# Gravitational waves through the cosmic web

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# Introduction

Cosmology is a purely observational science: we cannot put the Universe in a laboratory and reproduce its history at our will, tweaking parameters to see how its constituents react. It always has to rely on the detection of a signal, whether it is a photon, or since 2016 [1], a gravitational wave (GW). Basing its hypothesis exclusively on observations, cosmology attempts at providing a single description of the Universe, from its early stages to the present, in terms of only few parameters. In this sense, the first GW direct detection marked a breakthrough for the sciences of the Universe: it opened up a new observational window affected by an entirely different set of systematics compared to electromagnetic signals. The observation of the Cosmic Microwave Background (CMB) by WMAP and Planck [2–4], confirmed that the photons comprising this signal are described by a black-body spectrum with a temperature of 2.73 K, up to one part in $10^5$. Since the Universe is expanding, the temperature of photons increases as we go backward in time so that, at the time of the CMB emission, it was about $10^3$ K. One can think of extrapolating this increasing behavior to even earlier times, before the moment when nuclei and free electrons combined to form neutral atoms and the photon-baryon fluid was kept in thermal equilibrium by Compton scattering, or even before that, until one reaches the typical energy scale of the Universe at the moment of the formation of the first nuclei: $\sim 10^{1}0$ K [5]. So, the ultimate goal of a cosmological model is to build a unique description of the Universe across (at least) roughly 10 orders of magnitude of different energy regimes. This heroic effort has successfully produced the standard cosmological model, $\Lambda$CDM: gravity follows the laws of General Relativity, a cosmological constant $\Lambda$ embodies the *dark energy* driving the recent accelerated expansion of the Universe, and *cold dark matter* (CDM), an unknown form of matter electromagnetically neutral, is responsible for structure formation. The last two components make up almost 96% of the current energy budget of the Universe, while the remaining portion is composed of standard model particles, which in cosmology are divided into two categories: radiation and baryons, where in the latter leptons are included as well.

The ΛCDM model's success, lies in its ability to fit a wide range of observational data, not only the CMB, but also type Ia Supernovae (SNe) datas [6, 7], and the mapping of cosmic structures, with the baryon acoustic oscillation (BAO) peak [8]. Yet, ΛCDM still fosters in its dark components large areas of theoretical uncertainties. Enlarging the volume of data, with ongoing and upcoming galaxy and weak-lensing surveys [9–12], will necessarily subject the standard model to a new level of scrutiny. The rise of precision cosmology, has already seen the emergence of tensions between datasets when interpreted within the ΛCDM framework [13, 14], which could signal the first cracks in the model as we achieve a new level of precision in measuring its parameters [15]. Since one of the assumptions of ΛCDM is General Relativity, testing this model also mean testing the theory of gravity on cosmological scales: after passing a battery of tests in the Solar System and on galactic scales, we can investigate gravity at work at low energies.

Thanks to the LIGO-Virgo-Kagra (LVK) [16, 17] collaboration, we can now add GW observations to the pool of different datasets, giving us direct access to the dynamical regime of gravity. The waves observed by these interferometers are produced during the inspiral of a compact object binary. When two black holes or two neutron stars, or a black hole and a neutron star, orbit around each other, they emit gravitational waves. The energy lost through this channel makes the orbit of the binary shrink until the two objects merge into one. In the coming years, the next generation of ground-based interferometers is expected to provide us with a "dark map" of the Universe by observing tens of thousands of GW events [18, 19]. Comparing this map with those compiled by galaxy surveys, will boost our understanding of the electromagnetically neutral species of the Universe. For example, we may be able to determine whether all astrophysical GW sources are located inside galaxies. If so, this would suggest that GWs and galaxies trace the underlying dark matter gravitational potential wells equivalently, so that these two probes can be used jointly, maximizing the scientific return from the two missions. Alternatively, if some astrophysical GW sources are found to be outside of galaxies, it would open up new avenues for scientific inquiry [20]. The observation of exotic compact objects through GWs could have significant implications for cosmology. A particularly intriguing possibility is the existence of *primordial black holes* [21] potentially accounting for part of the Universe's dark matter, which might have formed from the collapse of large, small-scale inflaton perturbations in the early Universe, or through other exotic channels (see [20] and references therein).

The concept that GW observations can enhance our understanding of the early Universe is not new. The detection of *primordial tensor modes*, through a B-mode polarization pattern of the CMB, would have significant implications for our comprehension of the primordial Universe [22–24]. According to the current model, during the pre-standard model era, at least two quantum fields existed in the Universe: the metric and the inflaton, driving a phase of exponentially accelerated expansions. Their fluctuations were first stretched beyond the Hubble horizon, and then slowly re-entered during standard cosmic history setting, for instance, the initial seeds for the development of gravitational potentials, eventually leading to the formation of cosmic structures. Since the inflationary paradigm treats scalar and tensor perturbations on the same footing, primordial gravitational waves are a key prediction of it. These types of GWs are substantially different from astrophysical events seen by LVK. Since their source is a quantum process, they would constitute a *stochastic gravitational wave background* (SGWB), similar in nature to the CMB's photons. From the first GW detection, the literature on SGWB has expanded considerably, as there are a number of mechanisms thought to take place in the early Universe that can produce a diffuse GW signal. The SGWB can also be of astrophysical origin: if the single GW events are not distinguishable one from the other, they are expected to form a background too. Because of the incredible wealth of information, detecting the SGWB is a key science goal for the future GW missions, target of space-based interferometers and pulsar-timing arrays [25–31].

The waveform of a GW and the frequency spectrum of the SGWB depend on the specifics of their sources. Regardless of the generation mechanism, propagation effects can have a significant impact. For instance, a wave traveling through an expanding Universe is damped faster than one traveling through a static one, or objects along the way can cause various distortions. As a result, GWs also convey information about the dynamics of the Universe, including the late-time cosmic expansion, so dark energy, and the cosmic structures tracing the gravitational potential wells of dark matter. Propagation effects introduce an irreducible error, setting an upper limit on the precision with which we can measure the source parameters. However, one can turn this around and look at propagation effects as a rich resource of cosmological information, both at the background and perturbed level.

Photons undergo similar effects during propagation, making it possible to use both probes jointly. Joint observations of GWs and photons are particularly promising for testing proposals for dark energy, where the two messengers behave differently. Such proposals are widespread in the literature and, thanks to the degeneracy breaking between the two sectors, they can be thoroughly investigated in the near future. Despite the similarities, GWs and photons can also exhibit very different behaviors, especially when comparing them at different frequencies. The typical energy of a photon, whether it belongs to the CMB or if it was emitted by an astrophysical event,

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is much higher than the energy scale associated to any (known) object it may encounter during propagation. In particular, an obstacle can be a large-scale structure, roughly at the energy scale of $10^{-17}$ Hz, or a compact object, ranging from supermassive to stellar black holes in the $\sim 10^{-3} - 10^{1}$ Hz band. Even the low-energy CMB photons have frequencies of approximately 160 GHz. This implies that photons are always well described by the *ray-optics* limit. By contrast, there are no limitations for the wavelengths of GWs and, depending on the situation, *wave-optics* effects may arise. The validity of these two regimes, and the kind of description they allow, will be profusely discussed in this Thesis. However, we can already appreciate the fact that GWs not only carry new valuable cosmological information, but also they offer new theoretical challenges.

The main focus of this Thesis is characterizing the propagation effects affecting GWs in the late time Universe, both in the ray-optics limit and in the wave-optics one. We use the former to describe resolved astrophysical GWs, with frequencies higher than the mHz, when these travel through the large-scale structures of the Universe. In this case, we will explore their potentialities in testing particular models for dark energy called *scalar-tensor theories*. Finally, we will give up the high frequency approximation and, in General Relativity, explore the wave-optics effects, paying particular attention to the polarization content of the gravitational waves. The purpose of this Introduction is to provide the reader with all the necessary tools to understand the topics covered in the following Chapters.

## 1.1. The standard cosmological model

Two very important aspects which any proposed cosmological model has to describe are the expansion of the homogeneous and isotropic Universe and the growth of linear perturbations. In the two sections below, we describe their phenomenology in ΛCDM.

### 1.1.1. Friedmann equations

Observations suggest that on scales $\gtrsim$ 100 Mpc, the *cosmological principle* holds, namely that the properties of the Universe are the same for all observers comoving with the expansion. Observations also suggest that the Universe, on large scales, does not display a preferred direction. These two facts together, fixe the metric describing the Universe at these scales to

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = a^2(\tau)\left[-d\tau^2 + \frac{d\chi^2}{(1-\kappa\chi^2)} + \chi^2\Big(d\theta^2 + \sin^2\theta\, d\varphi^2\Big)\right], \tag{1.1}$$

also known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [32], where $\tau$ is the conformal time, $a(\tau)$ is the scale factor, describing how the Universe expands over time, $\kappa$ is the intrinsic curvature of the 3D spatial hypersurfaces, and it can take the discrete values $\kappa = 0, +1, -1$, representing *flat*, *positively* or *negatively curved* spatial slices. In the metric above, $\chi$ is the *comoving distance*, related to the scale factor as in Eq. (1.8). The FLRW metric (1.1) describes a Universe as *spatially homogeneous* and *isotropic*. The cosmological principle also fixes the form of the possible energy content of the Universe to that of a perfect fluid described by a stress-energy tensor of the form [32]

$$T_{\mu\nu} = \rho\, u_\mu u_\nu + P \Lambda_{\mu\nu}\,, \tag{1.2}$$

here $\Lambda_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$, is the orthogonal projector to the worldlines of the observers whose 4-velocity is $u^\mu$, while $\rho$ is the energy density and $P$ is the isotropic pressure of the fluid element. Then, in the standard cosmological model, the FLRW metric is a solution of Einstein's equations

$$G_{\mu\nu} = 8\pi G\left[T_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right], \qquad T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G}\, g_{\mu\nu} \tag{1.3}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is given by Eq. (1.2). In the expressions above, we have introduced $\Lambda$, the cosmological constant driving the late time cosmic expansion in the $\Lambda$CDM model. To close the system, it is necessary to supplement an equation of state relating pressure and energy density of the perfect fluid, usually in the form $P = w\rho$. In the $\Lambda$CDM model, the matter species contributing to the stress-energy tensor are: pressureless non-relativistic matter (CDM and baryons), described by the equation of state $w_m = 0$ and radiation supported by its pressure and characterized by $w_r = 1/3$ [32]. Also, $T^{\Lambda}_{\mu\nu}$ can be written in a similar fashion of Eq. (1.2), with equation of state $w_\Lambda = -1$ and $\rho_\Lambda = \Lambda/(8\pi G)$. The independent conservation of their stress-energy tensors, gives the continuity equations

$$\Omega_i' + 3\mathcal{H}\,\Omega_i(1 + w_i) = 0\,, \qquad \text{with} \qquad {}' = \partial_\tau\,, \tag{1.4}$$

where we defined the time dependent density parameters

$$\Omega_i(\tau) \equiv \frac{\rho_i(\tau)}{\rho_{crit}}\,, \qquad \rho_{crit} = \frac{3H_0^2}{8\pi G} \tag{1.5}$$

with $i = r, m, \Lambda$. Note that $\Omega^0_\Lambda = [\Lambda/(8\pi G)]/\rho_{crit} = \Lambda/(3H_0^2)$. According to the value of the equation of state parameters, $w_i$, the continuity equation takes the different solutions

$$\Omega_r(a) = \Omega^0_r/a^4\,, \qquad \text{and} \qquad \Omega_m(a) = \Omega^0_m/a^3\,, \qquad \text{and} \qquad \Omega_\Lambda(a) = \Omega^0_\Lambda\,, \tag{1.6}$$

where $\Omega^0_i$ are the values of the density parameters today, i.e. at $a = a_0 = 1$, choosing the spatial curvature to be zero, $\kappa = 0$. The different power laws dictate that the various species "dilute" at different rates, in such a way that we can consider one

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| $\chi_{\mathrm{M}}$ | $\chi_\Lambda$ | $\chi_*$ | $\chi_{\mathrm{M}\Lambda}$ | $\chi_i$ |
|---|---|---|---|---|
| $\left[H_0(\Omega_m^0)^{1/2}\right]^{-1}$ | $\left[H_0(\Omega_\Lambda^0)^{1/2}\right]^{-1}$ | $\left[H_0(\Omega_m^0)^{1/3}(\Omega_\Lambda^0)^{1/6}\right]^{-1}$ | $\chi_*-\chi_\Lambda$ | $3\chi_*-\chi_\Lambda$ |
| 0.027 | 0.017 | 0.023 | 0.0054 | 0.052 |

Table 1.1: List of special comoving distance values entering in Eq. (1.9). The first row contains their definition, while the numerical values are computed considering $H_0 = 67.3$, $\Omega_m^0 = 0.31$ and $\Omega_\Lambda^0 = 0.69$ [4].

element at a time to drive the expansion of the Universe. In order, we will have: radiation first, being the most abundant in the early stages of the Universe, succeeded by a matter dominated phase and finally the cosmological constant in the very recent time. Since we are interested in describing the late time Universe, in our discussion we will always neglect the contribution of radiation to the energy budget. This results in another simplification: we will treat CDM and baryons in the same way.

From Eq. (1.3), one can find Friedmann's equation, relating the scale factor to the energy density of the constituents

$$\mathcal{H}^2 = a^2 H_0^2\left(\Omega_m+\Omega_\Lambda\right), \tag{1.7}$$

where $\mathcal{H}\equiv a'/a$ is the Hubble parameter in conformal time. We solve analytically Friedmann's equation separately in the *matter* and $\Lambda$ *dominated* epochs, and match the solutions at the moment of their equivalence, $a_{\mathrm{M}\Lambda} = \sqrt[3]{\Omega_m^0/\Omega_\Lambda^0}$. For later convenience, we write the solution in terms of the comoving distance

$$\chi \equiv \tau_0-\tau = \int_a^1 \frac{d\tilde a}{(\tilde a\mathcal{H}(\tilde a))}, \tag{1.8}$$

where $\tau_0$ is the value of conformal time today, i.e. such that $a(\tau_0)=1$. From this definition, it is clear that $d\chi=-d\tau$. The scale factor and the comoving distance belong respectively to the range $[0,1]$ and $[0,\chi_i]$, where $\chi_i$ is the value of $\chi$ corresponding to $a=0$. Such initial value does not have any particular physical meaning, and it is merely a consequence of having neglected the radiation contribution. We find the solutions

$$a(\chi)=\begin{cases}\dfrac{\left[\chi-\chi_i\right]^2}{4\chi_{\mathrm{M}}} & \chi\in[\chi_{\mathrm{M}\Lambda},\chi_i]\\[2ex] \dfrac{\chi_\Lambda}{\chi_\Lambda+\chi} & \chi\in[0,\chi_{\mathrm{M}\Lambda}]\end{cases}, \qquad \mathcal{H}(\chi) = -\frac{1}{a}\frac{\partial a}{\partial\chi} = \begin{cases}\dfrac{2}{\chi_i-\chi} & \chi\in[\chi_{\mathrm{M}\Lambda},\chi_i]\\[2ex] \dfrac{1}{\chi+\chi_\Lambda} & \chi\in[0,\chi_{\mathrm{M}\Lambda}]\end{cases} \tag{1.9}$$

where all the quantities defined in the equation above can be found in the Table 1.1. We point out that a solution of Eq. (1.7) interpolating between matter and $\Lambda$ domination eras exists [33], however it is quite intricate and impractical, therefore in this Thesis we will use the explicit expressions given above when needed.

In the ΛCDM model, dark energy (DE) is modeled with the cosmological constant. To better understand the character of the accelerated expansion it drives, it is best to rewrite the scale factor in terms of cosmic time $dt \equiv a d\tau$. In this case, and neglecting the matter contribution in Eq. (1.7), it is easy to find that

$$\frac{\partial a}{\partial t} = \sqrt{\frac{\Lambda}{3}}\, a, \qquad \rightarrow \qquad a(t) = e^{\sqrt{\frac{\Lambda}{3}}\, t}. \tag{1.10}$$

Because of this exponential behavior of the scale factor, the cosmological constant is said to drive the *accelerated expansion* of the Universe in the very recent epoch, a phenomenon detected via the observation of Supernovae type Ia (SNe) which was worth a Nobel Prize [6, 7] .

Another important quantity in background cosmology is the *cosmological redshift*, related to the scale factor as

$$1 + z = \frac{1}{a}, \tag{1.11}$$

determining also the red-shifting of the wavelengths of photons and GWs as they propagate through the expanding Universe.

