This is life: some thoughts on self-organized structure formation in active liquids and biological systems

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CHAPTER 7

Full Derivation of Active Height Equation

This chapter accompanies Chapter 5 and Chapter 6. In the first part of this chapter we present a detailed derivation of the equations of motion used in these two chapters. Namely, starting from the Frank free energy, the Helfrich free energy, and the standard equations of active nematodynamics, we systematically derive the equations of motion governing the dynamics of an active liquid crystal coupled to an elastic, two-dimensional surface of arbitrary geometry. The resulting equations can be used to study activity-induced shape deformations of elastic surfaces. In the second part of this chapter we investigate these equations for an initially flat, disc-shaped elastic surface. We analyze how the presence of ± 1 and $\pm 1/2$ topological defects in such a system can drive buckling instabilities. We find that only defects with positive charge result in a buckled surface. In the third part of this chapter we investigate the case of a toroidal ground state geometry, where no defects are present. We present preliminary results on the instability of the ground state geometry in this case.

All laws, written, and unwritten, have need of interpretation.

T. Hobbes. *Leviathan.*

[B]ay is a noun only if water is dead. When bay is a noun, it is defined by humans, trapped between its shores and contained by the word. But the verb wiikwegamaa — to be a bay — releases the water from bondage and lets it live. "To be a bay" holds the wonder that, for this moment, the living water has decided to shelter itself between these shores, conversing with cedar roots and a flock of baby mergansers. Because it could do otherwise — become a stream or an ocean or a waterfall [...].

R. W. Kimmerer. *Braiding Sweetgrass.*

La crisi consiste appunto nel fatto che il vecchio muore e il nuovo non può nascere: in questo interregno si verificano i fenomeni morbosi più svariati.

A. Gramsci. *Quaderni del carcere.*

7.1 Introduction

The goal of this chapter is two-fold. First, we present in more detail the derivations of the equations of motion that govern the active liquid crystal coupled to an elastic surface. We used these equations in the two preceding chapters, Chapters 5, 6, to describe the dynamics of an elastic surface coupled to an active liquid crystal in the presence of topological defects. We focus on the mathematical aspects here and refer to the previous chapters for more details about the connection to biology and the motivation. Because of this, the present chapter is not independent and rather technical. (If it was not for its length, this chapter could have been an appendix for the preceding chapters.) In Chapter 5 we considered a surface that is initially a flat disc, and investigated how the presence of a +1 defect results can drive a buckling instability. In Chapter 6 we investigated a surface that initially had the geometry of a sphere. We considered both the case of a polar liquid crystal, in which case two +1 defects were present on the sphere, as well as a nematic liquid crystal with four +1/2 defects. In both chapters the governing equations of motion coupled the equilibrium elastic relaxation of the elastic membrane and liquid crystal with the activity and the resulting flow and pressure fields. The full derivation of this set of equations is rather lengthy and thus we present them here in a separate chapter. In the second part of this chapter we generalize the results of Chapter 5 on the buckling instability for a flat disc in the presence of a +1 defect to arbitrary defect charges. Lastly, in the third part of this chapter, we solve the equations for an active liquid crystal on a torus. Some basic concepts of differential geometry used here are explained in the introduction, Chapter 1.

7.2 The Model

The model we use here can be used to explain, for example, the morphodynamics of cell monolayers, even though the model can more generally be used to study activity-driven shape changes. We refer to Chapters 5, 6 for the connection between the equations derived below and biological systems. Here, we focus on the derivation of the equations, not on their (biological) interpretation and application. In this section we recapitulate the equations used to describe the dynamics of an active liquid crystal on a fixed, curved surface. Afterwards, we generalize the equations to allow for the surface itself to deform. We consider a two-dimensional active liquid crystal (either polar or nematic) with the director field denoted by $\boldsymbol{p}$. This liquid crystal field is constrained to be on the two-dimensional surface $\mathscr{M}$, that is the director does not have a component normal to the surface. The elastic energy of the liquid crystal is minimized if nearby directors are aligned, that is gradients in the director field are energetically unfavorable. This is quantified by the Frank free energy (Sec. 1.1.2, Refs. [1, 3])

$$F_{\mathrm{F}} = \frac{\kappa_{\mathrm{F}}}{2} \int_{\mathscr{M}} \mathrm{d}A \, |\nabla \boldsymbol{p}|^2 \ , \tag{7.1}$$

where κ_{F} is the Frank elastic constant and we assume the one-elastic-constant approximation. To model the dynamics of cells, which are inherently out of equilibrium, it is necessary to add activity. As explained, e.g., in Sec. 1.2.1, we choose the following expression for the active stress tensor [41, 42, 48, 55, 56]:

$$\sigma^{\mathrm{a}}_{ij} = \alpha \left(p_i p_j - \frac{1}{2} g_{ij} \right) \ . \tag{7.2}$$

Here, g_{ij} is the metric of the surface $\mathscr{M}$ which the director field is constrained to. In flat space it reduces to the Kronecker delta $g_{ij} = \delta_{ij}$ and in this case Eq. (7.2) reduces to the expression found most commonly in the literature. However, we anticipate that our ultimate goal is to work in curved space by writing the general expression for this stress tensor with a generic metric here. This active force can be found from the active stress tensor by taking the divergence. The active force can drive a fluid flow whose velocity we denote by v_i. We assume the flow to be incompressible which gives rise to a hydrodynamic pressure P_{h}. The flow can interact with the director field by reorienting it. For example, if the flow is sufficiently strong it can be energetically favorable for elongated particles to align with the direction of flow even if this entails that gradients in the director field result in an increasing Frank free energy. This is well-known from passive liquid crystal physics with essentially the only difference being that in the passive case the flow is externally driven while in the active case the flow is fueled by the activity of the system itself. The interaction is described by the Leslie-Ericksen equation (Sec. 1.1.4, Refs. [1–3, 26–28])

$$\frac{D}{Dt} p^i = \left(g^{ij} - p^i p^j \right) \left(\lambda u_{jk} p^k - \omega_{jk} p^k + \frac{\kappa_{\mathrm{F}}}{\Gamma} \nabla^2 p_j \right) \ . \tag{7.3}$$

Here, $D_t = \partial_t + v^i \nabla_i$ is the material derivative. $u_{ij} = \left(\nabla_i v_j + \nabla_j v_i \right)/2$ and $\omega_{ij} = \left(\nabla_i v_j - \nabla_j v_i \right)/2$ are the strain rate and vorticity tensors, respectively. λ is the so-called flow alignment parameter. Finally, $M_i = \delta F_{\mathrm{F}} / \delta p^i = \kappa_{\mathrm{F}} \nabla^2 p_i$ is the molecular field found from the Frank free energy Eq. (7.1). It is coupled to the rotational viscosity Γ and in the absence of flows results in a minimization of the Frank free energy, i.e., this term aligns the director field. Finally, an equation describing the dynamics of the velocity field is needed. We use the overdamped force balance (Sec. 1.1.4, Refs. [1–3, 26–28]):

$$\frac{D}{Dt} v_j = \nabla^i \left[\sigma^{\mathrm{d}}_{ij} + \sigma^{\mathrm{e}}_{ij} \right] \ , \tag{7.4a}$$

where

$$\sigma^{\mathrm{d}}_{ij} = -P_{\mathrm{h}} g_{ij} + 2\eta u_{ij} + \sigma^{\mathrm{a}}_{ij} \tag{7.4b}$$

is the deviatoric stress tensor, while σ^{e}_{ij} is an, as of yet undetermined, equilibrium stress tensor. We will derive an expression for it below. The first term of Eq. (7.4b) enforces incompressibility. The second term is the stress due to the strain rate with η the viscosity. These two terms are well known from the Navier-Stokes equation. The third term is the stress due to activity. In the absence of activity, $\alpha = 0$, and elastic stresses, Eq. (7.4a) thus reduces to the overdamped limit of the Navier-Stokes equation. In this case Eqs. (7.3) and (7.4a) are the equations commonly used to describe passive liquid crystal hydrodynamics. Note that all of the derivatives ∇_i

here are covariant derivatives such that the equations are valid on a curved surface $\mathscr{M}$.

So far we have assumed that the geometry of the surface $\mathscr{M}$ is given. We now turn towards including the dynamics of $\mathscr{M}$ itself. Here, we are interested in coupling the dynamics of the liquid crystal to an elastic surface. In particular, we want to investigate how the presence of activity can introduce curvature in an initially flat disc, or distort an initially spherical shell. The idea being that the presence of active forces can distort the elastic surface, resulting in curvature. To describe the elastic surface we use the Helfrich free energy (Sec. 1.3.3, Refs. [149, 160, 161])

$$F_{\mathrm{H}} = \int \mathrm{d}A \left[\gamma + \kappa_{\mathrm{B}} \left(H - H_0\right)^2 + \kappa_{\mathrm{G}} K_{\mathrm{G}}\right] . \tag{7.5}$$

Here, γ is the surface tension acting towards reducing the surface area. κ_{B} is the bending modulus coupled to the mean curvature H and the spontaneous mean curvature H_0. We assume the latter to be constant. κ_{G} is the Gaussian bending modulus coupled to the Gaussian curvature K_{G}.

With this we have introduced all the ingredients of our model. The goal of the first part of this chapter chapter is to derive the equations of motion describing an active liquid crystal coupled to an elastic, deformable surface, and to investigate if an initially flat state can become unstable due to the presence of activity. However, before we derive these equations, we briefly introduce some notation to describe topological defects. We denote by (Sec. 1.1.3, Refs. [3, 10])

$$\rho_D(\boldsymbol{r}) := 2\pi \sum_i s_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) \tag{7.6}$$

the density of defects of charge s_i at position $\boldsymbol{r}_i$. It will be convenient to rewrite the Frank free energy as follows. We follow Bowick and Giomi [10] and introduce a pair of tensors $\boldsymbol{E}_\alpha$ that are locally orthonormal, such that

$$\boldsymbol{E}_\alpha \cdot \boldsymbol{E}_\beta = E_{\alpha i} E^i_\beta = \delta_{\alpha\beta} \tag{7.7}$$

while

$$E_{\alpha i} E^\alpha_j = g_{ij} . \tag{7.8}$$

Here we denote by greek indices the coordinates on the surface $\mathscr{M}$ and by latin indices the coordinates of the embedding space $\mathbb{R}^3$. Consequently, $\alpha, \beta = 1, 2$ and $i, j = 1, 2, 3$. Latin indices are used interchangeably with bold-face characters, see Sec. 1.A. These tangent vectors can thus still be used to define the metric, but unlike the tangent vectors defined directly from the surface parametrization (see Eq. (1.43)), they are locally orthonormal. Given an arbitrary vector field $\boldsymbol{w}$, it can be expressed in this basis as $\boldsymbol{w} = w^\alpha \boldsymbol{E}_\alpha$, with $w^\alpha = w^i E^\alpha_i$. Its covariant derivative can then be written as

$$\nabla_i v_\alpha = E_{\alpha j} \nabla_i w^j = \partial_i w_\alpha + \Omega_{i\alpha\beta} w^\beta , \tag{7.9}$$

where we used $\nabla_i E_{\alpha j} = 0$ and where we introduce the spin connection

$$\Omega_{i\alpha\beta} = \boldsymbol{E}_\alpha \cdot \partial_i \boldsymbol{E}_\beta \tag{7.10}$$

which is antisymmetric in its greek indices as is straightforward to see from the definition of $\boldsymbol{E}_\alpha$. In two dimensions it is thus possible to write it as $\Omega_{i\alpha\beta} = \epsilon_{\alpha\beta}\Omega_i$. The spin connection is thus defined by a single vector $\boldsymbol{\Omega}$. It is possible to show (see App. 7.A) that this vector is related to the Gaussian curvature via

$$\nabla \times \boldsymbol{\Omega} = \epsilon^{ij}\nabla_i\Omega_j = K_\mathrm{G} \ . \tag{7.11}$$

Now, in the locally orthonormal frame $\{\boldsymbol{E}_1, \boldsymbol{E}_2\}$ the director field $\boldsymbol{p}$ can be written as

$$\boldsymbol{p} = \cos\Theta\boldsymbol{E}_1 + \sin\Theta\boldsymbol{E}_2 \ . \tag{7.12}$$

Using this, the derivative of the director field can be written as

$$\nabla_i p_j = E_j^\alpha\left(\partial_i p_\alpha + \Omega_{i\alpha\beta}p^\beta\right) = -\epsilon_{\alpha\beta}p^\beta E_j^\alpha\left(\partial_i\Theta - \Omega_i\right) \ , \tag{7.13}$$

where in the second step we used $\partial_i p_\alpha = -\epsilon_{\alpha\beta}p^\beta\partial_i\Theta$. Thus, the Frank free energy can be written as

$$F_\mathrm{F} = \frac{\kappa_\mathrm{F}}{2}\int_\mathscr{M} \mathrm{d}A\left(\partial_i\Theta - \Omega_i\right)\left(\partial^i\Theta - \Omega^i\right) \ . \tag{7.14}$$

Minimizing the energy one finds the Euler-Lagrange equation $\nabla_i\left(\partial^i\Theta - \Omega^i\right) = 0$. As is straightforward to check, this equation is solved by a scalar field that obeys the equation

$$-\epsilon^{ij}\nabla_j\chi = \partial^i\Theta - \Omega^i \ . \tag{7.15}$$

As the spin connection is related to the Gaussian curvature it is possible to define the Airy stress function χ as [10, 174, 264]

$$\nabla^2\chi := K_\mathrm{G} - \rho_D \ . \tag{7.16}$$

Here we used that $\oint \mathrm{d}\Theta = 2\pi s_i$, which, using Stokes theorem, can be written as $\epsilon^{ij}\nabla_i\nabla_j\Theta = \rho_D(\mathrm{r})$. The Airy stress function thus is the solution of a Poisson equation. In the absence of Gaussian curvature, $\nabla^2\chi = -\rho_D$ and the Airy stress function is a defect potential, similar to the electric potential, where the defect density corresponds to the density of electric charges. If the Gaussian curvature does not vanish, it screens the defect charge density. This means that if defects with positive (negative) charge are present in regions of positive (negative) Gaussian curvature the "effective defect charge" $K_\mathrm{G} - \rho_D$ is reduced.

The Frank free energy Eq. (7.1) can then be written in terms of the Airy stress function:

$$F_\mathrm{F} = \int_\mathscr{M} \mathrm{d}A\left(\nabla_i\chi\right)^2 = -\int_\mathscr{M} \mathrm{d}A\chi\nabla^2\chi \ , \tag{7.17}$$

where in the second step we integrated by parts and dropped the boundary term. We will now turn towards deriving the equations of motion.

