Nonabelian flows in networks

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Non-Abelian Flows in Networks

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Abstract

In this work we consider a generalization of graph flows. A graph flow is, in its simplest formulation, a labeling of the directed edges with real numbers subject to various constraints. A common constraint is conservation in a vertex, meaning that the sum of the labels on the incoming edges of this vertex equals the sum of those on the outgoing edges. One easy fact is that if a flow is conserving in all but one vertex, then it is also conserving in the remaining one. In our generalization we do not label the edges with real numbers, but with elements from an arbitrary group, where this fact becomes false in general. As we will show, graphs with the property that conservation of a flow in all but one vertex implies conservation in all vertices are precisely the planar graphs.

1 Introduction

A *graph* (or *network*) is a pair (V, E) where V is a finite set of *vertices* and the set of *edges* E is a subset of $\binom{V}{2}$, the set of all size 2 subsets of V. In this article we consider groups which are not required to be abelian and therefore write our group operation multiplicatively. With Γ a group and $G = (V, E)$ a graph, we call a map $f : V^2 \to \Gamma$ a Γ-*flow in* G if for all $u, v \in V$ we have $f(u, v) = f(v, u)^{-1}$, and $f(u, v) = 1$ if $\{u, v\} \notin E$. This definition agrees with the classical definition of a network flow when $\Gamma = \mathbb{R}$.

Non-abelian graph flows were first considered by M.J. DeVos in his PhD thesis [1] and later by A.J. Goodall et al. [2] and B. Litjens [3]. They consider graphs embedded on surfaces and ask whether flows exists which are nowhere trivial, i.e. $f(u, v) \neq 1$ if and only if $\{u, v\} \in E$. Although our main result involves planar embeddings of graphs, we instead ask to which extent Kirchhoff's law of conservation holds.

Let $G = (V, E)$ be a graph, Γ a group and f a Γ-flow in G. An *orientation* on G is a family $\rho = (\rho_v)_{v \in V}$, where ρ_v is a transitive permutation on the set of neighbours of v. We define the *excess* of f to be the map $e = e_{\rho,f}$ that sends $v \in V$ to the conjugacy class of $f(\rho_v^0(u), v) \cdot f(\rho_v^1(u), v) \cdots f(\rho_v^{-1}(u), v)$ for any choice of neighbour u of v. Since the excess is defined up to conjugacy, it does not depend on the choice of u. We say f is *conserving* in v if $e(v) = 1$. In the classical case, we have the following lemma.

Lemma 1.1. *Let Γ be an* abelian *group, let (V, E) be a graph with a Γ-flow f and orientation ρ and let $w \in V$. If f is conserving in all vertices of $V \setminus \{w\}$, then f is conserving in w.*

Proof. Since Γ is abelian all conjugacy classes consist of a single element and we may interpret e to be a map $V \to \Gamma$. Moreover, the orientation is irrelevant in computing e. We have

$$e_f(w) = \prod_{v \in V} e_f(v) = \prod_{(u,v) \in V^2} f(u, v) = \prod_{\{u,v\} \in E} f(u, v) f(v, u) = 1.$$

□

We will show that Lemma 1.1 can fail for non-abelian Γ. Given an orientation ρ on G we say f *leaks* if it is conserving in all but precisely one vertex. An embedding of G on some compact orientable surface induces an orientation ρ on G. We say an orientation is *planar* if it is induced by some planar embedding of G. We say f is *tractable* if for all $v \in V$ the group generated by $\{f(u,v) \mid u \in V\}$ is abelian. We say G is *leak-proof* if no tractable flow in G leaks.

Theorem 1.2. *Let G be a graph. The following are equivalent:*
(i) *G is leak-proof;*
(ii) *for all flows f in G there exists an orientation on G such that f does not leak;*
(iii) *there exists an orientation on G such that no flow in G leaks;*
(iv) *G is planar.*

We say a flow f in G has a *binary leak* at distinct vertices $u, v \in V$ if it is conserving in all vertices of $V \setminus \{u, v\}$ while $e(u) \neq e(v)^{-1}$. Here u and v can be thought of as a source and sink of the flow. We call G *binary leak-proof* if no tractable binary leaking flows exist in G. Analogously to Lemma 1.1 one can show that a flow cannot have a binary leak when the group is abelian. We call a graph $G = (V, E)$ *extra-planar* if for all pairs of distinct $u, v \in V$ the graph $(V, E \cup \{u, v\})$ is planar. We prove the following analogue to Theorem 1.2 in Section 5.