### 1.1.2. Cosmological perturbation theory

Departures from the homogeneous and isotropic configuration characterize *large-scale structures* (LSS) of the Universe. On scales between $\sim 100 - 50$ Mpc, these deviations are still small in amplitude, and thus can be treated with linear relativistic perturbation theory.

Prior to the emission of the CMB, photons and electrons were tightly coupled through Compton scattering, forming a unique *photon-baryon fluid*. As a result, ordinary non-relativistic matter was supported by the radiation pressure of photons, preventing it from collapsing under gravitational interaction. In contrast, the electromagnetically neutral and non-relativistic (pressureless) CDM could form clumps, preparing the gravitational potential wells for the baryons to fall into, after the moment of recombination, when the Universe became transparent to photons. This process eventually led to the formation of stars, galaxies, and galaxy clusters inside the wells and filaments of the dark matter distribution. Therefore, it is often said that galaxies *trace* dark matter introducing the concept of *galaxy bias*, as we will see later. In the late-time Universe, relativistic species, such as photons, constitute a subdominant part of the total energy density budget as they dilute faster than the other components (see Eq. (1.6)). Since this is the period we are most interested in, we neglect them when studying LSS and treat baryons and CDM jointly as a non-relativistic pressureless component.

The equations of motion ruling the growth of LSS can be found by linearizing the

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gravitational field equations around the FLRW background,

$$g_{\mu\nu} = a^2(\tau)\,\eta_{\mu\nu} + \epsilon\delta g_{\mu\nu}\,, \tag{1.12}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric and $\epsilon$ is the expansion parameter tracking the LSS. We expect that CMB temperature anisotropies are of the same order of magnitude of the metric perturbation, so that $\epsilon \sim 10^{-5}$. The FLRW background is symmetric under rotations in the 3D spatial hypersurfaces. One can exploit this fact and break down the metric perturbation $\delta g_{\mu\nu}$ into irreducible representations of the Euclidean rotations. This is the so-called *scalar-vector-tensor* decomposition. Due to the symmetry of FLRW, each subgroup decouples from the others to linear order, allowing an independent analysis of each type of modes [5, 34–38][1]. Considering the FLRW background fixed, then under an infinitesimal gauge transformation $x^\mu \to x^\mu + \epsilon\xi^\mu$, the metric perturbation transforms as

$$\delta g'_{\mu\nu} = \delta g_{\mu\nu} - \left(\bar\nabla_\mu \xi_\nu + \bar\nabla_\nu \xi_\mu\right)\,, \tag{1.13}$$

where $\bar\nabla_\mu$ are the covariant derivatives built with the FLRW metric. The metric perturbation, $\delta g_{\mu\nu}$, being a 4 symmetric tensor, has 10 independent components. The gauge freedom fixes 4 of them, leaving 6 modes to be fixed with Einstein's equations. A common gauge choice is the so-called Poisson's gauge [34, 39], in which the line element takes the form

$$ds^2 = a^2(\tau)\Big[-(1+2\epsilon\Phi)d\tau^2 + 2\epsilon\omega_i d\tau dx^i + (1-2\epsilon\Psi)d\mathbf{x}^2 + 2\epsilon\gamma_{ij}\,dx^i dx^j\Big]\,, \tag{1.14}$$

where

1. $\Phi$ and $\Psi$ are two scalar gravitational potentials (2 modes),

2. $\omega_i$ is the vector potential, such that $\partial^i\omega_i = 0$ (2 modes),

3. $\gamma_{ij}$ is the transverse and traceless tensor mode, such that $\eta^{ij}\gamma_{ij} = \partial^i\gamma_{ij} = 0$ (2 modes),

with the convention that spatial indices are raised and lowered with the Minkowski metric, $\eta_{ij}$. In Poisson's gauge, the two gravitational potentials $\Phi$ and $\Psi$ correspond to the gauge invariant Bardeen's potentials [40] and they describe the perturbation of the time direction and of the spatial hypersurfaces' curvature. Note that the tensor mode $\gamma_{ij}$ *does not* coincide with our definition of GWs, which will be given later.

[1] There are many references for cosmological perturbation theory. This section is based on [34]. However, we do change notation to be consistent with the rest of this Thesis, e.g. the two scalar gravitational potentials have exchanged names $\Phi \to \Psi$ and $\Psi \to \Phi$, we call $\mathcal{H}$ the conformal Hubble parameter instead of $\eta$.

Similarly to the metric, also the matter stress-energy tensor is decomposed as $T^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + \epsilon\delta T^{\mu}_{\nu}$, or expliciting the components out, as

$$
\begin{aligned}
T^0_0 &= -\bar{\rho} - \epsilon\delta\rho && (1.15)\\
T^i_0 &= -\epsilon(\bar{\rho}+\bar{P})\, v^i && (1.16)\\
T^0_i &= +\epsilon(\bar{\rho}+\bar{P})\,[v_i+\omega_i] && (1.17)\\
T^i_j &= +\bar{P}\delta^i_j + \epsilon\left(\delta P\,\delta^i_j + \Sigma^i_j\right) && (1.18)
\end{aligned}
$$

where $\bar{\rho} = \bar{\rho}(\tau)$ and $\bar{P} = \bar{P}(\tau)$ are the background energy density and the isotropic pressure, only time dependent. The anisotropic stress, $\Sigma^i_j$, accounts for velocity gradients related to irreversible processes, and we assume it is traceless by reabsorbing any bulk contribution in the definition of the pressure. Also, in the expression above, $v^i$ is the peculiar velocity, whose definition is

$$
u^{\mu} = \frac{1}{a}\left[(1-\epsilon\Phi),\, \epsilon v^i\right]. \tag{1.19}
$$

An important quantity is the energy density contrast

$$
\delta \equiv \frac{\delta\rho}{\bar{\rho}}, \tag{1.20}
$$

sourcing the gravitational potential wells. The peculiar velocity and $\Sigma^i_j$ can be decomposed into irreducible representations with respect to the spatial rotation as well

$$
v_i = v^T_i + \partial_i v, \qquad \Sigma_{ij} = \mathcal{D}_{ij}\Sigma + \partial_{(i}\Sigma_{j)} + \Sigma^{TT}_{ij}, \tag{1.21}
$$

with $\partial^i v^T_i = 0$ and $\partial^i \Sigma^{TT}_{ij} = 0$ and

$$
\mathcal{D}_{ij}\Sigma \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)\Sigma, \qquad \partial_{(i}\Sigma_{j)} \equiv \frac{1}{2}\left(\partial_i\Sigma_j + \partial_j\Sigma_i\right), \tag{1.22}
$$

and we called $\Delta = \partial_k\partial^k$. In the decomposition of the velocity, $v$ is called *velocity gradient*, and it will play an important role later. All these expressions are then plugged in Einstein Eqs. (1.3), yielding the linearized equations

$$
\begin{aligned}
&[G^0_0]: && -k^2\Psi_k - 3\mathcal{H}(\Psi'_k + \mathcal{H}\Phi_k) = 4\pi G a^2\,\bar{\rho}\,\delta_k, && (1.23)\\
&[G^0_i]_{\parallel}: && -\left(\Psi'_k + \mathcal{H}\Phi_k\right) = 4\pi G a^2(\bar{\rho}+\bar{P})\, v_k, && (1.24)\\
&[G^0_i]_{\perp}: && -k^2\omega_{k,i} = 16\pi G a^2(\bar{\rho}+\bar{P})\left[v^T_{k,i} + \omega_{k,i}\right], && (1.25)\\
&\left[G^i_i\right]: && \Psi''_k + \mathcal{H}\left(\Phi'_k + 2\Psi'_k\right) + (2\mathcal{H}' + \mathcal{H}^2)\Phi_k - \frac{k^2}{3}(\Phi_k - \Psi_k) = 4\pi G a^2\delta P_k, && (1.26)\\
&\left[G^i_{j\neq i}\right]_{\parallel}: && (\Psi_k - \Phi_k) = 8\pi G a^2\,\Sigma_k, && (1.27)\\
&\left[G^i_j\right]_{\perp}: && -(\partial_\tau + \mathcal{H})\, k_{(i}\,\omega_{k,j)} = 8\pi G a^2 k_{(i}\Sigma_{k,j)}, && (1.28)\\
&\left[G^i_j\right]_{TT}: && (\partial^2_\tau + 2\mathcal{H}\partial_\tau + k^2)\gamma_{ij} = 8\pi G a^2\Sigma^{TT}_{k,ij}, && (1.29)
\end{aligned}
$$

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in Fourier space (see Eq. (1.115) for notation). Note that $k^i$ is a 3D spatial vector and $k^2 \equiv \eta^{ij} k_i k_j$. The isotropy of the FLRW background guarantees that each perturbed quantity is exclusively a function of $(\tau, k)$ and, in particular, does not depend on the direction of the wave-vector. This point will become crucial in Chapter 6, and will come back into the discussion. As usual, using Eq. (1.24) into (1.23), one finds the Poisson's equation

$$\Psi_k = -\frac{3H_0^2\Omega_m^0}{2ak^2}\,\delta_m^C(a,k)\,, \qquad \delta_m^C(a,k) \equiv \delta_k - 3\mathcal{H}(\bar\rho + \bar P)\,v_k \tag{1.30}$$

where we have defined the gauge-invariant density contrast, $\delta_m^C(a,k)$. Eq. (1.30) clearly shows that $\Psi_k$ is affected by the instantaneous variations of its source (contrary to the retarded time for solutions of wave-like equations). Crucially, some equations among (1.23) - (1.29) are of first or zero order in time derivative. These are called *constraint equations*, and their role is to enforce particular relations between the field variables, such as the one between the scalar gravitational potential and the density contrast in Eq. (1.30).

In the ΛCDM model and in the late time Universe, we neglect the contribution to the stress-energy tensor given by radiation and focus only on baryons, cold dark matter and the cosmological constant. The former two are given in terms of a collisionless, non-relativistic gas of particles, with null adiabatic sound speed so that $\delta P = 0$ and negligible anisotropic stress, $\Sigma_{ij} = 0$. Not clustering, the cosmological constant only affects the background dynamics and does not contribute to the perturbation of stress-energy tensor. In this case, Eq. (1.27) becomes sourceless as the right-hand side vanishes in absence of anisotropic stress, so that

$$\Psi_k(\tau) = \Phi_k(\tau)\,. \tag{1.31}$$

As far as vector modes go, we note that also Eq. (1.25) is a Poisson equation, like (1.30), and that Eq. (1.28) shows that vector modes are redshifted away by the Hubble expansion unless supported by the anisotropic stress [37]. Hence, relying on the assumption $\Sigma_{ij} = 0$, one can set $\omega_i = v_i^T = 0$. The matter equations (continuity and Euler equations) can be found either via the conservation of the stress-energy tensor $\delta_\epsilon\left[\nabla_\mu T^\mu{}_\nu = 0\right]$, or via suitable combinations of the linearized Einstein equations (and their time derivatives) and using also the background Friedmann's Eq. (1.1.1). Overall, in a ΛCDM late time Universe, the geometry of the spacetime and of the energy-momentum density, at first order in $\epsilon$, is governed by the set of equations

$$\delta_k' - k^2 v_k = 3\Phi_k'\,, \tag{1.32}$$

$$v_k' + \mathcal{H} v_k = -\Phi_k\,, \tag{1.33}$$

$$\Phi_k' + \mathcal{H}\Phi_k = -\frac{3H_0^2\,\Omega_m^0}{2a}\,v_k\,, \tag{1.34}$$

$$\Phi_k'' + 3\mathcal{H}\Phi_k' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi_k = 0\,, \tag{1.35}$$

together with Eqs. (1.30), (1.31) and (1.29). An important consequence of these equations comes from evaluating the last one during matter domination. In this case, we neglect $\Omega_\Lambda$ in Friedmann Eq. (1.7), so that

$$\mathcal{H}^2 + 2\mathcal{H}' = \mathcal{H}^2 + H_0^2 \left(2a^2\mathcal{H}\Omega_m + a^2\Omega'_m\right) = 3\mathcal{H}^2 - 3\mathcal{H}^2 = 0\,, \tag{1.36}$$

using also Eq. (1.6). Because of this, Eq. (1.35) becomes $\Phi''_k + 3\mathcal{H}\Phi'_k = 0$, which admits as a solution

$$\Phi_k(\tau) \sim \frac{\mathcal{C}_k}{a^2} + \Phi^m_k\,, \tag{1.37}$$

meaning that the gravitational potential is supported by one decaying and one constant solution in matter domination. Neglecting the decaying branch, this leads to the important results that $\Phi_k(\tau) = \Phi^m_k$ , i.e. the gravitational potential wells are constant and do not grow during matter domination. Since we take $\delta^C_m$, $v$ and $\Phi$ to be solutions of Einstein's equation, we can relate them in a more compact form. To this end, we define the *matter transfer function*, $T_m(k)$, describing the behavior of the matter density contrast through the equality between radiation and matter dominated epochs, and the *matter growth factor*, $D_m(a)$, accounting for its late time evolution [5]. We write the field perturbations in terms of these two as

$$\begin{aligned}
\delta^C_m(a,k) &= -\frac{9}{10}T_m(k)\mathcal{G}_m(a,k)\Psi^{in}_k\,, && (1.38)\\
v(a,k) &= -\frac{9}{10}T_m(k)\,\mathcal{G}_v(a,k)\Psi^{in}_k\,, && (1.39)\\
\Phi(a,k) &= \frac{9}{10}T_m(k)\frac{\mathcal{G}_\Phi(a,k)}{a}\Psi^{in}_k\,. && (1.40)
\end{aligned}$$

where $\Psi^{in}_k$ is the primordial value of the gravitational potential and

$$\mathcal{G}_\phi = D_m\,,\qquad \mathcal{G}_m = \frac{2k^2}{3H_0^2\Omega^0_m}D_m\,,\qquad \mathcal{G}_v = f(a)\,\frac{\mathcal{H}}{k^2}\,\mathcal{G}_m\,. \tag{1.41}$$

In the expression above $f \equiv \mathrm{d}\ln D_m/\mathrm{d}\ln a$ is the growth rate. Ideally, one uses these forms of the density contrast, velocity gradient and gravitational potential into the linearized Einstein's equations, to find the evolution equation for $T_m(k)$ and $D_m(a)$. These are normally integrated numerically with Einstein-Boltzmann solvers codes, such as `CAMB`, `CLASS` (see [41, 42]). This in particular means that the form of $\{\mathcal{G}_\phi, \mathcal{G}_m, \mathcal{G}_v\}$ depends on the gravitational theory. Alternatively, one can use suitable fitting formulas for them [5, 43–45]. Since we are interested in the late time Universe, well after the end of the radiation epoch, the transfer function is nearly constant [5, 43, 44], while the growth factor for modes inside the horizon ($k \gg \mathcal{H}$) can be approximated as [5, 35, 46]

$$D_m(a) = \frac{5H_0^2\Omega^0_m}{2}\,\frac{\mathcal{H}(a)}{a}\int_0^a \frac{da}{[\mathcal{H}(a)]^3}\,. \tag{1.42}$$

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Note that in our notations we use $\mathcal{H} = (a'/a)(\tau)$ as the Hubble parameter in conformal time, while in [5] they use $H = (\dot{a}/a)(\tau)$, and the dot stands for derivative with respect to the cosmic time $dt = ad\tau$. A more handy fitting formula given in [35, 46] is

$$D_m(a) = \frac{5\Omega_m^0}{2a^2E^2(a)}\left[\left(\frac{\Omega_m^0}{a^3E^2(a)}\right)^{4/7} - \frac{\Omega_\Lambda^0}{E^2(a)} + \left(1+\frac{\Omega_m^0}{2a^3E^2(a)}\right)\left(1+\frac{\Omega_\Lambda^0}{70E^2(a)}\right)\right], \quad (1.43)$$

with $E^2(a) = \mathcal{H}^2/(a^2H_0^2) = (\Omega_m^0/a^3+\Omega_\Lambda^0)$. A quantity that will be used later in the text, is the *gravitational potential transfer function*, which is given by

$$\mathcal{T}_k^\Phi(a) = \frac{9}{10}T_m(k)\frac{\mathcal{G}_\Phi(a,k)}{a}, \quad (1.44)$$

as it can be understood from Eq. (1.40).

