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Gravitational waves through the cosmic web

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Part II

Ray-optics limit: distance duality relation and polarization tests

4

The gravitational waves' angular diameter distance

We analyze the propagation of high-frequency gravitational waves in scalar-tensor theories of gravity, with the aim of examining properties of cosmological distances as inferred from their measurements. By using symmetry principles, we first determine the most general structure of the GW linearized equations and of the GW energy momentum tensor, assuming that GW propagate at the speed of light. We then specialize to the case of GW propagating through a perturbed cosmological spacetime, deriving the expressions for the GW luminosity and angular diameters distances, proving the validity of the Etherington reciprocity law $d_L^{\rm GW} = (1+z)^2\, d_A^{\rm GW}$*. We find that, as in the case of the luminosity distance, also the GW angular diameter distance is explicitly modified compared to the electromagnetic one. We discuss implications of this result in the context of strong lensing time delay, showing that the effects of the scalar field representing dark energy compensate: lensed GW arrive at the same time as their lensed electromagnetic counterparts.*

Keywords: Gravitational waves, dark energy, geometric optics, angular diameter distance, distance duality relation, strong lensing

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4.1. Introduction

Cosmologists use various different definitions of *distance* depending on the context and the observables they are interested in [296, 297]. While usually definitions make use of light detected from distant sources, GW offer new tools for measuring cosmological distances. We have already seen an example of GW distance in Section 1.4.2: using Eq. (1.104) one can infer the GW luminosity distance. This definition of $\bar{d}_L^{\mathrm{GW}}(z)$ is simply given in analogy to the General Relativistic result (the GW's amplitude is inversely proportional to the luminosity distance), and it doesn't follow from rigorous definitions. The purpose of this Chapter is to formally derive the *gravitational waves cosmological distances*, in particular the *luminosity distance* in Section 4.5.2 and the *angular diameter distance* in Section 4.5.3, in a gravitational theory where dark energy is represented by a scalar field. Following early important works [197, 198, 204, 225, 234, 247], $d_L^{(\mathrm{GW})}$ is being recognized as a key observable to independently measure cosmological parameters by means of GW, as well as testing scalar-tensor theories of gravity, as we have seen in Chapters 2 and 3. Here, we wish to draw some more general statements about GWs in scalar-tensor gravity models, relaxing also the cosmological background assumption, namely that the background metric is Eq. (1.14). To this extent, the high-frequency approximation is rather useful: as discussed in Section 1.3.2, we can define GWs without specifying the background line element. We use symmetries as a guiding principle, in particular generalized coordinate invariance, for characterizing the scalar-tensor system and the behavior of propagating degrees of freedom, without choosing a specific model. To disentangle tensor and scalar waves, generically coupled when the propagation is considered over arbitrary spacetimes, we assume that the properties of the GW at emission are identical to those of General Relativity, and we identify physically reasonable conditions to decouple the evolution equations of these different sectors. Focusing on the propagation of tensor modes, we work out their stress-energy tensor at second order, and define a covariant conservation of the *graviton number density current*, which we use to formally define the *Gravitational wave distances*, d_L^{GW} and d_A^{GW}, in scalar-tensor theories. The effects of a dynamical dark energy factorize into an overall multiplication factor.

Only after this general results, we focus on the case of cosmological perturbed spacetimes, and we prove the validity of the Etherington's reciprocity law

$$d_L^{\mathrm{GW}} = (1+z)^2\, d_A^{\mathrm{GW}}, \tag{4.1}$$

within a scalar-tensor framework considered. Since this relation is at the basis for relating angular and luminosity distances in GW measurements, it is of crucial importance to understand whether it is valid or not in a general theory of gravity, for GW propagation on a general space-time. The definitions given in this Chapter are, of course, compatible with Eq. (1.104). Considering that d_L^{GW} can be modified with respect to the distance inferred through an electromagnetic signal, as shown in Eq. (1.105), then the validity of Eq. (4.1) implies that also angular diameter distances are rescaled by the same factor, i.e.

$$d_A^{\rm GW} = \frac{M_P(z)}{M_P(0)}\, d_A^{\rm EM}\,. \tag{4.2}$$

Finally, we investigate our results about $d_A^{\rm (GW)}$ in the context of strong lensing of GWs and their time delay, which depends on a combination of angular diameter distances [149]. Strong lensing of GW can be important in the future for providing alternative ways for determining cosmological parameters (see e.g. [298]). Since we are considering theories where GWs travel at the speed of light, these follow null-geodesics as photons, and we do not expect any different time delay between GW and EM signals. We show explicitly that this is the case in Section 4.5.5, where we rewrite the time delay formula, which is given in terms of the modified $d_A^{\rm (GW)}$, all in terms of the geometrical comoving distances.

4

4.2. Tensor and scalar waves

Even though we do not restrict ourselves to a specific Horndeski theory, we assume that the physical system under consideration derives from an action of the form

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{M_p^2}{2}R - \mathcal{L}(g_{\mu\nu},\varphi)\right), \tag{4.3}$$

where φ is the DE field. In Section 1.3.2, we addressed the subtle issue of defining the metric perturbation. In scalar-tensor theories, this problematic extends similarly also to the definition of the scalar field fluctuations, which we address here. The approach taken in this Chapter follows the definition of the field fluctuations typical of geometric optics techniques.

We base our considerations on a double perturbative expansion for the metric and the scalar field around quantities solving the background equations, as in [151, 152]. We expand metric and scalar fields as[1]

$$g_{\mu\nu}(t,\mathbf{x}) = \bar{g}_{\mu\nu}(t,\mathbf{x}) + \alpha\, h_{\mu\nu}(t,\mathbf{x})\,, \tag{4.4}$$

$$\phi(t,\mathbf{x}) = \bar{\varphi}(t,\mathbf{x}) + \alpha\,\delta\phi(t,\mathbf{x})\,, \tag{4.5}$$

and we are interested to study the dynamics of the metric and scalar perturbations $h_{\mu\nu}$ and $\delta\varphi$. We adopt the geometric optics arguments to define the field fluctuations: $h_{\mu\nu}$ and φ are small high-frequency fluctuations whose gradients are enhanced by a factor of ω with respect to the background. This parameter, defined

[1] Please note the notation: $\delta\phi$ corresponds to the scalar wave, while $\delta\varphi$ in, e.g., Eq. (1.110) is the large scale structure contribution of the DE clustering. To make this difference more apparent, we use two different expansion parameters: α and ϵ.

for the first time in Eq. (1.85), is given by

$$\frac{1}{\omega} = \frac{\lambda}{L} \ll 1\,, \tag{4.6}$$

controlling the ratio among the typical (small) wavelength λ of the high-frequency fields versus the (large) scale L of spatial variation of slowly-varying background quantities.

The general topic of identifying the propagating scalar and tensor degrees of freedom in theories such as (4.3) started in the classic papers [299, 300], considering a Minkowski background. It was then reconsidered, using a variety of methods, in [158, 182, 235, 281, 301] attempting to go beyond the flat hypothesis. The problem, technically speaking, arises because the generic background configuration $\{\bar{g}_{\mu\nu}, \bar{\varphi}\}$ allows for coupling between tensor and scalar modes and, thus, correctly identify their roles in the evolution equations can be subtle. The issue can be even more subtle in theories where scalar and metric fluctuations propagate with different speed, a phenomenon associated with spontaneous breaking of global Lorentz invariance by means, for instance, of a non-vanishing time-like or space-like gradient of the DE field. Note that these situations are the most interesting: they include cosmological and screenings settings. Here we develop a covariant approach to address the problem, more similar in spirit to the original works of Isaacson [151, 152], and to the effective field theory of inflation [82] and dark energy [86] (see e.g. [87] for a comprehensive review).

In our set-up, we assume to have an action in Eq. (4.3), invariant under generalized coordinate transformations, and that the background fields profile break spontaneously Lorentz symmetry, providing the preferred vector

$$v_\mu \equiv \partial_\mu \bar{\varphi}\,. \tag{4.7}$$

Under an infinitesimal spacetime translation, $x^\mu \to x^\mu + \xi^\mu$, the linearized fluctuations transform as

$$h'_{\mu\nu} = h_{\mu\nu} - (\bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu)\,, \tag{4.8}$$

$$\delta\phi' = \delta\phi - v^\mu \xi_\mu\,, \tag{4.9}$$

for infinitesimal vector ξ_μ and where $\bar{\nabla}_\mu$ is the covariant derivative associated to $\bar{g}_{\mu\nu}$. In order to actively apply the transformation in Eq. (4.8) and have that $h'_{\mu\nu}$, is still a small (first order in α) and high-frequency (its gradient of order ω), we assume that ξ_μ is a high-frequency field too and that its size is reduced by a factor of ω^{-1} with respect to $h_{\mu\nu}$ [150],

$$\mathcal{O}\left(\xi_\mu\right) \sim \frac{1}{\omega}\,\mathcal{O}\left(h_{\mu\nu}\right) \sim \frac{\alpha}{\omega}\,. \tag{4.10}$$

The gradients acting on the high-frequency ξ_μ in Eq. (4.8), enhance their contributions by a factor $\mathcal{O}(\omega)$, so that the result is of order $\alpha \times \mathcal{O}(\omega^{-1}) \times \mathcal{O}(\omega) = \alpha \times \mathcal{O}(\omega^0)$, i.e. of the same order of $h_{\mu\nu}$.

We note that, because of the spontaneously broken background, i.e. $v_\mu \neq 0$, the gauge transformations in Eqs. (4.8) and (4.9) mix the metric and scalar perturbations, in the sense that a gauge fixing on one will affect also the other and vice-versa. Even if DE fluctuations are not produced at the source as we are assuming here (for example thanks to some screening mechanism), they can be generated by metric fluctuations that are travelling from source to detection. We then expect that propagation effects are able to excite scalar modes with an amplitude suppressed by a factor of $\mathcal{O}(\epsilon)$ with respect to metric fluctuations:

$$\mathcal{O}(\varphi) \sim \omega^{-1}\,\mathcal{O}(h_{\mu\nu})\,. \tag{4.11}$$

This assumption makes compatible Eqs. (4.10) and (4.9) ($v^\mu \xi_\mu$ is of order α/ω). In any case, understanding the extent of the implications of such assumption, and providing more formal arguments for all the considerations just illustrated, is the topic of Chapter 5.