Theorem 1.3. *A graph is binary leak-proof if and only if it is extra-planar.*

Instead of studying leak-proof graphs, one could also study leak-proof groups, where we call a group Γ *leak-proof* if for all graphs $G = (V, E)$ no tractable flows $f : V^2 \to \Gamma$ in G leak. Theorem 1.2 shows that the decision problem 'Is this graph leak-proof?' can be decided in time $O(|V|)$, as Hopcroft and Tarjan gave an algorithm to test graph planarity in [4] of this complexity. For leak-proof groups, we prove the following in Section 6.

Theorem 1.4. *The decision problem 'Is this finite group leak-proof?' is decidable.*

The present work, in particular Theorem 1.3, was inspired by a problem the author encountered in his Master's thesis [5] on graded rings. Here a flow with a binary leak gives rise to an example (Example 2.17 of [5]) of an efficient ring grading with a non-abelian group that cannot be replaced by an abelian group.

2 Definitions and properties of (non-)planar graphs

We briefly go through some basic definitions. Let $G = (V, E)$ be a graph. We call a graph (W, F) a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. For $W \subseteq V$ we call $(W, \{\{u, v\} \in E \mid u, v \in W\})$ the subgraph of G *induced by* W. A *path from* $u \in V$ *to* $v \in V$ in G is a finite sequence of vertices $(x_0, \ldots, x_n)$ for some $n \in \mathbb{Z}_{\geq 0}$ such that $x_0 = u$, $x_n = v$ and $\{x_i, x_{i+1}\} \in E$ for all $0 \leq i < n$. We call this path *non-trivial* if $n > 0$ and *closed* if $x_0 = x_n$. We write $N(v) = N_G(v) \subseteq V$ for the set of neighbours of v. An edge $\{u, v\} \in E$ is called a *bridge* if all paths in G from u to v contain the edge $\{u, v\}$. A *forest* is a graph in which every edge is a bridge.

Definition 2.1. For $A, B \in \mathbb{R}^2$ write $\overline{AB}$ for the line $\{tA + (1-t)B \mid t \in (0,1)\}$. Let $G = (V, E)$ be a graph. A *planar embedding* of G is an injective map $\varepsilon : V \to \mathbb{R}^2$ such that for all $\{a, b\}, \{c, d\} \in E$ we have $\overline{\varepsilon(a)\varepsilon(b)} \cap \overline{\varepsilon(c)\varepsilon(d)} = \emptyset$ when $\{a, b\} \neq \{c, d\}$, and $\overline{\varepsilon(a)\varepsilon(b)} \cap \varepsilon[V] = \emptyset$. We call G *planar* if it has a planar embedding. For a planar embedding we define the *induced orientation* to be the clockwise permutation of the neighbours at each vertex.

The above definition of a planar embedding has been simplified for our purposes, which is justified by Fáry's Theorem [6].

Definition 2.2. Let $G = (V, E)$ be a graph with orientation ρ. A *boundary walk* of G with respect to this orientation is a non-trivial closed path $(x_0, x_1, \ldots, x_n)$ in G such that for all $i, j \in \mathbb{Z}/n\mathbb{Z}$ we have $x_{i+2} = \rho_{x_{i+1}}(x_i)$ and if $(x_i, x_{i+1}) = (x_j, x_{j+1})$, then $i = j$.

Lemma 2.3. *Let ε be a planar embedding of a graph $G = (V, E)$ and let $p = (u_1, u_2, \ldots, u_n)$ be a boundary walk. If $(u_i, u_{i+1}) = (u_{j+1}, u_j)$ for some $i, j \in \mathbb{Z}/n\mathbb{Z}$, then $\{u_i, u_j\}$ is a bridge.*

Proof. To show that $e = \{u_i, u_j\}$ is a bridge, it suffices to show that u_i and u_j are disconnected in the graph $G' = (V, E')$ with $E' = E \setminus \{e\}$. Note that $a, b \in V$ are connected in G' if and only if $\varepsilon(a)$ and $\varepsilon(b)$ are connected in the topological space $X = \varepsilon[V] \cup \bigcup_{\{x,y\} \in E'} \overline{\varepsilon(x)\varepsilon(y)}$. Hence it suffices by the Jordan curve theorem to show that there exists a loop C in $\mathbb{R}^2 \setminus X$ separating u_i and u_j, as any path from u_i to u_j must intersect this loop.

