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## Imperfect information variants of combinatorial games

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# Chapter 8

## Synchronized Push

In this chapter, based on joint work with Ronald Takken [19], we discuss some results on the synchronized versions of Push and Shove. Both games being separable, the results largely mirror those in Chapter 7 and support Conjecture 2.3.34. In this chapter, we consider both the combinatorial and synchronized version of Push in detail, showing that it behaves similarly but not identically to Hackenbush. In Section 8.4, we briefly consider the game of Shove, again showing some similar behavior.

### 8.1 Basics

The ruleset of Push, described in Example 2.1.20, provides us with separable games. If the players pick two pieces of which the moves do not influence each other, it is clear that the moves can be executed simultaneously in any order. If the move of one of the pieces would cause the second piece to be pushed, we perform the move on the second piece first.

**Example 8.1.1.** Consider the synchronized game

$$G = \boxed{\quad} \boxed{P} \boxed{\quad} \boxed{P}$$

Here, moving either piece does not affect the position of the other piece, so  $G^{LR} = G^{RL} = G^S$  is the only synchronized move of the game.

Next, consider

$$G = \boxed{\phantom{P}} \boxed{P} \boxed{P}$$

We now find that the blue piece is pushed by moving the red piece, so the only synchronized move amounts to first moving the blue piece out of the way, followed by moving the red piece:  $G^S = G^R = G^{LR}$ .  $\triangleleft$

Hence, by Corollary 2.3.11, all Push positions are numbers in combinatorial sense. The numerical values of some simple combinatorial Push positions are shown in [2, Problem 5.15], proven in [19].

**Theorem 8.1.2.** Writing  $\boxed{\phantom{P}}^n$  for  $n$  blank squares, we have

- (i)  $\boxed{\phantom{P}}^n \boxed{P} = n + 1$ ;
- (ii)  $\boxed{\phantom{P}}^n \boxed{P} \boxed{P} = 2 - \frac{1}{2^{n+1}}$ ;
- (iii)  $\boxed{\phantom{P}}^n \boxed{P} \boxed{\phantom{P}}^m \boxed{P} = m + 1$  for  $m > 0$ .

For Push and Shove, we are able to prove Conjecture 2.3.33.

**Theorem 8.1.3.** Let  $G$  and  $H$  be games of synchronized Push or Shove, and suppose  $T$  is terminal. Then  $v(G + T) = v(G) + v(T)$ .

*Proof.* We prove the statement for Push; for Shove, the proof is analogous. If  $G$  is also terminal, the result is trivial. Hence, suppose  $G$  is non-terminal. Furthermore, without loss of generality, suppose  $T \in \mathcal{L}$ , in which case we may represent  $T$  by  $T = \boxed{\phantom{P}}^n \boxed{P}$ .

Let  $G^R$  be any move of Left which moves the leftmost piece on any one strip of  $G$ , let  $G^R$  be any arbitrary move on  $G$ , and consider  $G_1 = G^{L+R} + T$  and  $G_2 = G^R + T^L = G^R + T - 1$ . First, if  $G^{L+R} = G^{LR} = G^R$ , i.e., if Left's chosen piece is pushed by Right's move, then

$$v(G_1) = v(G^{L+R}) + v(T) = v(G^R) + v(T) > v(G^R) + v(T) - 1 = v(G_2).$$

Next, suppose that Left's chosen piece is not pushed by the move of Right. Noting that the chosen piece is the leftmost blue one by assumption, the result is

$$\begin{aligned} G_1 &= \cdots \boxed{P} \cdots \boxed{P} \boxed{P} \boxed{\phantom{P}} \cdots + \boxed{\phantom{P}}^n \boxed{P} \\ G_2 &= \cdots \boxed{\phantom{P}} \boxed{P} \cdots \boxed{P} \boxed{P} \cdots + \boxed{\phantom{P}}^{n-1} \boxed{P} \end{aligned}$$