### 1.1.3. Statistical description of the large-scale structures

Describing the cosmic web is intrinsically a statistical task: we are interested in describing its average properties, rather than the exact shape and position of each gravitational potential's well or tensor perturbation.[2] Accordingly, we will treat $\delta g_{\mu\nu}(x)$ and $\delta T^\mu_\nu(x)$ as random fields with zero mean, and their observed configurations are a specific realization of the stochastic process, i.e. a particular member of the statistical ensemble. These random fields inhabit the cosmological Universe which, on large-scales, is homogeneous and isotropic, suggesting the idea of promoting these properties to a statistical level for the description of the cosmic inhomogeneities. Hence, given any cosmological perturbation, $\delta A(\tau,\mathbf{x})$, we will assume that it is *statistically homogeneous*, so that its mean and variance are independent of position, and *statistically isotropic*, implying that there is no preferred direction. Assuming also that the fields at large distances are uncorrelated [48], we get that these random fields are *Ergodic*, in the sense that we can exchange

$$\text{ensemble average} \quad \longleftrightarrow \quad \text{volume average}$$

since, in sufficiently far away portions of the Universe, $\delta A(\tau,\mathbf{x})$ and $\delta A(\tau',\mathbf{y})$ should be causally disconnected, making them two independent representative realizations of the stochastic process. In this sense, ensemble expectation values of field dependent observables are well approximated by volume averages, provided that the volume is large enough. A quantity which plays a major role in the description of LSS, is the *two-point autocorrelation function* of the random field

$$\xi^A(\tau,\mathbf{x};\tau',\mathbf{x}') \equiv \langle\, \delta A(\tau,\mathbf{x})\,\delta A(\tau',\mathbf{x}')\,\rangle, \quad (1.45)$$

[2]This section takes inspiration form [47].

where the average $\langle \ldots \rangle$ is the ensemble average, or the volume one. For a statistically homogeneous and isotropic field, the two-point function must be translation and rotation invariant, therefore

$$\xi^A(\tau,\mathbf{x};\tau',\mathbf{x}') = \xi^A(\tau,\tau',|\mathbf{x}-\mathbf{x}'|)\,. \tag{1.46}$$

Another important quantity is the *power spectrum*, namely the autocorrelation function in Fourier space

$$P^A(\tau,\mathbf{k};\tau',\mathbf{k}') \equiv \left\langle\, \delta A(\tau,\mathbf{k})\, \delta A(\tau',\mathbf{k}')\,\right\rangle = \int d^3x\, d^3x'\, \xi_A(\tau,\mathbf{x};\tau',\mathbf{x}')\, e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}\cdot\mathbf{x}'}\,. \tag{1.47}$$

When the real-space 2-point function depends only on $|\mathbf{x}-\mathbf{x}'|$, also the power spectrum takes a simplified form

$$\begin{aligned} P^A(\tau,\mathbf{k};\tau',\mathbf{k}') &= \int d^3x\, d^3x'\, \xi_A(\tau,\tau',|\mathbf{x}-\mathbf{x}'|)\, e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}\cdot\mathbf{x}'} = \\ &= (2\pi)^3\, \delta^3(\mathbf{k}+\mathbf{k}') \int_0^{+\infty} dr\, 4\pi\, \frac{\sin(kr)}{kr}\, r^2\, \xi_A(\tau,\tau',r) \end{aligned} \tag{1.48}$$

from which we read that, in the case of statistically homogeneous and isotropic random fields the power spectrum is such that

$$P^A(\tau,\mathbf{k};\tau',\mathbf{k}') = (2\pi)^3\, \delta^3(\mathbf{k}+\mathbf{k}')\, P^A(k,\tau,\tau')\,. \tag{1.49}$$

In particular, the assumption of homogeneity results into a diagonal power spectrum, so that the different Fourier modes are independent, while isotropy implies that it depends only on the modulus $k = |\mathbf{k}|$, of the wave-vector.

But what is the physical meaning of these quantities? This is best understood looking at the case of *Gaussian Random Fields* : fields whose probability density functional, dictating the stochastic properties, is given by

$$\mathscr{P}\left[\delta A(\tau,\mathbf{k})\right] = \prod_{\mathbf{k}} \frac{1}{\sqrt{2\pi}P(k,\tau)}\, \exp\left[-\frac{|\delta A(\tau,\mathbf{k})|^2}{2P(k,\tau)}\right]. \tag{1.50}$$

Therefore, for a Gaussian random field: each Fourier component is statistically independent of the others and follows a Gaussian distribution with variance given by the power spectrum (at equal time). $\mathscr{P}\left[\delta A(\tau,\mathbf{k})\right]$ can then be understood as the joint probability of having a specific realization for $\delta A(\tau,\mathbf{k})$ at each $\mathbf{k}$. Since the Fourier mode $\delta A(\tau,\mathbf{k})$ is complex, it is easy to understand that Eq. (1.50) actually implies that the *real amplitudes* of each mode are gaussian distributed, while the *phases* are drawn from a uniform distribution. Finally, in the case of Gaussian Random fields, all odd-number correlation functions vanish.

If the random field $\delta A(\tau,\mathbf{x})$ is defined on the sphere, it is convenient to work in harmonic space, rather than Fourier. This is the typical case, for instance, where we observe an incoming GW or photon: we are interested in the direction of arrival, while

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its comoving distance and time are related by the geodesic equation. In these cases, it is more convenient to decompose the field on the spherical harmonics basis, rather than the Fourier one, as

$$\delta A(\tau,\chi,\hat{n}) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} a^A_{\ell m}(\tau,\chi)\, Y_{\ell m}(\hat{n}), \tag{1.51}$$

where $\chi$ is the comoving distance and $\hat{n} = (\theta,\varphi)$ the coordinates on the unit 2-sphere, such that $\mathbf{x} = \chi\hat{n}$. By using the properties of the spherical harmonics (see Appendix A), we can relate $a^A_{\ell m}$ with the Fourier component of the field as

$$a^A_{\ell m}(\tau,\chi) = 4\pi i^\ell \int \frac{d^3k}{(2\pi)^3}\, \delta A(\tau,\mathbf{k})\; j_\ell(k\chi)\, Y^*_{\ell m}(\hat{n}), \tag{1.52}$$

$$\left[a^A_{\ell m}(\tau,\chi)\right]^* = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3}\, [\delta A(\tau,\mathbf{k})]^*\; j_\ell(k\chi)\, Y_{\ell m}(\hat{n}) \tag{1.53}$$

where $k = |\mathbf{k}|$ and $j_\ell(k\chi)$ is the spherical Bessel function. With these expressions, we can define the *angular power spectrum*

$$C^A_{\ell m;\ell' m'}(\tau,\chi;\tau'\chi') = \left\langle a^A_{\ell m}(\tau,\chi)\, a^A_{\ell' m'}(\tau',\chi')\right\rangle, \tag{1.54}$$

and relate it to the power spectrum. In the case of statistically homogeneous and isotropic fields, it is easy to compute that

$$C^A_{\ell m;\ell' m'}(\tau,\chi;\tau'\chi') = 4\pi\,\delta_{\ell\ell'}\delta_{mm'}\int_0^{+\infty}\frac{dk}{k}\, j_\ell(k\chi)\, j_\ell(k\chi')\left[\frac{k^3P(k,\tau,\tau')}{2\pi^2}\right], \tag{1.55}$$

from which we understand that the angular power spectrum, in this case, is given by

$$C^A_{\ell m;\ell' m'}(\tau,\chi;\tau'\chi') = \delta_{\ell\ell'}\delta_{mm'}\, C^A_\ell(\tau,\chi;\tau'\chi'). \tag{1.56}$$

With these definitions, we can understand the advantage of having introduced the transfer functions in Eqs. (1.38) - (1.40): they allow describing three random fields at the price of one. Let's take the example of the gauge invariant density contrast, $\delta^C_m(a,k)$. We promote it to a Gaussian random field, and its two point function in Fourier space will be the *matter power spectrum*, $P_m(k)$. This quantity is the target of multiple observational campaigns, ranging from those targeting redshift-space-distortions and galaxy clustering or weak lensing surveys. Using Eq. (1.38), we can write it as

$$P_m(a,k) = \left[\frac{9}{10}T_m(k)\mathcal{G}_m(a,k)\right]^2 P^\Psi_{in}(k), \tag{1.57}$$

because the transfer functions and growth factor incorporate only deterministic processes, and similarly for $v(a,k)$ and $\Phi(a,k)$. This means that we can describe the statistical properties of the three fields characterizing the LSS all in terms of one single power spectrum, $P^\Psi_{in}(k)$. In the expression above, $P^\Psi_{in}(k)$ is the *primordial scalar*

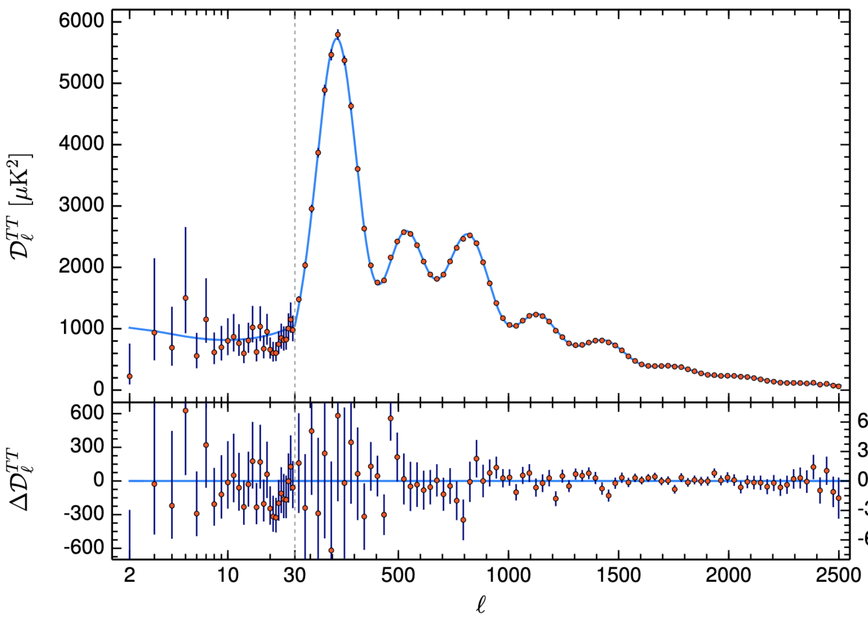


Figure 1.1: The angular power spectrum of the CMB temperature anisotropies, displaying the acoustic peaks of the photon-baryon fluid, from [4]. On the y-axis $\mathcal{D}_\ell = \ell(\ell+1)C_\ell$. The blue line is the prediction of ΛCDM with the best fit values of its parameters, while the red dots corresponds to data. The lower panel shows the residuals with respect to this model. For small $\ell$, the plateau corresponds to the integrated Sachs Wolfe effect (ISW), an effect that depends on the integrated time dependence of the gravitational potentials (see e.g. Eq. (1.114)). Since these are constant during matter domination, the ISW switches on during dark energy domination. In this region, though, the error is dominated by the irreducible cosmic variance.

*power spectrum*: it contains information about the initial conditions of the linear scalar perturbations. In the standard theory, these are set by the *inflaton*, a primordial spin-0 field which guides the Universe through an exponentially accelerated phase of expansion before the particles of the standard model were produced [5, 47]. The simplest inflationary theory predicts an almost scale-invariant primordial power spectrum

$$P_{in}^{\Psi}(k) = A_s \left[\frac{k}{k_*}\right]^{n_s-1}, \tag{1.58}$$

with $k_*$ a pivot scale, $A_s$ the amplitude of the scalar power spectrum and $n_s = 0.9649 \pm 0.0042$ the spectral index, both measured through observations of the CMB by the Planck satellite at the % precision [3]. So, from the matter power spectrum, by knowing the gravitational theory, we can extract information about the initial stages of the Universe in what would seem a remarkably straightforward way, at least on linear scales. This seemingly simple task is then complicated by the fact that we do not directly observe all the matter, as the majority of this is in the form of dark mat-

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ter, which does not emit electromagnetic radiation. This means that we do not have direct access to the full density contrast $\delta_m^C(a,k)$, but only to galaxies which should be located in the peaks of the CDM distribution. To account for this, cosmologist introduce the time-dependent *linear galaxy bias* as

$$\delta_g(z,k) = b(z)\,\delta_m^C(z,k)\,, \tag{1.59}$$

where $z$ is the redshift and $\delta_g$ is the galaxy overdensity field, to which we have direct access from a galaxy survey. Therefore, we can measure its power spectrum, and relate it to the one of matter as $P_g(z,k) = [b(z)]^2 P_m(z,k)$ [5, 47]. To further complicate the matter, one has to keep into account the so-called *Redshift Space distortions* (RSD): the peculiar velocity of the galaxy falling into the gravitational potential well adds a contribution to the measured redshift [49]. This contribution introduces a dependence on the cosine of the angle between the line-of-sight and the peculiar velocity, $\mu$, such that

$$P_g(z,k) = \left[\, b(z) + b_v f(z)\mu_k^2 \,\right]^2 P_m(z,k)\,, \tag{1.60}$$

where $b_v$ is the bias between the galaxy and matter velocity distributions and $f$ the growth rate already introduced.

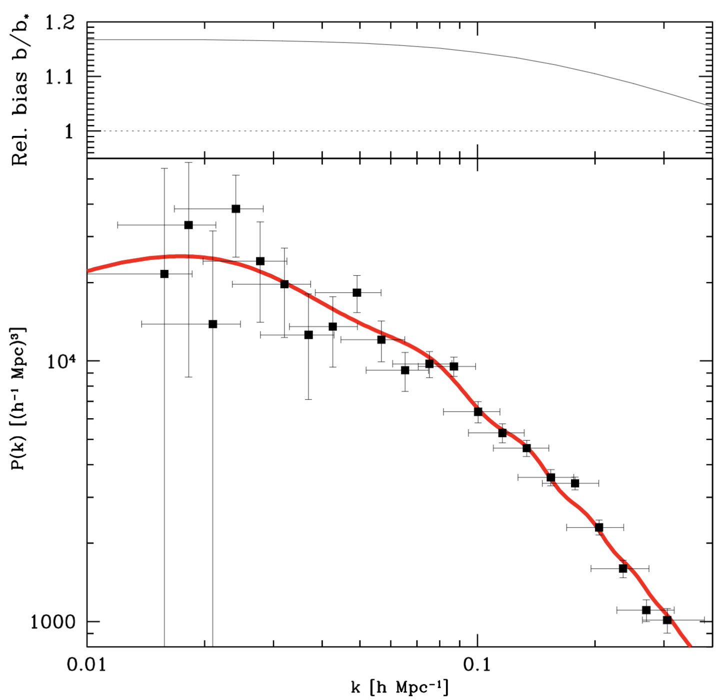


Figure 1.2: The galaxy power spectrum from [50]. The solid curve is the best fit linear ΛCDM model.

All of these considerations can be generalized to the case of more random fields. For instance, with two of them we can consider their correlation matrix

$$\xi^{AB}(\tau, \mathbf{x}; \tau', \mathbf{x}') \equiv \left\langle \delta A(\tau, \mathbf{x}) \delta B(\tau', \mathbf{x}') \right\rangle, \tag{1.61}$$

which allows defining the cross power spectrum $P^{AB}(k)$ through its 3D Fourier transform, or the angular cross correlations $C^{AB}(\ell)$ through its decomposition on the spherical harmonics basis. Once again, we can appreciate the great value of Eqs. (1.38) - (1.40): they clearly show that the different scalar tracers $(\delta_m^C, v, \Phi)$ actually have the same statistical properties, and hence it is worth considering their correlations. This possibility is very powerful as the measurements of the different fields are, in general, independent, therefore less subject to systematic errors. Additionally, since the growth factors $\{\mathcal{G}_m, \mathcal{G}_v, \mathcal{G}_\Phi\}$ in Eq. (1.41) depend differently on the cosmological model parameters, the mixed estimators $\xi^{AB}$, can feature a greater constraining power on the cosmological parameters, due to the breaking of degeneracies. As we will show in Section 1.4.3, also GWs and SNe can be used to trace the underlying dark matter distribution, in a way that depends on the background cosmology too. We will use GWs to build estimators to constrain the parameters of scalar-tensor theories in Chapter 2 in combination with SNe, and in Chapter 3 in combination with galaxy and weak lensing surveys.