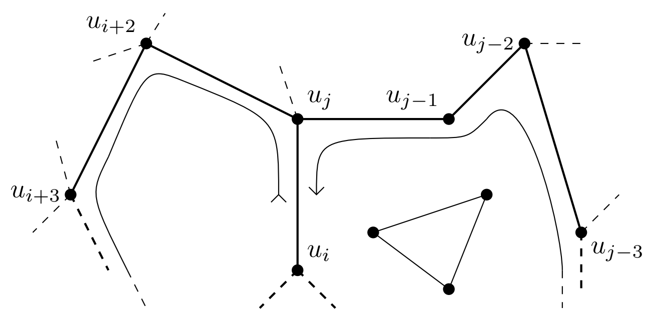

Figure 1: Boundary walk

We informally construct this loop as follows (see Figure 1). Place yourself at the midway point between u_i and u_j. Walk along the path p in G in the direction of u_j and while doing so draw a continuous curve C on your left hand side, being careful not to let C intersect itself or the graph. That this is possible follows from the definition of a boundary walk. Stop once you have reached your starting point for the first time again, and note that this time you are facing u_i by the assumption that $(u_i, u_{i+1}) = (u_{j+1}, u_j)$. Thus on your right hand side is the start of your curve C, and connect the endpoints, crossing $\overline{\varepsilon(u_i)\varepsilon(u_j)}$ once. Then C satisfies the requirements, so e is a bridge. □

Definition 2.4. Let $G = (V, E)$ be a graph. We call a subgraph $H = (W, F)$ of G a *spanning forest* if it is a forest and $W = V$. For a spanning forest $H = (W, F)$ of G we define $G_H = (C, D)$ to be the *contraction of H in G*, where C is the set of connected components of H and $D = \{\{X, Y\} \in \binom{C}{2} \mid (\exists\, u \in X, v \in Y)\ \{u, v\} \in E\}$. A graph M is a *minor* of G if it can be embedded in some contraction of G.

Write K_5 for the complete graph on 5 vertices and $K_{3,3}$ for the complete bipartite graph on 3 and 3 vertices.

Theorem 2.5 (Kuratowski, Theorem 4.4.6 in [7])**.** *A graph G is planar if and only if G does not have K_5 or $K_{3,3}$ as a minor.*

3 Non-planar graphs

First we show that non-planar graphs are not leak-proof, which is (i) ⇒ (iv) in Theorem 1.2. Recall that for tractable flows, the excess, and hence the concept of a leak, does not depend on the choice of orientation.

Lemma 3.1. *A graph is leak-proof if and only if all its subgraphs are leak-proof.*

Proof. Since each graph is its own subgraph, the implication ($\Leftarrow$) is trivial. Let $G = (V, E)$ be a graph with a subgraph $H = (W, F)$ and assume that there exists some group Γ with a leaking tractable Γ-flow $g : W^2 \to \Gamma$ of H. Then we consider $f : V^2 \to \Gamma$ by taking $f(u, v) = g(u, v)$ when $\{u, v\} \in F$ and $f(u, v) = 1$ otherwise. Then f is a leaking tractable flow in G, proving ($\Rightarrow$). $\square$

Proposition 3.2. *A graph is leak-proof if and only if all its minors are leak-proof.*

Proof. Let $G = (V, E)$ be a graph. By Lemma 3.1 it suffices to show that if a contraction of a spanning tree H in G admits a leaking tractable flow, then so does G. By induction, contracting a single edge at a time, we may even assume $H = (V, \{e\})$ for some edge $e = \{a, b\}$. Then $G_H \cong (W, F)$ with $W = (V \setminus e) \cup \{e\}$ under the natural isomorphism $e \mapsto e$ and $w \mapsto \{w\}$ for $w \in V \setminus e$. Assume (W, F) admits a leaking tractable flow $f : W^2 \to \Gamma$ for some group Γ. Note that it is possible but not necessary that $e_f(a) \neq 1$ or $e_f(b) \neq 1$. Let $X = N_G(a) \setminus e$ and $Y = N_G(b) \setminus (e \cup X)$. We define a flow $g : V^2 \to \Gamma$ such that for $u, v \in W$ it is given by

$$\begin{aligned} g(u, v) &= f(u, v) && u, v \notin e, \\ g(a, u)^{-1} = g(u, a) &= f(u, e) && u \in X, \\ g(v, b)^{-1} = g(b, v) &= f(e, v) && v \in Y, \\ g(b, a)^{-1} = g(a, b) &= \prod_{u \in X \setminus \{b\}} f(u, a), \end{aligned}$$

and $g(u, v) = 1$ otherwise. Note that g agrees with f outside of e and that the flow values on the edges pointing towards e have been divided among a and b. Thus g is tractable and $e_g(u) = e_f(u)$ for $u \notin e$. By definition of $g(a, b)$ we have that $e_g(a) = 1$ and $e_g(b) = e_f(e)$. Hence g is a leaking flow in G. $\square$

It now suffices by Theorem 2.5 to show that K_5 and $K_{3,3}$ admit a leaking tractable flow.

