S^\times$-torsor. The log structure, in particular the map $\alpha\colon \mathsf{M}_S \to \mathcal{O}_S$, equips this preimage with a map $\mathcal{O}_S(-m)^\times \to \mathcal{O}_S$.

**Example 2.3.1.** As a guiding example throughout this section, consider the example $\mathbb{A}^n = \mathrm{Spec}(k[\mathbb{N}^n])$ with the log structure induced by this monoid ring structure, which we have briefly seen in Example 2.2.1. That is, writing $x_1, ..., x_n$ for each of the generators of $k[\mathbb{N}^n]$, we take the log structure associated to the pre-log structure defined by $\mathbb{N}^n \to k[\mathbb{N}^n], \underline{v} \mapsto x^{\underline{v}}$ and so

$$\alpha\colon \mathsf{M}_{\mathbb{A}^n} \to \mathcal{O}_{\mathbb{A}^n}$$

is given on global sections by $k^* \oplus \mathbb{N}^n \to \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n), (u, \underline{v}) \mapsto ux^{\underline{v}}$.

Again, we can view this as the same log structure as the log structure associated to a boundary divisor, namely the divisor $\prod x_i = 0$ (the intersection of the coordinate hyperplanes). Explicitly, let $D_i = V(x_i)$ be the coordinate planes for $i = 1, ..., n$, then the associated log structure is given by

$$\mathsf{M}_{\mathbb{A}^n,D}(U) = \{f \in \mathcal{O}_{\mathbb{A}^n}(U) \mid f \text{ invertible outside } D_1, \ldots, D_n\}.$$

Functions in $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) = k[x_1, ..., x_n]$ that are invertible outside the coordinate hyperplanes include integer powers of the $x_i$.

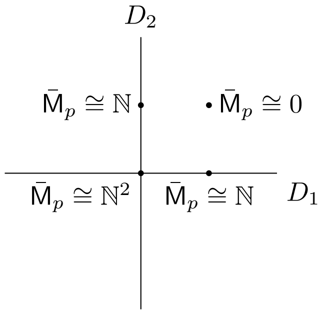



Figure 2.12: Sketch with stalks of the characteristic monoid sheaf for guiding example $\mathrm{Spec}(k[\mathbb{N}^n])$ for $n = 2$

Let $m \in \bar{\mathsf{M}}_{\mathbb{A}^n}(\mathbb{A}^n) = \mathbb{N}^n$, and in order to study the preimage along the map $q \colon \mathsf{M} \to \bar{\mathsf{M}}$, note that

$$\mathsf{M}(\mathbb{A}^n) \to \mathbb{N}^n$$

is given by $f \mapsto (\mathrm{ord}_{x_i} f)_{i=1,\ldots,n}$. Assuming $\mathrm{ord}_{x_i} m = m_i \in \mathbb{N}$, the preimage $\mathcal{O}_{\mathbb{A}^n}(-m)^\times$ of $m$ in $\bar{\mathsf{M}}_{\mathbb{A}^n}$ under $q$ is given by

$$q^{-1}m(U) = \left\{ f \in \mathcal{O}_{\mathbb{A}^n}(U) \mid \mathrm{div}\, f = \sum_{i=1}^n m_i D_i \right\},$$

which you can think of as the $\mathcal{O}_{\mathbb{A}^n}^\times$-torsor

$$\mathcal{O}_{\mathbb{A}^n}^\times \cdot x^m = \mathcal{O}_{\mathbb{A}^n}^\times \cdot x_1^{m_1} \cdots x_n^{m_n}.$$

Note that this $\mathcal{O}_{\mathbb{A}^n}^\times$-torsor is isomorphic to the familiar torsor

$$\mathcal{O}_{\mathbb{A}^n}^\times \left( -\sum_{i=1}^n m_i D_i \right)(U) = \left\{ f \in \mathcal{O}_{\mathbb{A}^n}(U) \mid \mathrm{div}\, f = \sum_{i=1}^n m_i D_i \right\}$$

obtained by removing the zero section from the line bundle

$$\mathcal{O}_{\mathbb{A}^n}\left(-\sum_{i=1}^{n} m_i D_i\right)(U) = \left\{ f \in \mathcal{O}_{\mathbb{A}^n}(U) \mid \operatorname{div} f - \sum_{i=1}^{n} m_i D_i \geq 0 \right\}.$$

The map to $\mathcal{O}_{\mathbb{A}^n}$ (which is induced by the log structure and is used in further constructions) is in this example simply the inclusion. ⧫

*Remark* 2.3.2. To explain why we consider this a guiding example, recall that in the case where we have a log structure induced by a strict normal crossings divisor, étale locally this situation applies: when a normal crossings divisor $D$ is, locally at a $p \in D$ with a set of local coordinates $g_i$, given by the vanishing of $g_1 \cdots g_e$ (that is, locally the intersection of $e$ coordinate planes), then we have an étale chart $\mathbb{N}^e \to \bar{\mathsf{M}}, e_i \mapsto g_i$. ⧫

**Step 2: Associating a line bundle**

From the restriction of the log structure map $\mathcal{O}_S(-m)^\times \to \mathcal{O}_S$, we build the following diagram

$$\begin{tikzcd} & \mathcal{O}_S \\ \mathcal{O}_S(-m) \arrow[ur] & \mathcal{O}_S(-m)^\times . \arrow[l] \arrow[u] \end{tikzcd}$$

The map on the $\mathcal{O}_S^\times$-torsor admits a unique $\mathcal{O}_S^\times$-equivariant extension to a map of line bundles $\mathcal{O}_S(-m) \to \mathcal{O}_S$, where we built $\mathcal{O}_S(-m)$ from $\mathcal{O}_S(-m)^\times$ by filling in the zero section. Formally, this means we consider a $\mathcal{O}_S^*$-action on the product $\mathcal{T} \times \mathcal{O}_S$ for a torsor $\mathcal{T}$, (act on $\mathcal{O}_S$ via multiplication by the inverse) and consider the line bundle resulting from quotienting by that action. The procedure is also given in [Sch18b, Prop. 1.29].

**Example 2.3.2** (continued)**.** In the guiding example of $\mathbb{A}^n = \operatorname{Spec}(k[\mathbb{N}^n])$, we can view the $\mathcal{O}^\times$-torsor as

$$\mathcal{O}_{\mathbb{A}^n}^\times \cdot x^m = \mathcal{O}_{\mathbb{A}^n}^\times \cdot x_1^{m_1} \cdots x_n^{m_n}.$$

Then filling in the zero-section yields $\mathcal{O}_{\mathbb{A}^n}(-m)$ which is the ideal generated by $x^m$. That is, we allow multiplication by other functions, allowing greater orders of vanishing.

We also showed that $\mathcal{O}^{\times}_{\mathbb{A}^n}(-m)$ equals the torsor $\mathcal{O}^{\times}_{\mathbb{A}^n}(-\sum_{i=1}^n m_i D_i)$ writing $D_i$ for the divisor defined by $x_i = 0$ for $i = 1, ..., n$. Then filling in the zero-section results in $\mathcal{O}_{\mathbb{A}^n}(-m) \cong \mathcal{O}_{\mathbb{A}^n}(-\sum_{i=0}^n m_i D_i)$. The map of line bundles $\mathcal{O}_{\mathbb{A}^n}(-m) \to \mathcal{O}_{\mathbb{A}^n}$ is simply the inclusion. ♦

**Step 3: Taking the dual line bundle with a section**

Dualising gives a map $\mathcal{O}_S \to \mathcal{O}_S(m) := \mathcal{O}_S(-m)^\vee$, and the image $\ell_m$ of the section 1 of $\mathcal{O}_S$ defines a section of $\mathcal{O}_S(m)$. This concludes the construction of the map

$$\mathcal{O}_S(-)\colon \bar{\mathsf{M}}_S(S) \to \mathrm{Div}(S).$$

**Example 2.3.2** (continued)**.** Because filling in the zero-section results in $\mathcal{O}_{\mathbb{A}^n}(-m) \cong \mathcal{O}_{\mathbb{A}^n}(-\sum_{i=0}^n m_i D_i)$, we quickly see that the dualising yields the line bundle $\mathcal{O}_{\mathbb{A}^n}(\sum_{i=0}^n m_i D_i) = \mathcal{O}_{\mathbb{A}^n}(m)$. The morphism $\mathcal{O}_{\mathbb{A}^n} \to \mathcal{O}_{\mathbb{A}^n}(m)$ is also simply an inclusion: the functions in $\mathcal{O}_{\mathbb{A}^n}(m)$ are allowed poles but need not have them. Therefore the image $\ell_m$ of the section 1 is simply the canonical section $s_D$ (notation as in [Ful84, Appendix B.4]) corresponding to 1 when viewing our line bundle $\mathcal{O}_{\mathbb{A}^n}(m)$ as sub-$\mathcal{O}_{\mathbb{A}^n}$-sheaf of $\mathcal{K}_{\mathbb{A}^n}$. In that perspective, $\mathcal{O}_{\mathbb{A}^n}(m)$ is the subsheaf generated by $1/f$ for $f = x_1 \cdots x_n \in \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ and thus our line bundle also corresponds to the effective Cartier divisor defined by $\{(\mathbb{A}^n, f)\}$ and its associated divisor class equals

$$\operatorname{div} \mathcal{O}_{\mathbb{A}^n}(m) = \sum \operatorname{ord}_{D_i}(f)[D_i] = \sum m_i[D_i].$$

♦

*Remark* 2.3.3. Note that both maps $\mathcal{O}_{\mathbb{A}^n}(-m) \to \mathcal{O}_{\mathbb{A}^n}$ and $\mathcal{O}_{\mathbb{A}^n} \to \mathcal{O}_{\mathbb{A}^n}(m)$ are injective maps of sheaves, but these are not universally injective. If we base change the resulting morphism might not be injective and so the image $\ell_m$ of 1 need not always non-zero. For example, starting with the affine plane $\mathrm{Spec}(k[\mathbb{N}^2])$ and mapping a point to the origin (or any point with non-trivial log structure), then one can verify that for the base change the image $\ell_m$ is 0. ♦

**Expressing the characteristic monoid in terms of boundary divisors**

In the above we described the resulting line bundle and section very explicitly in terms of the boundary divisors. In general in the cases we study, as suggested by the heading ‘Piecewise polynomials as polynomials in boundary divisors’ in the Section 3.3.2 or [HS22, Section 3.2], there is a relation between the global sections of the characteristic monoid sheaf and the boundary divisors.