7.3 Derivation of the Equations of Motion

We derive the equations of motion for an active liquid crystal coupled to an elastic surface. Following Salbreux and Jülicher [143] we write down the general force and torque balance equations on a membrane. They read:

$$\nabla_i \boldsymbol{\sigma}^i = -\boldsymbol{\Xi}^{\text{ext}} , \tag{7.18a}$$

$$\nabla_i \boldsymbol{m}^i = \boldsymbol{\sigma}^i \times \boldsymbol{e}_i . \tag{7.18b}$$

Here, $\boldsymbol{\sigma}_i$ is the surface stress tensor, $\boldsymbol{m}_i$ the surface moment tensor, and $\boldsymbol{\Xi}^{\text{ext}}$ the external force per unit area. In writing these equations we assumed that there are no inertial forces (overdamped dynamics), no external torques, and that there is no deviatoric contribution to the moments. We can decompose the quantities appearing in this equation into tangential and normal components as follows:

$$\boldsymbol{\sigma}_i = \left(\sigma^{\text{e}}_{ij} + \sigma^{\text{d}}_{ij}\right) \boldsymbol{e}^j + \left(\sigma^{\text{e}}_{\text{n},i} + \sigma^{\text{d}}_{\text{n},i}\right) \boldsymbol{n} , \tag{7.19a}$$

$$\boldsymbol{m}_i = m^{\text{e}}_{ij} \boldsymbol{e}^j + m^{\text{e}}_{\text{n},i} \boldsymbol{n} , \tag{7.19b}$$

$$\boldsymbol{\Xi}^{\text{ext}} = \Xi^{\text{ext}}_i \boldsymbol{e}^i + \Xi^{\text{ext}}_{\text{n}} \boldsymbol{n} . \tag{7.19c}$$

Here, σ_{ij} is the tangential component of the surface stress tensor while $\sigma_{\text{n},i}$ is its normal component. $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{n}\}$ is the coordinate system on $\mathscr{M}$ as explained in Sec. 1.3. By the index e we denote the equilibrium component while we use d for deviatoric terms. By plugging Eqs. (7.19) into the Eqs. (7.18) one easily sees that the force and torque balance equations can be decomposed into in-plane and out-of-plane equations. We find for the force balance:

$$\nabla_i \boldsymbol{\sigma}^i = \nabla_i(\sigma^{ij} \boldsymbol{e}_j + \sigma^i_{\text{n}} \boldsymbol{n}) = \left(\nabla_i \sigma^{ij} + K^j_i \sigma^i_{\text{n}}\right) \boldsymbol{e}_j + \left(\nabla_i \sigma^i_{\text{n}} - \sigma^{ij} K_{ij}\right) \boldsymbol{n} = -\boldsymbol{\Xi}^{\text{ext}} , \tag{7.20a}$$

and for the torque balance:

$$\begin{aligned} \nabla_i \boldsymbol{m}^i &= \nabla_i(m^{ij} \boldsymbol{e}_j + m^i_{\text{n}} \boldsymbol{n}) = \left(\nabla_i m^{ij} + K^j_i m^i_{\text{n}}\right) \boldsymbol{e}_j + \left(\nabla_i m^i_{\text{n}} - m^{ij} K_{ij}\right) \boldsymbol{n} \\ &= \sigma^{ij} \boldsymbol{e}_j \times \boldsymbol{e}_i + \sigma^i_{\text{n}} \boldsymbol{n} \times \boldsymbol{e}_i = \sigma^{ij} \epsilon_{ji} \boldsymbol{n} + \sigma^i_{\text{n}} \epsilon_i^{\ j} \boldsymbol{e}_j , \end{aligned} \tag{7.20b}$$

where we used the Gauss-Weingarten equations. Projecting both equations onto the in-plane and normal components, respectively, we thus have four independent equations. Note that the second fundamental form K_{ij} couples the normal component of the surface stress to the in-plane force balance. This very general system of equations is connected to our problem by specifying the components in Eqs. (7.19). We take the equilibrium components to be the contribution of the elastic membrane and the elasticity of the liquid crystal, while the deviatoric components are due to the presence of activity. I.e., in the limit of a passive system the deviatoric components vanish. We now outline first the derivation of the equilibrium components and after that comment on the deviatoric components.

7.3.1 Equilibrium stress tensor

In this subsection we summarize the derivation of the stress tensor from the free energies Eqs. (7.5), (7.17). More details of the derivations can be found in the Appendix. In this derivation we combine and summarize the work of Refs. [162, 165, 166, 264, 327, 328]. In particular, Refs. [162, 165, 166, 327, 328] developed the very elegant method we will use below for deriving the stress tensor for an elastic membrane described by the Helfrich energy, see also Ref. [149]. Santiago [264], building on this work and using the same general ideas, used this approach to derive the stress tensor of a liquid crystal on a surface with surface tension, but without bending modulus or spontaneous curvature. Here, we combine both, deriving the equilibrium stress tensors for a liquid crystal on an elastic surface. We show that, due to the linearity of the equations, the resulting stress tensors are simply the sum of the stress tensors derived before, and we proof that there are no additional terms. In the following subsection we combine these equilibrium stress tensors with the deviatoric stress tensors (which has not been done before) and derive a set of equations describing the model outlined in the previous section.

The equilibrium components of the stress tensor can be found from the variation of the total free energy $F = F_\mathrm{F} + F_\mathrm{H}$ with respect to the position vector $\boldsymbol{X}$ of the surface $\mathscr{M}$ (see Sec. 1.3 for the definition). That is, we want to find how the free energy transforms when the position vector is varied as $\boldsymbol{X} \to \boldsymbol{X}' = \boldsymbol{X} + \delta\boldsymbol{X}$. Computing the variation of the different terms in the free energy explicitly, we find that it is possible to write the variation of the free energy in the form

$$\delta F = \int \mathrm{d}A\, \boldsymbol{\mathcal{S}}[\delta\boldsymbol{X}] \cdot \delta\boldsymbol{X} + \int \mathrm{d}A\, \nabla_a \mathcal{Q}^a[\delta\boldsymbol{X}]\,, \tag{7.21}$$

with the Euler-Lagrange derivative $\boldsymbol{\mathcal{S}}[\delta\boldsymbol{X}]$ and the Noether current $\mathcal{Q}^a[\delta\boldsymbol{X}]$. The explicit expression is rather lengthy and for easier readability we do not write it here. The full expression, as well as the derivation, can be found in App. 7.C. We can use this expression to find the equilibrium stress and moment tensor by considering different variations of the position vector $\delta\boldsymbol{X}$. Namely, to find the moment tensor, we have to consider how the free energy behaves under a generic infinitesimal rotation. To find the stress tensor, on the other hand, we consider an infinitesimal translation, i.e., $\delta\boldsymbol{X} = \boldsymbol{a}$. Translation invariance of the free energy implies that $\delta F = 0$ such that locally

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$$\boldsymbol{\mathcal{S}}[\delta\boldsymbol{X}] \cdot \delta\boldsymbol{X} = -\nabla_i \mathcal{Q}^i[\delta\boldsymbol{X}]\,. \tag{7.22}$$

Substituting $\delta\boldsymbol{X} = \boldsymbol{a}$ into this expression, it is possible to write the resulting equation as $\boldsymbol{\mathcal{S}} = -\nabla_i \boldsymbol{\sigma}^{\mathrm{e},i}$ through which the stress tensor $\boldsymbol{\sigma}^{\mathrm{e},i}$ is defined. The explicit expression can then easily be read off. We find that it is possible to split the stress tensor into tangential and normal component such that it can be written as

$$\boldsymbol{\sigma}^{\mathrm{e},i} = \sigma^{\mathrm{e},ij}\boldsymbol{e}_j + \sigma^{\mathrm{e},i}_{\mathrm{n}}\boldsymbol{n}\,, \tag{7.23a}$$

where

$$\begin{aligned}\sigma^{\mathrm{e}}_{ij} =& -\kappa_{\mathrm{B}}\left(H-H_0\right)K_{ij} + (\gamma + \kappa_{\mathrm{B}}\left(H-H_0\right)^2)g_{ij}\\ &+\kappa_{\mathrm{F}}\chi\Delta\chi g_{ij} - \kappa_{\mathrm{F}}\chi\nabla_i\nabla_j\chi + \kappa_{\mathrm{F}}\nabla_i\chi\nabla_j\chi\ , \qquad (7.23\mathrm{b})\\ \sigma^{\mathrm{e}}_{\mathrm{n},i} =& \kappa_{\mathrm{B}}\nabla_i H + \kappa_{\mathrm{F}}[(Kg_{ij}-K_{ij})\chi + 2(K_{ij}-g_{ij}K)]\nabla^j\chi\ . \qquad (7.23\mathrm{c})\end{aligned}$$

See App. 7.D for the derivation and further details. The moment tensor can be derived similarity when considering how the energy transforms under a infinitesimal rotation $\delta\boldsymbol{X} = \boldsymbol{b}\times\boldsymbol{X}$. We find $\boldsymbol{m}^i$ to be

$$\boldsymbol{m}^i = m^{ij}\boldsymbol{e}_j + m^i_{\mathrm{n}}\boldsymbol{n}\ , \qquad (7.24\mathrm{a})$$

with

$$\begin{aligned}m^{\mathrm{e}}_{ij} &= m_{ij} = \kappa_{\mathrm{B}}\left(H-H_0\right)\epsilon_{ij} + (2\kappa_{\mathrm{G}} - 4\kappa_{\mathrm{F}}\chi)H\epsilon_{ij} + (2\kappa_{\mathrm{F}}\chi - \kappa_{\mathrm{G}})K^k_i\epsilon_{kj}\ , \qquad (7.24\mathrm{b})\\ m^i_{\mathrm{n}} &= \kappa_{\mathrm{F}}\chi\nabla_j\chi\epsilon^{ij}\ . \qquad (7.24\mathrm{c})\end{aligned}$$

Again, see App. 7.D for the details.

7.3.2 Equations of motion

We now have found all the stress and moment tensors we need and can write down the explicit form of the equations of motion, Eqs. (7.20). First, note that both the tangential and normal projection of the torque balance, Eq. (7.20b), are automatically fulfilled for the stress and moment tensors that we derived. See App. 7.E for the proof. Thus, of the originally four equations in Eqs. (7.20) only two are non-trivial and determine the dynamics of the system. We now derive these equations. First, the tangential projection of the force balance is found from Eq. (7.20a) and reads

$$\nabla^i\sigma_{ij} + K_{ij}\sigma^i_{\mathrm{n}} = -\Xi^{\mathrm{ext}}_j\ , \qquad (7.25)$$

where $\sigma_{ij} = \sigma^{\mathrm{e}}_{ij} + \sigma^{\mathrm{d}}_{ij}$, and $\sigma^i_{\mathrm{n}} = \sigma^{\mathrm{e},i}_{\mathrm{n}}$. For the equilibrium terms on the left-hand side we find, after some straightforward manipulations,

$$\nabla^i\sigma^{\mathrm{e}}_{ij} + \sigma^i_{\mathrm{n}}K_{ij} = \kappa_{\mathrm{F}}\left[(-2\rho_D + K_{\mathrm{G}}\chi)\nabla_j\chi - K_{\mathrm{G}}\chi\nabla_j\chi\right]\ , \qquad (7.26)$$

where we used $\nabla^i K_{ij} = \nabla_j K$, $R_{ij} = K_{\mathrm{G}}g_{ij} = KK_{ij} - K_{ik}K^k_j$, and $[\nabla_i,\Delta]\chi = -K_{\mathrm{G}}\nabla_i\chi$. Second, the normal projection of the force balance is given by:

$$\nabla_i\sigma^i_{\mathrm{n}} - \sigma^{ij}K_{ij} = -\Xi^{\mathrm{ext}}_{\mathrm{n}}\ . \qquad (7.27)$$

Using the Theorema Egregium $4H^2 - K^{ij}K_{ij} = 2K_{\mathrm{G}}$ we find that the equilibrium terms on the left-hand side can be written as

$$\sigma^{\mathrm{e}}_{ij}K^{ij} - \nabla_i\sigma^{\mathrm{e},i}_{\mathrm{n}} = f^{\mathrm{e}}_{\mathrm{n}} + f^{\mathrm{d}}_{\mathrm{n}}\ , \qquad (7.28)$$

where

$$f^{\mathrm{e}}_{\mathrm{n}} = 2\gamma H - \kappa_{\mathrm{B}}\left\{\nabla_i\nabla^i H - \left(H-H_0\right)\left[2H(H-H_0) - 4H^2 + 2K_{\mathrm{G}}\right]\right\} \qquad (7.29)$$

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is a force due to the Helfrich free energy, and

$$f_{\mathrm{n}}^{\mathrm{d}} = 2\kappa_{\mathrm{F}}(2Hg^{ij} - K^{ij})\nabla_i\nabla_j\chi + 2\kappa_{\mathrm{F}}(K^{ij} - Hg^{ij})\nabla_i\chi\nabla_j\chi \tag{7.30}$$

is due to the Frank free energy.

Finally, to close the system of equations, we need an equation of motion for the director field that appears in the deviatoric stress tensor σ_{ij}^{d}. We choose the Leslie-Ericksen equation adapted to curved geometries, Eq. (7.3). To summarize, we thus have the following three equations of motion:

$$\nabla^i\sigma_{ij}^{\mathrm{d}} - 2\kappa_{\mathrm{F}}\rho_D\nabla_j\chi = D_t v_j - \Xi_j^{\mathrm{ext}}\,, \tag{7.31a}$$

$$K^{ij}\sigma_{ij}^{\mathrm{d}} + f_{\mathrm{n}}^{\mathrm{e}} + f_{\mathrm{n}}^{\mathrm{d}} = \mathrm{d}\partial_t h + \Xi_{\mathrm{n}}^{\mathrm{ext}}\,, \tag{7.31b}$$

$$D_t p^i = \left(g^{ij} - p^i p^j\right)\left(\lambda u_{jk}p^k - \omega_{jk}p^k + \frac{M_j}{\Gamma}\right)\,. \tag{7.31c}$$

Note that, because we assumed incompressibility, we furthermore have the condition

$$\nabla_i v^i = 0\,. \tag{7.32}$$

Lastly, we have introduced two terms containing time derivatives. In Eq. (7.31a) the time derivative of the velocity field, as known from the common Navier-Stokes equation. In Eq. (7.31b) the time derivative of the height field with a drag coefficient d.

7.3.3 Discussion

This equilibrium stress tensor in the limit $\kappa_{\mathrm{F}} = 0$ is the same as found in Refs. [149, 328] with opposite sign convention. $\boldsymbol{\sigma}_i^{\mathrm{e}}$ does not depend on κ_{G} which enters only through the boundary conditions when solving the force balance equations to find the surface shape. In the absence of activity and external forces, $\boldsymbol{\sigma}_i^{\mathrm{e}}$ is covariantly conserved, and this is equivalent to the classical shape equation of elastic membranes if $\kappa_{\mathrm{F}} = 0$. The Eqs. (7.31a) and (7.31b) are the hydrodynamic equations for the velocity field of the active matter and the deformation of the membrane, respectively, whereas Eq. (7.31c) describes the dynamic of the director. Note that $f_{\mathrm{n}}^{\mathrm{d}} = 0$ is just the von Kármán equation. Furthermore, with this stress tensor and in the absence of equilibrium components, Eq. (7.31a) reduces to the incompressible Stokes equation, with a force resulting from the active stress, commonly used to describe active liquid crystals, see Sec. 1.1.4 and Refs. [1–3, 26–28]. This is the case where the geometry of the surface $\mathscr{M}$ is fixed, see the beginning of this chapter.