Definition 3.3. Let C_2 be the cyclic group with two elements. Let $n \in \mathbb{Z}_{>0}$ and consider the groups $N = C_2^{n+1} = \langle z, x_1, \ldots, x_n \rangle$ and $G = C_2^n = \langle x_{n+1}, \ldots, x_{2n} \rangle$. Define an action $\varphi : G \to \mathrm{Aut}(N)$ defined on the generators as

$$x_{n+i} \mapsto \big(x_j \mapsto x_j z^{\delta_{ij}}, \quad z \mapsto z\big) \quad \text{for all } 1 \leq i, j \leq n,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Then define the group $\mathrm{ES}_n = N \rtimes_\varphi G$, the semidirect product [8, p. 170] of N and G.

Equivalently, we can give ES_n as a presentation on the generators $z, x_1, \ldots, x_{2n}$, subject to the relations (1) that all generators have trivial square, (2) that z commutes with every generator, and (3) that x_i and x_j commute unless $|i - j| = n$, in which case $x_i x_j = z x_j x_i$. However, from this alternative definition of ES_n we may not immediately deduce that none of the generators are trivial. Although we will not use the fact, the ES_n are all *extraspecial* 2*-groups.*

Example 3.4. Consider the utility graph $K_{3,3} = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{u, v\} \mid u \in \{1, 2, 3\},\ v \in \{4, 5, 6\}\}$. We define a flow $f : V^2 \to \mathrm{ES}_2$ which we specify by an ES_2-valued (symmetric) matrix where the omitted entries are trivial:

$$f = \left(\begin{array}{ccc|ccc} & & & x_1 & x_2 & x_1x_2 \\ & & & x_4 & x_3 & x_4x_3 \\ & & & x_1x_4 & x_2x_3 & x_1x_4x_2x_3 \\ \hline x_1 & x_4 & x_1x_4 & & & \\ x_2 & x_3 & x_2x_3 & & & \\ x_1x_2 & x_4x_3 & x_1x_4x_2x_3 & & & \end{array}\right).$$

For the first 5 columns it is easy to see that multiplying the first two non-trivial entries yields the third. Thus for the first five vertices v we have $\langle f(u,v) \,|\, u \in V\rangle \cong C_2^2$, which is abelian, and $e_f(v) = 1$. For $v = 6$ we observe that $(x_1x_2)(x_4x_3)(x_1x_4x_2x_3) = z$ and thus $\langle f(u,6) \,|\, u \in V\rangle = \langle x_1x_2, x_4x_3, z\rangle \cong C_2^3$ is abelian, and $e_f(6) = z \neq 1$. Hence f is a tractable flow that leaks at 6 and $K_{3,3}$ is not leak-proof.

Example 3.5. Consider the complete graph $K_5 = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$. Now we consider $f : V^2 \to \mathrm{ES}_3$ given by

$$f = \begin{pmatrix} & x_1 & x_2 & x_3 & x_1x_2x_3 \\ x_1 & & x_6 & x_5 & x_1x_6x_5 \\ x_2 & x_6 & & x_4 & x_2x_6x_4 \\ x_3 & x_5 & x_4 & & x_3x_5x_4 \\ x_1x_2x_3 & x_1x_6x_5 & x_2x_6x_4 & x_3x_5x_4 & \end{pmatrix}.$$

For each of the first four columns one notes that its first three non-trivial elements commute pair-wise, while multiplying them yields the fourth. Thus for the first four vertices v the group $\langle f(u,v) \,|\, u \in V\rangle \cong C_2^3$ is abelian and $e_f(v) = 1$. For the last column, note that each pair (a, b) of entries is of the form $a = x_ix_jx_k$ and $b = x_ix_{j+3}x_{k+3}$ with $i, j, k, j + 3, k + 3 \in \mathbb{Z}/6\mathbb{Z}$ distinct. Hence $ab = x_i^2(x_jx_k)(x_{j+3}x_{k+3}) = x_i^2(x_{j+3}x_{k+3})(x_jx_k) = ba$, so each pair commutes. Finally, one computes $e_f(5) = (x_1x_2x_3)(x_1x_6x_5)(x_2x_6x_4)(x_3x_5x_4) = z \neq 1$. Thus f is a tractable leaking flow and thus K_5 is not leak-proof.

Both examples were found by starting with the free group F with symbols V^2 and dividing out the relations $N \trianglelefteq F$ required to make the obvious map $f : V^2 \to F/N$ a tractable flow that is conserving in $\#V - 1$ vertices. Adding the restriction that the generators have order 2 gives us the groups ES_2 and ES_3.