4

4.2.1. Decomposing the metric fluctuation

Assuming a time-like v_μ, we introduce the vector

$$X_\mu \equiv \frac{v_\mu}{\sqrt{2X}}\,, \qquad \text{such that} \qquad X^\mu X_\mu = -1\,, \tag{4.12}$$

where $X \equiv -(v^\mu v_\mu)/2$. We decompose the gauge vector ξ_μ into its orthogonal and parallel components with respect to v_μ,

$$\xi_\mu = \xi_\mu^{(T)} + X_\mu \xi^{(S)}\,, \qquad \text{with} \qquad X^\mu \xi_\mu^{(T)} = 0\,. \tag{4.13}$$

From this definition, and Eq. (4.9), it is clear that $\delta\phi$ transforms only under transformations generated by $\xi^{(S)}$. We also introduce the quantity

$$\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} + \bar{\nabla}_\mu H_\nu + \bar{\nabla}_\nu H_\mu\,, \qquad \text{with} \qquad H_\mu \equiv \frac{X_\mu}{\sqrt{2X}}\delta\phi\,. \tag{4.14}$$

Because of Eq. (4.11), the contributions to $\tilde{h}_{\mu\nu}$ are of the same order in the gradient expansion ω^{-1}, and it is easy to show that $\tilde{h}_{\mu\nu}$, transforms only under transformations generated by $\xi_\mu^{(T)}$ as

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} - \bar{\nabla}_\mu \xi_\nu^{(T)} - \bar{\nabla}_\nu \xi_\mu^{(T)}\,, \tag{4.15}$$

so that we can choose gauges for $\tilde{h}_{\mu\nu}$ and $\delta\phi$ independently. This procedure is equivalent to performing a Stuckelberg trick [53]. We define the orthogonal projection operator relative to the vector X^μ,

$$\Lambda_{\mu\nu} \equiv \bar{g}_{\mu\nu} + X_\mu X_\nu\,, \tag{4.16}$$

4

and decompose $\tilde{h}_{\mu\nu}$ as,

$$\tilde{h}_{\mu\nu} = X_\mu X_\nu h^{(S)} - \left(X_\mu h^{(V)}_\nu + X_\nu h^{(V)}_\mu \right) + h^{(T)}_{\mu\nu} , \tag{4.17}$$

with $h^{(S)} \equiv X^\rho X^\sigma \tilde{h}_{\rho\sigma}$, $h^{(V)}_\mu \equiv X^\rho \Lambda^\sigma_\mu \tilde{h}_{\rho\sigma}$ and $h^{(T)}_{\mu\nu} \equiv \Lambda^\rho_\mu \Lambda^\sigma_\nu \tilde{h}_{\rho\sigma}$Under a T-type transformation they transform as

$$h^{'(S)} = h^{(S)} , \tag{4.18}$$

$$h^{'(V)}_\mu = h^{(V)}_\mu - X^\rho \bar{\nabla}_\rho \xi^{(T)}_\mu , \tag{4.19}$$

$$h^{'(T)}_{\mu\nu} = h^{(T)}_{\mu\nu} - \left(\bar{\nabla}_\mu \xi^{(T)}_\nu + \bar{\nabla}_\nu \xi^{(T)}_\mu \right) - X^\rho \left(X_\mu \bar{\nabla}_\rho \xi^{(T)}_\nu + X_\nu \bar{\nabla}_\rho \xi^{(T)}_\mu \right) , \tag{4.20}$$

up to order $\mathscr{O}(\omega^0)$. Indeed, the gauge transformation in Eq. (4.15), produces also terms at orders $\mathscr{O}(\omega^{-1})$, which we do not consider here as we focus only on the geometric optics orders $\mathscr{O}(\omega)$ and $\mathscr{O}(\omega^2)$ (see discussion about geometric optics in Section 1.3.2). We note that $h^{(S)}$ is both $S-$ and T-gauge invariant at order $\mathscr{O}(\omega^0)$. For later purposes, we further decompose $h^{(T)}_{\mu\nu}$ as

$$h^{(T)}_{\mu\nu} = \gamma_{\mu\nu} + \frac{1}{3} \Lambda_{\mu\nu} h^{(\mathrm{tr})} \tag{4.21}$$

with $h^{(\mathrm{tr})} \equiv \Lambda^{\mu\nu} h^{(T)}_{\mu\nu}$ and $\Lambda^{\mu\nu}\gamma_{\mu\nu} = \bar{g}^{\mu\nu}\gamma_{\mu\nu} = 0$, and whose transformation laws are,

$$h^{'(\mathrm{tr})} = h^{(\mathrm{tr})} - 2\Lambda^{\mu\nu}\bar{\nabla}_\nu \xi^{(T)}_\mu , \tag{4.22}$$

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \left(\bar{\nabla}_\mu \xi^{(T)}_\nu + \bar{\nabla}_\nu \xi^{(T)}_\mu \right) + \frac{2}{3} \Lambda_{\mu\nu} \bar{\nabla}^\rho \xi^{(T)}_\rho - X^\rho \left(X_\mu \bar{\nabla}_\rho \xi^{(T)}_\nu + X_\nu \bar{\nabla}_\rho \xi^{(T)}_\mu \right) . \tag{4.23}$$

4.2.2. Gauge fixing

We first choose $\xi^{(S)}$ and $\xi^{(T)}_\mu$ such that

$$\delta\phi + \sqrt{2X}\xi^{(S)} = 0 , \qquad \rightarrow \qquad \delta\phi' = 0 , \tag{4.24}$$

$$h^{(V)}_\mu - X^\rho \bar{\nabla}_\rho \xi^{(T)}_\mu = 0 \qquad \rightarrow \qquad h^{'(V)}_\mu = 0 . \tag{4.25}$$

The last condition is compatible, at order $\mathscr{O}(\omega^0)$, with the orthogonality requirement $X^\mu h^{(V)}_\mu = 0$ as it can be checked by contracting with X^μ both sides. Eq. (4.25) leaves the residual T-gauge freedom $X^\rho \nabla_\rho \zeta^{(T)}_\mu = 0$, which we use to fix

$$\bar{\nabla}^\mu \gamma'_{\mu\nu} = 0 , \tag{4.26}$$

at order $\mathscr{O}(\omega^0)$, using Eq. (4.23). After such gauge choices, the quantity $\gamma'_{\mu\nu}$ is transverse and traceless; we identify it as the high-frequency GW and dub it

$$\gamma'_{\mu\nu} \equiv h^{(TT)}_{\mu\nu} . \tag{4.27}$$

We point out that is not possible to choose $h^{(\mathrm{tr})} = 0$, using the residual gauge freedom, left after the last transformation, if $h^{(\mathrm{tr})}$ depends on the coordinate in the direction of X_μ. For simplicity, we can exhaust the gauge freedom imposing $\nabla^\mu \xi^{(T)}_\mu = 0$, such that the trace $h^{(\mathrm{tr})}$ is gauge-invariant, while the transverse-traceless GW excitations $h^{(TT)}_{\mu\nu}$ are invariant under the residual transformation that can be read from Eq. (4.23):

$$h^{(TT)}_{\mu\nu} \to h^{(TT)}_{\mu\nu} - \bar{\nabla}_\mu \xi^{(T)}_\nu - \bar{\nabla}_\nu \xi^{(T)}_\mu . \tag{4.28}$$

After the gauge fixing procedure described, the metric perturbation reads

$$\tilde{h}_{\mu\nu} = X_\mu X_\nu h^{(S)} + \frac{1}{3} \Lambda_{\mu\nu} h^{(\mathrm{tr})} + h^{(TT)}_{\mu\nu} . \tag{4.29}$$

The quantity $\tilde{h}_{\mu\nu}$, before we make any gauge choice, has 10 non-vanishing components. Making gauge fixings as explained above, we imposed 6 conditions, since both $h^{(V)}_\mu$ and $h^{(TT)}_{\mu\nu}$ are by construction orthogonal to the vector X^μ. Hence, we are left with 4 independent metric components. In Section 4.3.1 we show that only 3 out of these 4 are independent propagating degree of freedom, while $h^{(S)}$ is a constrained field. We will decouple the evolution equations of The evolution equations of $h^{(\mathrm{tr})}$ and $h^{(TT)}_{\mu\nu}$ under physical assumptions on the velocities of the fields involved.

4.3. Equations of motion

Isaacson, working in the context of the geometric optics limit of General Relativity, showed that the original diffeomorphism invariance is preserved order-by-order in the gradient expansion [151, 152] in the equations of motion. In our scalar-tensor framework, we change perspective and *impose* the symmetry invariance at each order in the ω-expansion. This viewpoint allows us to write the most general structure for the equations governing the GW dynamics, and to encode the effects of the DE field in few physically transparent parameters.

4.3.1. Separating the evolution equations

We consider that the equations of motion of $\tilde{h}_{\mu\nu}$ in Eq. (4.29), can be obtained from the action (4.3). As usual in the context of geometric optics, we neglect the contribution of standard matter, considering that it does not have high-frequency excitations. The gravitational field equations can be expressed in terms of $\tilde{h}_{\mu\nu}$ as

$$G^{(1)}_{\mu\nu}\left[\tilde{h}_{\rho\sigma}\right] = T^{(1)}_{\mu\nu}\left[\tilde{h}_{\rho\sigma}\right] , \tag{4.30}$$

where $G^{(1)}_{\mu\nu}\left[\tilde{h}_{\rho\sigma}\right]$ is the linearized Einstein tensor, written in terms of $\tilde{h}_{\rho\sigma}$, and $T^{(1)}_{\mu\nu}$ represent any other contribution to the field equations. Note that we are choosing the gauge $\delta\phi = 0$, hence it does not appear in the equations above. Taking the trace of Eq. (4.30), we find that the left-hand-side is given by (minus) the first order Ricci scalar

$$R^{(1)} = -\bar{\Box} h^{(\mathrm{tr})} + \Lambda^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\left(h^{(S)} + \frac{1}{3}h^{(\mathrm{tr})}\right), \tag{4.31}$$

from which we see that, while the trace scalar $h^{(\mathrm{tr})}$ receives a kinetic contribution controlled by the d'Alembert operator $\bar{\Box}$, second derivatives acting on the scalar $h^{(S)}$ are weighted by the projector operator $\Lambda_{\mu\nu}$. Let us consider, as an example, the case of a background field configuration which is homogeneous and isotropic. In this case, $\bar{g}_{\mu\nu}$ is the FLRW metric and $v_\mu \propto \delta_{\mu 0}\varphi_0'$ so that $\Lambda^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{(S)} \propto \partial^i\partial_i h^{(S)}$. As a result, one finds that the kinetic contributions of $h^{(S)}$ coming from the Ricci scalar, are not sufficient for propagating this field. Indeed, $h^{(S)}$ plays a role analogous to the lapse function N in the ADM formalism (see e.g. [160]): not dynamical and whose equation of motion serves as a constraint equation. We also discuss in Appendix B an explicit, simple example where $h^{(S)}$ is manifestly non-dynamical. To proceed further, one has to understand whether the energy-momentum tensor in the right-hand-side of Eq. (4.30), can make $h^{(S)}$ dynamical by, for instance, containing a term such $\sim \bar{\Box} h^{(S)}$. If in the action (4.3), the couplings of dark energy scalar to the metric are expressed in a covariant form in terms of the metric, Riemann, and Ricci tensors, we claim this is not possible. Indeed, if this is the case, their contribution will have a form similar to the one of the Ricci in Eq. (4.31) and, consequently, the same considerations as above apply. We conclude that the role of $h^{(S)}$ is to fix certain conditions on the high-frequency modes.