7.3.4 Small-height approximation

Solving the system of equations Eqs. (7.31) in full generality is very hard. Instead of attempting this we will perform linear instability analysis. Here, we are interested in the linear instability of a flat surface. That is, at the onset of the instability curvatures are small and we can linearize the Eqs. (7.31) by working in the so-called small-height approximation of the Monge gauge [149, 329]. We introduce a coordinate system (x, y), where x and y are the usual two-dimensional cartesian coordinates in the plane, and describe the surface by a height function $h(x, y)$ above the flat reference plane. That is, we can write the position vector as

$$\boldsymbol{X}(x, y) = \begin{pmatrix} x \\ y \\ h(x, y) \end{pmatrix} . \tag{7.33}$$

It is then straightforward to compute the tangent vectors and from there one finds expressions for metric and curvature tensor. We provide the explicit expressions in App. 7.G. If deviations from the flat reference plane are small, i.e., if the curvature of the surface is small and $|\nabla h| \ll 1$, one can simplify the expressions significantly. At the linear level we find, for example, that the metric can be approximated by the flat metric, $g_{ij} = \delta_{ij} + \mathcal{O}(\nabla h^2)$, and the covariant derivative by the flat derivative. Furthermore, the second fundamental form reduces to $K_{ij} = -\partial_i \partial_j h + \mathcal{O}(\nabla h^2)$ such that $H = -\nabla^2 h/2 + \mathcal{O}(\nabla h^2)$ and $K_{\mathrm{G}} = \mathcal{O}(\nabla h^2)$. See, e.g., Ref. [149] and App. 7.G. Using this approximation we can simplify the equations of motion, Eqs. (7.31). Namely, we find that, at linear order in ∇h, Eq. (7.31a) can be written as

$$\eta \Delta v_i + f_i^{\mathrm{a}} - \nabla_i P_{\mathrm{h}} - 2\kappa_{\mathrm{F}} \rho_D \nabla_i \chi = \partial_t v_i - \Xi_i^{\mathrm{ext}} , \tag{7.34}$$

where Δ is the Laplacian, $f_i^{\mathrm{a}} = \nabla^j \sigma_{ij}^{\mathrm{a}}$ the active force, and where we used flow incompressibility, cf. Eq. (7.32). Furthermore, for the normal force balance, Eq. (7.31b), we find that

$$\begin{aligned} f_{\mathrm{n}}^{\mathrm{e}} &= -\gamma \nabla^2 h + \kappa_{\mathrm{B}} \left(\nabla^2 h\right) (\det \partial_i \partial_j h) + \frac{\kappa_{\mathrm{B}}}{4} \left(\nabla^2 h\right)^3 + \frac{\kappa_{\mathrm{B}}}{2} \nabla^2 \nabla^2 h \\ &= -\gamma \nabla^2 h + \frac{\kappa_{\mathrm{B}}}{2} \nabla^2 \nabla^2 h + \mathcal{O}(\nabla h^2) , \end{aligned} \tag{7.35}$$

as well as

$$\begin{aligned} f_{\mathrm{n}}^{\mathrm{d}} =& 4\kappa_{\mathrm{F}} \left[\partial_x \partial_y \chi - \partial_y \chi \partial_x \chi\right] \partial_x \partial_y h + \kappa_{\mathrm{F}} \left[(\partial_y \chi)^2 - (\partial_x \chi)^2 - 2\partial_y^2 \chi\right] \partial_x^2 h \\ &+ \kappa_{\mathrm{F}} \left[(\partial_x \chi)^2 - (\partial_y \chi)^2 - 2\partial_x^2 \chi\right] \partial_y^2 h + \mathcal{O}(\nabla h^2) . \end{aligned} \tag{7.36}$$

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For future reference, we summarize the equations above again here. In the small-height approximation, in the absence of external forces, and in the stationary limit, Eqs. (7.31) reduce to

$$\eta \Delta v_i + f_i^{\mathrm{a}} - \nabla_i P_{\mathrm{h}} - 2\kappa_{\mathrm{F}} \rho_D \nabla_i \chi = 0 , \tag{7.37a}$$

$$\sigma_{ij}^{\mathrm{d}} \nabla^i \nabla^j h + f_{\mathrm{n}}^{\mathrm{e}} + f_{\mathrm{n}}^{\mathrm{d}} = 0 , \tag{7.37b}$$

$$v^j \partial_j p^i = \left(\delta^{ij} - p^i p^j\right) \left(\lambda u_{jk} p^k - \omega_{jk} p^k + \frac{\kappa_{\mathrm{F}}}{\Gamma} \Delta p_j\right) , \tag{7.37c}$$

with $f_{\mathrm{n}}^{\mathrm{e}}$ and $f_{\mathrm{n}}^{\mathrm{d}}$ given by Eqs. (7.35) and (7.36), respectively. Lastly, the defining equation of the Airy stress function, Eq. (7.16), reduces to

$$\nabla^2 \chi = -\rho_D . \tag{7.38}$$

7.4 Defect-driven Buckling Instability

We want to study the dynamics of the membrane in a disc geometry with a topological defect at the center. As the disc is initially flat the spontaneous curvature vanishes, $H_0 = 0$. We write the director in polar coordinates as

$$p^i = \begin{pmatrix} \cos(\theta - \varphi) \\ \sin(\theta - \varphi) \end{pmatrix} , \tag{7.39}$$

with $\theta = s\varphi + \epsilon$, where ϵ is a constant and $2s \in \mathbb{Z}$ is the defect charge.

We then find after some manipulations that both components of the Leslie-Ericksen equation, that is Eq. (7.37c), yield the same equation for θ, namely:

$$\begin{aligned} v^j \nabla_j \theta = \frac{\partial_r (r v_\varphi) - \partial_\varphi v_r}{2r} + \frac{\lambda}{2} \Bigg[\sin(2\varphi - 2\theta) \left(r\partial_r \frac{v_r}{r} - \frac{\partial_\varphi v_\varphi}{r} \right) \\ + \cos(2\varphi - 2\theta) \left(r\partial_r \frac{v_\varphi}{r} + \frac{\partial_\varphi v_r}{r} \right) \Bigg] . \end{aligned} \tag{7.40}$$

Furthermore, in polar coordinates the active force appearing in Eq. (7.37a) has components

$$\boldsymbol{f}^{\mathrm{a}} = \begin{pmatrix} f_r^{\mathrm{a}} \\ f_\varphi^{\mathrm{a}} \end{pmatrix} = \frac{\alpha s}{r} \begin{pmatrix} \cos\left[2(s-1)\varphi + 2\epsilon\right] \\ \sin\left[2(s-1)\varphi + 2\epsilon\right] \end{pmatrix} . \tag{7.41}$$

We first consider a $+1$ defect, that is $s = 1$, as in this case the system is rotationally symmetric. Afterwards we consider other defect charges, in particular $s = \pm 1/2$.

7.4.1 Height equation: $+1$ defect

The simplest case is that of a $+1$ defect due to the rotational symmetry of the director field, that is, for $s = 1$ we have

$$p^i = \begin{pmatrix} \cos\epsilon \\ \sin\epsilon \end{pmatrix} . \tag{7.42}$$

We assume, without loss of generality, that $\chi(R) = 0$, and enforce no-slip boundary conditions $v_\varphi(r = R) = v_r(r = R) = 0$. Furthermore, we enforce that the height function h and all its derivatives vanish at $r = R$. Then the Airy stress function is easily found to be

$$\chi = -\ln \frac{r}{R} . \tag{7.43}$$

Assuming rotational symmetry of the velocity field, i.e., $v_\varphi = v_\varphi(r)$ and $v_r = v_r(r)$, the incompressibility condition Eq. (7.32) reads

$$\partial_r v_r(r) = 0 \tag{7.44}$$

and determines the radial velocity field up to a constant. Requiring the velocity field to be finite everywhere we thus find $v_r = 0$. Substituting this into Eq. (7.40) we find

$$0 = (1 + \lambda \cos 2\epsilon)(r\partial_r v_\varphi - v_\varphi) . \tag{7.45}$$

The solution to this equation is given by

$$\epsilon = \pm\frac{\arccos(-1/\lambda)}{2} , \tag{7.46}$$

i.e., the flow alignment parameter λ sets the geometry of the defect.

With this at hand we now turn to the generalized Stokes equation, Eq. (7.37a), to find an expression for the azimuthal velocity field. Note that the active force now reads

$$\boldsymbol{f}^{\mathrm{a}} = \begin{pmatrix} f_r^{\mathrm{a}} \\ f_\varphi^{\mathrm{a}} \end{pmatrix} = \frac{\alpha}{r}\begin{pmatrix} \cos 2\epsilon \\ \sin 2\epsilon \end{pmatrix} = \frac{\alpha}{r}\begin{pmatrix} -1/\lambda \\ \pm\sqrt{1-1/\lambda^2} \end{pmatrix} . \tag{7.47}$$

Because $P_{\mathrm{h}} = P_{\mathrm{h}}(r)$ and $\chi = \chi(r)$ we thus have two equations, namely the r- and φ-component of Eq. (7.37a), that read

$$0 = -\frac{\alpha}{\lambda r} - \partial_r P_{\mathrm{h}} - 2\rho_D \partial_r \chi , \tag{7.48}$$

$$0 = \eta \Delta v_\varphi \pm \frac{\alpha}{r}\sqrt{1-\frac{1}{\lambda^2}} . \tag{7.49}$$

Using $\partial_r \delta(r) = -\delta(r)/r$ we thus find the pressure field

$$P_{\mathrm{h}} = -\frac{\alpha}{\lambda}\ln\frac{r}{R} + \kappa_{\mathrm{F}} , \tag{7.50}$$

and the azimuthal velocity field

$$v_\varphi = \mp\frac{\alpha\sqrt{\lambda^2-1}}{2\eta\lambda} r \ln\frac{r}{R} , \tag{7.51}$$

where we used the boundary conditions to fix the integration constants. It remains to investigate the generalized shape equation, Eq. (7.37b). Note that in a rotationally symmetric system Eq. (7.36) reduces to

$$f_{\mathrm{n}}^{\mathrm{d}} = \kappa_{\mathrm{F}}\frac{(\partial_r\chi)^2 - 2\partial_r^2\chi}{r}\partial_r h - \kappa_{\mathrm{F}}\frac{2 + r\partial_r\chi}{r}(\partial_r\chi)(\partial_r^2 h) = -\kappa_{\mathrm{F}}\frac{\partial_r h}{r^3} + \kappa_{\mathrm{F}}\frac{\partial_r^2 h}{r^2} , \tag{7.52}$$

where we used Eq. (7.43) in the second step. Furthermore, in polar coordinates $\nabla_i\nabla_j h = \partial_i\partial_j h - \Gamma_{ij}^k \partial_k h$, with Γ_{ij}^k the Christoffel symbol associated with δ_{ij}. This is non-zero only if $i = j = r$, in which case $\nabla_r^2 h = \partial_r^2 h$, or if $i = j = \varphi$, then $\nabla_\varphi^2 h = r\partial_r h$. We thus find the following equation for the height function:

$$P_{\mathrm{h}}\nabla^2 h + \alpha\frac{\lambda-1}{2\lambda}\partial_r^2 h + \alpha\frac{\lambda+1}{2\lambda r}\partial_r h + \frac{\alpha}{2}\nabla^2 h - \frac{\kappa_{\mathrm{F}}\partial_r h}{r^3} + \frac{\kappa_{\mathrm{F}}\partial_r^2 h}{r^2} - \gamma\nabla^2 h + \frac{\kappa_{\mathrm{B}}}{2}\nabla^4 h = 0 \tag{7.53}$$

as $u_{ij}\nabla^i\nabla^j h = 0$. Hence, the velocity does not enter explicitly in the final equation. This equation can be written more compactly as

$$\frac{\kappa_{\mathrm{B}}}{2}\nabla^4 h - \gamma_{\mathrm{eff}}\nabla^2 h + \frac{\kappa_{\mathrm{F,eff}}}{r}\partial_r\left(\frac{\partial_r h}{r}\right) = 0 , \tag{7.54}$$

where $\gamma_{\mathrm{eff}} = \gamma - P_{\mathrm{h}}$ and $\kappa_{\mathrm{F,eff}} = \kappa_{\mathrm{F}} + \alpha r^2/(2\lambda)$.

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7.4.2 Height equation: Other defect charges

We now derive the shape equations for other defect charges as well. The derivation follows along the same lines as the one just presented for the +1 defect. However, we are no longer dealing with a rotationally symmetric system. Hence, we do not require the Leslie-Ericksen equation to be fulfilled in the following for simplicity, because it is not possible to find a stationary solution when including this equation. That is, there is not stationary solution if the flow is allowed to act back on the director field (backflow). Thus, we are only solving the height and Stokes equation simultaneously. We take $\theta = s\varphi$ with $2s \in \mathbb{Z}$, but $s \neq 1, 2$ for reasons seen during the calculation below. We set $\epsilon = 0$ now. Choosing a different value for ϵ just corresponds to an overall rotation of the system, without modifying the dynamics. This is different from the case of a +1 defect where this constant modifies the geometry of the defect non-trivially. For the Airy stress function we find the same expression as before:

$$\chi = -s \ln \frac{r}{R} \ . \tag{7.55}$$

The active force has components

$$\boldsymbol{f}^{\mathrm{a}} = \begin{pmatrix} f_r^{\mathrm{a}} \\ f_\varphi^{\mathrm{a}} \end{pmatrix} = \frac{\alpha s}{r} \begin{pmatrix} \cos\left[2(s-1)\varphi\right] \\ \sin\left[2(s-1)\varphi\right] \end{pmatrix} \tag{7.56}$$

in polar coordinates. Now we take the divergence of Eq. (7.37a). Due to incompressibility the velocity term vanishes and we find a Poisson equation for the pressure:

$$\nabla^2 P_{\mathrm{h}} = \nabla \cdot \boldsymbol{f}^{\mathrm{a}} - 2\kappa_{\mathrm{F}} \nabla^i \left(\rho_D \nabla_i \chi\right) \ , \tag{7.57}$$

and for $s \neq 1$ we find the pressure field to be

$$P_{\mathrm{h}} = -\frac{\alpha s}{2(s-1)} \cos[2(s-1)\varphi] + P^{(0)} + \kappa_{\mathrm{F}} \rho_D \ . \tag{7.58}$$

Here, $P^{(0)}$ is a solution of the Laplace equation $\nabla^2 P^{(0)} = 0$, i.e., $P^{(0)} = c_1 + c_2 \ln r$. Having found the pressure, we now turn towards finding the velocity field. First, note that incompressibility of the velocity field now yields the condition $\partial_r(r v_r) = -\partial_\varphi v_\varphi$. Using this and the expression for the pressure just derived, we find that the r-component of Eq. (7.37a) reads

$$3r\partial_r v_r + r^2 \partial_r^2 v_r + \partial_\varphi^2 v_r + v_r + \frac{\alpha r s}{\eta} \cos[2(s-1)\varphi] - \frac{c_2}{\eta} r = 0 \ . \tag{7.59}$$