Note that the amount of red pieces to the left of the moved blue piece may be zero, and that the position of the empty square in  $G_2$  may vary depending

on Right's move. Now, couple  $G_1$  and  $G_2$ . Let Left play her Nash strategy on  $G_2$ , and move the same piece on  $G_1$ . Conversely, let Right play his Nash strategy on  $G_1$ , moving the corresponding piece in  $G_2$ . Denote the values under these strategies by  $v^L(G_1)$  and  $v^R(G_2)$ , respectively. Let play continue until the leftmost blue piece in  $G_1$  is moved off the board, ending in

$$\begin{aligned} G'_1 &= \square \cdots + \square^n \boxed{P} \\ G'_2 &= \boxed{P} \cdots + \square^{n-1} \boxed{P} \end{aligned}$$

if this piece has not been pushed in  $G_2$ , or in

$$\begin{aligned} G'_1 &= \cdots + \square^n \boxed{P} \\ G'_2 &= \cdots + \square^{n-1} \boxed{P} \end{aligned}$$

if it has been pushed. In the second case, it is clear that  $v(G'_1) > v(G'_2)$ . In the first case, continue the coupling, connecting a move on the leftmost piece in  $G'_2$  to a move on  $H$  in  $G'_1$ . We then arrive in either

$$\begin{aligned} G''_1 &= \cdots + \square^{n-1} \boxed{P} \\ G''_2 &= \cdots + \square^{n-1} \boxed{P} \end{aligned}$$

or

$$\begin{aligned} G''_1 &= \cdots + \square^n \boxed{P} \\ G''_2 &= \cdots + \square^{n-1} \boxed{P} \end{aligned}$$

depending on whether this piece is pushed at some time. In both cases, it is clear that  $v(G''_1) \geq v(G''_2)$ . Hence, we find that  $v(G_1) \geq v^L(G_1) \geq v^R(G_2) \geq v(G_2)$ .

We thus conclude that moving on  $G$  dominates moving on  $T$  for Left. Hence, by domination and induction, we conclude that  $v(G + T) = v(G) + v(T)$ .  $\square$

In the sequel, we may write  $n$  and  $-n$  for integer Push and Shove games, and consider these separately from the other summands in sums of games.

## 8.2 Halves

We continue by considering Conjecture 2.3.34 in the light of Push games. To this end, we start by analyzing copies of the games

$$H = 2 + \boxed{P} \boxed{P} \quad \text{and} \quad H' = -1 + \boxed{P} \boxed{P}$$

which are both equal to  $\frac{1}{2}$  in combinatorial sense by Theorem 8.1.2. Let  $v_n = v(nH)$  and  $v'_n = v(nH')$ . We then find the following recurrence relations, akin to Theorem 7.2.1.

**Theorem 8.2.1.** *We have*

$$v_n = \begin{cases} 1 + \frac{1}{n}v_{n-1} + \frac{n-1}{n}v_{n-2} & n \geq 3, \\ v_1 = 1, v_2 = \frac{3}{2}, \end{cases}$$

and

$$v'_n = \begin{cases} \frac{1}{n}v'_{n-1} + \frac{n-1}{n}(v'_{n-2} + 1) & n \geq 3, \\ v'_1 = 0, v'_2 = \frac{1}{2}. \end{cases}$$

*Proof.* Computing  $v_1$  is trivial; for  $v_2$ , note that playing on either copy of  $H$  uniformly at random is a Nash equilibrium by Theorem 2.2.20, resulting in  $v_2 = \frac{1}{2}(2 - 1 + 2 - 1) + \frac{1}{2}(2 - 2 + 2 - 1) = \frac{3}{2}$ . For the recurrence step, again by Theorem 2.2.20, both players play on the same copy of  $H$  with probability  $\frac{1}{n}$ , to  $(n - 1)H + (2 - 1)$  with value  $v_{n-1} + 1$ ; or on different copies with probability  $\frac{n-1}{n}$ , to  $(n - 2)H + (2 - 2) + (2 - 1)$  with value  $v_{n-2} + 1$ .

Again, computing  $v'_1$  and  $v'_2$  is straightforward. For the recurrence step, once more by Theorem 2.2.20, the players play on the same copy of  $H'$  with probability  $\frac{1}{n}$ , to  $(n - 1)H' + (-1 + 1)$  with value  $v'_{n-1}$ ; or on different copies with probability  $\frac{n-1}{n}$ , to  $(n - 2)H' + (-1 + 1) + (-1 + 2)$  with value  $v'_{n-2} + 1$ .  $\square$

Just like for Hackenbush, the recurrence relations do not have a closed-form solution, but the differences  $d_n = v_n - v_{n-1}$  and  $d'_n = v'_n - v'_{n-1}$  are well-behaved.

