Logarithmic approach to the double ramification cycle

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Introduction

In order to study invariants and deformations of families of algebraic complex curves (1-dimensional varieties over $\mathbb{C}$) Deligne and Mumford in 1969 constructed the moduli space $\overline{\mathcal{M}}_g$ of genus g stable curves. That is, aside from the smooth curves of genus g in $\mathcal{M}_g$, this space $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ also parameterises curves with nodal singularities and only finitely many automorphisms; here nodal means that the curves may have singularities that are étale locally of the form

$$\operatorname{Spec} \mathbb{C}[X, Y]/(XY) \subset \mathbb{A}^2_{\mathbb{C}},$$

the meeting of the two coordinate axes. Similarly, the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points allows us to study invariants of genus g curves with n ordered specified points. This moduli space $\overline{\mathcal{M}}_{g,n}$ is a Deligne–Mumford stack (or orbifold) which is nonsingular, irreducible, and of (complex) dimension $3g - 3 + n$.

In this thesis, we study particular classes in the Chow group: the group formed by algebraic cycles (finite formal sums of k-dimensional subschemes) up to rational equivalence. The Chow group of $\overline{\mathcal{M}}_{g,n}$ is then extended to allow $\mathbb{Q}$-coefficients in the formal sums.

We study those Chow classes we are interested in (see next subsection) from the perspective of *logarithmic geometry.* Log geometry is concerned with adding to a scheme (or algebraic stack) a log structure, which is a sheaf of monoids that intuitively allows us to keep track of certain boundary divisors. That is, a *log scheme* $(X, \mathsf{M}_X, \alpha)$ is a scheme X with sheaf of monoids with respect to the étale topology denoted by M_X and a morphism $\alpha\colon \mathsf{M}_X \to \mathcal{O}_X$ (where $\mathcal{O}_X$ is seen as a sheaf of monoids with the multiplication of functions) such that α identifies the units, i.e. $\alpha\colon \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. For the log algebraic stacks that we are interested in, there exists a unique normal crossings divisor Z, called the *boundary divisor*, along which the characteristic monoid sheaf $\bar{\mathsf{M}}_X = \mathsf{M}_X/\mathcal{O}_X^*$ is non-trivial. Then indeed the log structure is associated to a certain boundary divisor and so 'keeps track' of a boundary.

The double ramification cycle

Let $A = (a_1, \ldots, a_n)$ be a vector of n integers satisfying

$$\sum_{i=1}^{n} a_i = 0$$

and for simplicity assume all $a_i \neq 0$. Working over $\mathbb{C}$ and supposing that $2g-2+n > 0$, we will introduce the idea behind the *double ramification cycle* with respect to vector A as a class in the Chow group of the moduli space $\mathcal{M}_{g,n}$. Let $\mathcal{Z}_g(A)$ be the locus in $\mathcal{M}_{g,n}$ parameterising curves $[C, p_1, \ldots, p_n]$ satisfying

$$\mathcal{O}_C \left(\sum_{i=1}^{n} a_i p_i \right) \simeq \mathcal{O}_C \tag{1}$$

where $\mathcal{O}_C$ denotes the trivial line bundle, and $\mathcal{O}_C \left(\sum_{i=1}^{n} a_i p_i\right)$ the twist by the divisor formed by the formal sum of the distinct marked points viewed as codimension 1 divisors. To explain the name double ramification cycle, we translate this line bundle condition (1) to a more classical description. Using the equivalence between line bundles with a section up to isomorphism and Cartier divisors up to rational equivalence, we may also describe the above algebraic condition (at least for smooth curves) as follows. As the trivial bundle corresponds to principal divisors, being isomorphic to the trivial bundle is the same as asking if there exists a rational function $f \in \mathcal{K}_C^*(C)$ such that the principal divisor $\mathrm{div}(f)$ is equal to $\sum_{i=1}^{n} a_i p_i$. This means that if condition (1) is satisfied for $[C, p_1, \ldots, p_n]$, then there is a rational function $f \colon C \to \mathbb{P}^1_{\mathbb{C}}$ of degree