If the log structure on the log algebraic stack $S$ is trivial over a schematically-dense open $U \subseteq S$ (for example, this holds if $S$ is log regular), then the given section of $\mathcal{O}_S(m)$ is trivial over $U$. Because $U$ is schematically dense, it defines an effective Cartier divisor on $S$ supported away from $U$, which we denote $\operatorname{div} \mathcal{O}_S(m)$. If $S$ is a quasi-compact regular log regular log algebraic stack, then the given section of $\mathcal{O}_S(m)$ is non-vanishing outside a certain boundary divisor $Z = \bigcup_{i \in I} D_i$, and so defines an effective Cartier divisor on $S$ supported at $Z$, which we denote $\operatorname{div} \mathcal{O}_S(m)$ and which naturally lies in $\langle D_i : i \in I \rangle$. This is formulated in this thesis in Lemma 3.3.7 or [HS22, Lemma 3.7], restated here.

**Lemma 2.3.4.** *Sending $m \mapsto \operatorname{div} \mathcal{O}_S(m)$ gives an isomorphism of monoids*

$$\bar{\mathsf{M}}_S(S) \to \langle D_i : i \in I \rangle .$$

For example, if the $D_i$ form a divisor with strict normal crossings, then we may even prove that there is an isomorphism

$$\operatorname{Sym}(\bar{\mathsf{M}}^{\mathsf{gp}}_S(S)) \xrightarrow{\sim} \mathbb{Z}[D_i : i \in I] \tag{2.3.1.3}$$

to the free commutative ring on the $D_i$.

### 2.3.2 Map from strict piecewise-polynomial functions to Chow in the simple case

Now we compose map (2.3.1.1) with the (operational) first Chern class to construct the group homomorphism

$$\Phi^1 \colon \bar{\mathsf{M}}^{\mathsf{gp}}_S(S) \to \mathsf{CH}^1_{\mathsf{op}}(S), \tag{2.3.2.1}$$

with image contained in the subgroup generated by Cartier divisors.

For example, taking the first Chern class for a line bundle $\mathcal{O}(D)$ defined by a Cartier divisor $D$ is simply given by the associated Weil divisor, see [Ful84, Section 2.5], and so in our guiding example we have:

$$c_1(\mathcal{O}(m)) = c_1(\mathcal{O}(\sum m_i D_i)) = \sum m_i [D_i] \in \mathsf{CH}(S).$$

(The guiding example is a log scheme, and so we may again simply consider the Chow group instead of the operational Chow with which we did all formal constructions.)

To extend this to a map on the strict piecewise-polynomial functions, so on the sheafification of the symmetric algebra of $\bar{\mathsf{M}}^{\mathsf{gp}}_S(S)$, we firstly define an extension of $\Phi^1$ for simple log stacks.

### Simple log algebraic stacks

We restate the definition [HS22, Definition 3.5] or in this thesis Definition 3.3.5.

**Definition 2.3.5.** If $S$ is a regular log regular log algebraic stack with boundary divisor[3] $Z = \bigcup_{i \in I} D_i$, we say $S$ is *simple* if for every $J \subseteq I$ the fibre product

$$D_J := \bigtimes_{j \in J, S} D_j$$

is regular and in addition the natural map on sets of connected components $\pi_0(D_J) \to \pi_0(S)$ is injective. The closed connected substacks $D_J$ are the *closed strata* of $S$.

Example 1, the projective plane with toric boundary structure, is an example of a simple log stack: the divisors $D_0, D_1, D_2$ are closed subschemes so the fibre product is simply the intersection which is a regular point which is connected. By similar reasoning for the boundary divisors $\bar{D}_0, \bar{D}_1, \bar{D}_2, E$, the blowup in example 2 is also a simple log stack. In fact, the fans of all regular toric varieties yield simple log structures. However, example 3 does not satisfy the conditions of a simple log stack. The boundary nodal cubic $D$ is not a strict normal crossings divisor, and the nodal point of $D$ yields a double point which is not-regular. Note that this simple condition is more restrictive than requiring the boundary divisor to be a strict normal crossings divisor; consider the union of a line and a smooth conic in $\mathbb{P}^2$ meeting at two points, then the intersection is not connected.

Important is also the subsequent lemma, [HS22, Lemma 3.6] or in this thesis Lemma 3.3.6, which can be interpreted as saying that doing enough subdivisions will yield a simple case.

**Lemma 2.3.6.** *Let $S$ be a log regular log algebraic stack. Then there exists a log blowup $\tilde{S} \to S$ such that $\tilde{S}$ is simple.*

### From $\Phi^1$ on piecewise-linear to piecewise-polynomial

Let $S$ be a simple log algebraic stack, smooth[4] over $k$. The operational Chow group $\mathsf{CH}_{\mathsf{op}}(S)$ has a commutative ring structure coming from composition of operations. As such, the map (2.3.2.1)

$$\Phi^1 \colon \bar{\mathsf{M}}_S(S) \to \mathsf{CH}^1_{\mathsf{op}}(S)$$

[3] Here we implicitly mean that the $D_i$ are reduced and irreducible substacks of pure codimension 1.

[4] If $k$ is a field of characteristic zero then being smooth is here equivalent to being locally of finite type (since simple implies regular).

extends uniquely to a ring homomorphism

$$\Phi' \colon \operatorname{Sym}(\bar{\mathsf{M}}_S(S)) \to \mathsf{CH}_{\mathsf{op}}(S). \tag{2.3.2.2}$$

This defines a map on $\operatorname{Sym}(\bar{\mathsf{M}}_S(S))$, however not on the global sections $(\operatorname{Sym} \bar{\mathsf{M}}_S)(S) = \mathrm{PP}_S(S)$. The difference between these two may seem confusing. The natural map of $\mathbb{Z}$-algebras

$$\operatorname{Sym}(\bar{\mathsf{M}}_S^{\mathsf{gp}}(S)) \to (\operatorname{Sym} \bar{\mathsf{M}}_S^{\mathsf{gp}})(S)$$

need not be surjective or injective. We have seen a similar warning while discussing the log structure on $\mathbb{P}^2_k$ given by the nodal cubic, but this example is not simple. However, the following easy example illustrates the difference between $\operatorname{Sym}(\bar{\mathsf{M}}_S(S))$ and $(\operatorname{Sym} \bar{\mathsf{M}}_S)(S) = \mathrm{PP}_S(S)$ for a simple log scheme.

**Example 2.3.7.** Consider $\mathbb{P}^1_k$ with the log structure given by the divisor consisting of two points $0 = (0 : 1)$ and $\infty = (1 : 0)$. This is an example of a regular toric variety, so we know it is simple, but also the intersection of $D_0$ and $D_\infty$ being empty makes the conditions trivially easy to check.


(a) Opens $U$ and $V$ around $0$ and $\infty$ in $\mathbb{P}^1_k$

(b) Toric fan for $\mathbb{P}^1_k$

Figure 2.13: Log structure $\mathbb{P}^1_k$

Let $U$ be an open neighbourhood of $0$ not containing $\infty$, say the standard open $U_{t_1 \neq 0}$ which equals $U_{\sigma_1}$, and let $V$ be an open neighbourhood of $\infty$ not containing $0$, say the standard open $U_{t_0 \neq 0}$ which equals $U_{\sigma_0}$, so that $U$ and $V$ cover $\mathbb{P}^1_k$. Then we have the following diagram of monoids and restriction maps

$$\begin{tikzcd}
\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(U) \cong \mathbb{Z}\langle a \rangle \arrow[dr, bend left] & \\
& \bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(U \cap V) \cong 0. \\
\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(V) \cong \mathbb{Z}\langle b \rangle \arrow[ur, bend right] &
\end{tikzcd}$$

Hence a piecewise-linear function on $\mathbb{P}^1_k$ is given by sections of $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(U)$ and $\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(V)$, that is $\alpha a$ and $\beta b$ for $\alpha, \beta \in \mathbb{Z}$. Because the log structure can be described as

$$\mathsf{M}_{\mathbb{P}^1_k}(U) = \{f \in \mathcal{O}_{\mathbb{P}^1_k}(U) \mid f \text{ invertible outside } 0, \infty\},$$

we may view generator $a$ of the groupified characteristic monoid at $U$ as a function in $k[t_0/t_1]$ that takes value 0 at point $0 = (0:1)$ and is invertible elsewhere (e.g. just $t_0/t_1$), and similarly $b$ as a function in $k[t_1/t_0]$ that takes value 0 at point $\infty = (1:0)$ and is invertible elsewhere.

The piecewise-linear functions $p = \{1 \cdot a, 0 \cdot b\}$ and $q = \{0 \cdot a, 1 \cdot b\} \in \bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(\mathbb{P}^1)$ generate all possible strict piecewise-linear functions. Their (non-zero) product $pq \in \mathrm{Sym}^2(\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(\mathbb{P}^1))$ maps under the morphism

$$\mathrm{Sym}(\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k}(\mathbb{P}^1_k)) \to (\mathrm{Sym}\,\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k})(\mathbb{P}^1_k),$$

to 0, however, as it is equal 0 on both the opens $U$ and $V$. Hence here the sheafification of $\mathrm{Sym}(\bar{\mathsf{M}}^{\mathsf{gp}}_S(U))$ to form $\mathrm{Sym}\,\bar{\mathsf{M}}^{\mathsf{gp}}_S = \mathrm{sPP}_S$ matters: the sheaf gluing conditions now make the function equal 0 as element of $(\mathrm{Sym}\,\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^1_k})(\mathbb{P}^1_k)$. ♦

The crux to the solution to navigating between global sections $\mathrm{Sym}(\bar{\mathsf{M}}_S(S))$ and $(\mathrm{Sym}\,\bar{\mathsf{M}}_S)(S) = \mathrm{PP}_S(S)$ can be found in the following two results in [HS22, Theorems 3.8 and lemma 3.9] or in this thesis Theorem 3.3.8 and Lemma 3.3.9, for quasi-compact simple log algebraic stacks.

**Theorem 2.3.8.** *Let $S$ be a quasi-compact simple log algebraic stack. Then the natural map of $\mathbb{Z}$-algebras*

$$\mathrm{Sym}(\bar{\mathsf{M}}^{\mathsf{gp}}_S(S)) \to (\mathrm{Sym}\,\bar{\mathsf{M}}^{\mathsf{gp}}_S)(S)$$

*is surjective.*

In other words, $\mathrm{sPP}_S(S)$ is a quotient of the symmetric algebra on $\bar{\mathsf{M}}_S(S)$; every global piecewise-polynomial function can be written *globally* as a polynomial in piecewise-linear functions.

**Lemma 2.3.9.** *Any element of the kernel of the surjective morphism* $\mathrm{Sym}^n(\bar{\mathsf{M}}^{\mathsf{gp}}_S(S)) \to (\mathrm{Sym}^n\,\bar{\mathsf{M}}^{\mathsf{gp}}_S)(S)$ *maps to* $0$ *in* $\mathsf{CH}_{\mathsf{op}}(S)$.