Assuming that we have solved the equation of $h^{(S)}$, we are left with $h^{(TT)}_{\mu\nu}$ and $h^{(\mathrm{tr})}$ as potentially propagating high-frequency degrees of freedom. The linearized gravitational field equations, can be decomposed as

$$G^{(1)}_{\mu\nu}\left[h^{(TT)}_{\rho\sigma}\right] + G^{(1)}_{\mu\nu}\left[h^{(\mathrm{tr})}\right] = T^{(T)}_{\mu\nu}\left[h^{(TT)}_{\rho\sigma}\right] + T^{(\mathrm{tr})}_{\mu\nu}\left[h^{(\mathrm{tr})}\right]. \tag{4.32}$$

We expect that second derivatives contributions to the scalar sector have a rich structure, due to the presence of $v^\mu \neq 0$. As a consequence, tensor and scalar fluctuations normally propagate with different velocities. We set the speed of GW to the one of light, given the strong experimental bounds on the GW velocity associated with the GW170817 event [302], and make the ansatzs

$$h^{(TT)}_{\mu\nu} = \mathcal{A}^{(T)}_{\mu\nu}\exp\left[i\omega\psi^{(TT)}\right], \tag{4.33}$$

$$h^{(\mathrm{tr})} = \mathcal{A}^{(\mathrm{tr})}\exp\left[i\omega\psi^{(\mathrm{tr})}\right]. \tag{4.34}$$

The amplitudes of both modes are slowly varying, while the phases are rapidly varying thanks to the factors of ω in the exponent. When plugging Eqs. (4.33) and (4.34) into Eq. (4.32), one gets a linear combination of terms with rapidly oscillating phases and slowly varying overall coefficients, with structure

$$\left(\ldots_{|\omega^2,\omega^1}\right)\exp\left[i\omega\psi^{(TT)}\right] + \left(\ldots_{|\omega^2,\omega^1}\right)\exp\left[i\omega\psi^{(\mathrm{tr})}\right] = 0 \tag{4.35}$$

where within the parenthesis we collect slowly varying contributions at order ω^2 and ω^1 in a gradient expansion. The ω^2 contributions depend on the derivative of the phases $\psi^{(TT)}$ and $\psi^{(\text{tr})}$: they control the dispersion relations for the two species of excitations, scalar and GW. Since in general $h_{\mu\nu}^{(TT)}$ and $h^{(\text{tr})}$ propagate with different speed, they are characterized by distinct dispersion relations, hence the phases $\psi^{(TT)}$ and $\psi^{(\text{tr})}$ are different. Eq. (4.35) is a linear combination of two contributions weighted by two distinct phases which rapidly oscillate over space and time: in order to satisfy it, we need to impose that the coefficients of each of these two terms separately vanish. Within the geometric optics limit, this procedure effectively separates the evolution of scalar modes (characterized by the phase $\psi^{(\text{tr})}$) and GW modes (characterized by the phase $\psi^{(TT)}$). Under all these assumptions illustrated, we consider Eq. (4.32) satisfied if

$$G_{\mu\nu}^{(1)}\left[h^{(\text{tr})}\right] = T_{\mu\nu}^{(\text{tr})}\left[h^{(\text{tr})}\right], \tag{4.36}$$

$$G_{\mu\nu}^{(1)}\left[h_{\rho\sigma}^{(TT)}\right] = T_{\mu\nu}^{(T)}\left[h_{\rho\sigma}^{(TT)}\right], \tag{4.37}$$

namely the two sectors solve individually their respective equations. As a result, the GW sector is decoupled from the scalar one at the linearized level.

4.3.2. The tensor mode equation

We now investigate the tensor mode equation in Eq. (4.37). The left-hand-side corresponds to the linearized Einstein tensor, evaluated in $h_{\rho\sigma}^{(TT)}$. Since the latter is transverse and traceless, we have that

$$G_{\mu\nu}^{(1)}\left[h_{\rho\sigma}^{(TT)}\right]_{|\omega^2,\omega^1} = -\frac{1}{2}\bar{\Box}h_{\mu\nu}^{(TT)}{}_{|\omega^2,\omega^1}. \tag{4.38}$$

The right-hand-side of Eq. (4.37) is theory dependent, nevertheless, symmetry considerations allow us to determine the general structure of $T_{\mu\nu}^{(1)}[h_{\rho\sigma}^{(TT)}]$, without relying on specific models. Considering Eq. (4.38) as left-hand-side of Eq. (4.37), and the invariance under residual $T-$ types gauge transformations, we see that the right-hand-side should be:

1. transverse and traceless,
2. orthogonal to v^μ at orders ω^2 and ω^1,
3. invariant under the transformation (4.28),
4. conserved at order ω^2: $\left[\nabla^\mu T_{\mu\nu}^{(T)}\right]_{\omega^2} = 0$,
5. containing at most second derivatives of $h_{\rho\sigma}^{(TT)}$: since we are considering an action of the form (4.3), stability requires that the corresponding equations of

motion are at most second order[2].

Additionally, we demand that it ensures that GW propagate at the speed of light, to be compatible with GW170817 [125]. The only allowed structure of the linearized $T^{(T)}_{\mu\nu}(h_{\rho\sigma})$ that satisfies all of these requirements at orders ω^2, ω is

$$T^{(T)}_{\mu\nu} \;=\; \tau_A \bar{\Box} h^{(TT)}_{\mu\nu} + \tau_B\, v^\rho\, \bar{\nabla}_\rho h^{(TT)}_{\mu\nu}\,, \tag{4.39}$$

where $\tau_{A,B}$ depend only on slowly varying fields. Calling the combination

$$\mathcal{T} = -\frac{2\tau_B}{1+2\tau_A}\,, \tag{4.40}$$

we can rewrite the GW evolution equation as

$$\left(\bar{\Box} h^{(TT)}_{\mu\nu}\right)_{|\omega^2,\omega^1} = \mathcal{T} \times \left(v^\rho \bar{\nabla}_\rho\, h^{(TT)}_{\mu\nu}\right)_{|\omega^1}\,. \tag{4.41}$$

The deviations from GR on the propagation of high-frequency GW only appear as a first-order gradient of the GW high-frequency fluctuation, proportional to the parameter $\mathcal{T}$ depending on slowly-varying fields. Such contribution can be thought as a 'friction term' for the GW, and is common in scalar-tensor theories with non-minimal couplings between scalar and metric degrees of freedom. In the context of gravitational wave cosmology, several groups explored the consequences of such term in specific cosmological models [114, 123, 124, 161, 183, 184, 186, 191, 211, 222, 223, 303–308], as we did in Chapters 2 and 3.

4.3.3. Amplitude evolution equation

We consider the eikonal anstaz in Eq. (4.34) where the gradient of the phase defines the GW 4-momentum as

$$k_\mu = \bar{\nabla}_\mu \psi^{(TT)}\,, \tag{4.42}$$

and plug it into the equation of motion (4.41). As usual in the context of geometric optics, we organize the equation obtained in this way in power of ω

$$\Big[\ldots\Big]\omega^2 + \Big[\ldots\Big]\omega + \cdots = 0\,, \tag{4.43}$$

and, since $\omega \gg 1$, we require that the coefficient of each order vanishes, in order to satisfy the equation. At order ω^2 we obtain,

$$k^\mu k_\mu = 0\,, \qquad k^\rho \bar{\nabla}_\rho k^\mu = 0\,, \tag{4.44}$$

[2] It could be possible that the equations of motion become second order only after having used specific constraint relations. This should be the case of DHOST theories, for instance.

which states that the GW 4-momentum is a null vector, propagating along null geodesics. Calling λ the affine parameter of the geodesics, we have that for any function f, the derivative along the geodesics is $df/d\lambda = k^\rho \bar{\nabla}_\rho f$. At order ω, we find the evolution equation for the amplitude

$$\left[2\, k^\rho \bar{\nabla}_\rho\, \mathcal{A}^{(T)} + (\bar{\nabla}_\rho k^\rho)\, \mathcal{A}^{(T)}\right] = \mathcal{T}\, k^\rho\, v_\rho\, \mathcal{A}^{(T)}\,, \tag{4.45}$$

where we defined $\mathcal{A}^{(T)}$ as $\mathcal{A}^{(T)} = \sqrt{\mathcal{A}^{(T)}_{\mu\nu}\,(\mathcal{A}^{(T)})^{\mu\nu}}$. Recalling that $v_\mu = \bar{\nabla}_\mu \bar{\varphi}$, the previous equation can be 'integrated' to

$$\bar{\nabla}_\rho \left(e^{-\int \mathcal{T}}\, k^\rho \left[\mathcal{A}^{(T)}\right]^2 \right) = 0\,, \tag{4.46}$$

where the schematic expression $\int \mathcal{T}$ denotes the following integral

$$\int \mathcal{T} \equiv \int_{\lambda_s}^{\lambda} \mathcal{T}\, \frac{d\bar{\varphi}}{d\lambda'}\, d\lambda'\,. \tag{4.47}$$

In the equation above, λ_s corresponds to the value of the affine parameter at the source position. The quantity in Eq. (4.47) represents a cumulative integration of the friction term in Eq. (4.41) over the GW geodesic's affine parameter. In integrating Eq. (4.45) we have chosen boundary conditions such that there are no scalar field effects at the source position $\lambda = \lambda_s$, as assumed throughout the work. Importantly, we do not need to demand that $\mathcal{T}$ is 'small' for writing Eq. (4.46).