$$D = \sum_{\substack{i \in \{1,..,n\} \\ a_i > 0}} a_i$$

with zeroes of orders specified by the a_i satisfying $a_i > 0$, and poles of orders specified by the a_i satisfying $a_i < 0$. In other words, a function with ramification profile A over $0, \infty \in \mathbb{P}^1_{\mathbb{C}}$. Conversely, also every such $f \colon C \to \mathbb{P}^1_{\mathbb{C}}$ (up to $\mathbb{C}^*$-scaling) determines an element of $\mathcal{Z}_g(A)$. Therefore, we may view $\mathcal{Z}_g(A) \subset \mathcal{M}_{g,n}$ also as the locus of curves admitting degree D maps $f \colon C \to \mathbb{P}^1_{\mathbb{C}}$ with ramification profile A over 0 and ∞. This ramification profile of f over two points is the reason for the term *double ramification*.

This algebraic condition defines a substack of $\mathcal{M}_{g,n}$. However, on the moduli space $\overline{\mathcal{M}}_{g,n}$, one needs to first describe a compactification $\bar{\mathcal{Z}}_g(A)$ of the locus $\mathcal{Z}_g(A)$ inside the moduli space $\overline{\mathcal{M}}_{g,n}$, and there are multiple ways to do so. A description of two common ways is given below. A Chow class associated to $\bar{\mathcal{Z}}_g(A)$ then yields what is commonly referred to as the *double ramification cycle*. For early results on the double ramification cycle, we refer to [BSSZ15, CMW12, FP05, GZ14, Hai13, MW13]. More recent study and applications, may be found in [BGR19, FP16, Hol19, HKP18, HPS19, MPS23, Sch18a, HMP+22, MR21, MW13, MW20, AW18].

This 'appropriate' compactification (a compactification that still correctly counts or describes the curves satisfying (1)) of the double ramification cycle from $\mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$ has different approaches. From the point of view of relative Gromov–Witten theory, the most natural compactification of the substack

$$\left\{(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} \mid \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \simeq \mathcal{O}_C\right\}$$

is in the space $\overline{\mathcal{M}}^{\sim}_{g,A}$ of stable maps to 'rubber': stable maps to $\mathbb{P}^1_{\mathbb{C}}$ relative to 0 and ∞ modulo the $\mathbb{C}^*$-action on $\mathbb{P}^1_{\mathbb{C}}$. This rubber moduli space carries a natural class, namely the virtual fundamental class $\left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\text{virt}}$ of (complex) dimension $2g-3+n$. The canonical morphism forgetting the map to $\mathbb{P}^1_{\mathbb{C}}$ and just remembering the (stabilisation of the) curve

$$\epsilon\colon \overline{\mathcal{M}}^{\sim}_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

then allows us to pushforward this natural class to obtain the double ramification cycle

$$\epsilon_* \left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\text{virt}} =: \mathsf{DR}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}).$$

This definition is the approach in for example [JPPZ17, JPPZ20], in which a formula for double ramification cycles and for double ramification cycles with target varieties is developed.

In this thesis, rather than using this approach with the virtual class of the moduli space of stable maps in Gromov–Witten theory, we use an alternative and perhaps more intuitive approach by partially resolving the classical Abel–Jacobi map. The method follows the path of [MW20, Hol19, HPS19]. The idea is to take condition (1) and deduce what this means on the level of morphisms from $\overline{\mathcal{M}}_{g,n}$ to the moduli space parameterising marked curves and also a given line bundle. To make this more precise, we introduce a moduli space $\mathcal{J}_{g,n}$

parameterising stable curves with line bundles of multidegree 0 (degree zero on each irreducible component of the curve): also called the universal Jacobian. Let $\mu\colon \mathcal{J}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the natural map forgetting the line bundle. The Abel–Jacobi map AJ_A on $\mathcal{M}_{g,n}$ is defined by

$$\begin{aligned} \mathsf{AJ}_A\colon \mathcal{M}_{g,n} &\to \mathcal{J}_{g,n}, \\ \mathsf{AJ}_A([C, p_1, \dots, p_n]) &= \mathcal{O}_C\left(-\sum_{i=1}^{n} a_i p_i\right). \end{aligned}$$