Hence this map $\Phi'$ descends to a unique ring homomorphism

$$\Phi \colon (\mathrm{Sym}\,\bar{\mathsf{M}}_S)(S) = \mathrm{sPP}_S(S) \to \mathsf{CH}_{\mathsf{op}}(S),$$

whose degree 1 part is $\Phi^1$.

*Remark* 2.3.10. The map $\Phi$ is in general not injective or surjective. For example, if $S$ has trivial log structure but a non-trivial Chow group (e.g. $\mathbb{P}^n$ with trivial log structure), the map is not surjective as $(\operatorname{Sym}\bar{\mathsf{M}}_S)(S)$ is trivial. For an example of the failure of injectivity, we can observe that for a log scheme $S$ the Chow group vanishes in degree higher that the dimension, whereas the strict piecewise-polynomial functions generally do not. Also example 1 below is not injective as all lines $D_0, D_1, D_2$ are linearly equivalent in $\mathbb{P}^2_k$. ♦

**Practical procedure for applying $\Phi$**

In the guiding example of $\mathbb{A}^n = \operatorname{Spec}(k[\mathbb{N}^n])$, we observed that $m \in \bar{\mathsf{M}}_{\mathbb{A}^n}(\mathbb{A}^n)$ yielded $\mathcal{O}_S(m)$ which mapped to $\sum m_i[D_i] \in \mathsf{CH}(\mathbb{A}^n)$. The reason for calling this example guiding, and studying it in so much detail, is that we can now see that the map $\Phi$ in practical situations reduces to 'the procedure of reading off the coefficients of the strict piecewise-linear function at the boundary divisors':

Starting with any section in $(\operatorname{Sym}\bar{\mathsf{M}}_S)(S)$, so any strict piecewise polynomial function, we write it as image of an element in $\operatorname{Sym}(\bar{\mathsf{M}}_S(S))$ by Theorem 2.3.8, and then we write it as a polynomial in strict piecewise-linear functions. If we have a log structure induced by a strict normal crossings divisor, we étale locally consider $\mathbb{N}^e$ with $e$ generators for the characteristic monoid, one for each branch $D_i$ of the boundary divisor in that neighbourhood. Hence, étale locally the situation of our guiding example $\mathbb{A}^n$ with the log structure from the coordinate planes as divisors applies. Then an $m \in \bar{\mathsf{M}}^{\mathsf{gp}}_S(S)$ is mapped to $\sum_{i=1}^n m_i[D_i]$ where each $m_i$ is the order of vanishing at branches $D_i$, meaning we only need to read the multiplicity of the strict piecewise-linear function with respect to the generator of $\bar{\mathsf{M}}^{\mathsf{gp}}_S(S)$ associated to the boundary divisors $D_i$ that appear in the log structure. We illustrate this procedure in our simple examples.

**Example** (Example 1)**.** The Chow ring of the projective plane from example 1 equals $\mathsf{CH}(\mathbb{P}^2_k) \cong \mathbb{Z}[H]/H^3$; in codimension 1 classes are $\mathbb{Z}$-multiples of $[H]$ the class of any line, and in codimension 2 these are $\mathbb{Z}$-multiples of $[\mathrm{pt}] = [H^2]$ as 2 general lines intersect in a point.

Given a strict piecewise-linear function, we use the procedure to compute which divisors appear with non-zero coefficient in the image under $\Phi$. For each divisor $D$ outside the boundary we associate the coefficient 0, that is $0\cdot[D] \in \mathsf{CH}^1(\mathbb{P}^2_k)$, and for each divisor in the boundary we evaluate the function at that divisor. In this case, the lines $D_0, D_1, D_2$ as codimension 1 divisors are contained in the boundary, and other lines are not. For a point $p \in D_1$ with

$p = (p_0 : 0 : 1)$ where $p_0 \neq 0$, we have seen the computation of

$$\bar{\mathsf{M}}^{\mathsf{gp}}_p = \bar{\mathsf{M}}^{\mathsf{gp}}(U_{\tau_1}) \cong \mathbb{Z}\langle p \rangle .$$

Then evaluating at $D_1$ means we are interested in the coefficient in $\mathbb{Z}\langle p \rangle$ which is $\alpha = \epsilon$. Similarly we see that to the function

$$\begin{Bmatrix} \alpha a & + & \beta b \\ \gamma x & + & \beta y \\ \alpha u & + & \gamma v \end{Bmatrix}$$

we associate $\beta[D_0] + \alpha[D_1] + \gamma[D_2] \in \mathsf{CH}^1(\mathbb{P}^2_k)$.

Returning to the example of a strict piecewise-polynomial function of degree 2 given by

$$\begin{Bmatrix} a & + & b \\ & & y \\ u & & \end{Bmatrix} \begin{Bmatrix} & & b \\ x & + & y \\ & & v \end{Bmatrix} = \begin{Bmatrix} ab + b^2 \\ xy + y^2 \\ uv \end{Bmatrix}$$

as element in $\mathrm{Sym}^2(\bar{\mathsf{M}}^{\mathsf{gp}}_{\mathbb{P}^2}(\mathbb{P}^2))$, the procedure is to first write it as a product of global strict piecewise-linear function. Conveniently, we already constructed the example this way. The images of both piecewise-linear functions under $\Phi$ are $[D_0]+[D_1]$ and $[D_0]+[D_2]$ in $\mathsf{CH}^1(\mathbb{P}^2_k)$. Then as $\Phi$ is a ring homomorphism, we may compute the image of the degree 2 piecewise-polynomial as the cap product

$$([D_0] + [D_1]) \cap ([D_0] + [D_2]),$$

which is equal to $4[\mathrm{pt}] \in \mathsf{CH}^2(\mathbb{P}^2_k)$. ♦

**Example** (Example 2)**.** In the example of $\tilde{S} = \mathrm{Bl}_{(1:0:0)}\mathbb{P}^2_k$ the codimension 1 Chow classes of $\tilde{S}$ are generated by the exceptional divisor and the transform of any line in $\mathbb{P}^2_k$ not through $(1 : 0 : 0)$, so

$$\mathsf{CH}^1(\tilde{S}) = \mathbb{Z}\langle E \rangle + \mathbb{Z}\langle \bar{D}_2 \rangle .$$

Applying the above procedure to the strict piecewise-linear function

$$\begin{Bmatrix} \alpha c & + & \beta d \\ \gamma e & + & \alpha f \\ \beta x & + & \delta y \\ \delta u & + & \gamma v \end{Bmatrix},$$

we know to check the coefficients for the boundary divisors $E, \bar{D}_0, \bar{D}_1, \bar{D}_2$. Similarly to the example $\mathbb{P}^2_k$, we get that the coefficient $\alpha$ in front of $c$ or $f$

corresponds to the image in $\bar{\mathsf{M}}(U_{\tau'})$ via the restriction maps, and so it corresponds to the coefficient in front of the class $[E]$. The coefficient $\beta$ corresponds to $\tau_2$ and $\bar{D}_2$, the coefficient $\gamma$ corresponds to $\tau_1$ and $\bar{D}_1$, and the coefficient $\delta$ corresponds to $\tau_0$ and $\bar{D}_0$.

Thus, we obtain the class of divisors

$$\alpha[E] + \beta[\bar{D}_2] + \gamma[\bar{D}_1] + \delta[\bar{D}_0] \in \mathsf{CH}^1(\tilde{S}).$$

Note that there are relations in the Chow group, such as $[\bar{D}_2] = [E] + [\bar{D}_0]$, and therefore the expression could have been further simplified.

We may now also consider the map $\mathrm{sPP}_S(S) \to \mathrm{sPP}_{\tilde{S}}(\tilde{S})$ induced by $f : \tilde{S} \to S$ in terms of the Chow classes. When introducing example 2, we described the map $f^*\bar{\mathsf{M}}_S \to \bar{\mathsf{M}}_{\tilde{S}}$ explicitly. On strict piecewise-linear functions, this induces the map $\mathrm{sPP}_S(S) \to \mathrm{sPP}_{\tilde{S}}(\tilde{S})$ given by

$$\begin{Bmatrix} \alpha a & + & \beta b \\ \gamma x & + & \beta y \\ \alpha u & + & \gamma v \end{Bmatrix} \mapsto \begin{Bmatrix} (\alpha+\beta)c & + & \beta d \\ \alpha e & + & (\alpha+\beta)f \\ \gamma x & + & \beta y \\ \alpha u & + & \gamma v \end{Bmatrix}.$$

Consider for example the piecewise-linear function

$$\begin{Bmatrix} a \\ 0 \\ u \end{Bmatrix}$$

(which is given by just the coefficient 1 for the ray corresponding to $\tau_1$) which we know to map to the divisor class $[D_1]$, and note that the map induced by the blowup sends it to

$$\begin{Bmatrix} c \\ e+f \\ 0 \\ u \end{Bmatrix}$$

(which has the coefficient 1 for the rays corresponding to $\tau_1$ and $\tau'$) which in turn gets mapped to $[E] + [\bar{D}_1]$ in Chow.

♦

### 2.3.3 Map from strict piecewise-polynomial functions to Chow in the non-simple case

We have yet to discuss what happens if $S$ is not simple in order to study the example with the divisorial log structure from the nodal cubic in $\mathbb{P}^2_k$. Let

$S$ be a quasi-compact log smooth log algebraic stack over $k$. As restated in Lemma 2.3.6, there exists a log blowup $\pi\colon \tilde{S} \to S$ with $\tilde{S}$ simple. We define

$$\Phi_S\colon (\mathrm{Sym}\,\bar{\mathsf{M}}_S)(S) = \mathrm{sPP}_S(S) \to \mathsf{CH}_{\mathsf{op}}(S) \tag{2.3.3.1}$$

as the composite

$$(\mathrm{Sym}\,\bar{\mathsf{M}}_S)(S) \to \mathrm{Sym}\,\bar{\mathsf{M}}_{\tilde{S}}(\tilde{S}) \xrightarrow{\Phi_{\tilde{S}}} \mathsf{CH}_{\mathsf{op}}(\tilde{S}) \xrightarrow{\pi_*} \mathsf{CH}_{\mathsf{op}}(S). \tag{2.3.3.2}$$

For any log regular $S$, this map $\Phi_S$ is independent of the choice of log blowup $\pi\colon \tilde{S} \to S$.