Using the ansatz $v_r = A(r)\cos[2(s-1)\varphi] + B(r)\sin[2(s-1)\varphi]$, with $A(r)$ and $B(r)$ arbitrary functions, we find $\partial_\varphi^2 v_r = -4(s-1)^2 v_r$ which can be used to solve the equation for v_r; we find:

$$v_r = \frac{s\alpha\cos[2(s-1)\varphi] - c_2}{4(s-2)s\eta} r + \left(r^{a(s)} c_3 + r^{b(s)} c_4\right)\left(\cos[2(s-1)\varphi] + \sin[2(s-1)\varphi]\right) \tag{7.60}$$

for $s \neq 2$. See App. 7.H for the explicit expressions of the constants $a(s)$ and $b(s)$. We will set the integration constants c_3 and c_4 to zero in the following. We also need to set $c_2 = 0$ in order for the solution to agree with our ansatz. From the incompressibility condition we then obtain v_φ. In summary, we thus find for the velocity fields:

$$v_r = \frac{s\alpha \cos[2(s-1)\varphi]}{4(s-2)s\eta} r \,, \tag{7.61a}$$

$$v_\varphi = -\frac{\alpha \sin[2(s-1)\varphi]}{4(2-3s+s^2)\eta} r \,. \tag{7.61b}$$

This velocity field does not vanish at the boundary of the disc, however. Thus, we need to find an additional velocity field $\boldsymbol{v}_0$ to enforce the no-slip boundary conditions. Namely, we require that at $r = R$:

$$v_r + v_{0,r} = 0 \qquad \text{and} \qquad v_\varphi + v_{0,\varphi} = 0 \,, \tag{7.62}$$

and this defines the velocity $\boldsymbol{v}_0$. See App. 7.H for the derivation of the expression of this velocity field. We can then compute the strain rate tensor associated with the total velocity $\boldsymbol{v}+\boldsymbol{v}_0$ and write down the height equation. After some straightforward manipulations we find

$$\alpha \frac{1-\mathcal{C}}{s-2} \sin[2(s-1)\varphi]\partial_r \frac{\partial_\varphi h}{r} + \kappa_{\mathrm{F,eff}} \frac{\nabla^2 h - 2\partial_r^2 h}{r^2} - \gamma_{\mathrm{eff}} \nabla^2 h + \frac{\kappa_{\mathrm{B}}}{2} \nabla^4 h = 0 \tag{7.63}$$

with the new effective surface tension

$$\gamma_{\mathrm{eff}} = \gamma - P_{\mathrm{h}} - P_0 \,, \tag{7.64}$$

and the new effective Frank elastic constant

$$\kappa_{\mathrm{F,eff}} = \left(\frac{\alpha(\mathcal{C}+s-1)}{2(s-2)} r^2 \cos[2(s-1)\varphi] + \kappa_{\mathrm{F}} s(s-2) \right) \,. \tag{7.65}$$

Here we defined the term

$$\mathcal{C} = \left(2s^2 - 3s\right) \left(\frac{R}{r}\right)^{2(s-1)} + \left(3s - 2s^2 - 1\right) \left(\frac{R}{r}\right)^{2s} \,, \tag{7.66}$$

which enters due to the no-slip boundary conditions, to write the equations more compactly. If one takes $\boldsymbol{v}_0 = 0$ in the derivation above this term vanishes identically. Furthermore,

$$P_0 = \frac{\alpha s(3-2s)}{2(2-3s+s^2)} \left(\frac{R}{r}\right)^{2(s-1)} \cos\left[2(s-1)\varphi\right] \tag{7.67}$$

is the pressure field due to the velocity field $\boldsymbol{v}_0$, i.e., the solution of the equation $\eta\Delta\boldsymbol{v}_0 = \nabla P_0$. See App. 7.H for details.

7.4.3 Analysis: +1 defect

We now turn towards analyzing the height equation, investigating whether the presence of activity results in shape deformations of the initially flat membrane. We first present the results for the +1 defect. This case has been discussed in detail in Chapter 5 and we refer to this chapter for more details. Here, we merely outline and repeat the arguments in anticipation of the discussion of the $\pm 1/2$ defects, where the equations are more complicated but have a similar structure. Thus, it is instructive to briefly repeat the discussion of the simpler case of the +1 defect here. We first consider the case where both bending modulus and activity vanish, $\kappa_{\mathrm{B}} = 0$, $\alpha = 0$. This corresponds thus to the passive instability, first considered by Frank and Kardar [270]. In this case the height equation simply reads

$$\gamma \nabla^2 h + \frac{\kappa_{\mathrm{F}}}{r} \partial_r \left(\frac{\partial_r h}{r} \right) , \tag{7.68}$$

which can be integrated to yield the equation

$$\left(r^2 - R_{\mathrm{c}}^2 \right) \partial_r h = 0 , \tag{7.69}$$

with the critical length scale

$$R_{\mathrm{c}} = \sqrt{\frac{\kappa_{\mathrm{F}}}{\gamma}} , \tag{7.70}$$

at which the prefactor of the above equation vanishes. Due to the boundary conditions for the height function h that we impose (namely, h and all its derivatives vanish at the boundary), this equation admits a nontrivial solution only if $R > R_{\mathrm{c}}$. Now, for non-zero activity, but still $\kappa_{\mathrm{B}} = 0$, the height equation can be written in the same form, but with a different critical length. Namely, the prefactor vanishes if

$$\frac{\kappa_{\mathrm{F}}}{R_{\mathrm{c}}^2} = \gamma - \frac{\alpha}{2\lambda} - \frac{\alpha}{\lambda} \ln \frac{R}{R_{\mathrm{c}}} \tag{7.71}$$

is fulfilled, defining the critical length scale. There is no exact solution to this transcendental equation. However, the last term on the right-hand side is of secondary importance when interested in the onset of the instability, where $R \sim R_{\mathrm{c}}$. Neglecting this term we thus find

$$R_{\mathrm{c}} = \sqrt{\frac{\kappa_{\mathrm{F}}}{\gamma - \frac{\alpha}{2\lambda}}} . \tag{7.72}$$

As in the previous case, we find that there is a non-trivial solution only if $R > R_{\mathrm{c}}$ in which case the flat conformation becomes unstable. Note that the activity renormalizes the surface tension. For $\lambda > 1$ the activity reduces (increases) the effective surface tension in the presence of contractile (extensile) stresses, and vice versa for negative λ. Thus, for positive flow alignment buckling is favored for extensile activity, while it is inhibited for contractile activity. Lastly, we note that reintroducing the bending stiffness κ_{B} does not modify the buckling instability qualitatively but energetically punishes diverging curvatures. We refer to Chapter 5 for a discussion of this case.

7.4.4 Analysis: $+1/2$ defect

For $s = +1/2$ the height equation Eq. (7.63) takes the form:

$$\frac{2\alpha(r+R)\sin\varphi}{3R}\partial_r\frac{\partial_\varphi h}{r} + \kappa_{\mathrm{F,eff}}\frac{\nabla^2 h - 2\partial_r^2 h}{r^2} - \gamma_{\mathrm{eff}}\nabla^2 h + \frac{\kappa_{\mathrm{B}}}{2}\nabla^4 h = 0\ , \tag{7.73}$$

with the new effective surface tension

$$\gamma_{\mathrm{eff}} = \gamma - \frac{\alpha(4r+3R)}{6R}\cos\varphi\ , \tag{7.74}$$

and the new effective Frank elastic constant

$$\kappa_{\mathrm{F,eff}} = \frac{\alpha(2r+R)}{6R}r^2\cos\varphi - \frac{3\kappa_{\mathrm{F}}}{4}\ . \tag{7.75}$$

First, we consider the passive case $\alpha = 0$. In this case the height equation Eq. (7.73) reduces to

$$-\gamma\nabla^2 h + \frac{\kappa_{\mathrm{B}}}{2}\nabla^4 h - \frac{3\kappa_{\mathrm{F}}}{4r^2}\left(\frac{\partial_\varphi^2 h}{r^2} - r\partial_r\frac{\partial_r h}{r}\right) = 0\ . \tag{7.76}$$

Again, we investigate the problem in the absence of bending stiffness, i.e., in the limit $\kappa_{\mathrm{B}} = 0$, such that the equation reduces to:

$$\gamma\nabla^2 h + \frac{3\kappa_{\mathrm{F}}}{4r^2}\left(\frac{\partial_\varphi^2 h}{r^2} - r\partial_r\frac{\partial_r h}{r}\right) = 0\ . \tag{7.77}$$

To solve this equation we us a mode expansion ansatz:

$$h(r,\varphi) = \sum_n h_n(r)\cos(n\varphi)\ . \tag{7.78}$$

We are interested in the linear instability of the flat state and the lowest modes will be the first to be excited. For $n = 0$, i.e., the radially symmetric term, we can integrate the equation that results from substituting $h(r,\varphi) = h_0(r)$ into Eq. (7.77) to find

$$\left(r^2 - \frac{3\kappa_{\mathrm{F}}}{4\gamma}\right)\partial_r h_0 = 0\ , \tag{7.79}$$

and thus we find a length scale

$$R_{\mathrm{c}} = \sqrt{\frac{3\kappa_{\mathrm{F}}}{4\gamma}}\ . \tag{7.80}$$

For $n \neq 0$ we have an equation which can be written as

$$-\frac{n^2 h_n}{r^3}\left(r^2 + \frac{3\kappa_{\mathrm{F}}}{4\gamma}\right) + \partial_r\left(\left[r^2 - \frac{3\kappa_{\mathrm{F}}}{4\gamma}\right]\frac{\partial_r h_n}{r}\right) = 0\ . \tag{7.81}$$

Only for $n = 1$ can this equation be integrated to yield

$$\left(r^2 - \frac{3\kappa_{\mathrm{F}}}{4\gamma}\right)\partial_r\frac{h_1\cos\varphi}{r} = 0\ . \tag{7.82}$$

Thus, we find two length scales in Eq. (7.81), but only one of them is real for positive elastic constants. Note that an exact solution for this equation can be found but it violates our initial assumption of small gradients of the height function, $|\nabla h| \ll 1$, as the solution is logarithmically divergent for small r.

In the active case, but still with $\kappa_{\mathrm{B}} = 0$, we have to solve

$$\frac{2\alpha(r+R)\sin\varphi}{3R}\partial_r\frac{\partial_\varphi h}{r} + \left(\frac{\alpha(2r+R)}{6R}r^2\cos\varphi - \frac{3\kappa_{\mathrm{F}}}{4}\right)\frac{\nabla^2 h - 2\partial_r^2 h}{r^2} - \left(\gamma - \frac{\alpha(4r+3R)}{6R}\cos\varphi\right)\nabla^2 h = 0\ . \tag{7.83}$$

Looking at the two lowest modes $n = 0, 1$ again and dropping terms of order 2φ and higher, we find an equation for which we can consider the equation for the zeroth and first mode separately. Integrating both of these equations we find

$$\frac{\alpha(r+R)r^2}{6R}\partial_r\frac{h_1}{r} + \frac{3\kappa_{\mathrm{F}} - 4r^2\gamma}{4r}\partial_r h_0 = 0\ , \tag{7.84a}$$

and

$$4r^3(r+R)\alpha\partial_r h_0 + 3R\left(3\kappa_{\mathrm{F}} - 4r^2\gamma\right)r^2\partial_r\frac{h_1}{r} = 0\ . \tag{7.84b}$$

Substituting the first into the second equation yields, after some small manipulations, the equation

$$\left[9R^2(3\kappa_{\mathrm{F}} - 4r^2\gamma)^2 - 8r^4(r+R)^2\alpha^2\right]\partial_r\frac{h_1}{r} = 0\ . \tag{7.85}$$

As before, to find the critical length scale R_{c} we are interested in, we consider the case where the prefactor vanishes. This condition can be written as

$$\frac{\kappa_{\mathrm{F}}}{R_{\mathrm{c}}^2} = \frac{4}{3}\left[\gamma + \frac{|\alpha|}{3\sqrt{2}}\right] + \frac{8}{81}|\alpha|\frac{R_{\mathrm{c}}}{R}\ . \tag{7.86}$$

A few comments are in order. First, note that the second term on the right-hand side is due to our choice of boundary conditions, through which the radius of the disc enters this equation. While it is possible to solve the above equation exactly for R_{c}, the resulting expression is rather lengthy and not very insightful. The main effect of the instability, however, is the renormalization of the surface tension in the first term on the right-hand side. To underline this message we can neglect the second term on the right-hand side, which is due to the boundary conditions, as we did for the +1 defect above, such that it is possible to write a simple expression for the critical radius:

$$R_{\mathrm{c}} = \sqrt{\frac{9\kappa_{\mathrm{F}}}{12\gamma + 2\sqrt{2}|\alpha|}}\ . \tag{7.87}$$

The term neglected here actually reduces the critical radius slightly, with the relevance of the term slightly increasing with increasing κ_{F} and decreasing γ. The main difference of the critical radius compared with the one found for a +1 defect

(Eq. (7.72)) is that only the absolute value of the activity enters here. Thus, the sign of activity is not relevant for the question if there is a buckling instability, only its magnitude. This can be understood as follows: Unlike for the +1 defect, where the pressure and velocity field were radially symmetric, now they are varying in azimuthal direction. Considering the hydrodynamic pressure, for example, which gives rise to the effective surface tension above, we have

$$P_{\mathrm{h}} = \frac{\alpha}{2} \cos\varphi \, . \tag{7.88}$$

Thus, the pressure is always positive in some region and negative in another, independent of the sign of activity. The sign only determines in which area the pressure is positive and negative. Thus, we can interpret the above results in this view in predicting that for a +1/2 defect activity, irrespective of sign, drives a buckling instability. The magnitude of activity determines the height of the buckled state. However, this buckling is not rotationally symmetric, and the sign of activity determines if the buckled region is on the side of the defects' head or tail. We find that for extensile activity, $\alpha < 0$, the pressure is positive for $\varphi \in [\pi/2, 3\pi/2]$, that is on the side of the defects' head. On the other hand, for contractile activity, $\alpha > 0$, the pressure is positive for $\varphi \in [-\pi/2, \pi/2]$, i.e., on the side of the defects' tail. In either case, this is the region towards which the velocity of the defect core is pointing.