4.4. Conservation laws

Eq. (4.45) shows that the DE field can introduce an additional damping term. However, the fact that this can be integrated, lead us to Eq. (4.46), giving us reasons to believe that we can actually still formulate a conservation law.

4.4.1. The energy momentum of GW

The presence of GWs can back-react on the background curvature. These effects were quantified, in the geometric optics limit and in General Relativity, by Isaacson [152], introducing the GW stress-energy tensor, which is at second order in the amplitude expansion regulated by α. Focusing only on the tensor modes $h^{(TT)}_{\mu\nu}$, we derive their associated stress-energy tensor in the generalized gravitational set-up outlined in the previous Sections. Again, we do so by changing perspective: we build the stress tensor bottom-up by using symmetry arguments, namely finding the only possible second-order tensor which is gauge invariant under the residual transformation in Eq. (4.28). Indeed, also Isaacson in [152] showed that the GW stress tensor in General Relativity is gauge invariant, at the geometric optics order, and here we use this symmetry property as a guiding principle to build $T^{(2),\mathrm{ST}}_{\mu\nu}$.

We start by that the stress-energy tensor must be quadratic in $h_{\mu\nu}^{(TT)}$, and contain two derivatives acting on the transverse-traceless GW excitations (by 'integration by parts', we can place one derivative per field). Given this information, the most general structure that $T_{\mu\nu}^{(2)}$ can have is

$$T_{\mu\nu}^{(2),\mathrm{ST}} = \frac{1}{32\pi\omega^2}\langle\bar{\nabla}_\mu h_{\alpha\beta}^{(TT)}\bar{\nabla}_\nu h_{\gamma\delta}^{(TT)}\rangle \mathcal{C}^{\alpha\beta\gamma\delta}, \tag{4.48}$$

where $\mathcal{C}^{\alpha\beta\gamma\delta}$ depends on slowly-varying fields and the symbol $\langle\ldots\rangle$ denotes the so-called Brill-Hartle spatial average [152, 160]. Being $h_{\gamma\delta}^{(TT)}$ traceless, we have that $\mathcal{C}^{\alpha\beta\gamma\delta} \not\propto \bar{g}^{\gamma\delta}$, otherwise the stress-energy tensor would vanish. Also, because $T_{\mu\nu}^{(2),\mathrm{ST}}$ is conserved at order ω^2 and $h_{\gamma\delta}^{(TT)}$ is transverse and traceless, the only way to arrange the free indices μ,ν is the one in Eq. (4.48). After fixing the gauge as discussed in Section 4.2.2, we are left with invariance under the transformation in Eq. (4.28). We select $\mathcal{C}^{\alpha\beta\gamma\delta}$ in Eq. (4.48) such that the stress-energy tensor is invariant under the same gauge transformations. Using Eq. (4.28), we see that the latter transforms as $T_{\mu\nu}^{(2),\mathrm{ST}} \to T_{\mu\nu}^{(2),\mathrm{ST}} + \delta T_{\mu\nu}^{(2),\mathrm{ST}}$ where

$$\delta T_{\mu\nu}^{(2),\mathrm{ST}} = -\frac{1}{16\pi\omega^2}\langle\bar{\nabla}_\mu\bar{\nabla}_\alpha\xi_\beta^{(T)}\bar{\nabla}_\nu h_{\gamma\delta}^{(TT)}\rangle \mathcal{C}^{\alpha\beta\gamma\delta}, \tag{4.49}$$

$$= \frac{1}{16\pi\omega^2}\langle\bar{\nabla}_\mu\xi_\beta^{(T)}\bar{\nabla}_\nu\bar{\nabla}_\alpha h_{\gamma\delta}^{(TT)}\rangle \mathcal{C}^{\alpha\beta\gamma\delta}, \tag{4.50}$$

which must vanish for any $\xi_\beta^{(T)}$, and for any choice of μ, ν. This can be achieved by choosing $\mathcal{C}^{\alpha\beta\gamma\delta} = \mathcal{C}\,\delta^{\alpha\gamma}\,C^{\beta\delta}$, for some function C and tensor $C^{\beta\delta}$ and upon using the transversality of the GW. Plugging this result in Eq. (4.48) we obtain

$$T_{\mu\nu}^{(2),\mathrm{ST}} = \frac{\mathcal{C}}{32\pi\omega^2}\langle\bar{\nabla}_\mu h_\beta^{(TT),\alpha}\bar{\nabla}_\nu h_{\alpha\delta}^{(TT)}\rangle \mathcal{C}^{\beta\delta}. \tag{4.51}$$

Since the EMT is symmetric in the indexes, the quantity $\mathcal{C}^{\beta\delta}$ is symmetric. Applying again the transformation (4.28), and integrating by parts, we find that the invariance of the stress-energy tensor also imposes $\mathcal{C}^{\beta\delta} \propto \delta^{\beta\delta}$. Therefore, symmetry arguments fixed the form of the GW stress-energy tensor, up to a multiplicative function, to

$$T_{\mu\nu}^{(2),\mathrm{ST}} = \frac{\mathcal{C}}{32\pi\omega^2}\langle\bar{\nabla}_\mu h_{\alpha\beta}^{(TT)}\bar{\nabla}_\nu h^{(TT)\alpha\beta}\rangle. \tag{4.52}$$

This result would be identical to the General Relativistic one in [152], is the function $\mathcal{C}$ of the slowly varying fields was $= 1$. In our case, we fix it by using the conservation equation[3].

$$\nabla^\mu T_{\mu\nu}^{(2),\mathrm{ST}} = 0, \tag{4.53}$$

[3]The GW stress-energy tensor enters the background gravitational field equations, of the form $G_{\mu\nu}^{(0)} = T_{\mu\nu}^{(2),\mathrm{ST}}[h^{(TT)}] + T_{\mu\nu}^{(2),\mathrm{ST}}[h^{(tr)}]$. Bianchi's identities guarantee the conservation of the right-hand-side of the latter equations. The assumption of having independent scalar and tensor sector guarantees that the two stress-energy tensors are conserved separately.

which, together with Eqs. (4.46) and (4.44) for the amplitude of the tensor modes, fixes $\mathcal{C}$ to the value $e^{-\int\mathcal{T}}$. Hence, we find that the second order GW energy-momentum tensor, in the geometric optics limit and at leading order, reads

$$T^{(2),\mathrm{ST}}_{\mu\nu} = \frac{e^{-\int\mathcal{T}}}{32\pi}\left[\mathcal{A}^{(T)}\right]^2 k_\mu k_\nu\,. \tag{4.54}$$

4.4.2. Conservation of the graviton number density current

The results of geometric optics of Section 4.3.3, allow us to interpret the GW stress-energy tensor in Eq. (4.54) and its conservation in terms of a graviton number density 4-current. We identify the quantity

$$\mathcal{N}_\mu \equiv k_\mu\left[\mathcal{A}^{(T)}\right]^2 e^{-\int\mathcal{T}} \qquad \rightarrow \qquad \bar{\nabla}_\mu\mathcal{N}^\mu = 0\,, \tag{4.55}$$

as the *graviton number density 4-current*, which is conserved by virtue of Eq. (4.46), and all the assumptions it is based on. We can express the GW stress-energy tensor in terms of the graviton number density as,

$$T^{(2),\mathrm{ST}}_{\mu\nu} = \frac{1}{16\pi}\,k_{(\nu}\mathcal{N}_{\mu)}\,, \tag{4.56}$$

where the parenthesis stands for the symmetrization in the μ,ν indices. Being able to express the GW stress-energy tensor as in Eq. (4.56), supports its definition found through only symmetry arguments: it's tensor whose components are related to the flux of the $\mu - th$ component of the energy-momentum density through a surface with x^ν constant coordinate (the vector k_ν is the wave-vector of the GW). This interpretation relies on the possibility of defining the rays, identified by the wave-vector k^μ, which make clear the concept of a *trajectory* for a gravitational wave.

We can give a further interpretation of these results in terms of the geometry of the cross-sectional area of the GW's ray bundle, $S(\lambda)$ in Figure 4.1, to further support the identification of $\mathcal{N}^\mu$ as the graviton number density 4-current. If λ is the affine parameter associated to the GW rays with 4-momentum k^μ, a geometric optics theorem (see [160], exercise 22.13) states that,

$$\frac{d\,S(\lambda)}{d\lambda} - \bar{\nabla}_\mu k^\mu\,S(\lambda) = 0\,, \tag{4.57}$$

which combined with Eq. (4.55) implies

$$\frac{d}{d\lambda}\left\{e^{-\int\mathcal{T}}\left[\mathcal{A}^{(T)}\right]^2 S(\lambda)\right\} = 0\,, \tag{4.58}$$

clearly showing that the combination $e^{-\int\mathcal{T}}\left[\mathcal{A}^{(T)}\right]^2$ is inversely proportional to the bundle's cross-sectional area, rather than only $\left[\mathcal{A}^{(T)}\right]^2$ as in General Relativity.

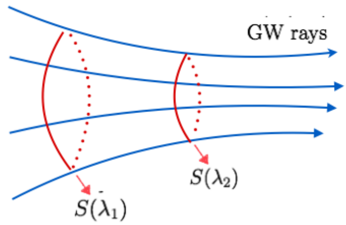

Figure 4.1: *Geometric optics representation of graviton number conservation. The flux of a stream of gravitons crossing the S-areas is conserved along the GW affine parameter. See Eq.* (4.58).

4

4.5. Gravitational wave distances and duality relation

In this Section we give the definitions of the *gravitational waves distances* by generalizing to the case of GWs in scalar-tensor theories the works [157, 213–217, 309] about fluctuations of the luminosity and angular diameter distances, both of photons and GWs. All of these works descend from the one done by Sasaki in [212] regarding photons, which we follow very closely in this Section. After giving these definitions, we prove the validity of *Etherington's reciprocity law*, also known as *distance duality relation*, between GW luminosity and angular distances, also in the scalar-tensor theories considered in this Chapter[4]. This law states that, in General Relativity, between the electromagnetic luminosity and angular diameter distance the following relation holds

$$d_L = (1+z)^2 \, d_A \,, \tag{4.59}$$

if: the spacetime is described by a pseudo-Riemannian manifold, photons propagate along null geodesics of the spacetime and their number density is conserved [149, 310, 311]. Such relation has been tested from several electromagnetic observations [312–316] and its role in the context of multi-messenger observation was explored as well in [185]. Etherington's reciprocity law, because it relies on very minimal assumptions, it provides a perfect playground to test the theory of gravity or the cosmological model [317], so one might wonder if we should expect a similar relation also for the GWs distances

$$d_L^{(\mathrm{GW})} = (1+z)^2 \, d_A^{(\mathrm{GW})} \,, \tag{4.60}$$

in light of the results obtained in Sections 4.3.3 and 4.4.1.