**Example** (Example 3)**.** Recall that in the example of the log structure on $S = \mathbb{P}^2_k$ given by the nodal cubic, the strict piecewise-polynomial function $ab \in \mathrm{sPP}_S(S)$ was not a product of strict piecewise-linear functions, as the product $\{\alpha(a+b)\}\{\beta(a+b)\}$ for $\alpha, \beta \in \mathbb{Z}$ will never yield $\{ab\}$. (Again, this demonstrates the above-emphasized difference between $\mathrm{Sym}(\bar{\mathsf{M}}_S(S))$ and $(\mathrm{Sym}\,\bar{\mathsf{M}}_S)(S) = \mathrm{PP}_S(S)$.) Then the first step (writing as product of strict piecewise-linear functions) in the procedure would fail, something that is only possible as this log algebraic stack is not simple. How to compute the Chow class in $\mathsf{CH}_{\mathsf{op}}(S)$ for this piecewise-polynomial function then?

The idea above is to blowup until the strict piecewise-polynomial function can be written as product of piecewise-linear functions. The theory tells us that for a blowup $\pi : \tilde{S} \to S$ that is simple, there will definitely be enough strict piecewise-linear functions to generate all strict piecewise-polynomial functions. However, we want to express $\{ab\}$ and we may not need to blowup to a simple stack, simply use the composition in (2.3.3.2) for a blowup $\tilde{S}$ where we have enough strict piecewise-linear functions to write $\{ab\}$ as a product of those. Consider the blowup $\tilde{S} = \mathrm{Bl}_P(S)$ as sketched in Figure 2.14. This blowup $\tilde{S}$ is not yet simple as the intersection of $E$ and $\bar{D}$ is not connected. If you want to do the computation via a blowup that is simple (for example, if you want to compute the image of $\{a^2b\}$), you need only do one more blowup in either of the singular points of the boundary of $\tilde{S}$. However, we will illustrate that this blowup $\tilde{S}$ is sufficient to compute the Chow class associated to the strict piecewise-polynomial function $\{ab\}$.

### Log structure on the blowup

We now consider the log structure associated to the divisor formed by both the exceptional divisor $E$ as well as the strict transform $\bar{D} \subset \tilde{S}$ of $D$. We may visualise the effect on the characteristic monoids as the subdivision of the ice

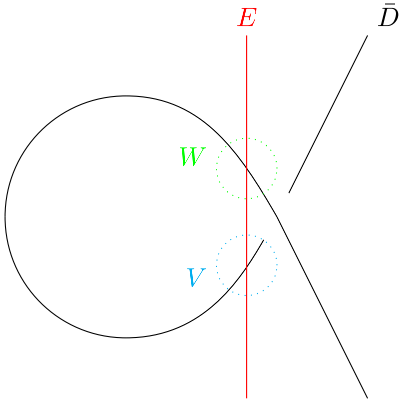
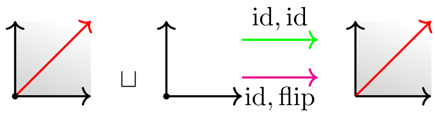
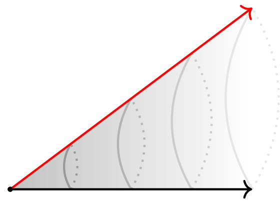



Figure 2.14: The blowup $\tilde{S} = \mathrm{Bl}_P(S)$.




(a) Colimit diagram



(b) Subdivision in ice cream cone

Figure 2.15: Visualising the subdivision that yields the characteristic monoid structure for $\tilde{S}$.

cream horn, as in Figure 2.15. Name the two intersection points of $\bar{D}$ and $E$ respectively $P_1, P_2$. Consider $W = \tilde{S} \setminus P_1$ and the open immersion $W \to \tilde{S}$ defining an open neighbourhood around $P_2$, and $V = \tilde{S} \setminus P_1$ and the open immersion $W \to \tilde{S}$ defining an open neighbourhood around $P_2$. Since these are both open immersions, we have an étale cover that is simply a Zariski cover. The log structure on $W$ and $V$ is given by two smooth divisors meeting at one point. By the Remark 2.2.7 or for example recalling the discussion of $\mathbb{P}^2_k$ with divisors for example $D_0$ and $D_1$, we obtain that the characteristic monoids on both covers are given by $\mathbb{N}^2$. Write $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(W) \cong \mathbb{Z}\langle x, y\rangle$ and $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(V) \cong \mathbb{Z}\langle u, v\rangle$.

Again, the morphism $\tilde{S} \to S$ contains the data of the map $\pi^*\mathsf{M}_S \to \mathsf{M}_{\tilde{S}}$ on the log structures. On the level of the characteristic monoid sheaves, we may simply describe it via the inclusion of the cones, as in the example of the blowup of the projective plane. Explicitly, on neighbourhood $U \subset S$ and $W, V \subset \tilde{S}$, the maps are

$$
\begin{aligned}
\bar{\mathsf{M}}^{\mathsf{gp}}_{S}(U) \cong \mathbb{Z}\langle a, b\rangle &\to \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(W) \cong \mathbb{Z}\langle x, y\rangle \\
a &\mapsto x \\
b &\mapsto x + y \\
\bar{\mathsf{M}}^{\mathsf{gp}}_{S}(U) \cong \mathbb{Z}\langle a, b\rangle &\to \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(V) \cong \mathbb{Z}\langle u, v\rangle \\
a &\mapsto u + v \\
b &\mapsto v.
\end{aligned}
$$

### Gluing maps

Because we simply have open neighbourhoods, we may describe the diagram of monoid restrictions as follows:

$$
\begin{tikzcd}
 & & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(W \setminus \bar{D}) \cong \mathbb{Z}\langle t\rangle & \\
 & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(W) \cong \mathbb{Z}\langle x, y\rangle \arrow[ur, "y\mapsto 0"'] \arrow[dr, "x\mapsto 0"] & & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S} \setminus \bar{D}) \cong \mathbb{Z}\langle t\rangle \arrow[ul] \arrow[ddl, bend left] \\
\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S}) \arrow[ur] \arrow[ddr] & & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(W \setminus E) \cong \mathbb{Z}\langle t\rangle & \\
 & & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(V \setminus \bar{D}) \cong \mathbb{Z}\langle t\rangle & \\
 & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(V) \cong \mathbb{Z}\langle u, v\rangle \arrow[ur, "u\mapsto 0"'] \arrow[dr, "v\mapsto 0"] & & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S} \setminus E) \cong \mathbb{Z}\langle t\rangle \arrow[uul, bend right] \arrow[dl] \\
 & & \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(V \setminus E) \cong \mathbb{Z}\langle t\rangle . &
\end{tikzcd}
$$

### Strict piecewise-polynomial functions

By the above diagram of gluing relations, when we define a strict piecewise-linear function via sections in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(W) \cong \mathbb{Z}\langle x, y\rangle$ and $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(V) \cong \mathbb{Z}\langle u, v\rangle$, the coefficient in the strict piecewise-linear function corresponding to $x$ must be equal to that of $v$ and similarly $y$ and $u$. Therefore strict piecewise-linear polynomials in $\mathrm{sPP}_{\tilde{S}}$ are of the form

$$\begin{Bmatrix} \alpha x & + & \beta y \\ \beta u & + & \alpha v \end{Bmatrix}$$

with $\alpha, \beta \in \mathbb{Z}$. To analyse what happened to strict piecewise-linear functions on $S$ in the blowup $\tilde{S}$, consider the map $\pi^* \bar{\mathsf{M}}_S \to \bar{\mathsf{M}}_{\tilde{S}}$ on characteristic monoid sheaves described above. The function $ab \in \mathrm{sPP}_S(S)$ corresponds to

$$\begin{Bmatrix} x & & \\ u & + & v \end{Bmatrix} \begin{Bmatrix} x & + & y \\ & & v \end{Bmatrix} = \begin{Bmatrix} x(x+y) \\ (u+v)v \end{Bmatrix}$$

in $\mathrm{sPP}_{\tilde{S}}(\tilde{S})$. Note that this *is* a product of two strict piecewise-linear functions (in contrast to $ab$), namely

$$\begin{Bmatrix} x & + & y \\ u & + & v \end{Bmatrix} \begin{Bmatrix} x \\ v \end{Bmatrix}.$$

To each of these strict piecewise-linear functions we may apply our procedure to establish the image under $\mathrm{sPP}_{\tilde{S}}(\tilde{S}) \to \mathsf{CH}(\tilde{S})$.

### Map to Chow

Firstly consider the strict piecewise-polynomial function

$$\begin{Bmatrix} x \\ v \end{Bmatrix}$$

on $\tilde{S}$. Following the diagram of restrictions of monoids, we see that this function is 1 on $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S} \setminus \bar{D})$ and 0 on $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S} \setminus E)$. Therefore this corresponds to the divisor $[E]$. (Other codimension 1 divisors do not appear in the boundary of this log structure so are automatically given coefficient 0.)

Then consider

$$\begin{Bmatrix} x & + & y \\ u & + & v \end{Bmatrix},$$

which is 1 on $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S} \setminus \bar{D})$ and 1 on $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\tilde{S} \setminus E)$. Therefore this corresponds to the divisor $[E] + [\bar{D}]$.

Finally, taking composition of maps yields

$$\begin{array}{rlll} (\mathrm{Sym}\, \bar{\mathsf{M}}_S)(S) & \to \mathrm{Sym}\, \bar{\mathsf{M}}_{\tilde{S}}(\tilde{S}) & \to \mathsf{CH}(\tilde{S}) & \to \mathsf{CH}(S) \\ ab & \mapsto \begin{Bmatrix} x & + & y \\ u & + & v \end{Bmatrix} \begin{Bmatrix} x \\ v \end{Bmatrix} & \mapsto [E] \cap ([\bar{D}] + [E]) & \mapsto [P]. \end{array}$$

The final step follows from the Chow intersection theory on $\tilde{S}$: we know that $[E]^2 = -[\mathrm{pt}]$ and $[E] \cap [\bar{D}] = 2[\mathrm{pt}]$, so the push of the sum equals to $[P]$.

Therefore, the 2-graded element $ab \in \mathrm{sPP}_S(S)$ corresponds to the divisor $[P]$ in $\mathsf{CH}(S)$. One may find it a relief that this results matches what one might intuitively think: if we view $a$ as the variable corresponding to one branch of the boundary divisor $D$ through the node, and $b$ as the other branch, then indeed their intersection yields the class of the node $[P]$.

♦

## 2.4 Piecewise-polynomial functions

So far we have only discussed what we mean by *strict* piecewise-polynomial functions. In [HS22], we work not just with the operational Chow group with rational coefficients, but with the log Chow group, see Definition 3.2.14 or [HS22, Definition 2.14], generalising an insight of [HPS19].