## 1.2. Alternative models for the late time Universe

General Relativity is the unique theory for an interacting, massless, spin-2 field in 4 dimensions [51]. It is based on the assumptions of diffeomorphism invariance as symmetry, the metric being the only field entering the gravitational action and that the latter must lead to equations of motion at most of second order. Any alternative to General Relativity, then, will in general introduce new degrees of freedom by abandoning any of these assumptions. This is true even if no new fields are explicitly added to the gravitational action. For example, having higher order equations of motion leads to more propagating degrees of freedom, requiring additional initial conditions. In such cases, one constraint equation of General Relativity could be promoted to a dynamical one [52]. Dropping diffeomorphism invariance also introduces new degrees of freedom as symmetries can be restored by adding new fields with suitable transformation laws under the broken generators, the so-called *"Stuckelberg fields"* [53]. In the simplest scenario, compatibly also with the broken time-translation of the expanding homogeneous and isotropic Universe, is to add one additional degree of freedom. This field can, eventually, drive the late time cosmic acceleration on large scales [52, 54]. For this reason, we address it generically as the "DE field". In order to satisfy some minimal stability requirements, *scalar-tensor theories* must lead to second order equations of motion for the propagating degrees of freedom, one massless tensor and one scalar. Scalar-tensor theories have a long history, and many famous gravitational theories belong to this class. Among them, for instance, we account Quintessence theories [55], Brans-Dicke gravity [56], Covariant Galileon cosmologies [57] and K-essence theories [58]. Covariant Galileons attracted much attention in cosmology because of the existence of a tracking solution which evolves into a de Sitter fixed point [59]. All of these theories belong to the so-called *Horndeski family* [60–62]. Initially considered the most general action of a scalar-tensor theory leading to a stable dynamics, this class of theories was subsequently enlarged in the beyond-Horndeski theories [63, 64], and then DHOST [65–70]. General Relativity, plus $\Lambda$, corresponds to the trivial case in which the DE field is constant, hence $\Lambda$CDM is always contained in the class of scalar-tensor gravities.

To be able to source the cosmic exponentially accelerated expansion, the DE field must have a mass of the order of magnitude of the Hubble scale, i.e. $\sim 10^{-33}$ eV. As a result, its Compton wavelength is large, and the field mediates a long range interaction. Because of this extra force, the laws of gravity appear modified above this scale [71–73]. Typically, this shows into a modification of the Poisson Eq. (1.30). Cosmological observables can be altered even below the scale of the Compton wavelength, since the DE field can still influence the dynamics of the homogeneous and isotropic Universe [74]. Another characteristic features of all scalar-tensor theories regards their small-scale behavior. If the DE field exists, it must hide itself in high-density region environments, such as the Solar System, where General Relativity has passed all tests performed so far. Thus, viable scalar-tensor theories must be equipped with a dynamical mechanism that make manifest the presence of the DE field only on large-scales (low density), such as the linear scales of cosmological perturbations, and suppresses otherwise. This *screening* can be achieved through non-linear interactions, which become prominent at small scales, with the result of effectively decoupling the DE field from matter (see [51] for a review).

### 1.2.1. The Horndeski subclass with luminal gravitational waves

The power of scalar-tensor theories, lies in the fact that they are a class. Practically, this means that one can treat many different alternatives at the same time, simply by leaving their coupling functions unspecified, rather than going on a one-to-one basis. Because of this, they are suited to build tests: instead of computing the phenomenology of each specific theory, we will have a continuous parametrized class of them, which we can constrain after gathering the necessary data. For instance, we have already commented that surveys mapping the galaxies' distribution have access to the growth of the linear gravitational potentials, through $P_g(a,k) = [b(a)]^2 P_m(a,k)$, and that how matter inhomogeneities grow depends on the theory of gravity, as it is shown in Eq. (1.57). In a dynamical dark energy scenario describing the late-time cosmic expansion, the initial condition set by $P^{\Psi}_{in}$ will be the same, but the transfer function and the growth factor will be changed, $T_m \to T_m^{DE}$ and

$\mathcal{G}_m \to \mathcal{G}_m^{DE}$, in a way that depends on the free coupling functions of the scalar-tensor theory. Another example comes from the tensor sector, as the amplitude, the propagation speed and the polarization content of the GWs can, in principle, be modified in a way which depends on the free scalar-tensor couplings.

For simplicity, in this Thesis we restrict our study to Horndeski models in which tensor modes propagate luminally [75–78] at all redshifts, satisfying the bound from GW170817 [79]. Therefore, we consider the action

$$\mathcal{S} = \mathcal{S}_G[g_{\mu\nu}, \varphi] + \mathcal{S}_M[g_{\mu\nu}, \chi_i], \tag{1.62}$$

with the gravitational part given by

$$\mathcal{S}_G = \int d^4x\sqrt{-g}\left(\frac{M_P^2[\varphi]}{2}R + G[\varphi, X]\Box\varphi + K[\varphi, X]\right), \tag{1.63}$$

where $\varphi$ is the DE field, $X \equiv -\partial^\mu\varphi\,\partial_\mu\varphi/2$ its kinetic term, and $M_P$, $G$, $K$ are free functions encoding possible self-couplings of DE field, and interactions with the spacetime metric. We assume that the matter action, $\mathcal{S}_M[g_{\mu\nu}, \chi_i]$, is universally coupled with the *Jordan frame* metric $g_{\mu\nu}$. This means that photons are not directly coupled to the DE field, and they interact with it only indirectly through gravitational interaction. This observation is crucial as it is the starting point for all the *multi - messenger* tests: the difference in the couplings between gravitational and electromagnetic waves to the DE field produced different phenomenologies in these two sectors, even when these two have similar frequencies. Because of the very different typical wavelength, though, it could be that photons and GWs are in different optical regimes and in this case DE effects can be degenerate with wave optics ones [80]. Note that sometimes it is convenient to perform a conformal transformation $g_{\mu\nu} = \Omega^2(\varphi)\tilde{g}_{\mu\nu}$, to remove the non-minimal coupling between the metric and the DE field. In this frame, also known as *Einstein frame*, one fixes $\Omega^2(\varphi)$ such that $M_P^2[\varphi]R[g_{\mu\nu}] \to m_0^2 R[\tilde{g}_{\mu\nu}]$ to remove the minimal coupling, and where $m_0^2 = (8\pi G)^{-1}$ is the Planck's mass. As a result $\mathcal{S}_M[g_{\mu\nu}, \chi_i] \to \mathcal{S}_M[\Omega^2(\varphi)\tilde{g}_{\mu\nu}, \chi_i]$, and the matter action becomes explicitly dependent on the DE field. This kind of description is useful in the context of screening scenarios (see e.g. [51]) and we use it in Chapter 5.

Starting from (1.62), one can derive the gravitational and scalar field equations by varying it with respect to $g_{\mu\nu}$ and $\varphi$. Effectively, the former can be recast in the form (1.3) with a particular stress-energy tensor,

$$G_{\mu\nu} = 8\pi G\left[\, T_{\mu\nu}^{(\mathrm{DE})}\left[g_{\alpha\beta}, \varphi\right] + T_{\mu\nu}\left[g_{\alpha\beta}, \chi_i\right]\right], \tag{1.64}$$

where the explicit form of $T_{\mu\nu}^{(\mathrm{DE})}$ depends on the Horndeski functions: $M_P, K, G$.

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### 1.2.2. Cosmology in alternative scenarios

Starting from Eq. (1.64), one can work out how cosmology is changed in the chosen extended theory. The presence of the DE field does not affect the cosmological principle, or the fact that the Universe is expanding, nor that matter is in the form of clumped structures on large scales. The effect of its presence is to, possibly, alter the dynamics of the scale factor and of the growth of LSS. Therefore, the approach to tackle the problem is the same as before: we choose the metric as in Eq. (1.12), and solve Eq. (1.64) perturbatively, instead of Eq. (1.3). The result is a set of equations similar to Eq. (1.7) and Eqs. (1.23) - (1.29), but with additional terms originating from $T^{(\mathrm{DE})}_{\mu\nu}$. The system of equations is also supplemented by the DE field equation of motion, both at the level of the homogeneous and isotropic configuration and its linear perturbations. Analogously to the metric and the matter stress-energy tensor, the DE field is decomposed as

$$\varphi(x) = \varphi_0(\tau) + \epsilon\delta\varphi(x)\,, \tag{1.65}$$

where we have chosen as background $\bar{\phi} = \varphi_0(\tau)$, namely a field configuration compatible with the symmetries of FLRW. Although conceptually simple, the exercise of working out cosmology in these extended theories is rather long and tedious, unless one chooses some specific form for the Horndeski functions $M_P, K, G$. Instead of choosing a specific model, one can opt for more agnostic explorations [81]. A powerful approach proposed in literature is the so-called Effective Field Theory of Dark Energy (EFT) [82–87]: a unifying framework able to give predictions about the expansion of the Universe and the growth of LSS.

At the center of the EFT approach is the idea that the observed time evolving profile of the expanding Universe is the result of a spontaneous symmetry breaking of time-translation. The Goldstone boson of the symmetry breaking, $\pi(x)$, is identified with the DE field via $\pi(x) = \delta\varphi(x)/\varphi_0'$, since the background expansion assures $\varphi_0' \neq 0$. Using then techniques of the effective approaches in quantum field theory, one can then write the most general action for the linear perturbations around the symmetry-breaking background. The EFT, then, starts by considering the gauge transformation of the perturbation of the DE field

$$\epsilon\delta\varphi' = \epsilon\delta\varphi - \xi^0\varphi_0'\,, \tag{1.66}$$

since $\varphi_0$ is exclusively time dependent, and choosing the *unitary gauge* : $\delta\varphi' = 0$. This way, the slices of constant time are identified with the hypersurfaces of uniform scalar field. As a result, one is left exclusively with the metric perturbation to construct the operators entering the second order action dictating the LSS linear dynamics, and they can be organized in power of derivatives. Since LSS are large-scale fluctuations, the most relevant operators dictating their dynamics contain fewer derivatives. The unitary gauge breaks time translation invariance, so that explicit functions of time are allowed in the EFT action. Additionally, the normalized vector orthogonal to the constant time hypersurfaces, in unitary gauge reads

$$n_\mu \equiv -\frac{\partial_\mu \varphi}{\sqrt{-(\partial\varphi)^2}} \to -\frac{\delta^0_\mu}{\sqrt{-g^{00}}}, \tag{1.67}$$

since $\partial_\mu\varphi = \delta^0_\mu$ if $\varphi$ is used as time coordinates. This means that, when building the EFT action we can contract tensors with $n_\mu$, terms with 0 free upper indices are allowed, such as $g^{00}$ or $R^{00}$. For more details on the construction of the EFT action, see [82, 85, 86, 88]. Imposing second-order equations of motion, and additionally a luminal speed of propagation for tensors, the resulting quadratic action reads

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big\{ & \frac{m_0^2}{2}\left[1+\Omega(\tau)\right] R + \Lambda(\tau) - c(\tau)a^2\delta g^{00} + \\ & + \gamma_1(\tau)\frac{m_0^2 H_0^2}{2}(a^2\delta g^{00})^2 - \gamma_2(\tau)\frac{m_0^2 H_0}{2}(a^2\delta g^{00})\delta K^\mu_\mu \Big\} + S_m[g_{\mu\nu},\chi_i]. \end{aligned} \tag{1.68}$$

where $m_0^2 = (8\pi G)^{-1}$ and $\delta g^{00}$, $\delta K^\mu_\mu$ are, respectively, the reduced Planck mass, the perturbations of the time-time component of the inverse metric, and the trace of the perturbations to the extrinsic curvature of constant-time hypersurfaces. The free functions of time $\Omega(a), \Lambda(a), c(a)$ and $\gamma_1(a), \gamma_2(a)$ are the EFT functions; the first three affect the dynamics both of the background and linear cosmological perturbations, while the latter two affect only perturbations. ΛCDM is included in this framework, and it corresponds to the choice $\Lambda(a) = const$, with the rest of EFT functions being zero. Different EFT functions correspond to different characteristics of the theory: the non-minimal coupling $\Omega(a)$, leads to a running Planck mass; the kineticity $\gamma_1(a)$, quantifies the independent dynamics of the scalar field; the braiding $\gamma_2(a)$, broadly signals a coupling between the metric and the scalar degree of freedom. Notice that we adopt the convention of [89, 90] for the EFT functions. The matter action is assumed to be universally coupled to the Jordan frame metric, as discussed previously.

The spirit of the EFT is to assign parametrization for the time dependency of the EFT functions, without relying on information coming from having chosen a specific model. Nonetheless, if one is interested in one specific realization of a scalar-tensor theories, the gravity model included in the EFT approach can be translated into the EFT language via particular mapping procedures. For reference: non-minimally coupled quintessence, f(R) gravity and Brans-Dicke theories would be characterized by non-trivial background EFT functions, while having $\gamma_1 = \gamma_2 = 0$; k-essence would further have $\gamma_1 \neq 0$ and k-mouflage correspond to all functions being nonzero. Alternative conventions are found in the literature, most commonly the so-called $\alpha_i$ parametrization [91, 92] in terms of $\{\alpha_M, \alpha_K, \alpha_B\}$ (while $\alpha_T = c_T^2 - 1$ is zero for our case) for which there is a simple direct correspondence with $\{\Omega, \gamma_1, \gamma_2\}$ (see e.g. [90, 93]). A typical feature of EFT of DE is that it allows to formulate a set of conditions that the EFT functions need to satisfy among them in order for the resulting theory to be viable and do not develop instabilities [86, 88, 92–95]. These can be quite powerful in restricting the parameter space of the EFT functions for which it makes sense to explore the phenomenology [96, 97] purely on theoretical grounds.

The modified equations of motion that follow from the EFT action are implemented in the code `EFTCAMB` [89, 90]: the extension of `CAMB` to scalar-tensor theories based on the EFT action (1.68). While we use this code to solve such equations numerically, it is still enlightening to understand how DE field modifies the equations of the cosmological model. For a thorough review of the EFT formalism and its applications to cosmological tests of gravity, we refer the reader to [93] and references therein.

### Modifications to the expansion history

From Eq. (1.68), it is clear that only $\Omega(a), \Lambda(a), c(a)$ can affect the expansion history of the Universe, as their presence in the EFT action results in a modification of Friedmann's equation (1.7). In order to deal with more familiar quantities, one can opt to work in the *designer* approach [85, 86, 90], and reverse the problem. In this approach, the expansion history is a given, and the two Friedmann equations are used to fix two of three EFT functions, in terms of the third one and $\mathcal{H}(a)$. To fix the expansion $\mathcal{H}(a)$, one can choose the equation of state of the DE field, $w_{\rm DE}$, since the background metric and DE field equations of motion can be rearranged to obtain

$$\mathcal{H}^2 = a^2 H_0^2 \left(\Omega_m + \Omega_\Lambda + \frac{\rho_{\rm DE}(a)}{\rho_{crit}}\right), \qquad \rho'_{\rm DE} + 3\mathcal{H}\,\rho_{\rm DE}(1 + w_{\rm DE}(a)) = 0, \tag{1.69}$$

and also two constraint equations giving $\Lambda$ and $c$ in terms of $\mathcal{H}(a)$ and the non-minimal coupling, $\Omega(a)$. Therefore, a model in the designer approach is fully specified by a choice $\Omega(a)$ and $w_{\rm DE}(a)$. While there is no preferred parametrization for their time dependency, a very common choice for the DE equation of state is the so-called "Chevallier-Polarski-Linder" (CPL) [98, 99] parametrization,

$$w_{\rm DE}(a) = w_0 + w_a(1-a), \tag{1.70}$$

where $w_0$ and $w_a$ are two constants. Choosing a parametrization has the effect of reducing the problem from constraining a time dependent function, throughout the entire history of the Universe, to constraining just a few constant parameters. Figure 1.3 shows an example of such constraints coming from the Planck mission [4], observing CMB temperature and polarization anisotropies, and the galaxy and weak lensing survey DES [100].