[4] We study the propagation of GW over a perturbed cosmological background and prove its validity up to first order in the perturbations

4.5.1. Raychaudhuri equation

Before giving the definitions of the GW distances, we perform some preliminary steps into further characterizing the GW's rays, which will become useful later. Our starting equations are the GW stress-energy tensor of Eq. (4.54), the amplitude evolution equation in Eq. (4.46) and the geodesic equation (4.44).

We perform a conformal transformation, defining the metric $\hat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu}/a^2$ and $\hat{g}^{\mu\nu} \equiv \bar{g}^{\mu\nu}/a^{-2}$ [5], mapping a null GW geodesics in $\bar{g}_{\mu\nu}$ into null geodesics in $\hat{g}_{\mu\nu}$ with rescaled affine parameter [32]

$$d\hat{\lambda} = a^{-2}\, d\lambda. \tag{4.61}$$

The amplitude evolution equation in the comoving frame then results

$$\frac{d}{d\hat{\lambda}}\left(e^{-\frac{1}{2}\int\mathcal{T}}\mathcal{A}^{(T)}\, a\right) + \frac{1}{2}\left(e^{-\frac{1}{2}\int\mathcal{T}}\mathcal{A}^{(T)}\, a\right)\hat{\theta} = 0 \qquad \text{with} \qquad \hat{\nabla}_\mu \hat{k}^\mu = \hat{\theta}, \tag{4.62}$$

where $\hat{\nabla}_\mu$ is the covariant derivative associated to the conformal metric and $\hat{k}_\mu \equiv k_\mu$, while $\hat{k}^\mu \equiv \hat{g}^{\mu\nu}\,\hat{k}_\nu$. The *expansion parameter* θ, satisfies the *Raychaudhuri equation*, which can be determined from Eq. (4.44), is

$$\frac{d\hat{\theta}}{d\hat{\lambda}} = -\hat{R}_{\mu\nu}\hat{k}^\mu\hat{k}^\nu - \frac{\hat{\theta}^2}{2} - 2\hat{\sigma}^2, \tag{4.63}$$

where $\hat{\sigma}^2 \equiv \hat{k}_{(\alpha;\beta)}\,\hat{k}^{(\alpha;\beta)}/2 - \hat{\theta}^2/4$ is the shear of the GW ray's bundle, and $\hat{R}_{\mu\nu}$ the conformal spacetime Ricci tensor [212]. The graviton number conservation (4.58) remains unchanged and reads

$$\frac{d}{d\hat{\lambda}}\left\{e^{-\int\mathcal{T}}\,\mathcal{A}^{(T)2}\, S(\hat{\lambda})\right\} = 0. \tag{4.64}$$

4.5.2. Gravitational wave luminosity distance

In the physical frame, we introduce an observer the 4-velocity u^μ, which measures the GW's energy flux given by

$$\mathcal{F}^\alpha = -\left[T^{(2),\mathrm{ST}}\right]^\mu{}_\nu\, \mathcal{P}^\alpha{}_\mu\, u^\nu, \tag{4.65}$$

$$= \mathcal{F}\, n^\alpha, \tag{4.66}$$

with the GW flux amplitude and frequency measured by the observer are

$$\mathcal{F} = \frac{e^{-\int\mathcal{T}}}{32\pi}\left[\mathcal{A}^{(T)}\right]^2 \nu^2, \qquad \nu = -k_\mu u^\mu, \tag{4.67}$$

and also

$$\mathcal{P}^\alpha_\mu = \delta^\alpha_\mu + u^\alpha u_\mu, \qquad n^\alpha = \frac{1}{\nu}\left(k^\alpha - \nu u^\alpha\right). \tag{4.68}$$

[5] From now onward, quantities in conformal frame are denoted with a hat

The notion of GW frequency allows us to define the redshift z at the value $\hat{\lambda}$ of the comoving GW geodesics affine parameter as

$$1 + z(\hat{\lambda}) = \frac{\nu(\hat{\lambda})}{\nu(0)}. \tag{4.69}$$

We assume that GW are emitted by an approximately spherically symmetric system, with characteristic radius R_s, see left panel of Figure 4.2, which we will $\to 0$ at the end of the computation. The flux amplitude $\mathcal{F}$ measured at the source position is related with the intrinsic source luminosity by the relation

$$\mathcal{F}(\hat{\lambda}_s) = \frac{\mathscr{L}_{GW}}{4\pi R_s^2}, \tag{4.70}$$

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with $\hat{\lambda}_s$ the conformal affine parameter at the source. We define *GW luminosity distance* $d_L^{(\mathrm{GW})}$ as the ratio of GW power emitted at source position (intrinsic GW luminosity), versus the GW flux at detector location [212]

$$d_L^{(\mathrm{GW})} \equiv \left[\frac{\mathscr{L}_{GW}}{4\pi\,\mathcal{F}(0)}\right]^{\frac{1}{2}} = \sqrt{\frac{\mathcal{F}(\hat{\lambda}_s)}{\mathcal{F}(0)}}\,R_s. \tag{4.71}$$

Substituting relation (4.67), we find the following expression

$$d_L^{(\mathrm{GW})} = \exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s}\mathcal{T}\right] \times \frac{\mathcal{A}^{(T)}(\hat{\lambda}_s)}{\mathcal{A}^{(T)}(0)} \times \left[1 + z(\hat{\lambda}_s)\right] \times R_s. \tag{4.72}$$

Note the role of the scalar field-induced friction term in the overall exponential factor, that encodes the interesting phenomenology of these theories, providing interesting observation prospects in case of multi-messenger events. By taking the definition of EM luminosity distance in [212], we find the ration between the two distances

$$\frac{d_L^{(\mathrm{GW})}}{d_L^{(\mathrm{EM})}} = \exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s}\mathcal{T}\,\frac{d\bar{\phi}}{d\lambda'}\,d\lambda'\right], \tag{4.73}$$

singling out the modified gravity contribution as an integral from $\lambda = 0$ (the position of the observer) to the source at $\hat{\lambda} = \hat{\lambda}_S$. This factor reduces to the effective Planck's mass of Eq. (1.105) for specific choices of $\mathcal{T}$, as we show explicitly in Appendix A.

4.5.3. Gravitational wave angular distance

The *GW angular distance* $d_A^{(\mathrm{GW})}$ is defined in terms of the ratio between the angular diameter d_s of the source located at conformal affine parameter $\hat{\lambda}_s$, and the source apparent angular size $\Delta\phi$ as measured by an observer at $\hat{\lambda} = 0$,

$$d_A^{(\mathrm{GW})} \equiv \frac{d_s}{\Delta\Omega}. \tag{4.74}$$

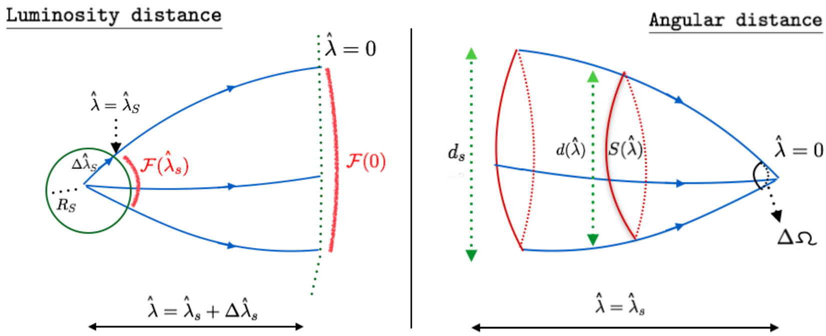

Figure 4.2: *Representative plot of the GW/photon rays from source to detector position. The path of the ray bundle in blue is parameterized by the affine parameter $\hat\lambda$.* **Left panel***: quantities entering the luminosity distance are associated with a bundle diverging from source to detector.* **Right panel***: quantities entering the angular distance are associated with a bundle converging from source to detector. See text for definitions.*

4

Following [212], it is convenient to reexpress $d_A^{(\mathrm{GW})}$ as

$$d_A^{(\mathrm{GW})} = \left(\frac{S(\hat\lambda_s)}{S(\Delta\hat\lambda_o)}\right)^{1/2} \frac{\mathbf{d}(\Delta\hat\lambda_o)}{\Delta\Omega}, \tag{4.75}$$

with $S(\hat\lambda)$ and $\mathbf{d}(\hat\lambda)$ the cross-section area and the diameter of the GW's ray bundle at $\hat\lambda$, and $\Delta\hat\lambda_o$ is the affine parameter in proximity of the observer (see Figure 4.2, right panel). In [212], it is shown that

$$\frac{\mathbf{d}(\Delta\hat\lambda_o)}{\Delta\Omega} = \frac{a^2[\tau(0)]\,\Delta\hat\lambda_o}{(1+z(\hat\lambda_s))\,a[\tau(\hat\lambda_s)]}, \tag{4.76}$$

connecting the ratio $\mathbf{d}(\Delta\hat\lambda)/\Delta\Omega$ with $\Delta\hat\lambda_o$. Thanks to Eq. (4.64), relating the amplitude of the GW with the area of the cross-section of the ray bundle, we can rewrite Eq. (4.75) as

$$d_A^{(\mathrm{GW})} = \exp\left[\frac{1}{2}\int_{\Delta\hat\lambda_o}^{\hat\lambda_s} \mathcal{T}\right] \times \left(\frac{\mathcal{A}^{(T)}(\Delta\hat\lambda_o)}{\mathcal{A}^{(T)}(\hat\lambda_s)}\right) \times \frac{\mathbf{d}(\Delta\hat\lambda_o)}{\Delta\Omega}. \tag{4.77}$$

4.5.4. Etherington's reciprocity law

We now prove the validity of Etherington's reciprocity law, connecting luminosity and angular GW distances. In the case of photons, the Etherington reciprocity law takes the form in Eq. (4.59) and it descends from very generic hypothesis: the spacetime is described by a Riemannian manifold, photons propagate along null geodesics of the spacetime and their number density is conserved [149, 310]. Among them, and in the case of GWs in geometric optics, our scalar-tensor scenario only affects the conservation of the graviton number density, via the $e^{-\int \mathcal{T}}$ in Eq. (4.64), standing for the friction induced by the DE scalar field. Nonetheless, we derived a modified conservation law, namely Eq. (4.55), hence there are good reasons to believe that the reciprocity law still holds for the GW distances, as defined in Sections 4.5.2 and 4.5.3.