**Definition 2.4.1.** Let $X$ be a log smooth stack of finite type over $k$. We define the (operational) log Chow ring of $X$ to be

$$\mathrm{LogCH}(X) = \mathrm{colim}_{\tilde{X}}\, \mathsf{CH}_{\mathsf{op}}(\tilde{X}),$$

where the colimit runs over monoidal alterations $\tilde{X} \to X$ with $\tilde{X}$ smooth over $k$.

Examples of monoidal alterations are log blowups, which we have already seen: the blowup of the projective plane with toric boundary log structure in the intersection of divisors $D_1, D_2$, and the blowup of the projective plane with log structure from nodal cubic $D$ in the node of $D$.

Similarly to looking at Chow groups of all possible log blowups, for piecewise-polynomial functions we look at strict piecewise-polynomial functions on all log blowups as follows, see Definition 3.3.15 or [HS22, Definition 3.15 and Lemma 3.16].

**Definition 2.4.2.** For a log algebraic stack $S$ we define the group of *piecewise-polynomial functions* as

$$\mathrm{PP}'(S) = \mathrm{colim}_{\tilde{S}\to S}\, \mathrm{sPP}(\tilde{S}),$$

where $\tilde{S} \to S$ runs over all log blowups of $S$.

**Lemma 2.4.3.** *The pullback* $\mathrm{sPP}(S) \to \mathrm{sPP}(\tilde{S})$ *is injective for* $\tilde{S} \to S$ *any log blowup, so the natural maps to the colimit are injective.*

We define the sheaf of piecewise-polynomials PP on the small strict étale site of $S$ as the sheafification of the presheaf of rings $\mathrm{PP}'\colon U \mapsto \mathrm{PP}'(U)$.

**Example** (Example 1 and 2)**.** In the example of the projective plane with toric boundary log structure, we saw that log-blowups could be realised by subdividing the fan. That yielded more strict piecewise-polynomial functions, as we added variables for each ray. Also, we described in example 2 how the blowup map $\tilde{S} \to S$ gives a map $\mathrm{sPP}(S) \to \mathrm{sPP}(\tilde{S})$ which is injective. ♦

**Example** (Example 3)**.** In the example the projective plane with log structure from nodal cubic $D$, the strict piecewise-polynomial functions on the blowup $\tilde{S}$ in the node of $D$, such as

$$\begin{Bmatrix} x & & \\ u & + & v \end{Bmatrix}, \begin{Bmatrix} x & + & y \\ & & v \end{Bmatrix}, \begin{Bmatrix} x(x+y) \\ (u+v)v \end{Bmatrix}$$

also determine a piecewise-polynomial function on $S$ itself. ♦

**Proposition 2.4.4.** *The maps* $\Phi$ *assemble into a ring homomorphism*

$$\Phi^{\mathsf{log}} \colon \mathrm{PP}(S) \to \mathrm{LogCH}(S). \tag{2.4.0.1}$$

The maps to the Chow groups that we have carefully illustrated in the last section, also lift to the level of piecewise-polynomial functions. The image of the above map $\Phi^{\mathsf{log}}$ plays a fundamental role in the definition of the *log tautological ring* in [HS22, Definition 3.18] (see Definition 3.3.18).

## 2.5 Example of $\overline{\mathcal{M}}_{1,2}$

The purpose of this section is to illustrate that we can use the above constructions and examples to understand piecewise-polynomial functions on actual moduli spaces of interest such as $\overline{\mathcal{M}}_{g,n}$, in particular the case of $\overline{\mathcal{M}}_{1,2}$.

### 2.5.1 Log stack $\overline{\mathcal{M}}_{1,2}$

Consider $\overline{\mathcal{M}}_{1,2}$, the stack of stable 2-marked genus 1 curves. A visualisation of $\overline{\mathcal{M}}_{1,2}$ is drawn in Figure 2.17, along with its universal curve and its boundary divisors $D_1$ and $D_2$ that mark the locus of singular curves inside the 2-dimensional space $\overline{\mathcal{M}}_{1,2}$. That is, above the point in $\overline{\mathcal{M}}_{1,2}$, the type of stable 2-marked genus 1 curve is drawn that such a point represents. The log structure on $\overline{\mathcal{M}}_{1,2}$ will be the log structure associated to the boundary divisor.

Now, this is an example that is not simply a scheme: it is a DM-stack, so we also have to consider the role of automorphisms. The most important example of a non-trivial automorphism is the automorphism $i$ of the stable curve (corresponding to the node of $D_1$) shown in Figure 2.16.

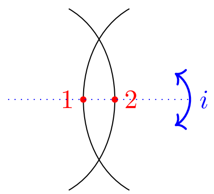



Figure 2.16: Automorphism $i$

#### Log structure

Again, there are two perspectives to describe the log structure on $\overline{\mathcal{M}}_{1,2}$ (that we want to consider here and which is in the literature the most common log structure on $\overline{\mathcal{M}}_{g,n}$): either via what is called the basic log structure on log curves, or via considering the log structure with respect to the boundary divisors $D_1, D_2$ forming $\partial\overline{\mathcal{M}}_{1,2}$.

To understand this log structure on the DM-stack $\overline{\mathcal{M}}_{1,2}$, or on any DM-stack, we refer to [Kat00]. As discussed in section 2 of that article, there exists a nice log structure on an $n$-pointed stable curve making it into a basic stable log curve. This log structure defined on the curve and the log structure it

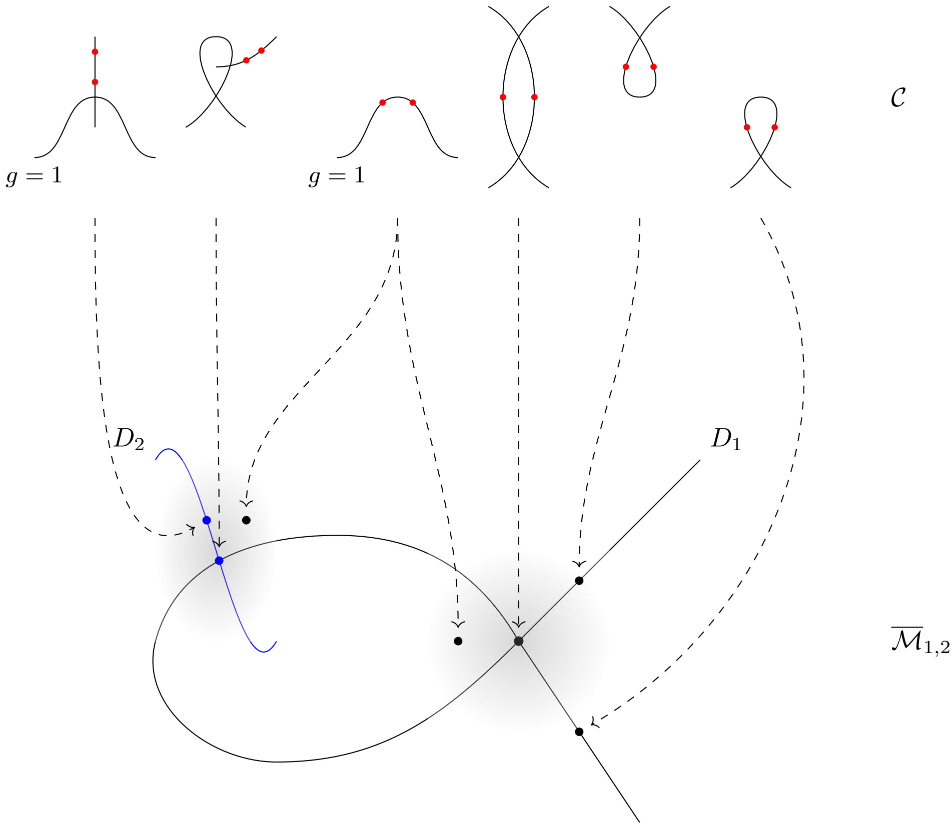



Figure 2.17: The universal curve $\mathcal{C}$ over $\overline{\mathcal{M}}_{1,2}$ and the boundary divisors $D_1$ and $D_2$.

defines on its base is analysed; these results we use in particular for the case of $\overline{\mathcal{M}}_{1,2}$ as the base with respect to the universal curve $\mathcal{C}$ over $\overline{\mathcal{M}}_{1,2}$. Then the basic log structure on the base is defined by the étale local expressions around the boundary divisors (that is, associated to the locus of singular curves in the universal curve). Hence, this basic log structure on curves ensures that $\overline{\mathcal{M}}_{1,2}$ is equipped with the log structure with respect to its boundary $\partial\overline{\mathcal{M}}_{1,2}$.

*Remark* 2.5.1. During the rest of the section, it may be confusing that we quickly switch perspectives between basic log structures on curves (which will be helpful in visualisations of the log structure) and the log structure only on the base $\overline{\mathcal{M}}_{1,2}$ with respect to its boundary divisors (which will allow us to use computations of (strict) piecewise-polynomial functions done in previous examples). The construction in section 2 of [Kat00] allows us to do this, but intuitively we can also explain this as follows. Remember that giving a map from a scheme $X$ to $\overline{\mathcal{M}}_{1,2}$ already contains the data of a genus 1 curve over base $X$ with 2 markings. So for a point $q\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ we have a curve $C$, and letting the horizontal maps in the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \operatorname{Spec}(k) \arrow[r, "q"] & \overline{\mathcal{M}}_{1,2} \end{tikzcd}$$

be strict, we may describe the log structure at a point $q$ in $\overline{\mathcal{M}}_{1,2}$ using properties of the curve $C \to \operatorname{Spec}(k)$. ♦

In section 3, Kato defines the meaning of giving a log structure on a stack: a log structure on $\overline{\mathcal{M}}_{1,2} \to \mathbf{Sch}$ is a factorisation $\overline{\mathcal{M}}_{1,2} \to \mathbf{LSch}$ such that for any two stable curves $c, c'$ and morphism $c \to c'$ the image is a strict morphism of log schemes. In section 4 it is then concluded that the basic log structure gives a log structure on $\overline{\mathcal{M}}_{1,2}$ that is natural in the sense that $\overline{\mathcal{M}}_{1,2}$ with that log structure represents the stack of stable log curves of type $(g, n)$. This is the reason this is the most commonly used log structure on $\overline{\mathcal{M}}_{1,2}$.

### Visualisation in terms of dual graphs

One visualisation of the characteristic monoid sheaf for the log structure on $\overline{\mathcal{M}}_{1,2}$ is giving lengths to edges of the dual graphs. Recall that the dual graph $\Gamma$ of a (stable) curve over an algebraically closed field has

- a vertex for each irreducible component of the curve, with a function $g\colon V \to \mathbb{Z}_{\geq 0}$ associating to each vertex the genus of that component,

- a leg for each marking, with a vertex assignment $L \to V$ for which component the marking belongs,
- an edge for each node between the components that meet.