### Modifications to the growth of structures

By varying the second order EFT action (1.68) with respect to the metric perturbation, one can find the equations of motion governing the linear growth of the cosmic structures. The results are implemented in the code EFTCAMB, which we use

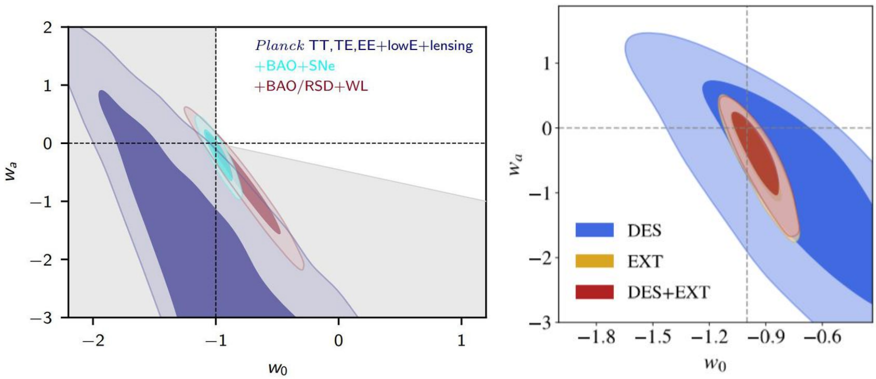


Figure 1.3: Marginalized constraints on $\{w_0, w_a\}$ from CMB observation [4] (left panel) and the galaxy survey DES [100] (right panel) alone or in combination with other probes. ΛCDM corresponds to $w_0 = -1$ and $w_a = 0$.

to compute the modified transfer functions and growth factors when needed. Despite the vast range of possibilities, including the DE field comes with some generic features, which can help in understanding the modified phenomenology of the linear scales. Typically, a dynamical dark energy field sources a scalar anisotropic shear stress, invalidating Eq. (1.31), and modifies the Poisson's Eq. (1.30). Detecting such deviations is the goal of galaxy and weak lensing surveys, such as DES [101, 102] and KiDS [103, 104], already delivering data, or the Stage IV missions: *Euclid* [105], the Vera C. Rubin Observatory [12, 106, 107] and Nancy Grace Roman Space Telescope [108]. All of these instruments are able to map LSS by analyzing properties of some tracers of it: the distribution of galaxies on large scales follows the gravitational potential, and the weak lensing induced distortions of their shapes depend on the foreground matter distribution (weak lensing is a typical propagation effect, as we will also see later). If one is not interested in the specific footprint of each operator in the EFT action (1.68) into the final observable, then it is also possible to opt to parametrize *directly* the gravitational field equations. Focusing only on the scalar sector, we rewrite Eqs. (1.31) and (1.30) as [109–111]

$$\Psi_k = \eta(a,k)\,\Phi_k \tag{1.71}$$

$$\Phi_k = -4\pi G a^2\,\mu(a,k)\,\bar{\rho}\,\delta_m^C(a,k)\,, \tag{1.72}$$

$$\Phi_k + \Psi_k = -8\pi G a^2\,\Sigma(a,k)\,\bar{\rho}\,\delta_m^C(a,k)\,, \tag{1.73}$$

and it is easy to prove the relation $\eta = 2\Sigma/\mu - 1$. These parametrizations are valid in the absence of shear anisotropic stress, otherwise the left-hand-side of the last two equations must be generalized [4, 109, 110]. The two functions $\eta(a,k)$ and $\Sigma(a,k)$ have specific physical interpretations[3]. The factor $\eta(a,k)$, signifying a difference be-

[3]We use notations of [4] and call $\eta(a,k)$ the slip parameters. In other references, such as [110] the slip is also referred to as $\gamma(a,k)$.

tween the two gravitational potential, is called *gravitational slip*. $\mu$ parametrizes modifications to the Newtonian gravitational potential, while $\Sigma$ regulates variations of the Weyl potential, $\Psi_W \equiv (\Phi+\Psi)/2$, affecting the geodesic equations of photons. While $\eta$ is not directly linked to an observable, $\mu$ and $\Sigma$ are, therefore, they can be probed with future observations [112–114]. A modified growth pattern of LSS, also causes changes in the velocities at which galaxies fall inside the gravitational potential wells, opening also *redshift space distortions* as another investigation channel for $\mu$, to be combined with weak lensing surveys probing $\Sigma$ [115, 116]. While the parametrizations in Eqs. (1.72) and (1.73) are incredibly handy to make contact with observations, their embedding into the theoretically motivated EFT functions is highly non-trivial. This fact is not to be underestimated: when dealing with the EFT functions (or equally the Horndeski functions), it is straightforward to formulate the stability conditions and explore viable theories, while it is not the case for $\mu(a,k), \Sigma(a,k)$. It is possible, to some extent, to translate these theoretical priors onto the phenomenologically motivated $\mu(a,k), \Sigma(a,k)$, and reduce the parameter space to investigate [95, 97, 117]. The parametrizations Eqs. (1.72) and (1.73) are also employed in parameterized approaches to cosmological perturbation theories, such as the Post-Friedmann Framework [52, 118–121].

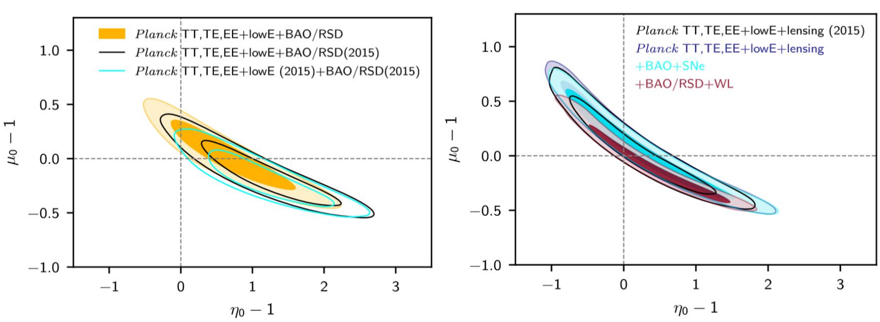


Figure 1.4: Marginalized posterior distributions of the values of $\mu$ and $\eta$ today from [4], from Planck alone or in combination with additional external data, neglecting any scale dependence. The CMB photons are sensitive to modifications of the cosmic structures' growth both through the integrated Sachs Wolfe effect, and the lensing inducing secondary anisotropies.

The dynamics of the tensor perturbations is, in general, affected by the DE field as well. Since we are considering theories where the tensor modes propagate luminally, Eq. (1.29) can only be modified as (see e.g. [122–124])

$$\gamma''_{ij} + 2\mathcal{H}(1-\delta(\tau))\gamma'_{ij} + k^2\gamma_{ij} = 0. \tag{1.74}$$

One can understand this as follows: being the DE field a scalar, it cannot produce a source for the linear order tensor modes. Therefore, the only way it can enter in their equation is through a modification of the time dependent coefficients already present in Eq. (1.29). Since we are requiring a luminal propagation speed, the ratio between the coefficients with the second derivatives must remain 1 (in our units $c = 1$), so that, effectively, the only term that is allowed to gain extra contribution is the damping one. For theories described by the action (1.68), it can be checked that

$$\delta(\tau) = -\frac{\partial \log M_P[\varphi_0]}{\partial \log a} = -\frac{M_P'[\varphi_0]}{\mathcal{H} M_P[\varphi_0]}\,, \tag{1.75}$$

where $M_P'[\varphi_0]$ is the time derivative of the running Planck's mass. This factor is commonly found in literature also under the name $\alpha_M$ [91]. As in the case of the scalar perturbations, one can opt for a phenomenological approach also in the case of GWs, instead of relating $\delta(\tau)$ to a specific theory, similarly to $\mu$ and $\Sigma$. In this case, deviations from General Relativity are usually parametrized by $\{\Xi_0, n\}$, as we will show in Eq. (1.108).

The fact that the DE field affects both scalars and tensors is a reflection of the assumption of having one, unique underlying theory describing the LSS: the same set of parameters enters the dynamics of all the perturbation modes. For this reason, it is possible to combine the information coming from *scalar sector probes*, such as galaxy clustering and lensing or the CMB angular power spectrum, with those from *tensor sector probes*, and exploit their joint power to pin down the effects of the DE field. GW170817 [125] is an example of such synergy: with one GW detection, the parameter space of the full Hordenski action has been restricted to three free functions, $M_P, K, G$[4], and the EFT functions to four, $\Omega(a), w_{\rm DE}(a), \gamma_1(a), \gamma_2(a)$ [76, 127, 128]. This is understandable since the equations of the tensor modes and of the gravitational slip both come from the spatial, traceless part of the gravitational field equations [124]. This drastic reduction of the parameter space, naturally, has caveats and loopholes [129, 130]. Other signatures of the DE field which affect both scalar and tensor modes have been investigated, for instance, in [110] and a more general review can be found in [78].

## 1.3. Gravitational waves

In Section 1.1.2 we have introduced large-scale structures and decomposed them into irreducible representations of the Euclidean spatial rotations. Among them, there was a tensor contribution, $\gamma_{ij}$, but we claimed that this is not what we identify as GWs in this Thesis. So what are GWs in our description? We define a GW, $h_{\mu\nu}$, as a linear perturbation, around a background configuration, $\bar{g}_{\mu\nu}$, of the spacetime metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + \alpha\, h_{\mu\nu}\,, \qquad \text{with} \qquad \alpha \ll 1\,, \tag{1.76}$$

where $\alpha$ is the expansion parameter used to keep track of the order of magnitude of the GW. Naturally, if one is considering scalar-tensor theories, also the DE field must

[4] See [126] for generalization to beyond Horndeski theories.

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be expanded as

$$\varphi = \bar{\varphi}(x) + \alpha\delta\phi\,. \tag{1.77}$$

In analogy with $h_{\mu\nu}$ being called GW, $\delta\phi$ is usually named *scalar wave* (SW)[5]. Note that in Eqs. (1.76) and (1.77), the background configurations are not necessarily homogeneous and isotropic and, as such, they can depend on the four spacetime coordinates, as indicated. If one takes a closer look at these two definitions and Eqs. (1.12) and (1.65) for LSS, it is easy to realize that these are formally the same, and the only difference is the background spacetime: in the case of LSS $\{\bar{g}_{\mu\nu}, \bar{\varphi}(x)\} = \{a^2(\tau)\eta_{\mu\nu}, \varphi_0(\tau)\}$, while these two can assume different forms in the case of GWs and SWs. This consideration, which might be argued a subtlety, has substantial consequences, both at the computational and interpretation level. Therefore, according to our definition, GWs are more similar to the whole LSS, $\delta g_{\mu\nu}$, rather than just its tensor subgroup and the study of their propagation is, in some way, similar to the study of the growth of cosmic structures. Note that we use two different expansion parameters to count the expansion in powers of the GW and the LSS: in the first case we use $\alpha$, while in the second $\epsilon$. This notation will be kept consistent throughout the entire Thesis, and can always be used as a guideline to understand which quantity is being used.

The linear dynamics of a GW, in the general theory of Eq. (1.64), can be found by considering

$$\delta_\alpha \left[ G_{\mu\nu} - 8\pi G\, T^{(\mathrm{DE})}_{\mu\nu}\left[g_{\alpha\beta}, \varphi\right] - 8\pi G T_{\mu\nu}\left[g_{\alpha\beta}, \chi_i\right] \right] = 0\,, \tag{1.78}$$

where $\delta_\alpha$ means linearization to first order in $\alpha$. The case of General Relativity is contained in these equations if one chooses $K = G = 0$ and $M_P(\varphi) = (8\pi G)^{-2}$ as forms of the Horndeski functions. As in the case of LSS, from the general coordinate covariance of the full theory, the linear metric perturbation inherits the gauge freedom

$$\alpha\, h'_{\mu\nu} = \alpha\, h'_{\mu\nu} - \left(\bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu\right), \tag{1.79}$$

with $\bar{\nabla}_\mu$ the covariant derivatives with respect to $\bar{g}_{\mu\nu}$. So, we are free to choose a gauge for $\alpha\, h_{\mu\nu}$, in the same way as we choose the Poisson's one to describe the cosmic structures in Eq. (1.14). Poisson's gauge was convenient in the case of LSS, because the symmetries of the FLRW background guaranteed the decoupling of the linear scalar, vector and tensor modes, as it can be clearly seen in Eqs. (1.23) - (1.29). In the case of a GW, where $\bar{g}_{\mu\nu}$ is, usually, either treated as unknown, or less symmetric than FLRW, the typical gauge choices performed in the context of LSS are not particularly convenient, and one usually opts for different ones. Nonetheless, one can already start appreciating the difficulty of the problem at hand: in cosmology it is clear which part of $\epsilon g_{\mu\nu}$ propagates ($\epsilon\gamma_{ij}$ in Eq. (1.29)) and which part corresponds

[5]Note that we are using the symbol $\alpha\delta\phi$ for the scalar wave and $\epsilon\delta\varphi$ for the DE clustering participating in the LSS.

to the gravitational potential ($\epsilon\Psi$ in Eq. (1.30)). What about $\alpha h_{\mu\nu}$? How can we separate, in the far zone, the static potential from the propagating modes (i.e. those modes which remain even when setting the sources to zero) when the background spacetime is highly non-symmetric? Engineering ways to go tackle this problem is the main focus of Chapter 6. As a general rule, when the background does not display any particular symmetry, it is convenient to go for a covariant gauge choice, such as the de-Donder gauge,

$$\bar{\nabla}^{\mu}\hat{h}_{\mu\nu}=0\,, \tag{1.80}$$

where $\hat{h}_{\mu\nu}=h_{\mu\nu}-h\bar{g}_{\mu\nu}/2$ is the trace-reversed metric perturbation. In this gauge, the first order Einstein tensor reads

$$\delta_{\alpha}G_{\mu\nu}=-\frac{\alpha}{2}\left[\bar{\Box}\hat{h}_{\mu\nu}+2\bar{R}_{\lambda\mu\alpha\nu}h^{\lambda\alpha}-h^{\lambda}_{\nu}\bar{R}_{\lambda\mu}-h^{\lambda}_{\mu}\bar{R}_{\lambda\nu}+h_{\mu\nu}\bar{R}-\bar{g}_{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\beta}\right]\,, \tag{1.81}$$

where all the barred quantities are computed with the background metric. Assuming also that the background fields are on-shell, namely satisfying the field equations (1.64), we rewrite equation (1.78) as

$$\bar{\Box}\hat{h}_{\mu\nu}+2\bar{R}_{\lambda\mu\alpha\nu}h^{\lambda\alpha}-\bar{g}_{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\beta}+8\pi G\left[\delta_{\alpha}\Theta_{\mu\nu}+\delta_{\alpha}\Theta^{\rm DE}_{\mu\nu}\right]=0\,, \tag{1.82}$$

where

$$\delta_{\alpha}\Theta_{\mu\nu}\equiv\frac{1}{2\alpha}\left[\bar{g}_{\mu\sigma}\left(\delta_{\alpha}T^{\sigma}_{\nu}\right)+\bar{g}_{\nu\sigma}\left(\delta_{\alpha}T^{\sigma}_{\mu}\right)\right]\,, \tag{1.83}$$

and $\delta_{\alpha}\Theta^{\rm DE}_{\mu\nu}$ the same for $T^{\rm (DE)}_{\mu\nu}$. These equations constitute the starting point for studying the propagation of GWs through the background spacetime, $\bar{g}_{\mu\nu}$.

### 1.3.1. Sources and wavelengths of GWs

Although it might look like GWs and electromagnetic waves are similar, there are some essential differences between the two. For instance, astronomical electromagnetic signals are usually an incoherent superposition of photons emitted from individual sources, while GWs can also be produced by coherent bulk motions of mass-energy or by nonlinear spacetime curvature features [131]. Moreover, the photon's wavelength is usually much smaller than the dimension of the source, allowing us to make pictures of the source and to treat them in *the ray-optics* limit. For GWs the situation is the opposite, and their frequency spectrum ranges from $\sim 10^3$ Hz downward for roughly 20 orders of magnitude, almost complementary to the range typical of astronomical electromagnetic radiation (from $10^7$ Hz upward) [132]. Even the low energy CMB photons have frequencies of the order of 160 GHz. Because of the variety of frequencies that GWs might have, they cannot always be treated in the ray (or geometrical) optics limit, as often it is necessary to opt for a treatment able to include *wave effects.*

Broadly speaking, GWs sources are divided into two main categories: astrophysical or cosmological. Among the first class, we include: continuous sources (rotating

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pulsars with intrinsic asymmetry from crust deformations), burst events (collapses of stars) and inspirals of compact objects binaries, by far the most famous as they are the sources of the GWs detected by LIGO-Virgo. The frequencies of these events depend on properties of the astrophysical sources, such as the masses of the black holes and neutron stars composing the binaries, and are typically in the high frequency side of the GW spectrum (>mHz). Belonging to the second class, instead, we find waves normally generated in the early Universe, such as the inflationary tensor modes [22–24, 133], or by quadrupolar collapse of large cosmic structures [134]. Various mechanisms can source GWs in the primordial Universe: cosmic strings and phase transitions [135–139], or second order scalar perturbation [140–148] are the most widely investigated. These waves are spread in a broad band of the frequency spectrum and are expected to form a stochastic background.