4 Planar graphs

Now we will prove that all planar graphs admit an orientation such that no flow leaks, which is (iv) $\Rightarrow$ (iii) of Theorem 1.2. Unsurprisingly, this will be the induced orientation. Recall that the excess is only defined up to conjugacy. For clarity we will write $\equiv$ for equality up to conjugacy.

Theorem 4.1. *Let $G = (V, E)$ be a graph with planar embedding ε and let $f : V^2 \to \Gamma$ be a flow in G. Let $u \in V$ and assume $e(v) \equiv 1$ with respect to the orientaion induced by ε for all $v \in V \setminus \{u\}$. Then $e(u) \equiv 1$.*

Proof. Firstly, if G is the singleton graph, then $e(u) \equiv 1$ is the empty product, so we are done. We now apply induction and thus assume that the statement holds for all strict subgraphs (W, F) of G with planar embedding $\varepsilon|_W$.

Secondly, we consider the case where G is not connected. Here we may apply the induction hypothesis to the induced subgraph of G with as vertex set the connected component of u to conclude that $e(u) \equiv 1$.

Thirdly, we consider the case where G is a forest. Then G has at least two vertices of degree 1, of which one, say v, is not u. Let $\{v, w\} \in E$ be the unique edge incident to v, and note that $f(w, v) \equiv e(v) \equiv 1$. Hence f is a flow in the subgraph H of G obtained by removing $\{v, w\}$. Note that ε is a planar embedding of H with the same round flow in each vertex, hence by the induction hypothesis we have $e(u) \equiv 1$.

Lastly we consider the case where G is connected and not a forest. Then G has an edge $\{v, w\} \in E$ that is not a bridge. Then by Lemma 2.3 the boundary walk $p = (x_0, \ldots, x_n)$ with $x_0 = v$ and $x_1 = w$ satisfies $(w, v) \neq (x_i, x_{i+1})$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Let $b : V^2 \to \{0, 1\}$ be the map such that for all $s, t \in V$

we have $b(s,t) = 1$ if and only if there exists some $i \in \mathbb{Z}/n\mathbb{Z}$ such that $(s,t) = (x_i, x_{i+1})$. Now consider $\gamma = f(v,w)$ and $g : V^2 \to \Gamma$ given by

$$(s,t) \mapsto \gamma^{b(t,s)} \cdot f(s,t) \cdot \gamma^{-b(s,t)}.$$

Firstly note that g is a flow in G: For all $s,t \in V$ we have

$$g(s,t)^{-1} = \gamma^{b(s,t)} \cdot f(s,t)^{-1} \cdot \gamma^{-b(t,s)} = g(t,s)$$

since f is a flow, and if $\{s,t\} \notin E$ we have $g(s,t) = f(s,t) = 1$ as $b(s,t) = b(t,s) = 0$. Secondly, we have that $g(v,w) = \gamma^0 \cdot \gamma \cdot \gamma^{-1} = 1$ by choice of $\{v,w\}$, so g is even a flow in the subgraph H of G obtained by removing $\{v,w\}$. We now show that $e_f = e_g$. Then by the induction hypothesis applied to H it follows that $e_f(u) \equiv 1$. Note that for all $s,t \in V$ we have by definition of b that $b(t,s) = b(s,\rho(t))$, where ρ is the induced orientation at s. Using this, we now simply verify for $\{s,t\} \in E$ and $n = \#N_G(s)$ that

$$\begin{aligned} e_g(s) &\equiv \prod_{k=0}^{n-1} g(\rho^k(t), s) \equiv \prod_{k=0}^{n-1} \gamma^{b(s,\rho^k(t))} \cdot f(\rho^k(t), s) \cdot \gamma^{-b(\rho^k(t),s)} \\ &\equiv \gamma^{b(s,t)} \left(\prod_{k=0}^{n-1} f(\rho^k(t), s) \gamma^{-b(\rho^k(t),s)} \gamma^{b(s,\rho^{k+1}(t))} \right) \gamma^{-b(s,\rho^n(t))} \\ &\equiv \gamma^{b(s,t)} \left(\prod_{k=0}^{n-1} f(\rho^k(t), s) \right) \gamma^{-b(s,t)} \equiv \prod_{k=0}^{n-1} f(\rho^k(t), s) \equiv e_f(s), \end{aligned}$$

as was to be shown. We conclude that the statement holds for all planar graphs by induction. □

An earlier proof of Theorem 4.1 was due to H.W. Lenstra. In his version he does not remove edges in the inductive step but contracts them in the sense of Definition 2.4. This proof turned out to be more difficult to formalize.