(Sometimes an edge is described using half-edges: then the set of half-edges $H$ comes with an involution $\iota\colon H \to H$ where an edge is a 2-cycle of $\iota$ and a leg is a fixed point of $\iota$). The genus of the original curve can be retrieved via $\sum_{v\in V} g(v) + h_1(\Gamma) = g$. By the classification of the basic log structure of a stable curve as in [Kat00], we know that for $q\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ given by curve $C$, there is a natural isomorphism $\bar{\mathsf{M}}_{\overline{\mathcal{M}}_{1,2},q} = \mathbb{N}^e$ for $e$ the number of edges. Then the length of an edge is the corresponding generator of $\mathbb{N}^e$, and these edge lengths built up the stalk of the characteristic monoid.

We briefly discuss all possible dual graphs in $\overline{\mathcal{M}}_{1,2}$. On the smooth locus, we have dual graphs consisting of one node (of genus 1) and 2 legs, but no edges, and so we have no edge lengths and indeed the smooth locus has the expected trivial log structure. However, for points in the boundary, there are edges. Firstly for $q\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ the intersection point of $D_1$ and $D_2$, the curve has dual graph given by Figure 2.18.

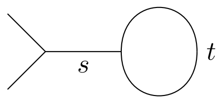



Figure 2.18: Dual graph of the curve represented by the intersection point of $D_1$ and $D_2$

Using the perspective of giving edge lengths (which we can later formally verify when we describe the characteristic monoid sheaf with respect to an étale cover), the local picture is

$$\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2},q} \cong \mathbb{Z}\langle s,t\rangle .$$

For $q\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ a point on $D_2$ unequal to the intersection with $D_2$, the curves have the dual graph shown in Figure 2.19. The stalk of the groupified characteristic monoid sheaf equals

$$\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2},q} \cong \mathbb{Z}\langle s\rangle .$$

For $q\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ a point on $D_1$ unequal to the node of $D_1$, the curves have the dual graph shown in Figure 2.20. The stalk of the groupified

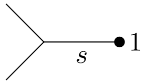



Figure 2.19: Dual graph of the curve represented by a point on $D_2$ not on $D_1$

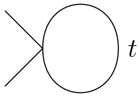



Figure 2.20: Dual graph of the curve represented by a point on $D_1$ not also in $D_2$ and not the nodal point of $D_1$

characteristic monoid sheaf equals

$$\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2},q} \cong \mathbb{Z}\langle t\rangle .$$

Finally, for $p\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ the node of $D_1$, the curve corresponds to the dual graph shown in Figure 2.21.

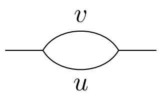



Figure 2.21: Dual graph of the curve represented by the nodal point of $D_1$

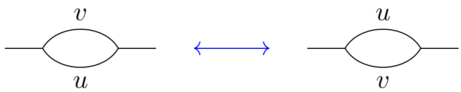
The stalk of the groupified characteristic monoid sheaf equals

$$\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2},p} \cong \bar{\mathsf{M}}^{\mathsf{gp}}_{\mathcal{U},p'} \cong \mathbb{Z}\langle u, v\rangle .$$

The data of all possible dual graphs of curves in $\overline{\mathcal{M}}_{1,2}$ with edge lengths yield a visualisation as shown in Figure 2.23 of the characteristic monoids as follows. Note that this visualisation is mostly a combination of pictures we have seen before: locally the intersection of $D_1$ and $D_2$ is like an affine patch of the first example, $\mathbb{P}^2_k$ with toric boundary structure, or as in the case of the guiding example $\operatorname{Spec}(k[\mathbb{N}^2])$.

Also, locally the nodal point of the boundary divisor $D_1$ gives a comparable characteristic monoid structure as the example of the nodal cubic in $\mathbb{P}^2_k$. However, it is important to note that in this case there are automorphisms involved. Formally, to give a sheaf on a stack, or to give a log structure on a stack, means you also have to specify what happens in case of the non-trivial automorphism $i$. In other words, locally at the node of $D_1$, we need to specify a sheaf not just on the point itself but on the quotient stack $[\mathrm{pt}/(\mathbb{Z}/2\mathbb{Z})]$: that is, a sheaf with a $\mathbb{Z}/2\mathbb{Z}$-action.

For the picture we are making, we note that automorphism $i$ shown in Figure 2.16 corresponds to swapping the nodes, and therefore can be naturally associated to interchanging the variables of the characteristic monoid, yielding 'switching of edges'. So on the level of dual graphs, the automorphism $i$ would correspond to the swapping




and therefore we are considering the colimit of the diagram Figure 2.22, where $i$ denotes $(u, v) \mapsto (v, u)$ (and again formally the colimit is taken in the category of cone stacks, see [CCUW20]). We will see this swapping map $i$ again in the subsection describing the gluing maps on an étale cover.

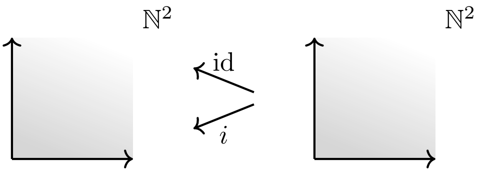



Figure 2.22: Colimit diagram for visualising characteristic monoid around node of $D_1$

Combining this automorphism with the lengths-on-the dual-graphs perspective, we obtain the visualisation in Figure 2.23 for the characteristic monoids of $\overline{\mathcal{M}}_{1,2}$. Be careful to note that this is simply a visualisation; one should formally take a colimit diagram and not think that the cones, such as the upper triangular cone and the lower cone actually embed in the same vector spaces.

### 2.5.2 Piecewise-polynomial functions on $\overline{\mathcal{M}}_{1,2}$

As in previous examples, we want to view piecewise-polynomial functions as functions on the stalks of the characteristic monoids (or sections of small enough opens) that glue appropriately. Hence, we describe $\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2}}$ on an appropriate étale cover.

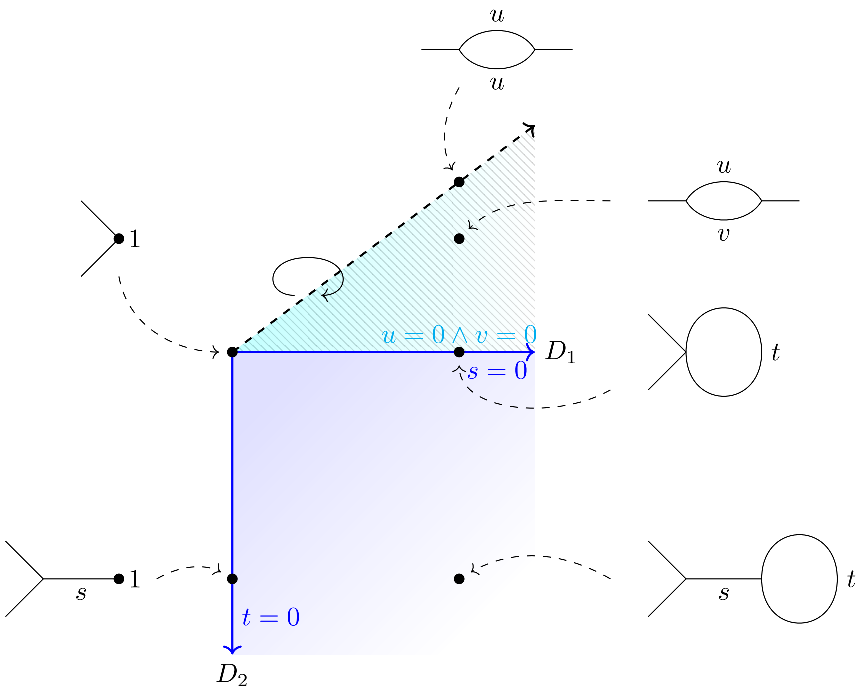



Figure 2.23: Visualisation of the characteristic monoids for $\overline{\mathcal{M}}_{1,2}$ by lengths-on-the-dual-graphs perspective.

Firstly, outside of $p\colon \mathrm{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ the nodal point of $D_1$, we are in the case of two smooth irreducible divisors specifying an easy log structure (as discussed in previous examples). Hence we can find some étale neighbourhood $\mathcal{V}$ (which need not be a scheme, simply $\overline{\mathcal{M}}_{1,2}$ without point $p$) of $\overline{\mathcal{M}}_{1,2}$ not covering $p$, on which we know the strict piecewise-polynomial functions by previous computations and theory. On this $\mathcal{V}$ the strict piecewise-linear functions are given by $\bar{\mathsf{M}}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{V}) \cong \mathbb{N}\langle s,t\rangle$ with generator $s$ corresponding to the boundary divisor $D_1$ and generator $t$ to $D_2$. (The reader may verify that this corresponds to the descriptions of the stalks in terms of edge lengths above.) This cover $\mathcal{V}$ is a simple stack, and so these strict piecewise-linear functions generate the strict piecewise-polynomial functions. Because étale locally around the intersection of $D_1$ and $D_2$ the strict piecewise-polynomial functions are given by a polynomial ring in two variables with no relations, as explained in (3.3.2.7), we also know that $\mathrm{sPP}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{V}) \cong \mathbb{Z}[s,t]$.

Secondly, we need to describe an étale cover around $p\colon \mathrm{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$ the nodal point of $D_1$. Note that describing an étale open neighbourhood $j\colon \mathcal{U} \to \overline{\mathcal{M}}_{1,2}$ also specifies the data of a curve over $\mathcal{U}$, and we can describe the map $j$ via the curve over $\mathcal{U}$. We want to give a neighbourhood of $p$ with which we (étale locally) describe the log structure from boundary $D_1$ as an affine plane with log structure given by the intersection of the coordinate axes. Therefore, consider $\mathcal{U} \subset \mathbb{A}^2_{u,v}$ an open in the affine plane with coordinates $u, v$ containing the origin, and let the map $j\colon \mathcal{U} \to \overline{\mathcal{M}}_{1,2}$ be given by the curve defined by $y^2 - ((x-1)^2 - u)((x+1)^2 - v) \in k[u,v][x,y]$ over $\mathcal{U}$. Note that we may possibly need to remove some points from $\mathcal{U}$, as we did in the cover in example 3, but we remain with a neighbourhood around the origin which corresponds to $y^2 - (x-1)^2(x+1)^2 = y^2 - (x^2-1)^2$ a curve with two components $y = \pm(x^2-1)$ and two nodes. (One may verify this yields an étale map, for example via asking SAGE for the $j$-invariant of this curve in terms of $u, v$ and concluding it is unramified in a neighbourhood of $u = v = 0$. ) This gives a genus one curve and we have two sections at infinity, which are for the weighted homogeneous equation $y^2 - ((x-z)^2 - uz^2)((x+z)^2 - vz^2)$ given by $z = 0, x = 1, y = \pm 1$, of which we choose an ordering so that we indeed have a morphism $j\colon \mathcal{U} \to \overline{\mathcal{M}}_{1,2}$.