7.4.5 Analysis: $-1/2$ defect

For $s = -1/2$ the height equation Eq. (7.63) takes the form:

$$\frac{2\alpha}{5}\left(1 - \frac{2r^3}{R^3} + \frac{3r}{R}\right)\sin 3\varphi \frac{\partial_\varphi h}{r} + \kappa_{\mathrm{F,eff}} \frac{\nabla^2 h - 2\partial_r^2 h}{r^2} - \gamma_{\mathrm{eff}} \nabla^2 h + \frac{\kappa_{\mathrm{B}}}{2} \nabla^4 h = 0 \, , \tag{7.89}$$

with the new effective surface tension

$$\gamma_{\mathrm{eff}} = \gamma + \frac{\alpha\left(5R^3 + 8r^3\right)}{30R^3} \cos 3\varphi \, , \tag{7.90}$$

and the new effective Frank elastic constant

$$\kappa_{\mathrm{F,eff}} = \frac{5\kappa_{\mathrm{F}}}{4} + \frac{\alpha r^2\left(3R^3 + 6rR - 4r^3\right)}{10R^3} \cos 3\varphi \, . \tag{7.91}$$

Again, we first analyze the passive case $\alpha = 0$, and assume a vanishing bending modulus $\kappa_B = 0$. In this limit the equation reduces to

$$\frac{\nabla^2 h - 2\partial_r^2 h}{r^2} - \frac{4\gamma}{5\kappa_{\mathrm{F}}} \nabla^2 h = 0 \, . \tag{7.92}$$

As before, a mode analysis can be employed to analyze this equation, $h(r,\varphi) = \sum_n h_n(r)\cos(n\varphi)$. Substituting this ansatz into the above equation yields

$$\left[\left(\frac{5\kappa_{\mathrm{F}}}{4\gamma} - r^2\right)\frac{n^2 h_n}{r^3} + \partial_r\left(\left[r^2 + \frac{5\kappa_{\mathrm{F}}}{4\gamma}\right]\frac{\partial_r h_n}{r}\right)\right]\cos n\varphi = 0 \, . \tag{7.93}$$

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For $n = 0$ and $n = 1$ the equation can be integrated. We find:

$$\left(r^2 + \frac{5\kappa_{\mathrm{F}}}{4\gamma}\right)\partial_r h_0 = 0\,, \qquad \left(r^2 + \frac{5\kappa_{\mathrm{F}}}{4\gamma}\right)\partial_r \frac{h_1\cos\varphi}{r} = 0\,. \tag{7.94}$$

Comparing with the equation we found for the $+1/2$ defect we see that, apart from the numerical prefactor, the sign is reversed. Defining the critical length scale as before we therefore find

$$R_{\mathrm{c}} = \sqrt{-\frac{5\kappa_{\mathrm{F}}}{4\gamma}} \tag{7.95}$$

which is always imaginary for positive elastic constants. Thus, there is no real critical length scale for the $-1/2$ defect. Unlike the positive $+1$ and $+1/2$ defects, there is thus no passive buckling instability. We now turn towards investigating whether the presence of activity can cause the flat surface to become unstable.

For non-zero activity, but with $\kappa_{\mathrm{B}} = 0$, the height equation reads:

$$\begin{aligned}&\frac{2\alpha}{5}\left(1 - \frac{2r^3}{R^3} + \frac{3r}{R}\right)\sin 3\varphi\frac{\partial_\varphi h}{r} - \left(\gamma + \frac{\alpha\left(5R^3 + 8r^3\right)}{30R^3}\right)\nabla^2 h\\ &+ \left(\frac{5\kappa_{\mathrm{F}}}{4} + \frac{\alpha r^2\left(3R^3 + 6rR - 4r^3\right)}{10R^3}\cos 3\varphi\right)\frac{\nabla^2 h - 2\partial_r^2 h}{r^2} = 0\,.\end{aligned} \tag{7.96}$$

We are interested in the modes $n = 0$ and $n = 3$, since these are the modes appearing in the equations above, e.g., the active force. Thus, we substitute the mode ansatz and consider modes up to order $n = 3$. We find that the resulting equations can be split into two independent sets of equations, two equations coupling h_0 and h_3, as well as two equations coupling h_1 and h_2. We can thus consider h_0 and h_3 independently, and set $h_1 = h_2 = 0$. Thus, our ansatz reads $h(r) = h_0 + h_3\cos 3\varphi$. We find the following set of equations:

$$\begin{aligned}0 &= \left(4\gamma r^2 + 5\kappa_{\mathrm{F}}\right)\frac{\partial_r h_0}{r} + \frac{2\alpha}{15}\left(\frac{r}{R}\right)^3\left\{9\left[3\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^3 - 2\right]h_3\right.\\ &\quad\left. + \left[9\left(\frac{R}{r}\right)^2 + 7\left(\frac{R}{r}\right)^3 - 2\right]r\partial_r h_3\right\}\,,\end{aligned} \tag{7.97}$$

$$\begin{aligned}0 = &\left\{\left[4r^2\gamma - 5\kappa_{\mathrm{F}}\right](9h_3 - r\partial_r h_3) - 4\left[4r^2\gamma + 5\kappa_{\mathrm{F}}\right]r^2\partial_r^2 h_3 + \right.\\ &\frac{4\alpha r^3}{15}\left(\frac{r}{R}\right)^3\left(\left[9\left(\frac{R}{r}\right)^2 + 2\left(\frac{R}{r}\right)^3 - 10\right]\partial_r h_0\right.\\ &\left.\left. - \left[9\left(\frac{R}{r}\right)^2 + 7\left(\frac{R}{r}\right)^3 - 2\right]r\partial_r^2 h_0\right)\right\}\cos 3\varphi\,.\end{aligned} \tag{7.98}$$

However, it does not seem possible to define a new activity-dependent critical radius as was possible for the $+1$ or $+1/2$ defects. Thus, it seems that either there is no activity-induced buckling, or this method used above to find the instability is not straightforwardly applicable here to find the instability. However, a closer analysis of this equations and a comparison with numerical simulations of either the above equations, or the phase field simulations used in Chapters 5, 6 are needed to make a more accurate and confident statement about buckling instabilities of negative defects. We note that we have presented here the equations for a $-1/2$ defect, but it is straightforward to repeat the same calculations for the -1 defect. Up to numerical factors we find the same result as for the $-1/2$ defect.

7.5 Active Liquid Crystal on Torus

Above and in Chapter 5 we have considered a disc as the ground state geometry. In Chapter 6, on the other hand, we have considered a spherical geometry to be the ground state. Now, in this section, we present the derivation of the shape equation for an initially toroidal surface on which an active liquid crystal is present. It turns out that the derivation that follows is similar to the case of a spherical surface which we considered in Chapter 6. One main difference between a spherical and a toroidal geometry is the different defect structure required by the Poincaré theorem. A sphere is of genus zero and thus the total defect charge on the surface must add up to two, $\sum_i s_i = 2$. On the other hand, the genus of a torus is one such that the sum of the charge of all defects must vanish, $\sum_i s_i = 0$. In particular, a defect-free ground state is available. We now first list a few geometric quantities. Afterwards, we solve the Leslie-Ericksen and the Navier-Stokes equation on the torus. Finally, we write down the resulting shape equation.

7.5.1 Geometry of torus

The surface of a torus can be parametrized as

$$\boldsymbol{X}(\theta, \varphi) = \begin{pmatrix} (a + b\cos\theta)\cos\varphi \\ (a + b\cos\theta)\sin\varphi \\ b\sin\theta \end{pmatrix} \tag{7.99}$$

with the angles $\theta, \varphi \in [0, 2\pi]$ and the two radii a and b, where $a > b > 0$. From this we find the tangent vectors

$$\boldsymbol{e}_\theta = b \begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} , \qquad \boldsymbol{e}_\varphi = \begin{pmatrix} -(a + b\cos\theta)\sin\varphi \\ (a + b\cos\theta)\cos\varphi \\ 0 \end{pmatrix} , \tag{7.100}$$

such that the metric components are

$$g_{\theta\theta} = b^2 , \qquad g_{\varphi\varphi} = (a + b\cos\theta)^2 , \qquad g_{\theta\varphi} = 0 . \tag{7.101}$$

The non-trivial Christoffel symbols are

$$\Gamma^{\varphi}_{\theta\varphi} = -\frac{b\sin\theta}{a + b\cos\theta} , \qquad \Gamma^{\theta}_{\varphi\varphi} = \left(\frac{a}{b} + \cos\theta\right)\sin\theta . \tag{7.102}$$

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From the surface normal

$$\boldsymbol{n} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta \end{pmatrix} \tag{7.103}$$

we find the second fundamental form to be

$$K_{ij} = \begin{pmatrix} b & 0 \\ 0 & \cos\theta(a + b\cos\theta) \end{pmatrix} . \tag{7.104}$$

In the coordinate system of the tangent vectors the director field can be written in terms of an angle Θ as

$$\boldsymbol{p} = \frac{\cos\Theta}{|\boldsymbol{e}_\theta|}\boldsymbol{e}_\theta + \frac{\sin\Theta}{|\boldsymbol{e}_\varphi|}\boldsymbol{e}_\varphi \tag{7.105}$$

where we added the absolute value of the tangent vectors to ensure that, the director is a unit vector. It therefore has components

$$p^\theta = \frac{\cos\Theta}{b} , \qquad p^\varphi = \frac{\sin\Theta}{a + b\cos\theta} . \tag{7.106}$$

Finally, the mean and Gaussian curvature are found to be

$$H = \frac{a + 2b\cos\theta}{2b(a + b\cos\theta)} , \qquad K_\mathrm{G} = \frac{\cos\theta}{b(a + b\cos\theta)} . \tag{7.107}$$

We can now proceed by solving the Leslie-Ericksen and Navier-Stokes equations.

7.5.2 Leslie-Ericksen and Navier-Stokes equations on torus

To write down an explicit expression of the Stokes equation (Eq. (7.31a)) in the stationary limit, we compute the following terms. From

$$A^j := \nabla_i \left(p^i p^j\right) = p^j \left(\partial_i p^i + \Gamma^i_{ki} p^k\right) + p^i \left(\partial_i p^j + \Gamma^j_{ki} p^k\right) \tag{7.108}$$

and

$$\nabla_i A^i = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} A^i\right) \tag{7.109}$$

we find

$$\nabla_i \nabla_j \left(p^i p^j\right) = \frac{-\cos 2\Theta}{b(a + b\cos\theta)} \cos\theta . \tag{7.110}$$

Furthermore, from the commutator

$$\left[\nabla_i, \nabla_j \nabla^j\right] v_k = g_{ik} \nabla^j \left(K_\mathrm{G} v_j\right) - \nabla_k \left(K_\mathrm{G} v_i\right) = g_{ik} v_j \nabla^j K_\mathrm{G} - \nabla_k \left(K_\mathrm{G} v_i\right) , \tag{7.111}$$

where we used incompressibility $\nabla_i v^i = 0$ in the second step, we find

$$\nabla^j \left(\nabla_i \nabla^i v_j + K_\mathrm{G} v_j\right) = 2 v_j \nabla^j K_\mathrm{G} = -2 \frac{a \sin\theta}{b(a + b\cos\theta)^2} v^\theta . \tag{7.112}$$

The incompressibility condition can be written as

$$\nabla_i v^i = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}v^i\right) = \partial_i v^i - \frac{b\sin\theta}{a+b\cos\theta}v^\theta \ . \tag{7.113}$$

Assuming that $v^i = v^i(\theta)$ we find from this equation:

$$v^\theta = 0 \ . \tag{7.114}$$

Furthermore, taking the divergence of the Navier-Stokes equation yields

$$\Delta P_{\mathrm{h}} = -\frac{\alpha\cos 2\Theta}{b(a+b\cos\theta)}\cos\theta \ . \tag{7.115}$$

From this equation, and assuming $P_{\mathrm{h}} = P_{\mathrm{h}}(\theta)$, we can find the pressure field to be

$$P_{\mathrm{h}} = \alpha\cos 2\Theta\ln(a+b\cos\theta) + c_1\frac{2}{\sqrt{b^2-a^2}}\operatorname{artanh}\left[\frac{a-b}{\sqrt{b^2-a^2}}\tan\frac{\theta}{2}\right] + c_2 \ . \tag{7.116}$$

We have thus found an expression for all terms appearing in the Navier-Stokes equation. Substituting these terms, we find from the θ-component of the equation that $c_1 = 0$. Finally, we find that the φ-component can be written as the equation

$$(a+b\cos\theta)\partial_\theta^2 v^\varphi - 3b\sin\theta\partial_\theta v^\varphi = \frac{\alpha b^2\sin\theta\sin 2\Theta}{(a+b\cos\theta)\eta} \ . \tag{7.117}$$

Integrating the equation and using the integration constant to cancel the divergent terms we find the velocity field

$$v^\varphi(\theta) = \frac{b^2\left(a^2-b^2+3a(a+b\cos\theta)\right)\sin\theta}{(2a^2+b^2)(a+b\cos\theta)^2}\frac{\alpha\sin 2\Theta}{2\eta} \ . \tag{7.118}$$

For the Leslie-Ericksen equation (Eq. (7.31c)) we find after some lengthy but straightforward algebraic manipulations that it is possible to write

$$\Delta p^i = -\frac{\sin^2\theta}{(a+b\cos\theta)^2}p^i \ . \tag{7.119}$$

For the strain rate and vorticity tensors we find that the non-trivial components are given by

$$u_{\varphi\theta} = u_{\theta\varphi} = \frac{(a+b\cos\theta)^2}{2}\partial_\theta v^\varphi \ , \tag{7.120a}$$

$$\omega_{\varphi\theta} = -\omega_{\theta\varphi} = b(a+b\cos\theta)^2\sin\theta v^\varphi - \frac{(a+b\cos\theta)^2}{2}\partial_\theta v^\varphi \ , \tag{7.120b}$$

with all other components vanishing identically. The $\theta-$component of the Leslie-Ericksen equation reads

$$v^\varphi\nabla_\varphi p^\theta = \left(g^{\theta j} - p^\theta p^j\right)\left(\lambda u_{jk}p^k - \omega_{jk}p^k\right) \ . \tag{7.121}$$

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Using the results for the individual terms we found above, we find after some straightforward manipulations that this equation can be written as

$$\Theta = \frac{1}{2}\arccos\left(-\frac{1}{\lambda}\right) . \tag{7.122}$$

Thus, the angle of the director field is determined by the flow alignment parameter λ. Thus, to summarize, the hydrodynamic quantities we found are

$$v^{\varphi}(\theta) = \frac{\alpha}{\lambda}\frac{b^2\left(a^2 - b^2 + 3a(a + b\cos\theta)\right)\sin\theta}{(2a^2 + b^2)(a + b\cos\theta)^2}\frac{\sqrt{\lambda^2 - 1}}{2\eta} , \tag{7.123a}$$

$$P_{\mathrm{h}}(\theta) = -\frac{\alpha}{\lambda}\ln(a + b\cos\theta) + c_2 . \tag{7.123b}$$

7.5.3 Shape equation for torus

We can now turn towards writing down an explicit expression for the shape equation. Substituting all the relevant terms we found in the previous subsection, we find that it is possible to write the shape equation (Eq. (7.31b)) as

$$K^{ij}\sigma^{\mathrm{d}}_{ij} + f^{\mathrm{e}}_{\mathrm{n}} + f^{\mathrm{d}}_{\mathrm{n}} = P , \tag{7.124}$$

with

$$f^{\mathrm{e}}_{\mathrm{n}} = 2\gamma H - \kappa_{\mathrm{B}}\left\{\nabla_i\nabla^i H - (H - H_0)\left[2H(H - H_0) - 4H^2 + 2K_{\mathrm{G}}\right]\right\} , \tag{7.125}$$

and

$$f^{\mathrm{d}}_{\mathrm{n}} = 2\kappa_{\mathrm{F}}(2Hg^{ij} - K^{ij})\nabla_i\nabla_j\chi + 2\kappa_{\mathrm{F}}(K^{ij} - Hg^{ij})\nabla_i\chi\nabla_j\chi , \tag{7.126}$$

where $\Delta\chi = K_{\mathrm{G}}$ as no defects are present. We find

$$K^{ij}\sigma^{\mathrm{d}}_{ij} = -2P_{\mathrm{h}}H + \frac{\alpha a}{2\lambda b(a + b\cos\theta)} , \tag{7.127}$$

and

$$\chi = -\ln(a + b\cos\theta) + c_1 , \tag{7.128}$$

with an integration constant c_1. From this we find

$$f^{\mathrm{d}}_{\mathrm{n}} = \kappa_{\mathrm{F}}\frac{a + 4b\cos^3\theta + 3a\cos 2\theta}{2b(a + b\cos\theta)^3} \tag{7.129}$$

such that the shape equation reads

$$\begin{aligned} P =& 2\left(\gamma + \frac{\alpha}{\lambda}\ln(a + b\cos\theta)\right)H + \kappa_{\mathrm{F}}\frac{a + 4b\cos^3\theta + 3a\cos 2\theta}{2b(a + b\cos\theta)^3} + \frac{\alpha}{2\lambda}\frac{a}{b(a + b\cos\theta)} \\ & - \kappa_{\mathrm{B}}\left\{\nabla_i\nabla^i H - (H - H_0\left[2H(H - H_0) - 4H^2 + 2K_{\mathrm{G}}\right]\right\} . \end{aligned} \tag{7.130}$$

This equation assumes that the toroidal shape is fixed but can still be used as a starting point for considering shape deformations of a torus. In particular, as the

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term $u_{ij}K^{ij}$ vanished identically, one can investigate the properties of the active pressure field to get a first idea of the active forces acting on the torus. For a more complete picture, it is necessary to consider perturbations of the toroidal shape and solving the resulting perturbed equations. This way it is possible to see if activity enhances or suppresses certain perturbations. For example, one could consider $b \to b + \delta b(\varphi)$. However, the complexity of the equations increases significantly and it is not clear if it is possible to find an analytical expression for $\delta b(\alpha)$, the activity-induced perturbation. As a first step one can neglect the corrections to the hydrodynamic equations due to $b \to b + \delta b(\varphi)$ and only consider the shape equation, hence assuming that, for example, velocity and director field are not modified by the perturbation of the outer radius. We will not perform a rigorous analysis of this case here but leave it for future work. We conclude this chapter with some summarizing remarks.