Our findings and definitions so far are valid in any background spacetime, provided that the background scalar field assumes a non-trivial profile and $v_\mu \neq 0$. Here we choose a perturbed cosmological background

$$\bar{g}_{\mu\nu} = a^2(\tau)\left[\eta_{\mu\nu} + \epsilon\,\delta\hat{g}_{\mu\nu}\right], \quad (4.78)$$

$$\bar{\varphi} = \varphi_0(\tau) + \epsilon\,\delta\varphi(x), \quad (4.79)$$

4

and derive the Etherington's relation up to first order in cosmological perturbations as in [212]. In the expressions above, $\eta_{\mu\nu}$ is the Minkowski metric, and we have introduced again ϵ as the parameter keeping track of the order of magnitude of the long wavelength metric and scalar field perturbations, describing the large-scale structures, opposed to α for the high-frequency fluctuations. Therefore, even if we chose the unitary gauge for the high-frequency scalar field $\alpha\,\delta\phi(x)$, the long wavelength DE field perturbation, $\epsilon\,\delta\varphi(x)$, is still present. Recalling that we defined the comoving wave-vector as $\hat{k}^\mu \equiv \hat{g}^{\mu\nu} k_\nu$, we introduce a null vector

$$\hat{K}^\mu \equiv -\frac{\hat{k}^\mu}{\nu(\hat{\lambda}_s)\,a[\tau(\hat{\lambda}_s)]}, \quad (4.80)$$

and, from now onward, $\hat{\lambda}$ will be the affine parameter associated to it. This vector is normalized such that $\left(\hat{g}_{\mu\nu}\hat{K}^\mu \hat{u}^\nu\right)_{\hat{\lambda}_s} = 1$, where $\hat{u}^\mu$ is the observer 4-velocity in the conformal frame. The introduction of the vector $\hat{K}^\mu$ is convenient to easily relate the physical size of the source with the affine parameter along the GW geodesics. In fact, as shown in [212], the characteristic size R_s of the source can be expressed as

$$R_s = a(\tau(\lambda_s))\,\Delta\hat{\lambda}_s, \quad (4.81)$$

with $\Delta\hat{\lambda}_s$ the infinitesimal affine parameter associated with the source size (see Figure 4.2, left panel).

We prove the distance duality relation by taking the following steps:

i. We use Raychaudhuri equation (4.63), to relate the expansion parameter, $\hat{\theta}$, to the comoving affine parameter, $\hat{\lambda}$. The integration in $\hat{\lambda}$ requires the choice of boundary conditions; these will be different between luminosity and angular diameter distance because of the different geometry of the ray's bundle, as shown in Figure 4.2.

ii. We use Eq. (4.62) to relate the GW's amplitude to $\hat{\theta}$ and, using the result of the previous step, to the comoving affine parameter. We plug these relations in

Eqs. (4.72) and (4.77), written in terms of the GW's amplitude, to have $d_L^{(GW)}$ and $d_A^{(GW)}$ in terms of the comoving affine parameter (and the perturbation of the expansion rate).

iii. We combined the expressions obtained and arrive to Eq. (4.60).

Step i.

We solve Raychaudhuri equation (4.63) perturbatively in ϵ, the expansion parameter which tracks the large-scale structures in Eq. (4.78). We expand the expansion parameter as

$$\hat{\theta}(\hat{\lambda}) = \hat{\theta}_0(\hat{\lambda}) + \epsilon\delta\hat{\theta}(\hat{\lambda}), \tag{4.82}$$

4

so that $\hat{\theta}_0(\hat{\lambda})$ would represent its value on a Minkowski spacetime (remember we have performed a conformal transformation). The affine parameter $\hat{\lambda}$ still has contributions at linear order in ϵ [156, 157]. By expanding Eq. (4.63) in ϵ, and solving it at each order (see also [212, 318] for details), it can be checked that $\hat{\theta}_0$ and $\delta\hat{\theta}$ are given by,

$$\hat{\theta}_0(\hat{\lambda}) - \hat{\theta}_0(\hat{\lambda}_b) = \frac{2}{\hat{\lambda}-\hat{\lambda}_b}, \tag{4.83}$$

$$\delta\hat{\theta}(\hat{\lambda}) - \delta\hat{\theta}(\hat{\lambda}_b) = -\left[\hat{\theta}_0(\hat{\lambda})\right]^2 \int_{\hat{\lambda}_b}^{\hat{\lambda}} d\lambda' \frac{1}{\left[\hat{\theta}_0(\lambda')\right]^2} \times \delta(\hat{R}_{\mu\nu}\hat{K}^\mu\hat{K}^\nu)(\lambda'), \tag{4.84}$$

where $\hat{\theta}_0(\hat{\lambda}_b)$, $\delta\hat{\theta}(\hat{\lambda}_b)$ are boundary conditions to be fixed at $\hat{\lambda}_b$. We choose different boundary conditions in the case of the luminosity or the angular diameter distance. Looking at the left panel of Figure 4.2, it is clear that, in the first case, the expansion parameter $\hat{\theta}$ is zero at the source position $\hat{\lambda}_s + \Delta\hat{\lambda}_s$, while from the right panel of Figure 4.2, we see that $\hat{\theta} = 0$ at $\hat{\lambda} = \Delta\hat{\lambda}_o$, namely the observer position, in the case of angular distances. The main difference between the two situations is the direction the GW ray's bundle is diverging: toward or away from the observer. Therefore, for the luminosity distance boundary conditions we have

$$\hat{\theta}_0^L(\hat{\lambda}) = \frac{2}{\hat{\lambda}-\hat{\lambda}_s-\Delta\hat{\lambda}_s}, \tag{4.85}$$

$$\delta\hat{\theta}^L(\hat{\lambda}) = \frac{1}{\left[\hat{\lambda}-\hat{\lambda}_s-\Delta\hat{\lambda}_s\right]^2} \int_{\hat{\lambda}}^{\hat{\lambda}_s} d\lambda' \left[\lambda'-\hat{\lambda}_s-\Delta\hat{\lambda}_s\right]^2 \times \delta(\hat{R}_{\mu\nu}\hat{K}^\mu\hat{K}^\nu), \tag{4.86}$$

while for the angular diameter distance boundary conditions

$$\hat{\theta}_0^A(\hat{\lambda}) = \frac{2}{\hat{\lambda}-\Delta\hat{\lambda}_o}, \tag{4.87}$$

$$\delta\hat{\theta}^A(\hat{\lambda}) = -\frac{1}{\left[\hat{\lambda}-\Delta\hat{\lambda}_o\right]^2} \int_{\Delta\hat{\lambda}_o}^{\hat{\lambda}} d\lambda' \left[\lambda'-\Delta\hat{\lambda}_o\right]^2 \times \delta(\hat{R}_{\mu\nu}\hat{K}^\mu\hat{K}^\nu). \tag{4.88}$$

Step ii.

As in [212], we can use the results above to integrate Eq. (4.62) and obtain the relation

$$\exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s}\mathcal{T}\right]\times\frac{\mathcal{A}^{(T)}(\lambda_s)\,a(\tau(\lambda_s))}{\mathcal{A}^{(T)}(0)\,a(\tau(0))}=\frac{\lambda_s+\Delta\hat{\lambda}_s}{\Delta\hat{\lambda}_s}\exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s}d\lambda\,\delta\theta_L(\lambda)\right]. \quad (4.89)$$

in the case of the luminosity distance boundary conditions, while

$$\exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s}\mathcal{T}\right]\times\frac{\mathcal{A}^{(T)}(\lambda_s)\,a(\tau(\lambda_s))}{\mathcal{A}^{(T)}(0)\,a(\tau(0))}=\frac{\Delta\lambda_o}{\hat{\lambda}_s}\exp\left[-\frac{1}{2}\int_{\Delta\lambda_o}^{\hat{\lambda}_s}d\lambda\,\delta\theta_A(\lambda)\right] \quad (4.90)$$

in the case of the angular diameter distance ones. We plug these two results into Eqs. (4.72) for $d_L^{\rm GW}$ and (4.77) for $d_A^{\rm GW}$, and obtain

$$d_L^{\rm (GW)}(\hat{\lambda}_s) = a[\tau(0)]\left[1+z(\hat{\lambda}_s)\right]\hat{\lambda}_s\times\exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s}d\lambda\,\delta\theta_L(\lambda)\right], \quad (4.91)$$

$$d_A^{\rm (GW)}(\hat{\lambda}_s) = \frac{a[\tau(0)]}{1+z(\hat{\lambda}_s)}\hat{\lambda}_s\times\exp\left[\frac{1}{2}\int_0^{\hat{\lambda}_s}d\lambda\,\delta\theta_A(\lambda)\right], \quad (4.92)$$

where we have used also the relation in Eq. (4.76) and sent $\Delta\hat{\lambda}_s, \Delta\hat{\lambda}_o \to 0$.

Notice that all the effects of scalar field-induced friction, are implicitly included in the expressions (4.89) and (4.90), which relate the affine parameter at the source position, $\hat{\lambda}_s$, with the remaining quantities. The compact expressions in Eqs. (4.91) and (4.92) (accompanied by relations (4.89) and (4.90)) include the effects of cosmological fluctuations implicitly. These can be made explicit by following the same procedure of [212]. Another possible approach is to use the *Cosmic Rulers* formalism, first developed in the context of photon propagation [156], then for GWs in General Relativity [157] and eventually in a scalar-tensor theory set up in [158]. This approach explicitly identifies contributions from peculiar velocities, weak lensing, Sachs-Wolfe effects, volume effects, and Shapiro time delay, and allows appreciating the contributions due to presence of the DE field, as in Eq. (1.112) of the Introduction. For the purpose of proving the validity of Etherington reciprocity law, Eqs. (4.91) and (4.92) are sufficient.

Step iii.