On the smooth locus, $\mathcal{M}_{g,n}$, the double ramification cycle may be defined by comparing the image of the zero section

$$\begin{aligned} e\colon \mathcal{M}_{g,n} &\to \mathcal{J}_{g,n}, \\ e([C, p_1, \dots, p_n]) &= \mathcal{O}_C. \end{aligned}$$

and the image of AJ_A. However, on $\overline{\mathcal{M}}_{g,n}$ the map AJ_A only defines a rational map (it only defines a morphism almost everywhere); in families of curves we may have line bundles of total degree 0, but when the curves degenerate to a stable curve which has multiple irreducible components, the bundle may not have multidegree 0, that is degree 0 on each irreducible component. This insight results in searching for a useful space over which AJ_A may make sense: in [Hol19], a 'modification' $\overline{\mathcal{M}}^{\lozenge}_{g,A}$ with a proper birational map $\overline{\mathcal{M}}^{\lozenge}_{g,A} \to \overline{\mathcal{M}}_{g,n}$ is constructed with inside that modification $\mathcal{U}^{\lozenge}_{g,A}$ the largest open on which the Abel–Jacobi map can be extended. Then, we will define the double ramification cycle via operations in intersection theory to the images of AJ_A on $\mathcal{U}^{\lozenge}_{g,A}$ and the image of the unit section (defined on all $\overline{\mathcal{M}}_{g,n}$). Namely, we consider the intersection product $\mathsf{AJ}^!_A([e])$ on $\mathcal{U}^{\lozenge}_{g,A}$ through the diagram

$$\begin{tikzcd} & & & \mathcal{J}_{g,n} \\ \mathcal{M}_{g,n} \arrow[r] \arrow[urrr, "\mathsf{AJ}_A"] & \mathcal{U}^{\lozenge}_{g,A} \arrow[r] \arrow[urr, "\mathsf{AJ}_A"'] & \overline{\mathcal{M}}^{\lozenge}_{g,A} \arrow[r] & \overline{\mathcal{M}}_{g,n}, \arrow[u, "e"] \end{tikzcd}$$

and push it forward to yield the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$.

A slight generalisation of this double ramification cycle is to consider, for a vector $A = (a_1, \dots, a_n)$ of integers satisfying

$$\sum_{i=1}^{n} a_i = k(2g-2),$$

the *twisted double ramification cycles*,

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}),$$

related to the classical loci

$$\left\{(C,p_1,\ldots,p_n) \in \mathcal{M}_{g,n} \mid \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right) \simeq \omega_C^k\right\},$$

which have been constructed in [Hol19, HKP18, MW20] for all $k \geq 1$. This thesis discusses several questions regarding the (twisted) double ramification cycle using tools from logarithmic geometry.

Chapter 1: Operational Chow and the universal double ramification cycle

One issue is to realise both the standard and the twisted double ramification cycle described above via one universal construction: namely the universal Abel–Jacobi construction of a double ramification cycle over *the Picard stack* discussed in [BHP+23], which is the basis of chapter 1 in this thesis. An object of the Picard stack of curves with line bundles $\mathfrak{Pic}_{g,n}$ over a base $\mathcal{S}$ is a flat family

$$\pi\colon \mathcal{C} \to \mathcal{S}$$

of prestable n-pointed genus g curves together with a line bundle $\mathcal{L} \to \mathcal{C}$. Since the degree of a line bundle is constant in flat families, $\mathfrak{Pic}_{g,n}$ is the disjoint union of $\mathfrak{Pic}_{g,n,d}$, the Picard stack of curves with degree d line bundles, for $d \in \mathbb{Z}$. Chapter 1 firstly treats foundational issues of intersection theory on algebraic stacks that are not necessarily Deligne–Mumford stacks, such as the Picard stack $\mathfrak{Pic}_{g,n}$.

Secondly, chapter 1 describes the construction of a *universal twisted double ramification cycle* in the operational Chow theory with $\mathbb{Q}$-coefficients of $\mathfrak{Pic}_{g,n,d}$,

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}),$$

given a vector of integers $A = (a_1,\ldots,a_n)$ with $\sum_{i=1}^n a_i = d$. Log geometry based on the stack of tropical divisors constructed in [MW20] plays a crucial role.