Regardless of the specifics of their source, we want to stress the vast range of frequencies characterizing GWs, contrary to the typical high ones for photons. This unique feature makes them the perfect test ground to probe some particular assumptions usually made in the context of wave propagation over curved spacetimes, well justified for the highly energetic electromagnetic signals but not necessarily for GWs. These assumptions are necessary in order to solve Maxwell's equations in curved spacetime (see e.g. [149]), or Eq. (1.82) in the case of GWs, otherwise too complicated. According to the frequency, different simplifying approaches are more or less convenient, as we will see.

### 1.3.2. Definition of a gravitational wave: optical regimes

Assuming that the theory we are working with admits general coordinate invariance, then its linearized version inherits the invariance under infinitesimal gauge transformations, such as in Eq. (1.79) for the metric perturbation. In the context of scalar-tensor theories, one also has the freedom to gauge the DE field according to

$$\alpha\delta\phi' = \alpha\delta\phi - \xi^{\mu}\partial_{\mu}\bar{\varphi}(x)\,. \tag{1.84}$$

The background $\bar{\varphi}(x^{\mu})$ is generic, and it can depend on all the four spacetime coordinates, in contrast to the LSS expansion and transformation in Eqs. (1.65) and (1.84). Due to this gauge freedom, it can soon be realized that the expansions in Eqs. (1.76) and (1.77) are easier said than done, and the problem of the *definition* of the GW and SW arises right away. Considering that the numerical components of a tensor can be made large or small by means of coordinate transformation, using the smallness of $\alpha$ as a criterion to distinguish the linear perturbation from the background spacetime, is not sufficient [150]. In other words, the splittings in Eqs. (1.76) and (1.77) are unambiguous only if supplemented by an additional coordinate independent criterion to distinguish $\bar{g}_{\mu\nu}$ from $h_{\mu\nu}$, and $\bar{\varphi}$ from $\delta\phi$. There are two main approaches to tackle this problem,

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1. linking the additional criterion to properties of the GWs and SWs such as their frequency: when these are much higher than the typical background time variation scale, the splitting in Eqs. (1.76) and (1.77) can be achieved by means of suitable averaging procedures.

2. assigning a priori a background $\{\bar{g}_{\mu\nu}, \bar{\varphi}\}$ and assuming it to be gauge invariant.

Note that the second avenue is the one adopted in standard cosmological perturbation theory for LSS, which suffers from the same problematics. In this case the FLRW background configuration of the metric, $\bar{g}_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$, and scalar field, $\bar{\varphi} = \varphi_0(\tau)$, are considered fixed and gauge independent.

According to the specific situation, one approach is more suitable than the other. We take the problem of the definition of the GW and SW at the heart of our discussion, and use this initial bifurcation to set up the two different *optical regimes* for GWs and SWs: *ray-optics*, where these two are defined according to the first method, and the *wave-optics* when using the second criterion. This assumption is going to be questioned and revisited throughout this entire dissertation, especially in Chapters 4, 5 and 6.

### Ray-optics definition

We can separate clearly what is « wave » from what is background when these two vary on two distinguishable scales. The literature of GWs in *ray-optics*, alternatively called *geometric optics* regime, starts from the pioneering works of Isaacson [151, 152], and then proceeds in many other papers, such as [153–155].

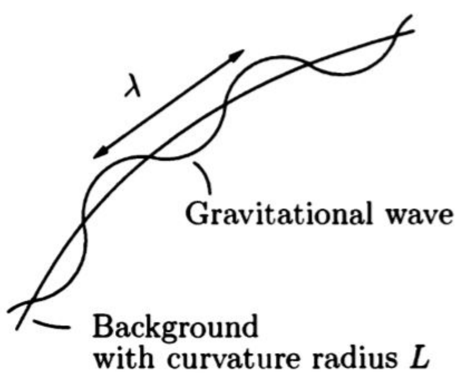


Figure 1.5: GW wavelength $\lambda$ compared to background curvature radius $L$. A GW can be distinguished from an unknown background when $\lambda \ll L$.

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In the geometric optics picture, one starts by introducing the parameter

$$\frac{1}{\omega} \equiv \frac{\lambda}{L} \ll 1\,, \tag{1.85}$$

where $\lambda$ is the wavelength of the GW and $L$ the typical scale of spatial variation of the background metric (see Figure 1.5). In the case where the background still varies on length scales smaller than the GW frequency, but it is practically static, one can use the different time profiles to separate GW and background, as in

$$\frac{1}{\omega} \equiv \frac{f_B}{f_{gw}} \ll 1\,, \tag{1.86}$$

where $f_{gw} = c/\lambda$ and $f_B$ is the frequency of the background. This is the typical situation in a GW detection through ground-based detectors: the size of their arms are smaller than the wavelength of the GW itself ($\lambda \simeq 500 - 50$ km corresponding to frequencies of $10^2 - 10^3$ Hz for typical waves detected by those interferometers) making the short-wave expansion as in Eq. (1.85) useless. Moreover, the variations of the gravitational field around the Earth due to its in-homogeneities are greater than the amplitude of the wave and occurs on small scales with respect to $\lambda$, but it is almost static [150]. Given this separation of scales (spatial or temporal), one can perform the splitting in Eq. (1.76) in a coordinate independent fashion, by defining an average $\langle \ldots \rangle$ such that the background metric is the "mean" metric: $\bar{g}_{\mu\nu} \equiv \langle\, g_{\mu\nu} \,\rangle$, and $h_{\mu\nu}$ is defined via $\langle\, h_{\mu\nu} \,\rangle = 0$. For scalar-tensor theories, the same goes for the SW and the background DE field profile. The description is completed by assigning a prescription on how to perform the averaging, and this is where the high-frequency nature of the GWs (and SWs) comes into play. The most widely used technique, also described in [151, 152], is the *ADM scheme*: we perform a space (or temporal) averages over volumes containing many periods of the GW, so that oscillatory perturbations average out to zero. Additionally, when the condition (1.85) is met, the background spacetime is practically constant in a wavelength and an almost plane-wave ansatz

$$h_{\mu\nu} = \mathrm{Re}\left[\mathcal{A}_{\mu\nu}\, e^{i\,\omega\,\theta}\right], \tag{1.87}$$

$$\delta\phi = \mathrm{Re}\left[\Xi\, e^{i\,\omega\,\vartheta}\right] \tag{1.88}$$

can be chosen. In the expressions above, Re is the real part, $\mathcal{A}_{\mu\nu}$ is the amplitude (tensorial) of the GW and $\Xi$ the one of the SW, $\theta$ and $\vartheta$ the phases of the waves. Note that the high frequency character is encoded in the $\omega \gg 1$. The amplitudes and the phases are assumed to be varying slowly, so they are almost constant within a period. Alternative names which can be found in literature for the ansatz above are: eikonal, geometric optics ansatz or WKB ansatz. The gradients of the metric perturbations chosen as in Eqs. (1.87) and (1.88) are enhanced by a factor $\omega$, coming from a derivative acting on the exponential: $\partial h_{\mu\nu}, \partial\delta\phi \sim 1/\omega \gg 1$.

Given these expressions, one proceeds in plugging them into the differential Eq. (1.82) which, then, can be organized in powers of $\omega$. Being second order, the equation has the schematic form

$$\omega^2\Big[\dots\Big]+\omega\Big[\dots\Big]+\Big[\dots\Big]=0\,, \tag{1.89}$$

since each derivative acting on the exponential brings down a factor $\omega$. Because $\omega \gg 1$, in order for this equation to be satisfied, the coefficients of each term in the $\omega$ expansion must vanish independently. From setting the coefficient of $\omega^2$ equal to zero, one finds the dispersion relation of the GW, while the first order gives an evolution equation for the amplitude of the wave. Additionally, it is usually assumed that the standard matter content does not have high frequency excitations [38]: $\delta_\alpha \Theta_{\mu\nu}[\chi_i] = 0$. This assumption will be widely commented and explored in Chapter 6, where we dubbed it *Classical Matter approximation.* In theories in which the DE field is dynamical, the situation is slightly more complicated (one has to decouple the kinetic terms of the degrees of freedom by performing a diagonalization), but conceptually analogous.

The ray-optics description is well suited to describe GWs in the bands observed by the ground- and space- based interferometers, considering their propagation through the large-scale structures of the Universe. These waves have frequencies $\gtrsim 10^{-3}$Hz, while the typical frequency associated to the linear matter structures lies in the CMB band, $\sim 10^{-16}$ Hz so that $\omega \lesssim 10^{-13}$. Nevertheless, the power of the geometric optics approximation is that it allows to draw general conclusions *regardless* of the spacetime of propagation, as long as Eq. (1.85) remains valid. Let us make an example in General Relativity. We plug the WKB ansatz in Eq. (1.87), into Eq.(1.82) with the choice $M_P = (8\pi G)^{-2}$ and $K = G = 0$, and find

$$\omega^2 \quad : \qquad \bar{g}_{\mu\nu}\, k^\mu\, k^\nu = 0\,, \tag{1.90}$$

$$\omega^1 \quad : \qquad \bar{\nabla}_\mu \left(\mathcal{A}^2\, k^\mu\right) = 0\,, \tag{1.91}$$

where $\mathcal{A}^2 \equiv \mathcal{A}_{\mu\nu}\mathcal{A}^{\mu\nu}$ and

$$k_\mu \equiv \partial_\mu \theta\,, \tag{1.92}$$

is the GW wave vector, or equally defined in terms of the covariant derivative. These equations allow the effective interpretation of a GW as a collection of particles, the *gravitons*, propagating with a wave vector $k^\mu$, which according to Eq. (1.90) is a null vector, and the wave's amplitude satisfies the continuity Eq. (1.91). The wave vector $k^\mu$ identifies the *rays* via $k^\mu = dx^\mu/d\lambda$, with $\lambda$ the affine parameter, which are geodesics of the background spacetime, since

$$k^\mu \bar{\nabla}_\mu k_\nu = k^\mu \bar{\nabla}_\mu(\partial_\nu \theta) = k^\mu \bar{\nabla}_\nu(\bar{\nabla}_\mu \theta) = \frac{1}{2}\bar{\nabla}_\nu(k^\mu k_\mu) = 0\,. \tag{1.93}$$

Note that Eqs. (1.90) and (1.91) are valid regardless of the form of $\bar{g}_{\mu\nu}$, as previously claimed. In this picture, the physical nature of the GW clearly arises (as opposed to gauge modes), with the possibility of defining a GW stress-energy tensor [152], accounting for the energy and momentum transport by the waves (more details can be found in Chapter 4). One can also understand this by making the following consideration: within a wavelength $\lambda$, space appears locally flat, and so the Riemann tensor describing curvature is gauge invariant. As long as $1/\omega \ll 1$, GWs do not have long wavelengths modes and the gauge invariant local behavior carries over the entire spacetime. Not only, because of the particle interpretation, one can adopt the techniques typical of photons, to treat GWs as well. Two relevant examples are: the *Cosmic Rulers formalism* [156–158], to describe projection effects induced by cosmic structures on the propagating GWs, and the Boltzmann equation [159], to describe the stochastic gravitational wave background (SGWB) similarly to the CMB.

Finally, the parameter $\omega$ is sometimes used to set up an expansion of the high-frequency perturbations' amplitudes. These additional terms are called the *beyond geometric optics corrections*, and they take the form

$$h_{\mu\nu} = \mathrm{Re}\left[\left(\mathcal{A}_{\mu\nu} + \omega^{-1}\mathcal{A}^1_{\mu\nu} + \omega^{-2}\mathcal{A}^2_{\mu\nu} + \cdots\right) e^{i\omega\theta}\right], \tag{1.94}$$

$$\delta\phi = \mathrm{Re}\left[\left(\Xi + \omega^{-1}\Xi^1 + \omega^{-2}\Xi^2 + \cdots\right) e^{i\omega\vartheta}\right]. \tag{1.95}$$

The beyond geometric optics order can contain valuable information and their role in lensing of GWs, in General Relativity, has been intensively investigated, e.g. in [154, 155, 160, 161].

### Wave-optics definition

Whenever Eq. (1.85) is not satisfied, the eikonal approximation cannot be chosen. This is the case, for instance, of stochastic backgrounds of gravitational waves (SGWB): it contains arbitrarily low frequencies, so it is preferable to have a description for it valid across its entire spectrum. In this case, we are forced to use the second approach to define waves: one must specify a fixed background fields configuration, and the field perturbations are simply defined as $\alpha h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ or $\alpha\delta\phi = \varphi - \bar{\varphi}$, in every gauge. We will use this approach in Chapter 6, to describe the SGWB without employing techniques which rely on the eikonal ansatz, typical of the ray-optics limit. This way, our formalism will be valid in every frequency range and can accommodate wave optics effects: interference and diffraction of the waves induced by masses situated along the path of the GW. These types of problems have been addressed for the first time in [162], and proceeded with many subsequent works [149, 163–166], showing that these become important when the mass of the lens is such that

$$M_L \lesssim 10^5 M_\odot \left(\frac{f}{Hz}\right)^{-1}, \tag{1.96}$$

where $f$ is the frequency of the GW. Such papers were then generalized to the case of multi-lens systems [167], or lenses composed of binary objects [168] and expanding backgrounds [169], and they are all in General Relativity. Wave-optics phenomena are expected to occur, for instance, in micro-lensing events due to substructures [170, 171] for GWs in the LIGO-Virgo frequency band [16]. Interestingly, it was proposed to use such events to discover unknown objects, such as intermediate mass black holes [172], more exotic forms of compact dark matter [173], or low-mass dark matter halos [174] and primordial black holes [175]. In the case of resolved GW events observed by LISA, it has been assessed that over $(0.1-1.6)\%$ of massive black hole binaries in the range of $10^5-10^{6.5}$ solar masses will display wave-optics effects [176, 177], while, in the frequency band of the ground-based detectors, it is expected that such events will be visible for sources up to redshift $z_s = 2-4$ with third generation observatories [178]. The condition in Eq. (1.96) can also be rewritten in terms of $\lambda$, the GW wavelength, and $L$, the background variation scale, as $\lambda \gtrsim L$: it is the opposite regime compared to the one of Eq. (1.85). Not surprisingly, wave-optics effects become important away from the regime where the effective description of a wave in terms of a stream of particles holds.