Combine Eqs. (4.91) and (4.92), we get

$$\tilde{d}_A^{\rm (GW)} = \frac{\tilde{d}_L^{\rm (GW)}}{(1+\tilde{z})^2}\exp\left[\frac{1}{2}\int_0^{\hat{\lambda}_s}\left(\delta\theta_A(\lambda)+\delta\theta_L(\lambda)\right)d\lambda\right], \quad (4.93)$$

$$= \frac{\tilde{d}_L^{\rm (GW)}}{(1+\tilde{z})^2}. \quad (4.94)$$

The explicit computational steps between Eqs. (4.93) and (4.94) can be found in [212]: since Eqs. (4.86) and (4.88) do not contain explicit DE-modifications, they are the same of the analogous computation for photons in General Relativity. The second line, Eq. (4.94), is the desired Etherington's relation, valid including first order perturbations.

Hence, we have proved that in the scalar-tensor framework discussed in this work, with the modified conservation of graviton number density in Eq. (4.55), luminosity and angular distances for GW are connected by the classic Etherington's law (4.94). A straightforward consequence of this result is that also the GW angular diameter distance satisfies an analogous relation to Eq. (4.73), namely

$$\frac{d_A^{(\mathrm{GW})}}{d_A^{(\mathrm{EM})}} = \exp\left[-\frac{1}{2}\int_0^{\hat{\lambda}_s} \mathcal{T}\frac{d\bar{\phi}}{d\lambda}\,d\lambda\right]. \tag{4.95}$$

Given the relevance of Eq. (4.73) in the context of multi-messenger events to test DE, Eq. (4.95) states that the same important role can be played by the angular diameter distances.

4.5.5. Implications for GW lensing

We discuss the implications of our findings for GW strong lensing: when a massive object is located between a source, emitting photon or GWs, and the observer, its gravitational field bends the messenger's path, resulting in a remapping of the source into multiple images. By comparing the arrival time between the images, it is possible to derive another distance measure, the so-called *time delay distance*, defined as

$$D_{\Delta t} = (1+z_l)\,\frac{d^A_{OL}\,d^A_{SO}}{d^A_{SL}}\,, \tag{4.96}$$

where z_l is the lens redshift and d^A_{OL}, d^A_{SO} d^A_{SL} are the angular diameter distances between observer-lens, source-observer and source-lens [149]. The time delay distance can be used to trace the distance-redshift relation and infer cosmological parameters [319, 320], similarly to what is done with the luminosity and angular diameter distances, or test the GW propagation properties [190]. Determining the value of the Hubble parameter today, H_0, through the observations of multi-lens system is a very promising avenue, and a great effort is being dedicated into making this tool more efficient and competitive [321]. Strong lensing of GWs hasn't been observed yet, however future interferometers such as LISA will likely detect lensed events [322], since it is able to probe high redshifts, so the literature of this topic is quite broad [164, 171, 178, 179, 226, 301, 322, 322–349]. One very promising application of these types of events is exactly that they can be used to test the distance duality relation: strong lensing events of standard distance indicators (SN [350] or

GWs [351]) can give us access both to luminosity distance and angular diameter distance.

We consider strong GW lensing from point-like lens in the geometric optics limit, valid when the GW wavelength is well shorter than the Schwarzschild radius of the lens. In this limit, we do not need to discuss interference effects, that will be the topic of Chapter 6. The goal of this section is to understand whether, in the scalar-tensor theory of gravity considered in this Chapter, the time delay between "light images" can differ from the time delay between "GW images", in a multi-messenger detection. Indeed, multi-messenger time delay can prove to be a powerful test for cosmology [333, 347]. The works [346, 348] show conclusively that GW and EM lensed signals arrive at the same time at the detector, provided that both waves propagate at the same speed and are emitted at the same time. In the geometric optics limit, this is expected when photons and GW travel through null geodesics, since by definition both sectors cover the minimal possible distance from source to detector. Causality arguments based on Fermat principle allow one to prove this statement in full generality and [348] also argues that the same result should be valid in any theory of gravity.

4

As for photons, the GW time delay $\Delta t^{(\mathrm{GW})}$ can be expressed as [149]

$$\Delta t^{(\mathrm{GW})} = (1+z_l)\,\frac{d_{OL}^{(\mathrm{GW})}\,d_{SO}^{(\mathrm{GW})}}{2\,d_{SL}^{(\mathrm{GW})}}\,|\theta-\theta_S|^2 + t_\Phi^{(\mathrm{GW})}\,, \tag{4.97}$$

where z is the lens's redshift, $d_{OL}^{(\mathrm{GW})}$ the GW angular distance as measured from the observer to the lens, $d_{SO}^{(\mathrm{GW})}$ the one from source to the observer, and $d_{SL}^{(\mathrm{GW})}$ from source to lens. In Eq. (4.97), θ is the observed angular position of the source, θ_S the would-be angular position of the source in absence of the lens. The first contribution in Eq. (4.97) is the *geometrical time delay*, and its derivation can be found in Appendix C, while the second contribution, t_Φ, is the *Shapiro time delay*, due to the due to the gravitational field of the lens. This contribution is similar to the one found in Eq. (1.112), and it is the same for GW and EM observations in a scalar-tensor framework, as it can be checked by also considering the same term in Eq. (1.114). In other words $t_\Phi^{(\mathrm{GW})} = t_\Phi^{(\mathrm{EM})}$. The geometrical contribution to Eq. (4.97), though, depends on the GW angular diameter distance, which can be modified compared to the EM ones, as Eq. (4.95) states. The corresponding EM-time delay, $\Delta t^{(\mathrm{EM})}$, can be found by substituting $d_A^{(\mathrm{EM})}$ in the same time delay expression [149]. Hence, even if apparently $\Delta t^{(\mathrm{GW})} \neq \Delta t^{(\mathrm{EM})}$, because of the different angular diameter distances, we will prove that the two time delay coincides, in line with the causality arguments previously mentioned. Using Eq. (4.95), we can write

$$\begin{aligned}
\Delta t_{\rm geo}^{\rm (GW)} &= (1+z)\,\frac{d_{OL}^{\rm (GW)}\,d_{SO}^{\rm (GW)}}{2\,d_{SL}^{\rm (GW)}}\,|\theta-\theta_S|^2\,,\\
&= \left(\frac{d_{OL}^{\rm (GW)}}{d_{OL}^{\rm (EM)}}\right)\left(\frac{d_{SO}^{\rm (GW)}}{d_{SO}^{\rm (EM)}}\right)\left(\frac{d_{SL}^{\rm (EM)}}{d_{SL}^{\rm (GW)}}\right)\Delta t_{\rm geo}^{\rm (EM)}\,,\\
&= \left(\exp\left[-\frac{1}{2}\int_{\lambda_O}^{\lambda_L}\mathcal{T}\,\frac{d\bar\phi}{d\lambda}\,d\lambda-\frac{1}{2}\int_{\lambda_S}^{\lambda_O}\mathcal{T}\,\frac{d\bar\phi}{d\lambda}\,d\lambda+\frac{1}{2}\int_{\lambda_S}^{\lambda_L}\mathcal{T}\,\frac{d\bar\phi}{d\lambda}\,d\lambda\right]\right)\Delta t_{\rm geo}^{\rm (EM)}\,,\\
&= \Delta t_{\rm geo}^{\rm (EM)}\,. \qquad (4.98)
\end{aligned}$$

We can see that integrals in the exponential carefully compensate, so that the geometric part of the time delays are equal, $\Delta t_{\rm geo}^{\rm (GW)} = \Delta t_{\rm geo}^{\rm (EM)}$. Together with the fact that the Shapiro contribution is the same for photons and GWs, the result extends to the full time delay, as in Eq. (4.97).

4.6. Discussion and Conclusions

In this Chapter, we studied the propagation of high-frequency gravitational waves in scalar-tensor theories of gravity, with the aim of examining properties of cosmological distances as inferred from GW measurements. We first developed a bottom-up, covariant approach to describe the dynamics of the high-frequency perturbations, based on the principle of coordinate invariance. Symmetry considerations allowed us to extract transverse-traceless components of the high-frequency scalar-tensor fluctuations, identified with GW. In scenarios where scalar and tensor components propagate at different speeds, we argued that the two sectors decouple at the linearized level around an arbitrary background, and the evolution of high-frequency GW and scalar modes can be studied independently. We then determined the most general structure of the GW linearized equations, namely Eq. (4.41) and of the GW energy momentum tensor in Eq. (4.48), where the presence of a dynamical DE scalar field is encoded in the slowly varying factor $\int\mathcal{T}$. Following the guide of [212], we defined the *gravitational waves distances,* $d_L^{\rm GW}$ and $d_A^{\rm GW}$, which descend from the GW's stress-energy tensor, obtainable exclusively because of the geometric optics assumption which allows for a simultaneous definition of wave-vector, k^μ, and trajectory via $k^\mu = dx^\mu/d\lambda$. Both GW luminosity and angular distances can be modified with respect to General Relativity, as shown in Eqs. (4.73) and (4.95), in a way that Etherington's reciprocity law (4.60) still holds, in a perturbed universe and within a scalar-tensor framework. We discussed implications of this result for gravitational lensing, focussing on time-delays of lensed GW. Compatibly with causality arguments, we showed that the time delay between EM images, $\Delta t^{\rm (EM)}$, corresponds to the same in terms of GW images, $\Delta t^{\rm (GW)}$, because we assumed that these were traveling on null geodesics.