**Theorem 8.2.2.** *We have*

$$d_n = \begin{cases} 1 - \frac{n-1}{n}d_{n-1} & n \geq 2, \\ d_1 = 1, \end{cases}$$

and

$$d'_n = \begin{cases} \frac{n-1}{n}(1 - d'_{n-1}) & n \geq 2, \\ d'_1 = 0. \end{cases}$$

The solutions are

$$d_n = \frac{2n + (-1)^{n-1} + 1}{4n} \quad \text{and} \quad d'_n = \frac{2n + (-1)^n - 1}{4n}.$$

*Proof.* We expand:

$$\begin{aligned}
 d_n &= v_n - v_{n-1} \\
 &= 1 + \frac{1}{n}v_{n-1} + \frac{n-1}{n}v_{n-2} - v_{n-1} \\
 &= 1 - \frac{n-1}{n}(v_{n-1} - v_{n-2}) \\
 &= 1 - \frac{n-1}{n}d_{n-1}.
 \end{aligned}$$

and

$$\begin{aligned}
 d'_n &= v'_n - v'_{n-1} \\
 &= \frac{1}{n}v'_{n-1} + \frac{n-1}{n}(v'_{n-2} + 1) - v'_{n-1} \\
 &= \frac{n-1}{n} + \frac{n-1}{n}v'_{n-2} - \frac{n-1}{n}v'_{n-1} \\
 &= \frac{n-1}{n}(1 - d'_{n-1}).
 \end{aligned}$$

The solutions may be verified by substituting them into the proven recurrences.  $\square$

**Corollary 8.2.3.** *We have*

$$\lim_{n \rightarrow \infty} \frac{v(nH)}{n} = \lim_{n \rightarrow \infty} \frac{v(nH')}{n} = \frac{1}{2}.$$

Hence, we see that for these Push positions, Conjecture 2.3.34 holds. Note that, just as for  $d_n$  in Section 7.2, the parity of  $d_n$  is crucial to its value. Here,  $d_{2k} = d'_{2k+1} = \frac{1}{2}$  for  $k \in \mathbb{N}$ ,  $d_{2k+1}$  converges to  $\frac{1}{2}$  from above and  $d'_{2k}$  from below, as shown in Figure 8.1. We will encounter a similar pattern again later on.

Next, we consider games of the form  $mH - nH$ . Some computed values can be found in Table 8.1.

We again follow a path similar to that in Chapter 7, mirroring Conjecture 7.4.2, Lemma 7.4.6 and Theorem 7.4.5.

**Conjecture 8.2.4.** *Let  $v_{m,n} = v(mH - nH)$ . Then*

- (1) *For all  $n \geq 2$ , we have  $v_{n,n-1} = \frac{3}{4}$ .*
- (2) *For all  $n$ , we have  $\lim_{m \rightarrow \infty} (v_{m,n} - v_{m-1,n}) = \frac{1}{2}$ .*
- (3) *For all  $m$ , we have  $\lim_{n \rightarrow \infty} (v_{m,n} - v_{m,n-1}) = -\frac{1}{2}$ .*
- (4) *For all  $m$ , the limit  $\lim_{n \rightarrow \infty} v_{m+n,n}$  exists.*

**Lemma 8.2.5.** *For all  $m, n \geq 0$ , we have  $v_{m,n+1} + 1 \geq v_{m+1,n}$ .*

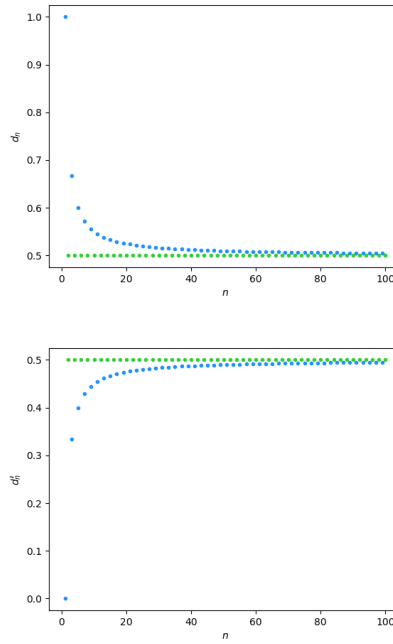


Figure 8.1: The convergence of  $d_n$  (left) and  $d'_n$  (right). The values for  $d_n$  for  $n$  even and  $d'_n$  for  $n$  odd are shown in green; the other values in blue.