Proof of Theorem 1.2. (iii) ⇒ (ii) Immediate from reordering quantifiers. (ii) ⇒ (i) Tractable flows do not depend on the choice of orientation. (i) ⇒ (iv) A non-planar graph has either K_5 or $K_{3,3}$ as minor by Theorem 2.5. Both K_5 and $K_{3,3}$ are not leak-proof by Example 3.5 respectively Example 3.4, so by Proposition 3.2 neither are the non-planar graphs. (iv) ⇒ (iii) Let $G = (V,E)$ be a planar graph and fix an orientation induced by a planar embedding. Then we are done by Theorem 4.1. □

5 Extra-planar graphs

In this section we will prove Theorem 1.3, classifying the binary leak-proof graphs. To do this we first prove a 'Kuratowski's Theorem' for extra-planar graphs. Write K_5^- and $K_{3,3}^-$ for the graphs obtained from K_5 respectively $K_{3,3}$ by removing a single edge, which by symmetry we do not have to specify.

Theorem 5.1. *A graph G is extra-planar if and only if G does not have K_5^- or $K_{3,3}^-$ as a minor.*

Proof. (⇒) This follows directly from Kuratowski's Theorem: If K_5^- or $K_{3,3}^-$ is a minor of G, then we may add a single edge to G such that K_5 respectively $K_{3,3}$ becomes a minor of this new graph, which is then non-planar.

(⇐) We proceed by contraposition, so assume that G is not extra-planar. Let $u, v \in V$ be such that $G^+ = (V, E \cup \{\{u,v\}\})$ is non-planar and let $H^+ = (V,F)$ be a spanning forest of G^+ such that K_5 or $K_{3,3}$ embeds into G^+_H. Consider the spanning forest $H = (V, F \setminus \{\{u,v\}\})$ of G. Then H has the same connected components as H^+ with the exception that if H^+ has a connected component containing both u and v, it might have been split into two. Let T_u and T_v be the connected components of u respectively v in H.

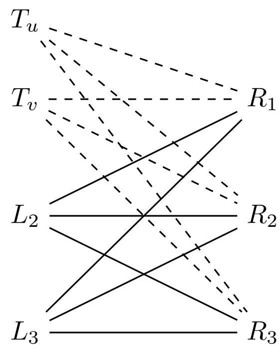

Figure 2: Case $K_{3,3}$

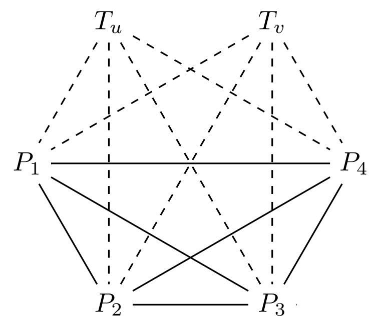

Figure 3: Case K_5

Case $K_{3,3}$: First consider the case where $K_{3,3}$ embeds into $G^+_{H^+}$, meaning there is a subset $C = \{L_1, L_2, L_3, R_1, R_2, R_3\}$ of size 6 of the set of connected components of H^+ such that $S^+ = (C, \{\{L_i, R_j\} \,|\, i, j \in \{1, 2, 3\}\})$ is a subgraph of $G^+_{H^+}$. If all elements of C are also connected components of H, then G_H has the graph S^+ minus possibly a single edge induced by $\{u, v\}$ as subgraph, hence G has $K^-_{3,3}$ as a minor. Otherwise, for some $X \in C$ we have $X = T_u \sqcup T_v$ and without loss of generality $X = L_1$. Then the subgraph S of G_H induced by $\{T_u, T_v, L_2, L_3, R_1, R_2, R_3\}$ is as in Figure 2, where the dashed lines indicate edges which are possibly present. Merging T_u and T_v in S yields $S^+ \cong K_{3,3}$, hence for each $i \in \{1, 2, 3\}$ the edge $\{T_u, R_i\}$ or $\{T_v, R_i\}$ is present. Thus T_u or T_v has degree at least 2, which without loss of generality is T_v. It follows that $K^-_{3,3}$ embeds into the subgraph of G_H induced by $\{T_v, L_2, L_3, R_1, R_2, R_3\}$, so $K^-_{3,3}$ is a minor of G.