4

Appendices

A. Comparison with the literature

To make contact with literature, here we show that Eq. (4.73), when specialized for a FLRW Universe, coincides with the standard expression. We choose $ds^2 = a^2(\tau)\eta_{\mu\nu}\,dx^\mu dx^\nu$ and $\bar\varphi = \varphi_0(\tau)$, so that $v_\mu = (\bar\varphi_0', 0, 0, 0)$. For definiteness, we compare our results with the notation of [249], in which the evolution of the amplitude of high-frequency tensor modes is given by

$$h'' + 2\mathcal{H}\,(1-\delta(\tau))\,h' - \nabla^2 h = 0\,, \tag{4.99}$$

where $\nabla^2 = \partial^i\partial_i$ and the ratio of the luminosity distances is written as

$$\frac{d_L^{(\mathrm{GW})}}{d_L^{(\mathrm{EM})}} = \exp\left[-\int_0^z \frac{\delta(z')}{1+z'}\,dz'\right]. \tag{4.100}$$

Evaluating Eq. (4.41) on the homogeneous and isotropic gives

$$h'' + 2\mathcal{H}\left(1 - \frac{\mathcal{T}\varphi_0'}{2\mathcal{H}}\right)h' - \nabla^2 h = 0\,. \tag{4.101}$$

Comparing this equation with Eq. (4.99), we identify

$$\delta(\tau) = \frac{\mathcal{T}\varphi_0'}{2\mathcal{H}}\,, \tag{4.102}$$

so that

$$\begin{aligned}\frac{d_L^{(\mathrm{GW})}}{d_L^{(\mathrm{EM})}} &= \exp\left[-\int_0^{z_s}\frac{\delta(z)}{1+z}\,dz\right] = \exp\left[-\int_{t_0}^{t_s}\delta(t)\,H\,dt\right] = \exp\left[-\int_{\tau_0}^{\tau_s}\delta(\tau)\,\mathcal{H}\,d\tau\right]\\ &= \exp\left[-\int_{\tau_0}^{\tau_s}\frac{\mathcal{T}\bar\phi'}{2\mathcal{H}}\mathcal{H}\,d\tau\right] = \exp\left[-\frac{1}{2}\int_{\tau_0}^{\tau_s}\mathcal{T}\bar\phi'\,d\tau\right]\\ &= \exp\left[-\frac{1}{2}\int_0^{\lambda_s}\mathcal{T}\frac{d\bar\phi}{d\lambda}d\lambda\right],\end{aligned} \tag{4.103}$$

which is Eq. (4.73). In the derivation above, we also used $d\tau/dt = 1/a$, $\mathcal{H} = H/a$, $1+z = a(0)/a(t)$. Using the relation between $\delta(\tau)$ and the running Planck's mass [122]

$$\delta(\tau) = \frac{\partial \ln M_P[\varphi_0(\tau)]}{\partial \ln a} \tag{4.104}$$

it is also straightforward to check that

$$\frac{d_L^{(\mathrm{GW})}}{d_L^{(\mathrm{EM})}} = \exp\left[-\frac{1}{2}\int_0^{\lambda_s}\mathcal{T}\frac{d\bar\phi}{d\lambda}d\lambda\right] = \exp\left[-\int_0^{z_s}\frac{\delta(z)}{1+z}\,dz\right] = \frac{M_P(z)}{M_P(0)}\,, \tag{4.105}$$

recovering Eq. (1.105).

B. A simple example: $F(\varphi)\,R$

Let us make a specific, simple example of the friction-term contributions found in our general formula of Eq. (4.41), which arises in models characterized by a time-varying Planck mass controlled by the dark energy scalar field φ. We consider the following non-minimal kinetic coupling between scalar φ and metric

$$\mathcal{L} = F(\varphi)R\,, \tag{4.106}$$

which can be considered a part of the classic Brans-Dicke action [56]. We linearize the gravitational field equations following this action and decompose them in terms of the high-energy fluctuations, focusing on orders ω^2 and ω^1 in the gradient expansion, as described in Section 4.3.1. We find that GW modes, as defined in Eq. (4.27), obey the equation

$$\Box h_{\mu\nu}^{(TT)} = \frac{2F_{,\varphi}}{F}\, v^{\lambda}\,\bar{\nabla}_{\lambda} h_{\mu\nu}^{(TT)}\,, \tag{4.107}$$

where $F_{,\varphi} = \partial F/\partial\varphi$. An evolution equation governing scalar modes can be determined by taking the trace of the Einstein equations

$$\bar{\Box} h^{(\mathrm{tr})} - \Lambda_{\alpha\beta}\,\bar{\nabla}^{\alpha}\bar{\nabla}^{\beta}\left(h^{(S)} + \frac{1}{3}h^{(\mathrm{tr})}\right) = -\frac{3\sqrt{2X}F_{,\varphi}}{F}\, X^{\lambda}\bar{\nabla}_{\lambda}\left(h^{(S)} + h^{(\mathrm{tr})}\right)\,, \tag{4.108}$$

where the vector X_μ is defined in Eq. (4.12), and the projector $\Lambda_{\mu\nu}$ in Eq. (4.16). These equations have the structure described in Section 4.3.1. Comparing the GW evolution equation (4.107), with the general expression in Eq. (4.41), we notice that the former has a friction term $\mathcal{T} = 2F_{,\phi}/F$. Upon renaming $F[\varphi] = M_P^2[\varphi]/2$, one can realize that this is the usual friction term. Using the results of section 4.4.1, we find that the stress-energy tensor at second order in the transverse-traceless fluctuations reads

$$\begin{aligned}
T_{\mu\nu}^{(2),\mathrm{ST}} &= \epsilon^2\,\frac{e^{-\int\mathcal{T}}}{32\pi}\,\langle\nabla_\mu h_{\rho\sigma}^{(TT)}\,\nabla_\nu h^{(TT)\,\rho\sigma}\rangle \\
&= \epsilon^2\,\frac{1}{32\pi}\,\exp\left[\int_{\varphi_{\mathrm{in}}}^{\varphi}\frac{d\ln F}{d\tilde{\varphi}}\,d\tilde{\varphi}\right]\langle\nabla_\mu h_{\rho\sigma}^{(TT)}\,\nabla_\nu h^{(TT)\,\rho\sigma}\rangle \\
&= \epsilon^2\,\frac{F(\varphi)}{32\pi}\,\langle\nabla_\mu h_{\rho\sigma}^{(TT)}\,\nabla_\nu h^{(TT)\,\rho\sigma}\rangle\,,
\end{aligned} \tag{4.109}$$

where we chose the extreme of integration φ_{in} such that $F(\varphi_{\mathrm{in}}) = 1$. This tensor has the expected structure associated with the Lagrangian density of Eq. (4.106). We can apply these findings to cosmology, and consider the case of GW propagating through a FLRW Universe, with metric $ds^2 = a^2(\tau)\,\eta_{\mu\nu}\,dx^\mu dx^\nu$, and for a homogeneous scalar field $\bar{\varphi} = \varphi_0(\tau)$. Then Eq (4.107) turns into

$$h_{\mu\nu}^{(TT)''} + 2\mathcal{H}\left(1 - \frac{F_{,\varphi}\varphi_0'}{F\mathcal{H}}\right)h_{\mu\nu}^{(TT)'} - \nabla^2 h_{\mu\nu}^{(TT)} = 0\,, \tag{4.110}$$

where $\mathcal{H}$ is the conformal Hubble parameter. The effect of the friction term due to the non-minimal scalar-tensor couplings has the expected structure and is manifest within the parenthesis of the previous expression.

C. The geometric time-delay

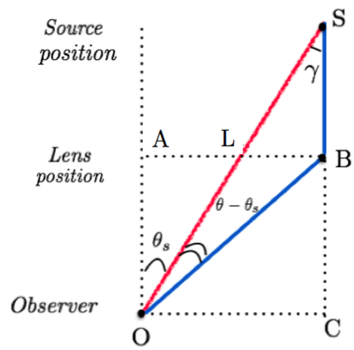

Figure C.3: *The configuration we consider.*

We derive the expression for the geometric time delay, i.e. the first term in Eq. (4.97). Since we are considering GWs propagating at the speed of light, these follow null geodesics whose affine parameter is the comoving distance, which we denote ℓ. For example, ℓ_{AL} is the length of the line that joins point A with point L in Figure C.3. Angular distances are defined as ratios between lengths and angles they subtend with respect to who observes them (which we write as the first letter, remember the definition of d^A using Figure 4.2, right panel). We will have

$$D_{OL} = \ell_{AL}/\theta_s, \qquad D_{SL} = \ell_{LB}/\gamma, \qquad D_{SO} = \ell_{OC}/\gamma. \tag{4.111}$$

We work in the limit of infinitesimal angles, so that

$$\ell_{OL}\sin\theta_s = \ell_{AL} \Rightarrow \ell_{OL} \sim D_{OL}, \tag{4.112}$$

$$\ell_{OB}\cos\theta = \ell_{OL}\cos\theta_s \Rightarrow \ell_{OB} \sim \ell_{OL} \sim D_{OL} \tag{4.113}$$

Considering GWs traveling at the speed of light, the geometrical time delay can be computed as

$$\Delta t = \ell_{SB} + \ell_{OB} - \ell_{SO}. \tag{4.114}$$

Since the triangles LSB and OSC are similar, we can write the equality

$$\frac{\ell_{LS}}{\ell_{OS}} = \frac{\ell_{LB}}{\ell_{OC}} = \frac{D_{SL}}{D_{SO}}, \tag{4.115}$$

so that

$$\ell_{OS} = \ell_{OL} + \ell_{LS} = \ell_{OL} + \ell_{OS}\,\frac{D_{SL}}{D_{SO}}\,. \tag{4.116}$$

implying

$$\ell_{OS} = D_{OL}\left(1 - \frac{D_{SL}}{D_{SO}}\right)^{-1} = \frac{D_{OL}D_{SO}}{D_{SO} - D_{SL}}\,. \tag{4.117}$$

Moreover, the law of cosines ensures that

$$\ell_{SB}^2 = \ell_{OB}^2 + \ell_{OS}^2 - 2\,\ell_{OB}\ell_{OS}\cos(\theta - \theta_s)\,. \tag{4.118}$$

Expanding the cosine for small angles, we can reassemble the previous formula as

$$\ell_{SB} \simeq (\ell_{OS} - \ell_{OB})\sqrt{1 + \frac{\ell_{OB}\ell_{OS}}{(\ell_{OB} - \ell_{OS})^2}|\theta - \theta_s|^2}\,, \tag{4.119}$$

$$\simeq (\ell_{OS} - \ell_{OB})\left(1 + \frac{\ell_{OB}\ell_{OS}}{2(\ell_{OB} - \ell_{OS})^2}|\theta - \theta_s|^2\right). \tag{4.120}$$

Then the time delay reads

$$\Delta t = \frac{\ell_{OB}\ell_{OS}}{2(\ell_{OS} - \ell_{OB})}|\theta - \theta_s|^2 = \frac{D_{OL}}{2}\,\frac{D_{OL}D_{SO}}{D_{SO} - D_{SL}}\,\frac{1}{\frac{D_{OL}D_{SO}}{D_{SO}-D_{SL}} - D_{OL}}\,|\theta - \theta_s|^2\,,$$

$$= \frac{D_{OL}D_{SO}}{2\,D_{SL}}\,|\theta - \theta_s|^2\,, \tag{4.121}$$

which is the formula used in Eq.(4.97) of the main text, with the GW angular diameter distance.

Note: My contribution to the paper this Chapter is based on regards all the scientific aspects, especially, but not only, the theoretical computations of the first part. I also had an active role in writing.