The basic compatibility of this new operational class with the standard double ramification cycle is as follows. Let $A = (a_1,\ldots,a_n)$ be a vector of integers with $\sum_{i=1}^n a_i = k(2g-2)$. Let $\varphi_{\omega_\pi^k}\colon \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,k(2g-2)}$ be the map determined by the data of the universal curve $\pi\colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ and relative dualising sheaf ω_π^k of the morphism π. The action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ corresponding to the universal data is compatible with the previously defined twisted double ramification cycle:

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\omega_\pi^k})\Big([\overline{\mathcal{M}}_{g,n}]\Big) = \mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

for all $k \geq 1$. The main theorem of chapter 1, and the first main theorem in [BHP+23], states the above compatibilities of the universal double-ramification cycle with the standard and twisted double ramification cycles:

Theorem 1. *Let $g \geq 0$ and $d \in \mathbb{Z}$. Let $A = (a_1, \ldots, a_n)$ be a vector of integers satisfying*

$$\sum_{i=1}^{n} a_i = d\,.$$

Logarithmic compactification of the Abel–Jacobi map yields a universal twisted double ramification cycle

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

which is compatible with the standard double ramification cycle

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

in case $d = k(2g-2)$ for $k \geq 0$.

Chapter 2: Piecewise-polynomial functions and divisors

Another commonly studied question with regard to cycles on $\overline{\mathcal{M}}_{g,n}$ is whether or not a class is provably in the tautological ring. That is, whether a class lies in a subring generated by 'computable' known classes. The fact that the double ramification cycle is tautological is already known from [FP05], and there is a specific formula to compute it [JPPZ17]. However, in [HS22], we are able to show more classes of interest are tautological, via describing or deciding what tautological should be in logarithmic geometry. Key to describing 'logarithmically tautological' is a log geometric approach to describing divisors and classes, namely using *piecewise-polynomial functions*. Formally, the sheaf of strict piecewise-polynomial functions is constructed as the sheaf of symmetric algebras over the groupified characteristic monoid sheaf $\bar{\mathsf{M}}^{\mathsf{gp}}$. Intuitively, these functions are polynomials in the irreducible components of the boundary divisor that is associated to the log structure.

The purpose of chapter two is to provide illustrations of piecewise-polynomial functions and their relation to classical divisor classes in the Chow group. It is a section explaining some concisely stated content of [HS22], which is aimed to help better understand the proofs and statements of that paper. We introduce strict piecewise-polynomial functions, whose definition generalises similar theory for toric varieties, which may be found [Pay06, Bri96] and its references. (As illustrated in those examples in that section that are toric varieties, these may be seen as functions on the toric fan.) However, also for non-toric examples, or actually for any regular log regular log algebraic stack S over a field k, we construct a map from strict piecewise-polynomial functions to the Chow group

$$\Phi_S \colon \mathrm{sPP}_S(S) \to \mathsf{CH}_{\mathsf{op}}(S). \tag{2}$$

Then, we consider the log Chow ring, which is the colimit of the Chow rings of all log blowups (an iterated blowup in boundary strata where the log structure is not trivial). Also, we take strict piecewise-polynomial functions on all log blowups which give us what is called the piecewise-polynomial functions, and we obtain the following statement.

Proposition 2. *The maps (2) assemble into a ring homomorphism*

$$\Phi^{\mathsf{log}} \colon \mathrm{PP}(S) \to \mathrm{LogCH}(S).$$

Chapter 3: Logarithmic intersections of the double ramification cycle

One other question with respect to the double ramification cycle, is: what is a good definition of a double-double ramification cycle, where we consider two ramification profiles simultaneously? That is, suppose we have two vectors $A, B \in \mathbb{Z}^n$, then the *double-double ramification cycle* $\mathsf{DR}^{\mathsf{op}}(A, B)$ measures the locus of marked curves where both

$$\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \simeq \mathcal{O}_C, \text{ and } \mathcal{O}_C\left(\sum_{i=1}^{n} b_i p_i\right) \simeq \mathcal{O}_C.$$