7.6 Conclusion and Outlook

In the first part of this chapter we have derived a set of equations that can be used to describe shape deformations of elastic surfaces in the presence of active liquid crystals. In our model, the director field is taken to be a two-dimensional vector field that is confined to the two-dimensional surface. We did not consider the case of a surface with a finite thickness or of a three-component director field that is not necessarily constrained to be on the surface. As mentioned in the introduction, we did not include explicit coupling between the extrinsic curvature and the director field in our model. This results in a minimal model that is nevertheless quite complex. The main source of complexity is due to the shape equation which, even in the passive case, is notoriously difficult to solve. As a first step we thus applied the model to one of the simplest possible problems, an initially flat disc with a single topological defect being present. We derived an explicit expression for the equations of motion in this case for an arbitrary defect charge s. These were the equations used in Chapter 5 in the case $s = +1$. In this chapter, we analyzed the equations for defect charges $\pm 1/2$. We presented preliminary results pointing towards a buckling instability for a positive charge and a stable flat surface for a negative charge. However, more numerical work is needed to substantiate these claims. Apart from an initially flat surface, it is possible to consider different ground state geometries. The case of a sphere was investigated in detail in chapter Chapter 6. Here, we presented preliminary results for a toroidal geometry. We solved the respective equations in the unperturbed state and thus found the velocity and director field on an undeformed torus. This can be taken as the ground state from which, using linear stability analysis, it is possible to investigate if the toroidal shape is unstable due to the presence of activity, even though no defects are present.

7.A Spin Connection and Gaussian Curvature

We derive in this section the relation between the curl of the spin connection and the Gaussian curvature. We follow Bowick and Giomi [10]. The Riemann tensor is defined as the commutator of the covariant derivatives. Namely, for an arbitrary vector w^a:

$$[\nabla_a, \nabla_b] w_c = R^d_{abc} w_d \; . \tag{7.131}$$

In the infinitesimal version, this can be transformed into a statement about how a vector is transformed as it is transported parallelly along an infinitesimal square loop of sides dx and dy:

$$\Delta w^a = R^a_{bcd} w^b \mathrm{d}x^c \mathrm{d}x^d \; , \tag{7.132}$$

where $\Delta w^a = w'^a - w^a$ is the difference between the original vector w^a and the vector after parallel transport, w'^a, both of which are situated at the same point of the manifold. Using the locally orthonormal tangent vectors $\{\boldsymbol{E}_\alpha\}$ this can be written as

$$\Delta w^\alpha = R^\alpha_{bc\beta} w^\beta \mathrm{d}x^c \mathrm{d}x^d \; , \tag{7.133}$$

by simply multiplying both sides with the tangent vectors, thereby projecting the vector w^a onto the local orthonormal coordinate system. Here, $R^\alpha_{bc\beta}$ is the curvature tensor associated with the spin connection. In two dimensions, where $\Omega_{i\alpha\beta} = \epsilon_{\alpha\beta}\Omega_i$, it can be written as

$$R_{ab\alpha\beta} = \partial_a \Omega_b \epsilon_{\alpha\beta} - \partial_b \Omega_a \epsilon_{\alpha\beta} + \Omega_a \Omega_b \epsilon_{\alpha\gamma} \epsilon^\gamma_\beta - \Omega_b \Omega_a \epsilon_{\alpha\gamma} \epsilon^\gamma_\beta = \left(\partial_a \Omega_b - \partial_b \Omega_a\right) \epsilon_{\alpha\beta} \; . \tag{7.134}$$

On the other hand, the two Riemann tensors are related through the projection operators:

$$R_{ab\alpha\beta} = R_{abcd} E^c_\alpha E^d_\beta \; . \tag{7.135}$$

We thus have the relation

$$R_{ab\alpha\beta} = R_{abcd} E^c_\alpha E^d_\beta = \left(\partial_a \Omega_b - \partial_b \Omega_a\right) \epsilon_{\alpha\beta} \; . \tag{7.136}$$

Using that in two dimensions the Riemann tensor is given by

$$R_{abcd} = K_{\mathrm{G}} \left(g_{ac} g_{bd} - g_{ad} g_{bc}\right) \tag{7.137}$$

we have

$$\left(\partial_a \Omega_b - \partial_b \Omega_a\right) \epsilon_{\alpha\beta} = K_{\mathrm{G}} \left(g_{a\alpha} g_{b\beta} - g_{a\beta} g_{b\alpha}\right) \; . \tag{7.138}$$

Contracting both sides with the respective inverse metric yields

$$\left(\partial_a \Omega_b - \partial_b \Omega_a\right) \epsilon^{ab} = K_{\mathrm{G}} \; . \tag{7.139}$$

7.B Deformations

To find the equations of a membrane equipped with nematic structure in equilibrium we can look at the deformation of its free energy. In this section we derive some general formulas we will need in the following. A variation of the position vector $\boldsymbol{X} \to \boldsymbol{X}' = \boldsymbol{X} + \delta\boldsymbol{X}$ can be written as $\delta\boldsymbol{X} = \Phi^a \boldsymbol{e}_a + \Phi\boldsymbol{n}$, with the two functions Φ^a

and Φ quantifying the tangential and normal deformation, respectively. From this the variation of the tangent vector $\boldsymbol{e}_a$ is found to be

$$\delta\boldsymbol{e}_a = \delta\left(\partial_a\boldsymbol{X}\right) = \partial_a(\delta\boldsymbol{X}) = (\nabla_a\Phi^b)\boldsymbol{e}_b - K_{ab}\Phi^b\boldsymbol{n} + (\nabla_a\Phi)\boldsymbol{n} + \Phi K_{ab}g^{bc}\boldsymbol{e}_c \ , \quad (7.140)$$

where $[\delta, \partial_a] = 0$ was used in the first step and the Gauss-Weingarten equations in the second step. For simplicity, to keep the expressions shorter, it will be useful below to consider the tangential variations $\delta_{||}\boldsymbol{X} = \Phi^a\boldsymbol{e}_a$ and the normal variations $\delta_{\perp}\boldsymbol{X} = \Phi\boldsymbol{n}$ separately. For the tangent vectors we thus have

$$\delta_{||}\boldsymbol{e}_a = (\nabla_a\Phi^b)\boldsymbol{e}_b - K_{ab}\Phi^b\boldsymbol{n} \ , \quad (7.141a)$$

$$\delta_{\perp}\boldsymbol{e}_a = (\nabla_a\Phi)\boldsymbol{n} + \Phi K_{ab}g^{bc}\boldsymbol{e}_c \ . \quad (7.141b)$$

On the other hand, the deformation of the surface normal is easily found from using the product rule as well as the relations $\boldsymbol{e}_a \cdot \boldsymbol{n} = 0$ and $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ to be

$$\delta_{||}\boldsymbol{n} = K_{ab}\Phi^a g^{bc}\boldsymbol{e}_c \ , \quad (7.142a)$$

$$\delta_{\perp}\boldsymbol{n} = -(\nabla_a\Phi)g^{ab}\boldsymbol{e}_b \ . \quad (7.142b)$$

From the variation of the tangent vectors we can find the variation of the metric $g_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{e}_b$ to be $\delta g_{ab} = (\delta\boldsymbol{e}_a) \cdot \boldsymbol{e}_b + \boldsymbol{e}_a \cdot (\delta\boldsymbol{e}_b)$. Using Eqs. (7.141) and $\boldsymbol{e}_a \cdot \boldsymbol{n} = 0$ it is straightforward to see that

$$\delta_{||}g_{ab} = \nabla_a\Phi_b + \nabla_b\Phi_a \ , \quad (7.143a)$$

$$\delta_{\perp}g_{ab} = 2K_{ab}\Phi \ . \quad (7.143b)$$

For the inverse metric we can use its definition to find $\delta\left(g^{ab}g_{bc}\right) = 0$ such that

$$\delta_{||}g^{ab} = -\nabla^a\Phi^b - \nabla^b\Phi^a \ , \quad (7.144a)$$

$$\delta_{\perp}g^{ab} = -2K^{ab}\Phi \ . \quad (7.144b)$$

From this it is found that the area element $\mathrm{d}A = \sqrt{\det g_{ab}}\mathrm{d}s_1\mathrm{d}s_2$ transforms as

$$\delta_{||}\mathrm{d}A = \mathrm{d}A\nabla_a\Phi^a \ , \quad (7.145a)$$

$$\delta_{\perp}\mathrm{d}A = \mathrm{d}AK\Phi \ . \quad (7.145b)$$

We now turn towards computing the variation of the Ricci scalar and the mean curvature. For this we first need to compute the variation of the Christoffel symbols. In general, we have

$$\delta\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\nabla_b\delta g_{ad} + \nabla_a\delta g_{bd} - \nabla_d\delta g_{ab}) \ . \quad (7.146)$$

Substituting Eqs. (7.143) and using the Codazzi-Mainardi equation $\nabla_a K_{bc} = \nabla_c K_{ab}$ we find, after some straightforward steps, that the variations of the Christoffel symbol can be written as:

$$\delta_{||}\Gamma^c_{ab} = \frac{1}{2}([\nabla_b, \nabla^c]\Phi_a + [\nabla_a, \nabla^c]\Phi_b + \{\nabla_a, \nabla_b\}\Phi^c) \ , \quad (7.147a)$$

$$\delta_{\perp}\Gamma^c_{ab} = K^c_a\nabla_b\Phi + K^c_b\nabla_a\Phi - K_{ab}\nabla^c\Phi + (\nabla_a K^c_b)\Phi \ . \quad (7.147b)$$

We use this results to compute the variation of the Ricci scalar. Again, the derivation is straightforward but rather lengthy. As a final result we find:

$$\delta_{\|} R = \Phi^a \nabla_a R \;, \tag{7.148a}$$
$$\delta_{\perp} R = -2R_{ab}K^{ab}\Phi + 2\nabla_a[(K^{ab} - g^{ab}K)\nabla_b\Phi] \;. \tag{7.148b}$$

Lastly, for the trace of the curvature tensor K we find from its definition after using Eqs. (7.141) and (7.143) that

$$\delta_{\|} K = \Phi^b \nabla^a K_{ab} \;, \tag{7.149a}$$
$$\delta_{\perp} K = -\nabla^2\Phi + (R - K^2)\Phi \;. \tag{7.149b}$$

The variation of the mean curvature H is found from remembering that, in our convention, $H = K/2$.

It remains to consider the variation of the Airy Function χ and of the defect charge density ρ_D. The Airy function is defined as $\nabla^2\chi = K_{\mathrm{G}} - \rho_D$, thus $\delta\nabla^2\chi = \delta K_{\mathrm{G}} - \delta\rho_D$. The total defect charge density is conserved under a small variation, that is

$$0 = \delta \int \mathrm{d}A\rho_D = \int (\delta \mathrm{d}A)\, \rho_D + \int \mathrm{d}A\delta\rho_D \;, \tag{7.150}$$

and it follows from Eqs. (7.145) that

$$\delta_{\|} \rho_D = -\rho_D \nabla_a \Phi^a \;, \tag{7.151a}$$
$$\delta_{\perp} \rho_D = -\Phi^a K \rho_D \;. \tag{7.151b}$$

These are all the expressions for the variation of quantities we will need to derive the stress tensor form the variation of the free energy. However, it will turn out that it is convenient to compute the commutator of the variation and the Laplace operator as well. Before turning to the variation of the free energy we will briefly state these results. We denote by J the commutator $J := [\delta, \nabla^2]f$, where f is a generic scalar function. We have,

$$J = [\delta, \nabla^2]f = \delta\left(g^{ab}\nabla_a\nabla_b f\right) - g^{ab}\nabla_a\nabla_b\delta f = \left(\delta g^{ab}\right)\nabla_a\nabla_b f - g^{ab}\left(\delta\Gamma^c_{ab}\right)\nabla_c f \;, \tag{7.152}$$

where we used $\delta f = 0$, $[\delta, \partial_a]f = 0$, $\nabla_a f = \partial_a f$, and $\nabla_a\nabla_b f = \partial_a\partial_b f - \Gamma^c_{ab}\partial_c f$. Substituting Eqs. (7.143), (7.144), and (7.147), as well as using the Theorema Egregium $R_{ab} = K_{\mathrm{G}} g_{ab}$, and $[\nabla_a, \nabla_b]f = 0$ we find[1]:

$$J_{\|} = [\delta_{\|}, \nabla^2]f = -2\nabla^a\Phi^b\nabla_a\nabla_b f - K_{\mathrm{G}}\Phi^a\nabla_a f - \nabla^2\Phi^a\nabla_a f \;, \tag{7.153a}$$
$$J_{\perp} = [\delta_{\perp}, \nabla^2]f = -2K^{ab}\Phi^a\nabla_a\nabla_b f + \nabla_a[(Kg^{ab} - 2K^{ab})\Phi^a]\nabla_b f \;. \tag{7.153b}$$

These are all the preliminary results we will need. We can now turn towards the variation of the free energies, namely we want to compute the variation of the Frank free energy

$$F_{\mathrm{F}} = -\int_M \mathrm{d}A\chi\nabla^2\chi \;, \tag{7.154}$$

[1]Note that there is a typo in Santiago Ref. [264] in the expression for $J_{\|}$ which we correct here.