When choosing the wave-optics definition for the GW, there is no simplifying ansatz for the waves and one has to attempt solving Eq. (1.82) with the chosen background configuration (changing gauge for the GW if needed). Recalling that this approach is also the one used in the context of LSS, whenever we take $\{\bar{g}_{\mu\nu}, \bar{\varphi}\}$ to be compatible with FLRW and its symmetries, then studying the propagation of GWs and SWs is formally equivalent to doing cosmological perturbation theory, and the only difference is the name of the actors: $\alpha$ plays the role of $\epsilon$, $\alpha h_{\mu\nu}$ the one of $\epsilon\delta g_{\mu\nu}$ and $\alpha\delta\phi$ the one of $\epsilon\delta\varphi$. Any other situation, must be evaluated case by case. For instance, in the literature of wave-optics effects in gravitational lensing [179], addressed in General Relativity ($\delta_\alpha \Theta^{\rm DE}_{\mu\nu} = 0$), the chosen background is the one describing a static, Newtonian source,

$$d\bar{s}^2 = -\Big(1 + 2\epsilon\, U(\boldsymbol{x})\Big) dt^2 + \Big(1 - 2\epsilon\, U(\boldsymbol{x})\Big) d\boldsymbol{x}^2\,, \tag{1.97}$$

where $U$ represent the gravitational potential well of the lens. In this case, Eq. (1.82) reduces simply to $\bar{\Box}\hat{h}_{\mu\nu} = 0$, outside the lens[6]. Then, usually one proceeds by neglecting polarization effects on the GW [179]. To this end, the GW is decomposed as $h_{\mu\nu} = h\, e_{\mu\nu}$, where $h$ is the amplitude and $e_{\mu\nu}$ the polarization tensor which is considered constant. This way, the wave equation is transformed into an equation only for the amplitude: $\partial_\mu(\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}\partial_\nu h)$, where $\bar{g}_{\mu\nu}$ is given by Eq. (1.97). Expanding it to first order in $U$ one finds [179]

$$\Big(\Delta + \omega^2\Big)\, h(\omega, \boldsymbol{x}) = 4\omega^2\, \epsilon\, U(\boldsymbol{x})\, h(\omega, \boldsymbol{x})\,, \tag{1.98}$$

where $h(t, \boldsymbol{x}) = e^{-i\omega t}\, h(\omega, \boldsymbol{x})$ and $\Delta$ is the Laplacian operator in spherical coordinates $(r, \theta, \varphi)$ with $r \equiv \sqrt{x^2 + y^2 + z^2}$. Without the lens ($U = 0$), the amplitude of the wave would fall inversely proportional to the distance as $h^{NL} = Ae^{i\omega r}/r$. At this

[6] The authors also neglect the contribution proportional to the Riemann tensor by assumption.

point, there are two main strategies to solve Eq. (1.98): the first involving the so-called *Diffraction Integral* (see e.g. [179]) while the second using the *Green's function method* (see e.g. [165]). We review only the second one because we will use a similar method in Chapter 6. In the Green's function method, one sets up a solution of Eq. (1.98) by considering that the effect of the gravitational potential must be small, since $\epsilon$ is a small parameter. Therefore, the full solution must be similar to the unlensed ($U = 0$) wave

$$h(\omega, \boldsymbol{x}) = h^{(0)}(\omega, \boldsymbol{x}) + \epsilon h^{(1)}(\omega, \boldsymbol{x})\,. \tag{1.99}$$

Then, one plugs this expansion into Eq. (1.98) and matches order by order in $\epsilon$. This way $h^{(0)}$ is a solution of Eq. (1.98) with $U = 0$ and $h^{(1)}$ satisfies

$$\left(\Delta + \omega^2\right) h^{(1)}(\omega, \boldsymbol{x}) = 4\omega^2\, U(\boldsymbol{x})\, h^{(0)}(\omega, \boldsymbol{x})\,. \tag{1.100}$$

As in [165], we solve this equation Using the Green's function of the Helmholtz equation, $e^{i\omega|\mathbf{x}-\mathbf{x}'|}/|\mathbf{x}-\mathbf{x}'|$, and obtain

$$h^{(1)}(\omega, \boldsymbol{x}) = -\frac{\omega^2}{\pi} \int d^3x'\, \frac{e^{i\omega|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, U(\boldsymbol{x}')\, h^{(0)}(\omega, \boldsymbol{x}')\,. \tag{1.101}$$

The solution of $h^{(1)}$ encodes the diffraction and interference pattern typical of the wave-optics regime, which are frequency dependent. Moreover, since the frequency of a GW produced during the inspiral of a compact object binary increases while going toward the merger, monitoring in time a lensed GW event will allow observing the change in the diffraction pattern. This frequency dependecy of the observed lens pattern can be used to break degeneracies between the lens parameters and infer properties of the lens objects with an increased level of detail and precision [180, 181]. For instance, the so-called *mass-sheet* degeneracy can be lifted [149].

## 1.4. GWs in ray-optics limit: relativistic effects

We have seen that the ray-optics definition of a GW allows for a treatment which is independent of the background spacetime metric. This is clearly visible in Eqs. (1.90) and (1.91), which are valid for any $\bar{g}_{\mu\nu}$, as long as $\omega^{-1} \ll 1$. In this Section, we merge all the knowledge gained from the three previous ones. First, we generalize Eqs. (1.90) and (1.91) in the case of the Horndeski theory (1.62), promoting again $\varphi$ to a dynamical variable. Then we choose as background field configuration $\{\bar{g}_{\mu\nu}, \bar{\varphi}\}$ the cosmological perturbed solution, namely Eq. (1.14) for the metric and Eq. (1.65) for the DE field. This means that we will have a double parameter expansion: $\alpha$ keeping track of the GW and SW, as in Eqs. (1.76) and (1.77), and $\epsilon$ for the LSS included in the background field profiles. We will show that GWs propagating on such spacetimes are damped (their amplitude is inversely proportional to the luminosity distance) and that LSS source the so-called *Relativistic effects*, modulating the amplitude of the signals.

### 1.4.1. Ray-optics with a DE field

When considering the Horndeski action (1.62), the linearized gravitational field equations become (1.82), with $\delta_\alpha \Theta^{\rm DE}_{\mu\nu} \neq 0$. If one plugs the geometric optics ansatz into the equation of motion, then Eqs. (1.90) and (1.91) result modified. Since the action (1.62) is tailored to have GWs propagating at the speed of light, then the dispersion relation remains unchanged, and the wave vector $k^\mu$, defined as the gradient of the phase of the WKB ansatz in Eq. (1.92), satisfies

$$\bar{g}_{\mu\nu}\, k^\mu\, k^\nu = 0\,. \tag{1.102}$$

In these extended gravitational theories, GWs are still described in terms of gravitons propagating along null geodesics of $\bar{g}_{\mu\nu}$. It has been shown in [158, 182][7], that the amplitude of the tensor modes of the GW, $\mathcal{A}^T$, satisfies

$$\bar{\nabla}_\rho \Big[\, k^\rho\, (\mathcal{A}^T)^2 \ln M_P^2[\bar{\varphi}(x)]\, \Big] = 0\,, \tag{1.103}$$

in the Horndeski theory (1.62) and with $\bar{\varphi}$ the generic, background DE field configuration. Eq. (1.91) follows simply by choosing $M_P[\bar{\varphi}] = (8\pi G)^{-2}$. The result of having promoted the cosmological constant to a dynamical scalar field described by the action (1.62), is encoded in a modification to the GW tensor modes amplitude's conservation equation. This is clear from the extra factor $\ln M_P^2[\bar{\varphi}(x)]$ in Eq. (1.103).

### 1.4.2. Standard sirens: the GW luminosity distance

The cosmological background is an expanding spacetime. Because of this, propagating waves are damped. This is the so-called *Hubble friction*, encoded in the factor $\mathcal{H}$ of Eq. (1.29) or the modified one $\mathcal{H}(1-\delta(\tau))$ in Eq. (1.74). Naturally, these considerations are contained also in Eq. (1.103). Indeed, it is easy to check that, if one chooses $\{\bar{g}_{\mu\nu}, \bar{\varphi}\} = \{a^2(\tau)\eta_{\mu\nu}, \varphi_0(\tau)\}$, then Eq. (1.103) is compatible with Eq. (1.74).

In case of a homogeneous and isotropic background, the amplitude evolution Eq. (1.103) can be integrated [158] to give

$$\mathcal{A}^T(z) = \frac{\mathcal{Q}^T\,(1+z)^2}{\bar{d}_L^{\rm GW}(z)} \tag{1.104}$$

where $\mathcal{Q}^T$ is an integration constant that depends on the properties of the source, $z$ is the redshift, related to the scale factor as $1+z = a^{-1}$. In the solution above, we have introduced the *gravitational wave's luminosity distance*, $\bar{d}_L^{\rm GW}(z)$, defined as

$$\bar{d}_L^{\rm GW}(z) \equiv \frac{M_P(z)}{M_P(0)}\, \bar{d}_L^{\rm EM}(z)\,, \tag{1.105}$$

[7] The procedure is actually slightly different, and we do not report it here. In particular, it requires decomposing the GW on a polarization basis, and focusing only on its tensor modes (in our definition, the GW $\alpha h_{\mu\nu}$ is more similar to the LSS, $\epsilon\delta g_{\mu\nu}$, rather than only its tensor modes $\epsilon\gamma_{ij}$). In the equation, $\mathcal{A}^T$ is the amplitude of the tensorial part of $\alpha h_{\mu\nu}$.

where the *electromagnetic luminosity distance* is

$$\bar{d}_L^{\mathrm{EM}}(z) = (1+z)\bar{\chi}(z) = \frac{(1+z)}{H_0}\int_0^z \frac{dz'}{E(z')}, \tag{1.106}$$

with $\bar{\chi}(z)$ the comoving distance and $E(z) \equiv \mathcal{H}(z)/(a^2 H_0) = \Omega_m + \Omega_\Lambda + \rho_{\mathrm{DE}}(a)/\rho_{crit}$ as in Friedmann's equation (1.69). Eq. (1.104) is a fundamental result in the literature of GWs in scalar-tensor theories. From it, together with Eq. (1.105), we see that the effect of the modification to the Hubble friction in the GW's tensor modes equation, is to produce a difference between the luminosity distances as inferred from electromagnetic signals and those from the amplitude of GWs. Since photons are contained in the universally coupled matter action, $\mathcal{S}_M[g,\chi]$, they are not directly affected by the DE field, and the Hubble drag they feel is not modified. The presence of a dynamical DE field, in this case, enters only implicitly the expansion rate $E(z)$. This different behavior between the GW and EM sector is a golden resource for testing scalar-tensor theories aimed at describing the late-time accelerated expansion. It allows to directly investigate the non-minimal coupling between the DE field and the curvature, also known as running Planck's mass, $M_P(\varphi)$ (see e.g. [183–192]). Eq. (1.105) is also renowned in literature with different notations, especially in terms of $\delta(\tau) = -\partial \ln M_P/\partial \ln a$, or $\alpha_M = -\delta(\tau)$. In this case

$$\frac{M_P(z)}{M_P(0)} = \exp\left[\int_0^z dz' \frac{\delta(z')}{1+z'}\right]. \tag{1.107}$$

This expression has two clear behaviors: when $z \to 0$, the integral becomes trivial and no modification of the luminosity distance occur. On the other hand, when $z \gg 1$ then, we enter matter domination, and we expect $\delta(z) \to 0$ since the DE field becomes very subdominant, and the results of General Relativity should be recovered. In this case, the ratio $M_P(z)/M_P(0)$ reaches a constant. Another very common parametrization for Eq. (1.105), more similar in spirit to the phenomenological approach, can be found in [122, 183, 184] and sees the introduction of the two parameters $\Xi_0$ and $n$ as

$$\frac{\bar{d}_L^{\mathrm{GW}}(z)}{\bar{d}_L^{\mathrm{EM}}(z)} = \Xi_0 + \frac{1-\Xi_0}{(1+z)^n}. \tag{1.108}$$

Such parametrization reproduces the two expected behaviors: when $z \to 0$ the ratio goes to 1 and when $z \gg 1$, the second term can be neglected and the same ratio approaches the constant value $\Xi_0$. Up to now, we do not have tight constraints on the value of $\Xi_0$ [193–195] and large deviations are still allowed by the data, possibly making its effect dominant over the one due to a modified expansion history [183, 184]. Interestingly, it has been proposed to use strongly lensed GW observations to put constraints on $\Xi_0$, confirming that we will be able to put constraints at the percent level with $\sim 4600$ events [196].

Before moving on to the relativistic corrections, we wish to stress one last point. The results of this Section are of utmost importance even when the prefactor

$M_P(z)/M_P(0) = 1$. Eq. (1.104) leads in any case to the important observational fact that a GW detection provides a direct measurement of the luminosity distance of their source. These events are, thus, standard distance indicator, dubbed *Standard Sirens* [197], similarly to SNe, also known as *Standard Candles*. Because of this, one can perform, with GW observations, the same distance-redshift tests which allowed the discovery of the late time accelerated expansion with SNe observations [195, 198, 199]. In this type of tests, one uses the knowledge of both the luminosity distance and redshift to pin down the only unknown left in Eq. (1.106): $H_0 E(z)$. Thus, in this way we constrain the expansion history and the Hubble parameter today, $H_0$ [200–202], the latter being one of the two parameters of the standard cosmological model under the spotlight. Indeed, there is a discrepancy of about $\sim 5\sigma$ between the measurements of $H_0$ inferred from the CMB anisotropies power spectrum (assuming $\Lambda$CDM) and the one from SNe (see [15] for a review and references therein). Standard sirens, thus, provide a third and independent way, affected by entirely different systematics, to constrain such parameter and could prove to be crucial into breaking the tie between the various determinations of $H_0$ [203]. Unfortunately, standard sirens come with a catch: GW observations carry information exclusively about the luminosity distance, and the redshift determination has to be obtained either through a direct detection of an electromagnetic counterpart [197, 198, 204], as it was for GW170817 [205], or via the individualization of the host galaxy of the source or through statistical methods [188, 206–210].

### 1.4.3. Relativistic corrections

After having described the solution of Eq. (1.103) on FLRW Universe, we can include the effect of LSS. We further simplify the situation, considering the so-called *restricted Poisson's gauge*, a subcase of Eq. (1.14) where $\epsilon\, w_i = \epsilon\, \gamma_{ij} = 0$. This choice is suitable in the late time Universe if there are no free tensor modes [39] and it is compatible with having a scalar dynamical DE field: being a scalar, it cannot source vector and tensor modes at linear order. With this assumption, the background field's configurations are

$$d\bar{s}^2 = a^2(\tau)\Big[-(1+2\epsilon\Phi)d\tau^2 + (1-2\epsilon\Psi)d\mathbf{x}^2\Big]\,, \tag{1.109}$$

$$\bar{\varphi}(x) = \varphi_0(\tau) + \epsilon\,\delta\varphi(x)\,, \tag{1.110}$$

where $\Phi, \Psi$ are the two gravitational potentials in Poisson's gauge and $\delta\varphi$ is the DE field fluctuation.

Before getting into the details of the computation, let us sketch what we expect to find. On the homogeneous and isotropic background, the luminosity distances depend only on redshift, leading to the standard distance-redshift relation. Because the LSS break the rotational symmetry in the 3D spatial slices, we expect inhomogeneities in the Universe to induce a dependence also on the direction of observa-

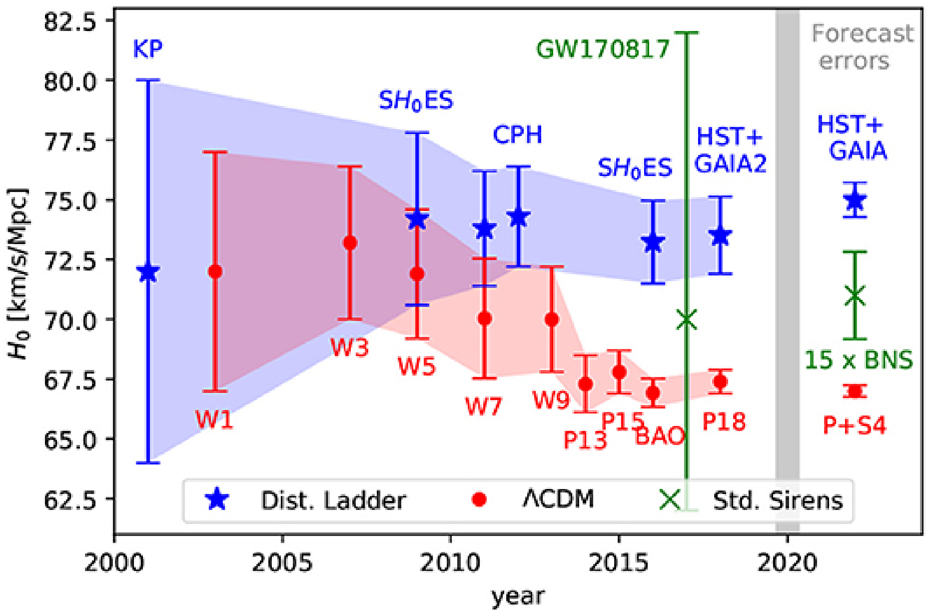


Figure 1.6: From [211]. Determinations of the Hubble constant $H_0$ from local Universe measurements (blue), and from CMB assuming ΛCDM (red). The green line corresponds to its measurement from the bright event GW170817 [205].

tion

$$d_L(\hat{\theta}, z) = \bar{d}_L(z) + \epsilon\, \Delta d_L(\hat{n}, z)\,. \tag{1.111}$$

The average luminosity distance, $\bar{d}_L(z)$, in the equation above, theoretically corresponds to Eq. (1.106) for photons and Eq (1.105) for GWs, since $\epsilon\delta g_{\mu\nu}$ has zero mean. Fluctuations in the electromagnetic luminosity distance $\Delta d_L$, which we introduced in Eq. (1.114), constitute an important probe for linear cosmology and have been well studied [212–216], while the case of GWs has been addressed in General Relativity in [217–223]. Relativistic effects, thus, can be used to probe the linear structures of the Universe. If one is not interested in them, and only wish to use standard distance-redshift tests, they must still be accounted for as they introduce irreducible errors in the determinations of the parameters [224–226]. Including relativistic effects can also have impact in the searches for the electromagnetic counterparts [218].