### Gluing maps

On the cover $\mathcal{V}$, we are in the situation of a log scheme as computed in previous examples, and therefore all restriction maps are what is to be expected.

However, for covering $\mathcal{U}$, note that if we form the fiber product

$$\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U}$$

we are dealing with a fiber product for stacks.

That is, over a point $q\colon \operatorname{Spec}(k) \to \overline{\mathcal{M}}_{1,2}$, the fiber product contains a point $q_1\colon \operatorname{Spec}(k) \to \mathcal{U}$ over $q$, a point $q_2\colon \operatorname{Spec}(k) \to \mathcal{U}$ over $q$, and an isomorphism $f$ between $j(q_1)$ and $j(q_2)$. In $\overline{\mathcal{M}}_{1,2}$, over the point $p$ that is the nodal point of the boundary divisor $D_1$, there is a non trivial automorphism $i$ described above. On the level of $\mathcal{U}$ and the curve

$$y^2 - ((x-1)^2 - u)((x+1)^2 - v)$$

over $\mathcal{U}$, the automorphism $i$ corresponds to $u \leftrightarrow v$ and $x+1 \leftrightarrow x-1$. Let $q_1\colon \operatorname{Spec}(k) \to \mathcal{U}$ be a point given by coordinates $(u_0, v_0)$, and write for the swapped point $q_2\colon \operatorname{Spec}(k) \to \mathcal{U}$, that is, the point given by $(v_0, u_0)$. We know that $\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U}$ contains at least the points

$$(q_1, q_1, \mathrm{id}), (q_1, q_2, i), (q_2, q_1, i), (q_2, q_2, \mathrm{id})$$

(of which if $u_0 = v_0$ some may be the same). But over each projection map $\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \to \mathcal{U}$, say the first projection, we have at least two distinct points in the fiber, namely

$$(q_1, q_1, \mathrm{id}), (q_1, q_2, i).$$

Therefore, we have an étale cover with two points over every fiber, and this leads to the claim that $\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \cong \mathcal{U} \sqcup \mathcal{U}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{U} \sqcup \mathcal{U} \arrow[dr] \arrow[drr, "{(\mathrm{id},\mathrm{id})}"] \arrow[ddr, "{(\mathrm{id},i)}"'] & & \\
& \mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "j"] \\
& \mathcal{U} \arrow[r, "j"] & \overline{\mathcal{M}}_{1,2}.
\end{tikzcd}$$

The map $j$ is étale and of degree 2: this is an open property and may be checked over $u = v = 0$. Counting the automorphisms of $y^2 - (x^2-1)^2$ that preserve the markings yields there are only 2 such automorphisms (there is a non-trivial automorphism on each component which results in swapping the nodes). Hence, the maps $\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \to \mathcal{U}$ and $\mathcal{U} \sqcup \mathcal{U} \to \mathcal{U}$ are both étale degree 2, and the induced map $\mathcal{U} \sqcup \mathcal{U} \to \mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U}$ is injective. Then we may conclude that indeed $\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \cong \mathcal{U} \sqcup \mathcal{U}$ and the restriction maps are given by $(\mathrm{id}, \mathrm{id})$ and $(\mathrm{id}, i)$.

**Strict piecewise-polynomial functions**

To give a strict piecewise-linear function on $\overline{\mathcal{M}}_{1,2}$ means specifying a section in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{V})$ which is given by $\{\alpha s+\beta t\} \in \mathbb{Z}\langle s,t\rangle$, and a section in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{U})$ which is of the form $\{\gamma u+\delta v\}$ for $\gamma,\delta \in \mathbb{Z}$. By invariance under gluing maps on $\mathcal{U}$ we know $\gamma u+\delta v = \gamma v + \delta u$, giving that a section in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{U})$ is more specifically of the form $\{\gamma(u+v)\}$ for $\gamma \in \mathbb{Z}$. By gluing $\mathcal{U}$ and $\mathcal{V}$ (which is similar to the log scheme scenario as we exclude node $p$), we obtain that a strict piecewise-linear function on $\overline{\mathcal{M}}_{1,2}$ is given by

$$\begin{Bmatrix} \alpha s & + & \beta t \\ \alpha(u & + & v) \end{Bmatrix}$$

with $\alpha, \beta \in \mathbb{Z}$.

In this case, in contrast to to example 3 of the nodal cubic, we will see that all strict piecewise-polynomial functions in $\mathcal{U}$ are symmetric in $u$ and $v$. The description of the fiber product $\mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \cong \mathcal{U} \sqcup \mathcal{U}$ means we can decompose $\mathrm{sPP}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{U} \sqcup \mathcal{U}) \cong \mathrm{sPP}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{U}) \times \mathrm{sPP}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{U})$, and $\mathrm{sPP}_{\overline{\mathcal{M}}_{1,2}}(\mathcal{U}) \cong \mathbb{Z}[u,v]$ is the polynomial ring in 2 variables. Then the restriction maps on $\mathcal{U}$ are given by

$$\begin{tikzcd} \mathbb{Z}[u,v] \arrow[r, bend left, "{\substack{u\mapsto(a,c)\\ v\mapsto(b,d)}}"] \arrow[r, bend right, "{\substack{u\mapsto(a,d)\\ v\mapsto(b,c)}}"'] & \mathbb{Z}[a,b]\times\mathbb{Z}[c,d]. \end{tikzcd}$$

The degree 2 function $uv \in \mathbb{Z}[u,v]$ maps to

$$(a,c)(b,d) = (ab,cd)$$

via one map and

$$(a,d)(b,c) = (ab,cd)$$

via the other. Hence, an example of a degree 2 strict piecewise-polynomial function may be

$$\begin{Bmatrix} s^2 \\ uv \end{Bmatrix}$$

which is again not a product of strict piecewise-linear functions. Therefore, in the next section, we will need to blowup $\overline{\mathcal{M}}_{1,2}$ to determine its image in Chow.

Note that higher degree strict piecewise-polynomial functions are required to be symmetric: for example the degree 3 function $u^2v \in \mathbb{Z}[u,v]$ maps to

$$(a,c)^2(b,d) = (a^2b, c^2d)$$

via one map and

$$(a,d)^2(b,c) = (a^2b, cd^2)$$

via the other which are not equal. Similar reasoning yields that indeed strict piecewise-polynomial functions are required to be symmetric.

### 2.5.3 Map to Chow for strict piecewise-polynomial functions on $\overline{\mathcal{M}}_{1,2}$

Once again, as in Example 3, we are not dealing with a simple log stack. In that case there may be a strict piecewise-polynomial function that is not the product of strict piecewise-linear functions, and we again have to apply an appropriate log blowup to $\overline{\mathcal{M}}_{1,2}$ to describe what the map to Chow for that strict piecewise-polynomial functions looks like. Again, we want to blowup at some ideal $I$ corresponding to the blowup in the node of $D_1$, obtaining the situation sketched in Figure 2.24. In particular, because blowing up commutes with flat (and so also étale) base change, we may visualise the blowup via the blowup $\tilde{\mathcal{U}}$ of $\mathcal{U}$ in the origin (in $\mathbb{A}^2_{u,v}$);

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r, "\mathrm{Bl}_{j^{-1}I}"] \arrow[d] & \mathcal{U} \arrow[d, "j"] \\ \overline{\mathcal{M}}^{\sim}_{1,2} \arrow[r, "\mathrm{Bl}_I"] & \overline{\mathcal{M}}_{1,2} \end{tikzcd} \tag{2.5.3.1}$$

We study the blowup $\overline{\mathcal{M}}^{\sim}_{1,2}$ via the cover $\tilde{\mathcal{U}}$, leaving $\mathcal{V}$ unchanged. (That is, we translate our cover $\mathcal{V}$ around $D_2$ to a cover in the neighbourhood $\bar{D}_2$. Its strict piecewise-polynomial functions remain essentially unchanged.) Now $\tilde{\mathcal{U}}$ is the blowup of $\mathcal{U}$ in the origin, and so it has three boundary divisors $D_v$, originally the $v$-axis, $D_u$ originally the $u$-axis, and the exceptional divisor $E$. There are the two intersection points: one of $D_v$ and $E$ which we denote $P_1$ and one of $D_u$ and $E$ which we denote $P_2$. On open neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ around $P_1$ and $P_2$ respectively, the log structure is given by two smooth divisors (see Remark 2.2.7) so the characteristic monoid around each are given by $\mathbb{N}^2$ with a generator for each boundary divisor. Say

$$\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{\mathcal{U}}}(\mathcal{U}_1) = \mathbb{Z}\langle x, y\rangle$$

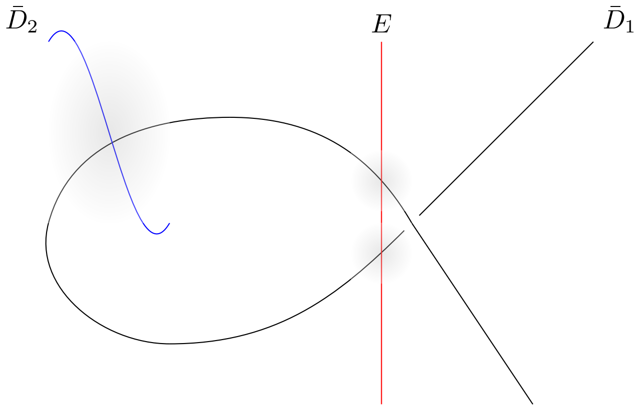



Figure 2.24: Blowup $\widetilde{\overline{\mathcal{M}}_{1,2}}$ of $\overline{\mathcal{M}}_{1,2}$.

where $x$ corresponds to $D_v$ and $y$ to $E$, and

$$\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{\mathcal{U}}}(\mathcal{U}_2) = \mathbb{Z}\langle X, Y\rangle$$

where $X$ corresponds to $D_u$ and $Y$ to $E$.