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as well as the Helfrich free energy

$$F_{\mathrm{H}} = \int \mathrm{d}A \left[\gamma + \kappa_{\mathrm{B}} \left(H - H_0\right)^2 + \kappa_{\mathrm{G}} K_{\mathrm{G}}\right] . \tag{7.155}$$

It is easy to see that a generic variation of the Frank free energy can be written as

$$\delta F_{\mathrm{F}} = -\int_M (\delta \mathrm{d}A)\chi\nabla^2\chi + \int_M \mathrm{d}A\chi(J + 2\delta\rho_D - 2\delta K_{\mathrm{G}}) , \tag{7.156}$$

where we integrated by parts (dropping boundary terms) and used $J = [\delta, \nabla^2]\chi$. Similarly, the variation of the Helfrich free energy can be written as

$$\delta F_{\mathrm{H}} = \int (\delta \mathrm{d}A)[\gamma + \kappa_{\mathrm{B}} \left(H - H_0\right)^2 + \kappa_{\mathrm{G}} K_{\mathrm{G}}] + \int \mathrm{d}A[\kappa_{\mathrm{B}} \left(H - H_0\right)^2 \delta K + \frac{\kappa_{\mathrm{G}}}{2}\delta R] , \tag{7.157}$$

where we used $\delta\left(H^2\right) = 2H\delta H$, $2H = K$, and $R = 2K_{\mathrm{G}}$. However, finding an explicit expression involves lengthy expressions and thus we consider parallel and normal variations separately in the following section.

7.C Deformation of the Energy and Shape Equation

We first consider the parallel deformation of the free energy, that is we want to find an explicit expression for

$$\delta_{\|} F = \delta_{\|} F_{\mathrm{F}} + \delta_{\|} F_{\mathrm{H}} . \tag{7.158}$$

To this end, we start from Eqs. (7.156) and (7.157). We then substitute Eqs. (7.145), (7.153), (7.151), (7.148), and (7.149). This yields a lengthy expression for the variation of the free energy. However, it is possible to compactly write it in the from

$$\delta_{\|} F = \int \mathrm{d}A \mathcal{S}^a \Phi_a + \nabla_a \mathcal{Q}^a_{\|} . \tag{7.159}$$

To achieve this it is necessary to integrate terms which contain a derivative of Φ^a by parts. The resulting boundary terms can be written as an area integral over a divergence and hence the expression takes the above form. Writing $\mathcal{S}^a = \mathcal{S}^a_{\mathrm{H}} + \mathcal{S}^a_{\mathrm{F}}$ and $\mathcal{Q}^a_{\|} = \mathcal{Q}^a_{\|,\mathrm{H}} + \mathcal{Q}^a_{\|,\mathrm{F}}$ we find the following explicit expressions:

$$\mathcal{S}^a_{\mathrm{H}} = 0 , \tag{7.160a}$$

$$\mathcal{S}^a_{\mathrm{F}} = 2\kappa_{\mathrm{F}}\rho_D\nabla^a\chi , \tag{7.160b}$$

$$\mathcal{Q}^a_{\|,\mathrm{H}} = [\gamma + \kappa_{\mathrm{B}} \left(H - H_0\right)^2 + \kappa_{\mathrm{G}} K_{\mathrm{G}}]\Phi^a , \tag{7.160c}$$

$$\mathcal{Q}^a_{\|,\mathrm{F}} = \kappa_{\mathrm{F}} \left\{-\chi(\rho_D + K_{\mathrm{G}})\Phi^a - \chi\Phi^b\nabla^a\nabla_b\chi + \Phi^b\nabla^a\chi\nabla_b\chi - \chi\nabla^a\Phi^b\nabla_b\chi\right\} . \tag{7.160d}$$

Note that the contribution from the Helfrich energy is just a boundary term because the energy is reparametrization invariant as is expected for the energy of a fluid membrane. The Frank free energy, on the other hand, is not and thus there is a bulk term contribution.

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Similarly, the perpendicular perturbation can be written in the form

$$\delta_\perp F = \int \mathrm{d}A\mathcal{S}\Phi + \int \mathrm{d}A\nabla_a \mathcal{Q}^a_\perp \tag{7.161}$$

with

$$\mathcal{S}_\mathrm{H} = 2H\gamma - \kappa_\mathrm{B}\left\{\nabla_i\nabla^i H - (H-H_0)\left[2H(H-H_0) - 4H^2 + 2K_\mathrm{G}\right]\right\}\ , \tag{7.162a}$$

$$\mathcal{S}_\mathrm{F} = \kappa_\mathrm{F}(2K^{ab} - Kg^{ab})\nabla_a\chi\nabla_b\chi + 2\kappa_\mathrm{F}(Kg^{ab} - K^{ab})\nabla_a\nabla_b\chi\ , \tag{7.162b}$$

$$\mathcal{Q}^a_{\perp,\mathrm{H}} = \kappa_\mathrm{B}(\nabla^a(H-H_0)\Phi - (H-H_0)\nabla^a\Phi) + \kappa_\mathrm{G}(K^{ab}\nabla_b\Phi - 2H\nabla^a\Phi)\ , \tag{7.162c}$$

$$\mathcal{Q}^a_{\perp,\mathrm{F}} = \kappa_\mathrm{F}[(Kg^{ab} - 2K^{ab})\chi + 2(K^{ab} - g^{ab}K)]\Phi\nabla_b\chi - 2\kappa_\mathrm{F}\chi(K^{ab} - g^{ab}K)\nabla_b\Phi\ . \tag{7.162d}$$

In summary, it is possible to write the variation of the free energy as

$$\delta F = \int \mathrm{d}A\boldsymbol{\mathcal{S}}\cdot\delta\boldsymbol{X} + \int \mathrm{d}A\nabla_a\mathcal{Q}^a\ , \tag{7.163}$$

where $\boldsymbol{\mathcal{S}} = (\mathcal{S}_\mathrm{H} + \mathcal{S}_\mathrm{F})\boldsymbol{n} + \mathcal{S}^a_\mathrm{F}\boldsymbol{e}_a$. The Euler-Lagrange equations found from this free energy show that in equilibrium we have $\boldsymbol{\mathcal{S}} = 0$ and therefore the tangential and normal components must vanish independently. We obtain two equations:

$$\begin{aligned}0 = \kappa_\mathrm{F}[(2K^{ab} - Kg^{ab})\nabla_a\chi\nabla_b\chi + 2(Kg^{ab} - K^{ab})\nabla_a\nabla_b\chi] + 2H\gamma\\ - \kappa_\mathrm{B}\left\{\nabla_i\nabla^i H - (H-H_0)\left[2H(H-H_0) - 4H^2 + 2K_\mathrm{G}\right]\right\}\end{aligned} \tag{7.164}$$

and $2\kappa_\mathrm{F}\rho_D\nabla^a\chi = 0$. However, instead of the equilibrium equations of motion we need to find the stress and torque tensor. Their derivation is outlined in the following section.

7.D Stress Tensor and Torque Tensor

To find the equilibrium stress tensor we consider how the free energy transforms under an infinitesimal translation $\boldsymbol{a}$, i.e., $\delta\boldsymbol{X} = \boldsymbol{a}$. Thus, $\Phi = \boldsymbol{a}\cdot\boldsymbol{n}$, $\Phi^a = \boldsymbol{a}\cdot\boldsymbol{e}^a$, and $\nabla_b\Phi^a = \nabla_b(\boldsymbol{a}\cdot\boldsymbol{n}) = \boldsymbol{a}\cdot K^c_b\boldsymbol{e}_c$. Then we want to find the stress tensor in the from $\boldsymbol{\mathcal{S}} = -\nabla_a\boldsymbol{\sigma}^{\mathrm{e},a}$. Since we can write the energy deformation as

$$\delta F = \int \mathrm{d}A\boldsymbol{\mathcal{S}}\cdot\delta\boldsymbol{X} + \nabla_a\mathcal{Q}^a \tag{7.165}$$

and invariance under translations implies that $\delta F = 0$ so that locally

$$\boldsymbol{\mathcal{S}}\cdot\delta\boldsymbol{X} = -\nabla_a\mathcal{Q}^a\ . \tag{7.166}$$

We can thus find the stress tensor $\boldsymbol{\sigma}^{\mathrm{e},a}$ by substituting $\Phi^a = \boldsymbol{a}\cdot\boldsymbol{n}$ etc. into $\mathcal{Q}^a$ and we can thus write the stress tensor as

$$\begin{aligned}\boldsymbol{\sigma}^{\mathrm{e},a} &= \sigma^{\mathrm{e},ab}\boldsymbol{e}_b + \sigma^{\mathrm{e},a}_\mathrm{n}\boldsymbol{n} = \left(\sigma^{ab}_{\perp,\mathrm{F}} + \sigma^{ab}_{\perp,\mathrm{H}} + \sigma^{ab}_{\parallel,\mathrm{F}} + \sigma^{ab}_{\parallel,\mathrm{H}}\right)\boldsymbol{e}_b\\ &\quad + \left(\sigma^{a}_{\perp,\mathrm{F}} + \sigma^{a}_{\perp,\mathrm{H}} + \sigma^{a}_{\parallel,\mathrm{F}} + \sigma^{a}_{\parallel,\mathrm{H}}\right)\boldsymbol{n}\ ,\end{aligned} \tag{7.167a}$$

where

$$\sigma^{ab}_{\perp,\mathrm{F}} = 2\kappa_{\mathrm{F}}\chi K_{\mathrm{G}} g^{ab} \,, \tag{7.167b}$$
$$\sigma^{ab}_{\perp,\mathrm{H}} = -\kappa_{\mathrm{B}}\left(H - H_0\right) K^{ab} - \kappa_{\mathrm{G}} K_{\mathrm{G}} g^{ab} \,, \tag{7.167c}$$
$$\sigma^{ab}_{\parallel,\mathrm{F}} = -\kappa_{\mathrm{F}}\chi(\rho_D + K_{\mathrm{G}})g^{ab} - \kappa_{\mathrm{F}}\chi\nabla^a\nabla^b\chi + \kappa_{\mathrm{F}}\nabla^a\chi\nabla^b\chi \,, \tag{7.167d}$$
$$\sigma^{ab}_{\parallel,\mathrm{H}} = (\gamma + \kappa_{\mathrm{B}}\left(H - H_0\right)^2 + \kappa_{\mathrm{G}} K_{\mathrm{G}})g^{ab} \,, \tag{7.167e}$$
$$\sigma^{a}_{\perp,\mathrm{F}} = \kappa_{\mathrm{F}}[(Kg^{ab} + 2K^{ab})\chi + 2(K^{ab} - g^{ab}K)]\nabla_b\chi \,, \tag{7.167f}$$
$$\sigma^{a}_{\perp,\mathrm{H}} = \kappa_{\mathrm{B}}\nabla^a H \,, \tag{7.167g}$$
$$\sigma^{a}_{\parallel,\mathrm{F}} = \kappa_{\mathrm{F}} K^{ab}\chi\nabla_b\chi \,, \tag{7.167h}$$
$$\sigma^{a}_{\parallel,\mathrm{H}} = 0 \,. \tag{7.167i}$$

To find the torque tensor we need to look at infinitesimal rotations instead, $\delta\boldsymbol{X} = \boldsymbol{b}\times\boldsymbol{X}$ such that

$$\Phi = \boldsymbol{b}\cdot\boldsymbol{X}\times\boldsymbol{n} \,, \qquad \Phi_a = \boldsymbol{b}\cdot\boldsymbol{X}\times\boldsymbol{e}_a \,, \tag{7.168}$$

so that $\nabla^b\Phi = \boldsymbol{b}\cdot(\epsilon^{ab}\boldsymbol{e}_a + K^{ab}\boldsymbol{X}\times\boldsymbol{e}_a)$ and $\nabla_b\Phi_a = \boldsymbol{b}\cdot(\epsilon_{ba}\boldsymbol{n} - K_{ab}\boldsymbol{X}\times\boldsymbol{n})$, where $\epsilon_{ab} = \sqrt{g}\varepsilon_{ab}$. The deformation of the energy is then given by

$$\delta F = \int \mathrm{d}A\boldsymbol{\mathcal{S}}\cdot(\boldsymbol{b}\times\boldsymbol{X}) + \int \mathrm{d}A\nabla_a\boldsymbol{\mathcal{T}}^a \,, \tag{7.169}$$

where $\boldsymbol{\mathcal{T}}^a = \boldsymbol{m}^a - \boldsymbol{X}\times\boldsymbol{\sigma}^{\mathrm{e},a}$ is the associated Noether current. We find $\boldsymbol{m}^a$ to be

$$\boldsymbol{m}^a = m^{ab}\boldsymbol{e}_b + m^a_{\mathrm{n}}\boldsymbol{n} = \boldsymbol{X}\times\mathcal{Q}^a[\Phi = \boldsymbol{n}, \Phi_a = \boldsymbol{e}_a] - \mathcal{Q}^a[\Phi = \boldsymbol{X}\times\boldsymbol{n}, \Phi_a = \boldsymbol{X}\times\boldsymbol{e}_a] \,, \tag{7.170a}$$

with

$$m^{ab} = \kappa_{\mathrm{B}}\left(H - H_0\right)\epsilon^{ab} + (2\kappa_{\mathrm{G}} - 4\kappa_{\mathrm{F}}\chi)H\epsilon^{ab} + (2\kappa_{\mathrm{F}}\chi - \kappa_{\mathrm{G}})K^a_c\epsilon^{cb} \,, \tag{7.170b}$$
$$m^a_{\mathrm{n}} = \kappa_{\mathrm{F}}\chi\nabla_b\chi\epsilon^{ab} \,. \tag{7.170c}$$

7.E Torque Balance

The tangential projection of the torque balance Eq. (7.20b) reads

$$\nabla_i m^{\mathrm{e},i}_k + m^{\mathrm{e},i}_{\mathrm{n}} K_{ik} = (\sigma^{\mathrm{e}}_{\mathrm{n},i} + \sigma^{\mathrm{d}}_{\mathrm{n},i})\epsilon^i{}_k \,. \tag{7.171}$$

We find that this relation is fulfilled for the following components of the stress and torque tensors which we found in the previous section: $m^{\mathrm{e},i}_{\mathrm{n}} = \kappa_{\mathrm{F}}\chi\nabla^j\chi\epsilon_{ji}$, $\sigma^{\mathrm{d}}_{\mathrm{n},i} = 0$, and

$$\sigma^{\mathrm{e}}_{\mathrm{n},i} = \kappa_{\mathrm{B}}\nabla_i H + \kappa_{\mathrm{F}}[(Kg^i_j - 2K^j_i)\chi + 2(K^j_i - g^j_i K)]\nabla_j\chi + \kappa_{\mathrm{F}} K^j_i\chi\nabla_j\chi \,. \tag{7.172}$$

To see this note that

$$\begin{aligned}\nabla_i m^{ij} + K^j_i m^i_{\mathrm{n}} =& (\kappa_{\mathrm{B}} + 2\kappa_{\mathrm{G}})\epsilon^{ij}\nabla_i H - \kappa_{\mathrm{G}}\epsilon^{kj}\nabla_i K^i_k - 4\kappa_{\mathrm{F}}\epsilon^{ij}\nabla_i(\chi H)\\ &+ 2\kappa_{\mathrm{F}}\epsilon^{kj}\nabla_i(\chi K^i_k) + \kappa_{\mathrm{F}}\epsilon^{ki}K^j_i\chi\nabla_k\chi \,,\end{aligned} \tag{7.173}$$

and

$$\sigma_{\mathrm{n},i}\epsilon^{ij} = \kappa_{\mathrm{B}}\epsilon^{ij}\nabla_i H + \kappa_{\mathrm{F}}\epsilon^{ij}\left[(Kg_i^k - 2K_i^k)\chi + 2(K_i^k - g_i^k)K\right]\nabla_k\chi + \kappa_{\mathrm{F}}\epsilon^{ij}K_i^k\chi\nabla_k\chi \,. \tag{7.174}$$

Then, multiplying both by $-\epsilon_{jl}$, using $\epsilon_{jl}\epsilon^{ij} = -\delta_l^i$, and subtracting the result, we find,

$$\begin{aligned}&\epsilon_{jl}\epsilon^{ij}\sigma_{\mathrm{n},i} - \epsilon_{jl}\nabla_i m^{ij} - \epsilon_{jl}K_i^j m_{\mathrm{n}}^i \\ &= -4\nabla_l(\chi H) + 2\nabla_i(\chi K_l^i) - \epsilon_{jl}\epsilon^{ki}K_i^j\chi\nabla_k\chi - \chi K\nabla_l\chi + \chi K_l^k\nabla_k\chi \\ &\quad - 2K_l^k\nabla_k\chi + 2K\nabla_l\chi = 0 \,,\end{aligned} \tag{7.175}$$

where we used $\epsilon_{jl}\epsilon^{ki} = \delta_j^k\delta_l^i - \delta_j^i\delta_l^k$.