Case K_5: Now consider the case K_5 embeds into $G^+_{H^+}$, meaning there is a subset $C = \{P_1, \ldots, P_5\}$ of the set of connected components of H^+ such that the subgraph of $G^+_{H^+}$ induced by C is isomorphic to K_5. As before, the only interesting case is where $P_5 = T_u \sqcup T_v$. Then the subgraph S of G_H induced by $\{T_u, T_v, P_1, P_2, P_3, P_4\}$ is as in Figure 3. Since merging T_u and T_v in S yields K_5, for each $i \in \{1, \ldots, 4\}$ the edge $\{T_u, P_i\}$ or $\{T_v, P_i\}$ is present. If both T_u and T_v have degree 2, then without loss of generality S contains the edges $\{T_u, P_3\}$, $\{T_u, P_4\}$, $\{T_v, P_1\}$ and $\{T_v, P_2\}$. Now note that S contains a $K^-_{3,3}$ which partitions its vertices as $\{\{T_u, P_1, P_2\}, \{T_v, P_3, P_4\}\}$. Hence G contains $K^-_{3,3}$ as a minor. Otherwise, without loss of generality T_u has degree at least 3 in S and the subgraph of G_H induced by $\{T_u, P_1, \ldots, P_4\}$ is either K_5 or K^-_5. Hence G has K^-_5 as a minor.

As G has $K^-_{3,3}$ or K^-_5 as a minor, the claim follows. □

We are now able to prove Theorem 1.3.

Proof of Theorem 1.3. ($\Leftarrow$) Let $G = (V, E)$ be an extra-planar graph and let $f : V^2 \to \Gamma$ be a tractable flow in G such that there are distinct $u, v \in V$ with $e_f(w) = 1$ for all $w \in V \setminus \{u, v\}$. Consider the graph $H = (V, E \cup \{\{u, v\}\})$, fix a planar embedding of H and interpret f as a flow in H. Now let $g : V^2 \to \Gamma$ be the map such that $g(s, t) = f(s, t)$ if $\{s, t\} \neq \{u, v\}$ and $g(u, v) = g(v, u)^{-1} = f(u, v)e_f(v)^{-1}$, where $e_f(v)$ is computed by starting from the vertex right after u in the ordering of $N_H(v)$. Then g is a (not necessarily tractable) flow in H such that $e_g(w) = 1$ for $w \in V \setminus \{u\}$. From $g(v, u) = e_f(v)f(v, u)$ it follows that $e_g(u)$ differs from $e_f(u)$ by a factor $e_f(v)$ when starting the multiplication at v. By Theorem 4.1 we have $1 \equiv e_g(u) \equiv e_f(u)e_f(v)$ and thus $e_f(u)e_f(v) = 1$. Hence G is binary leak-proof.

($\Rightarrow$) If G is not extra-planar, then it has K^-_5 or $K^-_{3,3}$ as minor by Theorem 5.1. It is straightforward to generalize Proposition 3.2 to show that a graph is binary leak-proof if and only if all its minors are too. It therefore suffices to show that K^-_5 and $K^-_{3,3}$ have a binary leaking flow. Simply take the flow f as defined in Example 3.4 which leaks at vertex 6 of $K_{3,3}$ and consider $K^-_{3,3}$ as the $K_{3,3}$ with the edge $\{3, 6\}$ removed. Then the flow f^- in $K^-_{3,3}$ which equals f except for $f^-(3, 6) = f^-(6, 3) = 1$ has a binary leak at 3 and 6. Using Example 3.5 for K^-_5 can be done analogously. □

6 Leak-proof groups

In this section we prove Theorem 1.4 and give some computational results. We recall some definitions from group theory. For a family $\mathcal{A} = (A_i)_{i\in I}$ of abelian groups the *direct sum* [8, p. 308] is the group

$$\bigoplus_{i\in I} A_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} A_i \,\middle|\, x_i \neq 1 \text{ for only finitely many } i \right\}$$

with coordinate-wise multiplication. For a group homomorphism $\varphi : G \to H$ the *image* of φ is the subgroup $\operatorname{im}(\varphi) = \{\varphi(g) \mid g \in G\}$ of H.

Definition 6.1. Let Γ be a (not necessarily finite) group. Write $V(\Gamma)$ for the set of maximal abelian subgroups of Γ. We define the group

$$F(\Gamma) = \left\{ (f_{u,v})_{(u,v)} \in \bigoplus_{(u,v)\in V(\Gamma)^2} (u \cap v) \,\middle|\, (\forall\, u,v)\ f_{u,v} = f_{v,u}^{-1},\ (\forall\, v)\ f_{v,v} = 1 \right\}$$

and the homomorphism

$$e_\Gamma : F(\Gamma) \to \bigoplus_{v\in V(\Gamma)} v, \quad (f_{u,v})_{(u,v)} \mapsto \Big(\prod_{u\in V(\Gamma)} f_{u,v} \Big)_{v\in V(\Gamma)}.$$

One can think of $V(\Gamma)$ as the vertex set of a complete graph, $F(\Gamma)$ the set of tractable flows in this graph, and $e_\Gamma(f)$ to be the excess for such flow $f \in F(\Gamma)$. However, $V(\Gamma)$ need not be finite. For example $\Gamma = \mathrm{GL}_2(\mathbb{R})$ has a maximal abelian subgroup $\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in \mathbb{R},\ a \neq 0\}$ with infinitely many conjugates.