Then we want a rational function f with ramification profile A and a rational function g with ramification profile B to exist. Intuitively, over $\mathcal{M}_{g,n}$, this would of course just be the intersection of the cycles associated to the intersection of the loci where such f and g respectively exist. Thus we would be looking at the intersection of the corresponding cycles $\mathsf{DR}_{g,A}$ and $\mathsf{DR}_{g,B}$. The key insight of [HPS19] was that this naive intersection is the 'wrong' way to extend this class to $\overline{\mathcal{M}}_{g,n}$. Instead, one should construct a new class $\mathsf{DR}(A, B)$ for the product, which over $\mathcal{M}_{g,n}$ is just the intersection of $\mathsf{DR}_{g,A}$ and $\mathsf{DR}_{g,B}$, but which in general will not equal the intersection product of the classes of the two factors:

$$\mathsf{DR}(A, B) \neq \mathsf{DR}_{g,A} \cdot \mathsf{DR}_{g,B}.$$

In this chapter, which is the paper [HS22], given line bundles $\mathcal{L}_1, \mathcal{L}_2$ (which in the above are the line bundles of the shape $\mathcal{L}_1 = \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right)$), using a logarithmic approach, we construct a $\mathsf{DR}^{\mathsf{op}}(\mathcal{L}_1, \mathcal{L}_2)$ (a class for the locus where the line bundles become trivial) in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ such that the formula

$$\mathsf{DR}^{\mathsf{op}}(\mathcal{L}_1, \mathcal{L}_2) = \mathsf{DR}^{\mathsf{op}}(\mathcal{L}_1, \mathcal{L}_1 \otimes \mathcal{L}_2) \tag{3}$$

does hold for arbitrary families in $\overline{\mathcal{M}}_{g,n}$ and not only over $\mathcal{M}_{g,n}$. Equality (3) is a particular instance of a $\mathrm{GL}_2(\mathbb{Z})$-invariance property for the double-double ramification cycles:

Theorem 3 (GL($\mathbb{Z}$)-invariance of DDR)**.** *If $M \in \mathrm{GL}_r(\mathbb{Z})$ and*

$$M[\mathcal{L}_1, \ldots, \mathcal{L}_r] = [\mathcal{F}_1, \ldots, \mathcal{F}_r],$$

then

$$\mathsf{DR}^{\mathsf{op}}(\mathcal{L}_1, \ldots, \mathcal{L}_r) = \mathsf{DR}^{\mathsf{op}}(\mathcal{F}_1, \ldots, \mathcal{F}_r).$$

The question of whether the r-fold DR-cycle is tautological is proven in the first main theorem of [HS22]:

Theorem 4. *Let g, n be non-negative integers, r a positive integer, and $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be line bundles on the universal curve over $\overline{\mathcal{M}}_{g,n}$. Then the r-fold double ramification cycle*

$$\mathsf{DR}^{\mathsf{op}}(\mathcal{L}_1, \ldots, \mathcal{L}_r)$$

lies in the tautological subring of the Chow ring of $\overline{\mathcal{M}}_{g,n}$.

This theorem is also proven in [MR21] with an independent approach by studying the virtual strict transforms of the DR cycle.

The key technical result of this chapter lies in a logarithmic approach via a log double ramification cycle in the log Chow group. The first step in the construction of $\mathsf{DR}(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ is realising that $\mathsf{DR}(\mathcal{L})$ naturally lives on a log blowup of $\overline{\mathcal{M}}_{g,n}$. We define a log Chow ring, the colimit of the Chow rings of all log blowups, and then construct $\mathsf{LogDR}(\mathcal{L}) \in \mathrm{LogCH}(\overline{\mathcal{M}}_{g,n})$. The product of $\mathsf{LogDR}(\mathcal{L}_i)$ is well behaved, and pushing it forward to the usual Chow ring yields the definition of the r-fold double ramification cycle $\mathsf{DR}(\mathcal{L}_1, \ldots, \mathcal{L}_r)$.

Our proof of the fact that double-double ramification cycles are tautological runs via showing that $\mathsf{LogDR}(\mathcal{L})$ is log tautological; a concept that we define using the piecewise-polynomial functions illustrated in chapter 2.

Theorem 5. LogDR *lies in the tautological subring of* $\mathrm{LogCH}(\mathfrak{Pic}_{g,n})$.