On the other hand, for the normal projection we find that, using the symmetry of σ_{ij} and K_{ij}, and the antisymmetry of ϵ_{ij}:

$$0 = \nabla^i m_{\mathrm{n},i} - m_{ij}K^{ij} = \kappa_{\mathrm{F}}\nabla^i(\chi\nabla^j\chi)\epsilon_{ji} - (-\kappa_{\mathrm{G}}K_i^k\epsilon_{kj} + 2\kappa_{\mathrm{F}}\chi K_i^k\epsilon_{kj})K^{ij} = 0 \,, \tag{7.176}$$

where all three terms vanish identically in the last step. Thus, for our choice of stress and moment tensors the torque balance equations are trivially fulfilled.

7.F Boundary Conditions

We only consider surfaces with vanishing spontaneous mean curvature in this section, $H_0 = 0$. This is assumed for simplicity and because all surfaces with $H_0 \neq 0$ that we consider (sphere and torus) do not have a boundary. Only the initially flat disc has a boundary but in this case the surface has no spontaneous mean curvature. To determine the boundary condition we work in the Darboux frame [149, 264]. That is at the boundary we have the frame $\{\boldsymbol{t}, \boldsymbol{l}, \boldsymbol{n}\}$ where $\boldsymbol{l} = \boldsymbol{t} \times \boldsymbol{n} = \boldsymbol{e}_a l^a$. is the outward pointing tangent normal vector. We can also write $\boldsymbol{t} = t^a\boldsymbol{e}_a$. On the edge $g_{ab} = t_a t_b + l_a l_b$ such that $\boldsymbol{e}_a = t_a\boldsymbol{t} + l_a\boldsymbol{l}$. The curvature tensor then has components $K_\perp = K_{ab}l^a l^b$, $K_\parallel = K_{ab}t^a t^b$, and $K_{\perp\parallel} = K_{ab}l^a t^b$ in this coordinate system. $K_{\perp\parallel}$ is called the geodesic torsion. The directional derivatives on the edge are $\nabla_\perp = l_a\nabla^a$ and $\nabla_\parallel = t^a\nabla_a$. There are the relations

$$\dot{\boldsymbol{t}} = -K_{\mathrm{G}}\boldsymbol{l} - k_n\boldsymbol{n} \,, \qquad \dot{\boldsymbol{l}} = K_{\mathrm{G}}\boldsymbol{t} - \tau_g\boldsymbol{n} \,, \qquad \dot{\boldsymbol{n}} = \tau_g\boldsymbol{l} + k_n\boldsymbol{t} \,, \tag{7.177}$$

where the dot indicates $\nabla_\parallel$, $k_n = K_\parallel$, $\tau_g = K_{\perp\parallel}$, and $K_{\mathrm{G}} = -l_b\nabla_\parallel t^b = -\boldsymbol{l}\cdot\nabla_\parallel\boldsymbol{t}$. The variation of the edge can be written as $\delta\boldsymbol{Y} = \phi\boldsymbol{t} + \psi\boldsymbol{l} + \Phi\boldsymbol{n}$, where $\psi = l^a\Phi_a$ and $\phi = t^a\Phi_a$. Therefore, for the deformation of the unit tangent we have

$$\delta\boldsymbol{t} = \delta(t^a\boldsymbol{e}_a) = (\nabla_\parallel\Phi^b)(t_b\boldsymbol{t} + l_b\boldsymbol{l}) - t^a K_{ab}\Phi^b\boldsymbol{n} + (\nabla_\parallel\Phi)\boldsymbol{n} + t^a\Phi K_{ab}(t^b\boldsymbol{t} + l^b\boldsymbol{l}) \,. \tag{7.178}$$

From $(\nabla_\parallel\Phi^a)t_a = \dot{\phi} + \psi K_{\mathrm{G}}$ and $\dot{\psi} = \dot{\psi} - \phi K_{\mathrm{G}}$ we find

$$\delta\boldsymbol{t} = \dot{\phi}\boldsymbol{t} + \dot{\psi}\boldsymbol{l} + \dot{\Phi}\boldsymbol{n} + \phi\dot{\boldsymbol{t}} + \psi\dot{\boldsymbol{l}} + \Phi\dot{\boldsymbol{n}} \,. \tag{7.179}$$

Furthermore,

$$\delta \oint \mathrm{d}s = \oint \mathrm{d}s \boldsymbol{t} \cdot \delta \boldsymbol{t} = \oint \mathrm{d}s(\dot{\phi} + \psi K_{\mathrm{G}} + \Phi k_n) = \Delta\phi + \oint \mathrm{d}s(\psi K_{\mathrm{G}} + \Phi k_n) \ , \quad (7.180)$$

with $\Delta\phi = 0$ for closed curves. Finally, we write

$$\mathcal{Q}^a_{\perp,\mathrm{F}} = M^{ab}\nabla_b\Phi + M^a\Phi \ , \qquad \mathcal{Q}^a_{\parallel,\mathrm{F}} = N^a_b\Phi^b + N_b\nabla^a\Phi^b \ , \quad (7.181)$$

and

$$\delta F = \oint \mathrm{d}s[l_a\mathcal{Q}^a_{\mathrm{H}} + l_a\mathcal{Q}^a_{\mathrm{F}} + \sigma_b(K_{\mathrm{G}}\psi + k_n\Phi)] \ , \quad (7.182)$$

where we added a term $\sigma_b \oint \mathrm{d}s$ as the line tension of the boundary to the energy. Then we find the boundary conditions of a free edge to be

$$0 = l_a l_b N^{ab} + N_b \nabla_\perp l^b + \gamma + \kappa_{\mathrm{B}} H^2 + \kappa_{\mathrm{G}} K_{\mathrm{G}} + \sigma_b K_{\mathrm{G}} \ , \quad (7.183a)$$

$$0 = l_a M^a - \nabla_\parallel [l_a t_b M^{ab}] + \kappa_{\mathrm{B}} \nabla_\perp H - \kappa_{\mathrm{G}} \nabla_\parallel \tau_g + \sigma_b k_n \ , \quad (7.183b)$$

$$0 = l_a l_b M^{ab} - \kappa_{\mathrm{B}} H - \kappa_{\mathrm{G}} K_\parallel \ , \quad (7.183c)$$

$$0 = l^b N_b \ , \quad (7.183d)$$

$$0 = t^b N_b \ , \quad (7.183e)$$

where

$$M^{ab} = 2\kappa_{\mathrm{F}}(K g^{ab} - K^{ab})\chi \ , \quad (7.183f)$$

$$M^a = \kappa_{\mathrm{F}}[(K g^{ab} - 2K^{ab})\chi + 2(K^{ab} - g^{ab}K)]\nabla_b\chi \ , \quad (7.183g)$$

$$N^{ab} = \kappa_{\mathrm{F}}\left[\nabla^a\chi\nabla^b\chi - \chi\nabla^a\nabla^b\chi - g^{ab}(\rho_D + K_{\mathrm{G}})\chi\right] \ , \quad (7.183h)$$

$$N^a = -\kappa_{\mathrm{F}}\chi\nabla^a\chi \ . \quad (7.183i)$$

7.G Monge Gauge

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To investigate the buckling instability of a flat disc, we use the small-height approximation of the Monge gauge. In this parametrization, the surface is described by a height function $h(x, y)$ above a flat reference plane with coordinates $\{x, y\}$. Here, we present the results for a cartesian coordinate system. It is straightforward to find the equivalent expression for other coordinate systems on the reference plane, e.g., polar coordinates. The surface parametrization can then be simply written as [149]

$$\boldsymbol{X}(x, y) = \begin{pmatrix} x \\ y \\ h(x, y) \end{pmatrix} . \quad (7.184)$$

From this the tangent vectors and normal vector are found to be

$$\boldsymbol{e}_x = \begin{pmatrix} 1 \\ 0 \\ \partial_x h \end{pmatrix} , \qquad \boldsymbol{e}_y = \begin{pmatrix} 0 \\ 1 \\ \partial_y h \end{pmatrix} , \qquad \boldsymbol{n} = \frac{\boldsymbol{e}_x \times \boldsymbol{e}_y}{\sqrt{g}} = \frac{1}{\sqrt{g}} \begin{pmatrix} -\partial_x h \\ -\partial_y h \\ 1 \end{pmatrix} . \quad (7.185)$$

Here, g is the determinant of the metric

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{pmatrix} 1+(\partial_y h)^2 & -\partial_x h \partial_y h \\ -\partial_x h \partial_y h & 1+(\partial_x h)^2 \end{pmatrix} . \tag{7.186}$$

The area element of the surface is then

$$\mathrm{d}A = \sqrt{g}\mathrm{d}x\mathrm{d}y = \sqrt{1+(\nabla h)^2}\mathrm{d}x\mathrm{d}y , \tag{7.187}$$

where $\nabla = \{\partial_x, \partial_y\}$ is the derivative on the flat reference plane. Finally, the curvature tensor is given by

$$K_{ij} = -\frac{1}{\sqrt{g}} \begin{pmatrix} \partial_x^2 h & \partial_x \partial_y h \\ \partial_x \partial_y & \partial_y^2 h \end{pmatrix} \tag{7.188}$$

such that the mean and Gaussian curvature are

$$H = -\frac{1}{2}\nabla \cdot \left(\frac{\nabla h}{\sqrt{g}}\right) , \qquad K_\mathrm{G} = \frac{\left(\partial_x^2 h\right)\left(\partial_y^2 h\right) - \left(\partial_x \partial_y h\right)^2}{g^2} . \tag{7.189}$$

These are the most important expressions written in the Monge gauge. Now, as we are interested in the buckling instability of a flat disc, we only consider small perturbations of the flat reference state. Thus, we can work in the so-called small-height approximation. In this approximation we assume $|\nabla h| \ll 1$ such that we can linearize the above expressions the expressions. We find that the metric can be approximated by the flat metric,

$$g_{ij} = \delta_{ij} + \mathcal{O}(\nabla h^2) , \tag{7.190}$$

and the covariant derivative on the surface is reduces to the flat derivative. Thus, the determinant of the metric is trivial, $g = 1+\mathcal{O}(\nabla h^2)$, and the second fundamental form reduces to the simple expression

$$K_{ij} = -\partial_i \partial_j h + \mathcal{O}(\nabla h^2) \tag{7.191}$$

such that mean and Gaussian curvature are simply given by

$$H = -\frac{1}{2}\nabla^2 h + \mathcal{O}(\nabla h^2) , \qquad K_\mathrm{G} = \mathcal{O}(\nabla h^2) . \tag{7.192}$$

In particular, the linearized Gaussian curvature vanishes. Lastly, note that expressions that can be written in terms of ∇ without explicit reference to ∂_x and ∂_y are coordinate-independent and thus also valid for, e.g., polar coordinates.

7.H No-slip Boundary Conditions

First, we note that the exponents not written in the main text are

$$a(s) = r^{-2\sqrt{\frac{1}{-4(s-2)s-3}}\sqrt{-4(s-2)s-3}|s-1|-1} , \tag{7.193}$$

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and

$$b(s) = r^{2\sqrt{\frac{1}{-4(s-2)s-3}}\sqrt{-4(s-2)s-3}|s-1|-1} . \tag{7.194}$$

However, since we set the corresponding integration constants to zero they will not be relevant in the following.

We now present in some more detail the derivation of the height equation for defect charges $s \neq 0$. As mentioned above, we impose no-slip boundary conditions at the the boundary of the disc, $r = R$. To find the velocity field that fulfills these boundary conditions we use the stream function

$$\psi = \left[\mathcal{A} r^{2(1-s)} + \mathcal{B} r^{2(2-s)}\right] \sin[2(1-s)\varphi] , \tag{7.195}$$

where $\mathcal{A}$ and $\mathcal{B}$ are constants to be determined. This stream function is in turn used to define a velocity

$$\boldsymbol{v}_0 = \begin{pmatrix} -\partial_y \psi \\ \partial_x \psi \end{pmatrix} . \tag{7.196}$$

Note that this vector field is divergence-free such that the velocity field $\boldsymbol{v}_0$ is incompressible by construction. The corresponding pressure field is found from the velocity field by solving

$$\eta \Delta \boldsymbol{v}_0 = \nabla P_0 . \tag{7.197}$$

We can add this new velocity and pressure field to the solution already found and, due to the linearity of the Stokes equation, this is a solution of the equation as well. In this way we can easily construct the velocity field $\boldsymbol{v}_0$ to enforce the boundary conditions. Namely, requiring at $r = R$ that

$$v_r + v_{0,r} = 0 \qquad \text{and} \qquad v_\varphi + v_{0,\varphi} = 0 , \tag{7.198}$$

with v_r and v_φ given by Eq. (7.61), it is straightforward to find

$$\mathcal{A} = \frac{\alpha R^{2s}}{8(2-s)\eta} , \qquad \mathcal{B} = \frac{\alpha s R^{2(s-1)}}{8(s^2 - 3s + 2)\eta} . \tag{7.199}$$

The components of the resulting velocity field can be written as

$$v_{0,r} = \left[(s-1)\left(\frac{R}{r}\right)^{2s} - s\left(\frac{R}{r}\right)^{2(s-1)}\right] \frac{\alpha r \cos[2(s-1)\varphi]}{4(s-2)\eta} , \tag{7.200a}$$

$$v_{0,\varphi} = \left[(s-1)\left(\frac{R}{r}\right)^{2s} - \frac{s-2}{s-1} s\left(\frac{R}{r}\right)^{2(s-1)}\right] \frac{\alpha r \sin[2(s-1)\varphi]}{4(s-2)\eta} . \tag{7.200b}$$

Lastly, from Eq. (7.197) we find the pressure to be

$$P_0 = \frac{\alpha s(3-2s)}{2(2-3s+s^2)} \left(\frac{R}{r}\right)^{2(s-1)} \cos[2(s-1)\varphi] . \tag{7.201}$$