Lemma 6.2. *Let Γ be a group. For $u \in V(\Gamma)$ and $\gamma \in u$ let $[\gamma]_u \in \bigoplus_{v\in V(\Gamma)} v$ be the vector consisting of all-ones except for a γ at coordinate u. We write $\Gamma^\bullet = (\bigoplus_{v\in V(\Gamma)} v)/\operatorname{im}(e)$. Then the map $d : \Gamma \to \Gamma^\bullet$ given by $\gamma \mapsto [\gamma]_v$ for any choice of v containing γ, does not depend on the choice of v.*

Proof. Let $\gamma \in \Gamma$ and suppose $u, v \in V(\Gamma)$ are such that $\gamma \in u$ and $\gamma \in v$. Then $\gamma \in u \cap v$, and $f = (f_{s,t})_{(s,t)\in V(\Gamma)^2}$, with $f_{u,v} = f_{v,u}^{-1} = \gamma$ and $f_{s,t} = 1$ for $\{s,t\} \neq \{u,v\}$, is an element of $F(\Gamma)$. We have $e(f) = [\gamma]_v \cdot [\gamma]_u^{-1}$, so $[\gamma]_u$ is equivalent to $[\gamma]_v$ in the quotient $\Gamma^\bullet$. □

An example one can consider is where Γ is abelian. Then $V(\Gamma) = \{\Gamma\}$ and $\Gamma^\bullet = \Gamma$ and d is the identity. Note that d is (in general) not a group homomorphism.

Proposition 6.3. *A group Γ is leak-proof if and only if $d(\gamma) = 1$ implies $\gamma = 1$.*

Proof. Suppose Γ is leak-proof and $d(\gamma) = 1$ for some $\gamma \in \Gamma$. Then there is some $u \in V(\Gamma)$ and $f \in F(\Gamma)$ such that $[\gamma]_u = e(f)$. Note that $E = \{\{u,v\} \in V(\Gamma) \mid f_{u,v} \neq 1\}$ and $V = \{u \mid \{u,v\} \in E\}$ are finite. Now f is a Γ-flow in (V,E) which is preserving in all vertices except possibly u. Since Γ is leak-proof, f is also preserving in u and $1 = e(f) = [\gamma]_u$, so $\gamma = 1$.

Conversely, suppose f is a tractable Γ-flow in some graph (V,E). Pick some map $c : V \to \Gamma(V)$ such that for all $v \in V$ we have $\langle f(u,v) \mid u \in V \rangle \subseteq c(v)$. Then f induces a tractable Γ-flow f' in the complete graph with vertex set $\{c(v) \mid v \in V\}$ where $f'(s,t) = \prod_{u:c(u)=s} \prod_{v:c(v)=t} f(u,v) \in s \cap t$. Hence $f' \in F(\Gamma)$. Moreover, if f leaks, then so does f'. Assume f' is preserving in all vertices except potentially $v \in V$. Then $e(f') = [\gamma]_v$ for some $\gamma \in v$ and $d(\gamma) = 1$. If $d(\gamma) = 1$ implies $\gamma = 1$, we obtain that f' and hence f does not leak, so Γ is leak-proof. □

Similarly, one can consider binary leak-proof groups. With a proof analogous to that of Proposition 6.3 one obtains that Γ is binary leak proof if and only if d is injective.

Proof of Theorem 1.4. Simply note that for finite Γ the corresponding group $\Gamma^\bullet$ is finite abelian and can thus be computed explicitly. In particular, we can decide for each $\gamma \in \Gamma$ whether $d(\gamma) = 1$. The theorem thus follows from Proposition 6.3. □

From Lemma 1.1 it follows that abelian groups are leak-proof, but they are hardly the only ones. By computer search we found the two extraspecial groups of order 32 to be the only smallest leaking groups, one of which we encountered in Example 3.4. The smallest leaking groups of order greater than 32 occur at order 64. That there are groups of order 64 that leak was to be expected, because a group leaks when it has a leaking subgroup. The smallest leaking symmetric group is the S_6 and the smallest leaking alternating group is the A_7. That for sufficiently large n the group S_n leaks is to be expected by Cayley's theorem, but interestingly no strict subgroup of S_6 leaks. It would be interesting to have a classification of leak-proof groups or to know whether there is some equivalent, better understood property of groups which is equivalent to being leak-proof like planarity is to graphs.

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