For the gluing maps on $\tilde{\mathcal{U}}$ we extend diagram (2.5.3.1) to

$$\begin{tikzcd}
\tilde{\mathcal{U}} \sqcup \tilde{\mathcal{U}} \arrow[r, "\mathrm{Bl}_{(j^{-1}I,\, j^{-1}I)}"] \arrow[d, "\wr"] \arrow[dd, bend right=40, "{(\mathrm{id},\mathrm{id})}"'] \arrow[dd, bend right=60, "{(\mathrm{id},i)}"'] & \mathcal{U} \sqcup \mathcal{U} \arrow[d, "\wr"] \arrow[dd, bend left=40, "{(\mathrm{id},\mathrm{id})}"] \arrow[dd, bend left=60, "{(\mathrm{id},i)}"] \\
\tilde{\mathcal{U}} \times_{\widetilde{\overline{\mathcal{M}}_{1,2}}} \tilde{\mathcal{U}} \arrow[r, "\mathrm{Bl}_{\mathrm{pr}_i^{-1} j^{-1} I}"] \arrow[d, shift right, "\mathrm{pr}_2"'] \arrow[d, shift left, "\mathrm{pr}_1"] & \mathcal{U} \times_{\overline{\mathcal{M}}_{1,2}} \mathcal{U} \arrow[d, shift right, "\mathrm{pr}_2"'] \arrow[d, shift left, "\mathrm{pr}_1"] \\
\tilde{\mathcal{U}} \arrow[r, "\mathrm{Bl}_{j^{-1}I}"] \arrow[d] & \mathcal{U} \arrow[d, "j"] \\
\widetilde{\overline{\mathcal{M}}_{1,2}} \arrow[r, "\mathrm{Bl}_I"] & \overline{\mathcal{M}}_{1,2}
\end{tikzcd}$$

where the horizontal blowup maps are constructed by base change with étale maps. Because of the identities $\mathrm{pr}_1^{-1} j^{-1} I = \mathrm{pr}_1^{-1} j^{-1} I$ as $j \circ \mathrm{pr}_1 = j \circ \mathrm{pr}_2$ and $j \circ (\mathrm{id}, i) = j \circ (\mathrm{id}, \mathrm{id})$, we choose the simplest map to describe which ideal we blowup. The commutative diagram then yields that also for $\tilde{\mathcal{U}}$ the restriction maps are given by

$$(\mathrm{id}, i), (\mathrm{id}, \mathrm{id}) \colon \tilde{\mathcal{U}} \sqcup \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}.$$

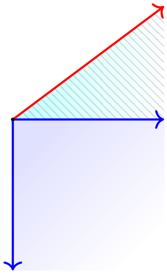


Figure 2.25: Visualisation of the subdivision of characteristic monoids for $\widetilde{\overline{\mathcal{M}}_{1,2}}$.

On $\mathcal{U}$ the map $i$ swapped $u \leftrightarrow v$, and so on $\tilde{\mathcal{U}}$ the map $i$ swaps $D_u$ and $D_v$ and flips $E$ onto itself. The map $\widetilde{\overline{\mathcal{M}}_{1,2}} \to \overline{\mathcal{M}}_{1,2}$ on the level of characteristic monoids may be visualised as subdividing the upper triangle, regaining two triangles glued at edges, see Figure 2.25.

### Strict piecewise-polynomial functions on the blowup

A strict piecewise-linear function on the blowup $\tilde{\mathcal{U}}$ is defined by giving sections in $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{\mathcal{U}}}(\mathcal{U}_1) \cong \mathbb{Z}\langle x, y\rangle$ and $\bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{\mathcal{U}}}(\mathcal{U}_2) \cong \mathbb{Z}\langle X, Y\rangle$, where the coefficient of the strict piecewise-linear function corresponding to $y$ must be equal to that of $Y$ as both correspond to the divisor $E$, and the coefficient in front of $x$ must be equal to that of $X$ by invariance under the swapping map. Therefore strict piecewise-linear polynomials in $\mathrm{sPP}_{\tilde{\mathcal{U}}}$ are of the form

$$\begin{Bmatrix} \alpha x & + & \beta y \\ \alpha X & + & \beta Y \end{Bmatrix}$$

with $\alpha, \beta \in \mathbb{Z}$.

To analyse what happened to strict piecewise-linear functions on $\overline{\mathcal{M}}_{1,2}$ in the blowup $\widetilde{\overline{\mathcal{M}}_{1,2}}$, recall that the map $\pi^*\bar{\mathsf{M}}_{\overline{\mathcal{M}}_{1,2}} \to \bar{\mathsf{M}}_{\widetilde{\overline{\mathcal{M}}_{1,2}}}$ is simply induced by the inclusion of cones. Explicitly, on the neighbourhood $\mathcal{U} \subset \overline{\mathcal{M}}_{1,2}$ and

$\mathcal{U}_1, \mathcal{U}_2 \subset \widetilde{\overline{\mathcal{M}}_{1,2}}$, the maps are

$$\begin{aligned}
\bar{\mathsf{M}}^{\mathsf{gp}}_S(\mathcal{U}) \cong \mathbb{Z}\langle u, v\rangle &\to \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\mathcal{U}_1) \cong \mathbb{Z}\langle x, y\rangle \\
u &\mapsto x \\
v &\mapsto x + y \\
\bar{\mathsf{M}}^{\mathsf{gp}}_S(\mathcal{U}) \cong \mathbb{Z}\langle u, v\rangle &\to \bar{\mathsf{M}}^{\mathsf{gp}}_{\tilde{S}}(\mathcal{U}_2) \cong \mathbb{Z}\langle X, Y\rangle \\
u &\mapsto X + Y \\
v &\mapsto Y.
\end{aligned}$$

**Map to Chow**

Via composing the induced map on strict piecewise-polynomial functions with the practical procedure for the map to Chow, we obtain the map to Chow

$$(\mathrm{Sym}\,\bar{\mathsf{M}}_{\overline{\mathcal{M}}_{1,2}})(\overline{\mathcal{M}}_{1,2}) \to \mathrm{Sym}\,\bar{\mathsf{M}}_{\widetilde{\overline{\mathcal{M}}_{1,2}}}(\widetilde{\overline{\mathcal{M}}_{1,2}}) \xrightarrow{\Phi_{\widetilde{\overline{\mathcal{M}}_{1,2}}}} \mathsf{CH}_{\mathsf{op}}(\widetilde{\overline{\mathcal{M}}_{1,2}}) \xrightarrow{\pi_*} \mathsf{CH}_{\mathsf{op}}(\overline{\mathcal{M}}_{1,2}).$$

For the example

$$\begin{Bmatrix} s^2 \\ uv \end{Bmatrix},$$

we first consider $uv$ and its image under the described map on strict piecewise-linear functions to the blowup. There $uv$ corresponds to

$$\begin{Bmatrix} x & & \\ X & + & Y \end{Bmatrix} \begin{Bmatrix} x & + & y \\ & & Y \end{Bmatrix} = \begin{Bmatrix} x(x+y) \\ (X+Y)Y \end{Bmatrix}.$$

So our example is the product of two strict piecewise-linear functions in the blowup;

$$\begin{Bmatrix} s & & \\ x & + & y \\ X & + & Y \end{Bmatrix} \begin{Bmatrix} s \\ x \\ Y \end{Bmatrix}.$$

To each of these strict piecewise-linear functions we may apply our procedure to establish the image under $\mathrm{sPP}_{\widetilde{\overline{\mathcal{M}}_{1,2}}}(\widetilde{\overline{\mathcal{M}}_{1,2}}) \to \mathsf{CH}(\widetilde{\overline{\mathcal{M}}_{1,2}})$.

Now, consider the strict piecewise-polynomial function

$$\begin{Bmatrix} s \\ x \\ Y \end{Bmatrix}.$$

Following the diagram of restrictions of monoids, we see that this function is 0 at the coefficient corresponding to $\bar{D}_1$, 1 at the coefficient corresponding to $E$, and, as the the coefficient in front of $t$ is zero, also 0 on $\bar{D}_2$. Therefore this corresponds to the divisor $[E]$. (Other codimension 1 divisors do not appear in the boundary giving the log structure so are automatically given coefficient 0.)

Consider

$$\left\{\begin{matrix} s & & \\ x & + & y \\ X & + & Y \end{matrix}\right\},$$

which is 1 on $\bar{D}_1$, 1 on $E$, and 0 on $\bar{D}_2$. Therefore this corresponds to the divisor $[E] + [\bar{D}_1]$.

Finally, taking composition of maps yields

$$\begin{array}{ccccc} (\operatorname{Sym}\bar{\mathsf{M}}_S)(S) & \to \operatorname{Sym}\bar{\mathsf{M}}_{\tilde{S}}(\tilde{S}) & & \to \mathsf{CH}(\tilde{S}) & \to \mathsf{CH}(S) \\ \left\{\begin{matrix} s^2 \\ uv \end{matrix}\right\} & \mapsto \left\{\begin{matrix} s & & \\ x & + & y \\ X & + & Y \end{matrix}\right\} & \left\{\begin{matrix} s \\ x \\ Y \end{matrix}\right\} & \mapsto [E] \cap ([\bar{D}_1] + [E]) & \mapsto [p]. \end{array}$$

The final step follows from the Chow intersection theory on the blowup $\widetilde{\overline{\mathcal{M}}_{1,2}}$: we know that $[E]^2 = -[\mathrm{pt}]$ and $[E] \cap [\bar{D}] = 2[\mathrm{pt}]$, so the push of the sum equals to the class $[p]$ of the point $p$ which is the nodal point point of boundary divisor $D_1$. Therefore, we have combined our computations in previous examples to give a strict piecewise-polynomial that maps to the class of the node $p$.

### Determining a preimage for a divisor

Note that the map to the Chow group as constructed, or the map to the log Chow group, is not necessarily injective or surjective. However, one might wonder whether a certain divisor, or a certain multiple of a divisor, is the image of a piecewise-polynomial function. When a divisor is easily expressed in boundary divisors, then the original piecewise-polynomial might be derived or guessed from the structure of the boundary divisors, but in general there is no algorithm to decide whether a class is an image of a piecewise-polynomial function.

An example of a question that we can answer with our computations of this section is: for which $\lambda \in \mathbb{Z}$ does the class $\lambda A$ for the stratum $A$ depicted in Figure 2.26 lie in the image of the map $\Phi$ from piecewise-polynomials to Chow.

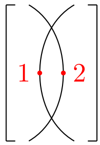
$A =$ 


Figure 2.26: Example of a Chow class of a boundary stratum

By our computations, we know that a piecewise-polynomial function with the expression $uv$ is a good candidate to have this image. Indeed the above described function

$$\begin{Bmatrix} s^2 \\ uv \end{Bmatrix}$$

maps to $\lambda = 1$ times the boundary stratum.

As a final remark, it is good to know that piecewise-polynomial functions may appear in different forms in the literature. Also it is possible to write piecewise-polynomial functions in a form that is more recognisable as relating to the tautological ring with its $\kappa$ and $\psi$-classes. When we view a piecewise-polynomial function is a polynomial in boundary divisors or branches of boundary divisors as above, and then relate these to $\psi$ classes, we would get such a form. For this, and also results such as giving a Pixton's formula as in [JPPZ20] in terms of piecewise-polynomial functions, we refer to [HMP$^+$22].