Mathematically, the problem at hand requires plugging Eqs. (1.109) and (1.110) into Eq. (1.103) and solve for $\mathcal{A}^T$. Naturally, this turns out to be very complicated and one resorts to perturbative schemes to solve the equations. Such methods were first developed in the context of photon propagations, and in particular go under the name of *Cosmic Rulers* [156, 227, 228] or also *line-of-sight approaches* [216, 229, 230]. Indeed, starting from Maxwell's equations, an evolution law identical to (1.91) can be found also for the amplitude of an electromagnetic signal [149, 212]. These types of computations were then generalized to the case of GWs in geometric optics in [157, 217] in General Relativity, and in [158] in the Horndeski theories of the form (1.62). This is the principal reason why we restrict our studies to scalar-tensor theories with luminal GWs: the form of the relativistic effects is not known otherwise. The Cosmic Rulers formalism also accounts for the projections effects induced by the fact that we observe in the *observer-frame*, different from the *real-frame* because, when observing, we use coordinates that flatten the past light-cone of the GW. Regardless of the method, the general philosophy is to expand (1.103) up to first order in LSS ($\epsilon$) and define $\mathcal{A}^T = \bar{\mathcal{A}}^T(1+\epsilon\Delta\ln\mathcal{A}^T)$, where $\bar{\mathcal{A}}^T$ satisfies the equations at order $\epsilon^0$, while $\Delta\ln\mathcal{A}^T$ those at order $\epsilon^1$ (or equivalently in terms of the luminosity distance knowing that $\mathcal{A}^T \propto (d_L^{\rm GW})^{-1}$). The logic is very similar to what we have done to find the wave-optics solution in Eq. (1.101), but starting from (1.103) and using Eqs. (1.109) and (1.110) instead of a background representing a static Newtonian lens.

In [158], it can be found that the relativistic effects on GW's amplitude, induced by propagation effects through the cosmic web, in the theory (1.62), are

$$\begin{aligned}\frac{\Delta d_L^{\rm GW}(\hat{n},z)}{\bar{d}_L^{\rm GW}(z)} =& -\kappa-(\Phi+\Psi)+\frac{1}{\chi}\int_0^{\chi}d\tilde{\chi}\,(\Phi+\Psi)+\Phi\left(\frac{1}{\mathcal{H}\chi}-\frac{M_P'[\varphi_0]}{\mathcal{H}M_P[\varphi_0]}\right)\\ &+\left(1-\frac{1}{\mathcal{H}\chi}+\frac{M_P'[\varphi_0]}{\mathcal{H}M_P[\varphi_0]}\right)\left[v_\parallel-\int_0^{\chi}d\tilde{\chi}\,(\Phi'+\Psi')\right]+\frac{M_{P,\varphi}[\varphi_0]}{M_P[\varphi_0]}\delta\varphi\,,\end{aligned} \tag{1.112}$$

where a prime indicates differentiation w.r.t. conformal time, $\kappa$ denotes the weak lensing convergence, $\chi$ the comoving distance to the source, $\Phi$ the Newtonian potential, $\Psi$ the intrinsic spatial curvature potential, $v_\parallel$ the component along the line of sight of the peculiar velocity of the source, and $M_{P,\varphi}$ is the derivative with respect to the scalar field: all in the restricted Poisson gauge and following the conventions of [158]. The physical effects contributing to $\Delta d_L^{\rm GW}$, in order, are: weak lensing convergence, volume dilation and a Shapiro time delay, that are only indirectly influenced by the DE field; Sachs-Wolfe (SW), Doppler shifts, and Integrated Sachs-Wolfe (ISW), showing an additional explicit decay that depends on the time evolution of $M_P[\varphi_0(\tau)]$; damping due to DE field inhomogeneities, $\epsilon\delta\varphi(x)$. The weak lensing convergence field, namely $\kappa$ in Eq. (1.112), is given by

$$\kappa(\hat{n}) = -\frac{1}{2}\nabla_\theta^2\Phi_L(\hat{n}) = -\int_0^{\chi}\frac{d\chi'}{\chi'}\int_{\chi'}^{\infty}d\chi_*\left(\frac{\chi_*-\chi'}{\chi_*}\right)\nabla_\theta^2\Psi_W(\chi'\hat{n},z') \tag{1.113}$$

where $\Psi_W$ is the Weyl potential and $\nabla_\theta^2$ indicates the 2D Laplacian with respect to the angle between the image and optical axis [149, 231]. Note that we are neglecting the shear deformations of the signal since these are subdominant in linear perturbation theory where WL mainly affects the magnification of the GWs.

1

In presence of the DE field, the GW luminosity distance generally differs from the one traced by electromagnetic signals. This is clear at the unperturbed level from Eq. (1.105). Not surprisingly, this is also true for their large-scale fluctuations as GWs follows Eq. (1.103), where the Horndeski function $M_P$ is explicitly present, while photons are not directly affected by the DE field. As already mentioned, it is possible to find a WKB solution for Maxwell's equation, and this would look like Eq. (1.91) also in the presence of a dynamical DE field. Therefore, the luminosity distance fluctuations as inferred from an EM detection, in the Horndeski theory (1.62), are given by

$$\frac{\Delta d_L^{\rm EM}(\hat{n},z)}{\bar{d}_L^{\rm EM}(z)} = -\kappa - (\Phi+\Psi) + \frac{1}{\chi}\int_0^{\chi} d\tilde{\chi}\,(\Phi+\Psi) + \frac{\Phi}{\mathcal{H}\chi} + \left(1-\frac{1}{\mathcal{H}\chi}\right)\left[v_\parallel - \int_0^{\chi} d\tilde{\chi}\,(\Phi'+\Psi')\right], \tag{1.114}$$

as it can be derived also by taking Eq. (1.112) and discarding terms in which $M_P[\varphi_0]$ appears explicitly. We clarify that Eqs. (1.114) and (1.112) carry an implicit dependence on all the Horndeski functions $M_P, K, G$, as the expansion history of the Universe and the growth of cosmic structures obey the modified equations described in Section 1.2.2. In a parametrized approach, these would correspond to Eqs. (1.69), (1.72) and (1.73) (see e.g. [232] for an explicit example). When opting for this kind of approach, though, one must be careful that the forms chosen for $w_{\rm DE}, \mu(a,k)$ and $\Sigma(a,k)$ are compatible with having started from the Horndeski theory in action (1.62), namely with luminal tensor modes speed [233].

Therefore, the luminosity distance fluctuations provide direct access to the linear structures of the Universe. In non-minimally coupled scalar-tensor theories, some of the effects building these signals are different for distances inferred from GWs or from electromagnetic signals, offering a new way to directly probe the effects of a dynamical DE field. The way that this information can be used is similar to what done in Section 1.1.3: promoting the field fluctuations to random variables, one computes their correlation functions. Thanks to Eqs. (1.114) and (1.112), we can relate the power spectra of the luminosity distance fluctuations, which we can obtain through GW and SNe observations, to those of the gravitational potentials, $\Phi$ and $\Psi$, and DE clustering $\epsilon\delta\varphi$, and investigate them.

## 1.5. This Thesis

This dissertation is divided in three parts, each of which tackles a specific aspect of the main theme: the propagation of GWs through cosmic structures. In the first two parts, we will use GWs in the ray optics regime to test scalar-tensor models of gravity aimed at describing the late time cosmic expansion. In the last part, we will demote the DE field back to a cosmological constant and address the matter of wave-optics effects. Each Chapter picks up the thread of thoughts from the previous one, investigating the extent of the assumptions made and possibly generalizing them. Keeping the detection prospect in high regard, the goal is not only to produce testable predictions, but also to understand various aspects of gravitational wave propagation that are yet to be fully grasped.

**Part I: Ray-optics limit: beyond the homogeneous and isotropic Universe**

We dedicate this part to the exploration of the potentialities of the luminosity distance fluctuations in Eqs. (1.112) and (1.114) in constraining cosmological parameters of various scalar-tensor models contained within the action (1.62). This part aims, therefore, at building tests using the *amplitude* of the GWs as signal. In particular,

- **Chapter 2**: we will investigate the power spectrum of the luminosity distance fluctuations of GWs observations, in two scalar-tensor models. After a first exploration, we will build an estimator which combines GWs and SNe observations with the goal of picking up the *dark energy clustering* contribution to the covariance of the observation. In other words, we attempt at singling out the contribution of $\epsilon\delta\varphi$ in Eq. (1.112), by comparing GW and electromagnetic observations. This signal is otherwise buried under the dominant contribution of the weak lensing convergence $\kappa$. We will conclude that, although picking up the DE clustering signal is very challenging, the estimator we built still provides a smoking gun proof for theories with a running Planck's Mass $M_P[\varphi]$, which can be used after gathering enough GWs and SNe data.

  **Based on:** *Detecting Dark Energy Fluctuations with Gravitational Waves*
  A. Garoffolo, M. Raveri, A. Silvestri, G. Tasinato, C. Carbone, D. Bertacca, S. Matarrese,
  Phys.Rev.D 103 (2021) 8, 083506, e-Print: 2007.13722 [astro-ph.CO]

- **Chapter 3**: considering that weak lensing gives the greatest contribution to the GW luminosity distance fluctuations, we investigate its role in constraining cosmological parameters. Hence, in this Chapter we focus on the contribution from $\kappa$ in Eq. (1.112), and study its potentialities both alone and in combination with galaxy surveys (both clustering and weak lensing). Without choosing a specific GW mission, we assess the number of GW events and the precision with which the luminosity distance must be determined, in order for GW observations to become competitive with galaxies in constraining cosmological parameters.

  **Based on:** *Prospects of testing late-time cosmology with weak lensing of gravitational waves and galaxy surveys*
  A. Balaudo, A. Garoffolo, M. Martinelli, S. Mukherjee, A. Silvestri,
  e-Print: 2210.06398 [astro-ph.CO]

**Part II: Ray-optics limit: distance duality relation and polarization tests**

After having investigated the potentialities of Eqs. (1.112), in this part we turn to other possible tests, always in the ray-optics limit. Indeed, from a GW detection, one receives more information than only its luminosity distance. In particular, we will take a step back and asks which other quantities related to their amplitude are modified in scalar-tensor theories, and then we will start looking at the polarization content.

- **Chapter 4**: distance measures have always played a prime role in building tests for the cosmological model. The fact that the luminosity distance, as inferred from a GW detection, can be modified compared to the electromagnetic one, is a golden opportunity to tests non-minimally coupled scalar-tensor theories. Nevertheless, in cosmology there are multiple notions of distances which can play an equivalently important role to $d_L^{\rm GW}$ in producing tests for the gravitational theory. In this Chapter, after deriving the equations of motion of a GW in a scalar-tensor set-up where these propagate at the speed of light, we take a proper look at the definition of *GW cosmological distances*, showing that, in the ray-optics regime, on top of $d_L^{\rm GW}$, one can also define an angular diameter distance, $d_A^{\rm GW}(z)$. To achieve this, we derive from a general principle the GW stress-energy tensor, under the assumption that the GW and SW propagate at different speeds. We will prove the validity of the *Etherington's reciprocity law*, namely $d_L^{\rm GW}(z) = (1+z)^2 d_A^{\rm GW}(z)$, implying that also the GW angular diameter distance is modified compared to the electromagnetic one. Finally, we investigate the implications of our findings in the context of strong lensing time delays.

  **Based on:** *Gravitational-wave cosmological distances in scalar-tensor theories of gravity*

  G. Tasinato, A. Garoffolo, D. Bertacca, S. Matarrese

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- **Chapter 5**: in order to derive the equations of motion of the GW and SW, in the previous Chapter we assumed that the amplitude of the SW was a factor $\omega^{-1}$ smaller than the GW. We justified this assumption by asking that at the moment of emission the SW is not sourced, so that it is produced by propagation effects only. In this Chapter, we elaborate on this assumption and thoroughly revisit the two definitions of GW given: as in geometric-optics and in wave-optics. We will see that, depending on the assumptions, different conclusions can be drawn about the amplitude of the SW. We investigate, then, whether the SW can be directly detected in light of screening mechanisms. In the context of two of them, we show that this should not be the case regardless of the definition used for the GW.

  **Based on:** *Unifying gravitational waves and dark energy*

A. Garoffolo, O. Contigiani,
e-Print:2110.14689 [astro-ph.CO]

**Part III: Wave-optics limit: the stochastic background and its polarization**:

In this last part, we assume that dark energy is described by the cosmological constant, $\Lambda$, as in the standard model of cosmology. We also switch the definition of the GW and opt to investigate the wave-optics limit, showing that the interaction with matter structures during propagation, in this limit, can produce scalar and vector polarization modes.

- **Chapter 6**: we study the propagation of GWs in a perturbed cosmological Universe without relying on tools typical of ray-optics techniques. This way, we can easily account for wave-optics effects. Similarly to Eq. (1.101), we obtain a perturbed solution for the waves, though accounting carefully for their polarization content instead of treating them as scalar fields. We work under the *classical matter approximation*, namely that the effect of the waves on the matter inhomogeneities is negligible. Our result shows that the interaction with matter structures can produce scalar and vector components in the GW, on top of tensor ones. We build the two point correlation function of the tensor modes, and introduce the Stokes parameters. In the case of an unpolarized, Gaussian, statistically homogeneous and isotropic initial background, we show that the interaction with matter does not generate a net difference between left- and right- helicity tensor modes, as expected, but it also does not produce Q- and U- polarization modes.

  **Based on:** *Wave-optics limit of the stochastic gravitational wave background*
  A. Garoffolo,
  e-Print: 2210.05718 [astro-ph.CO]

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# Appendices

## A. Special functions, Fourier and Harmonic transformations

In our convention, the 3D Fourier transform and anti-transform are given by

$$f(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, f_{\mathbf{k}}(\tau)\,, \tag{1.115}$$

$$f(\tau,\mathbf{k}) = \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, f(\tau,\mathbf{x})\,. \tag{1.116}$$

Similarly, a function can be expanded on a spherical harmonics basis using

$$f(\tau,\chi,\hat{n}) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\tau,\chi)\, Y_{\ell m}(\hat{n})\,, \tag{1.117}$$

$$a_{\ell m}(\tau,\chi) = \int_{\mathcal{S}^2} d^2\hat{n}\, f(\tau,\chi,\hat{n})\, Y^*_{\ell m}(\hat{n})\,, \tag{1.118}$$

with $\mathbf{x} = \chi\hat{n}$, where $\hat{n}$ is the vector on the unit sphere, i.e. $\hat{n} = (\theta,\phi)$ represents the angular coordinates. The spherical harmonics $Y_{\ell m}(\hat{n})$ are eigenfunctions of the angular part of the Laplacian

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_{\ell m}(\theta,\varphi) = -\ell(\ell+1)\, Y_{\ell m}(\theta,\varphi) \tag{1.119}$$

and their explicit expression takes the form

$$Y_{\ell m}(\hat{n}) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi} \times \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases}, \tag{1.120}$$

where $P_{\ell m}(\mu)$ are the associated Legendre functions, satisfying

$$P_{\ell m}(\mu) = (-1)^m\, (1-\mu^2)^{m/2}\, \frac{d^2}{d\mu^2}\, P_\ell(\mu)\,, \tag{1.121}$$

and $P_\ell(\mu)$ is the Legendre polynomial. These are $\ell$th-order polynomials in $\mu \in [-1,1]$. The first polynomials are

$$P_0(\mu) = 1\,, \qquad P_1(\mu) = \mu\,, \qquad P_2(\mu) = \frac{3\mu^2-1}{2}\,, \tag{1.122}$$

while the higher ones can be found by using the recursion relation

$$(\ell+1)P_{\ell+1}(\mu) = (2\ell+1)\,\mu\, P_\ell(\mu) - \ell P_{\ell-1}(\mu)\,. \tag{1.123}$$

The spherical harmonics satisfy the orthonormality relations

$$\int_{\mathcal{S}^2} d^2\hat{n}\, Y^*_{\ell m}(\hat{n})\, Y_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\,\delta_{mm'} \tag{1.124}$$

and they are related to Legendre polynomials via

$$P_\ell(\hat{n}\cdot\hat{n}') \quad = \quad \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{n})\, Y_{\ell m}(\hat{n}')\,. \tag{1.125}$$

Another class of important special functions are the spherical Bessel functions, $j_\ell(x)$, solutions of the differential equation

$$\frac{d^2 j_\ell}{dx^2} + \frac{2}{x}\frac{dj_\ell}{dx} + \left[1 - \frac{\ell(\ell+1)}{x^2}\right] j_\ell = 0\,. \tag{1.126}$$

They cover a relevant role because of the relation

$$e^{i\mathbf{k}\cdot\mathbf{x}} \quad = \quad \sum_{\ell=0}^{\infty} i^\ell\,(2\ell+1)\, j_\ell(k\chi)\, P_\ell(\hat{k}\cdot\hat{n})\,, \tag{1.127}$$

where we used again $\mathbf{x} = \chi\,\hat{n}$ and similarly $\mathbf{k} = k\,\hat{k}$.