$m \setminus n$	0	1	2	3	4	5	6	7
0	0.0000							
1	1.0000	0.0000						
2	1.5000	0.7500	0.0000					
3	2.1667	1.3571	0.7500	0.0000				
4	2.6667	1.9935	1.3338	0.7500	0.0000			
5	3.2667	2.5415	1.9411	1.3338	0.7500	0.0000		
6	3.7667	3.1333	2.4923	1.9236	1.3338	0.7500	0.0000	
7	4.3381	3.6608	3.0719	2.4721	1.9220	1.3338	0.7500	0.0000

Table 8.1: The Nash value  $v(mH - nH)$  rounded to four decimal places. Note that  $v(mH - nH) = -v(nH - mH)$ .

*Proof.* We couple  $G_1 = 1 - H + mH - nH$  and  $G_2 = H + mH - nH$  as in Lemma 7.4.6, again denoting the values under the coupled strategies by  $v^L(G_1)$  and  $v^R(G_2)$ , respectively. If Left and Right both not play on  $mH - nH$ , the result is 0 in both  $G_1$  and  $G_2$ . If only Left does not play on  $mH - nH$ , the result is the same. If only Right does not play on  $mH - nH$ , then the result is 1 in both  $G_1$  and  $G_2$ . In any case, we conclude that  $v^L(G_1) \geq v^R(G_2)$ , so that also  $v(G_1) \geq v^L(G_1) \geq v^R(G_2) \geq v(G_2)$ .  $\square$

**Theorem 8.2.6.** *For all  $m, n \in \mathbb{N}$  we have*

$$\lim_{k \rightarrow \infty} \frac{v_{km, kn}}{k} = \frac{m - n}{2}.$$

*Proof.* Using Theorem 8.2.2 and computational software, we find that

$$v_n = \frac{n}{2} + \frac{1}{4} \left( \psi \left( \left\lceil \frac{n}{2} \right\rceil + \frac{5}{4} \right) - \psi \left( \frac{5}{4} \right) \right).$$

The result then follows from Lemma 8.2.5 and a computation along the lines of the proof of Theorem 7.4.5.  $\square$

### 8.3 Quarters

In this section, we further explore the Nash values of synchronized Push positions, continuing with copies of the position

$$H_2 = 2 + \boxed{\phantom{P}} \boxed{P} \boxed{P}$$

which has value  $\frac{1}{4}$  in combinatorial sense. From  $H_2$ , Left plays to

$$H_2^L = 2 + \boxed{P} \boxed{\phantom{P}} \boxed{P}$$

while Right plays to

$$H_2^R = 2 + \boxed{P} \boxed{P} \boxed{\phantom{P}} = H$$

which is also the result of both players playing on  $H_2$  simultaneously. In the previous section, we analyzed the result of playing on multiple copies of  $H_1$ , and showed that in the limit the Nash value converges to the combinatorial value. To ultimately prove the same behaviour of multiple copies of  $H_2$ , we start by analyzing  $H_2^L$  in more detail.

**Lemma 8.3.1.** *For all  $n_1, n_2 \in \mathbb{N}$ , we find  $v(n_1H + n_2H_2^L) = v(n_1H)$ .*

*Proof.* By Theorem 2.2.20, the probability of a player playing on one given copy of either of the games is equal across all these copies. This allows us to rewrite the game as a two-dimensional zero-sum game. We continue by induction on the birthday, the base cases for  $n_1, n_2 \leq 2$  being straightforward computation.

If both players play on the same copy of  $H$ , the result is  $(2 - 1) + (n_1 - 1)H + n_2H_2^L = 1 + (n_1 - 1)H + n_2H_2^L$ . If the players play on different copies of  $H$ , the result is  $(2 - 1) + (2 - 2) + (n_1 - 2)H + n_2H_2^L = 1 + (n_1 - 2)H + n_2H_2^L$ . Hence, by induction, writing  $v_n = v(nH)$  as before, the expected result of both parties playing on copies of  $H$  is

$$\frac{1}{n_1}(1 + v_{n_1-1}) + \frac{n_1-1}{n_1}(1 + v_{n_1-2}) = 1 + \frac{1}{n_1}v_{n_1-1} + \frac{n_1-1}{n_1}v_{n_1-2} = v_{n_1},$$

where the last equality follows from Theorem 8.2.1. If Left plays on a copy of  $H$ , while Right plays on a copy of  $H_2^L$ , the result is  $(2 - 2) + (n_1 - 1)H + H + (n_2 - 1)H_2^L = n_1H + (n_2 - 1)H_2^L$ , with value  $v_{n_1}$  by induction. If, conversely, Left plays on a copy of  $H_2^L$  and Right on a copy of  $H$ , the result is  $(2 - 1) + (n_1 - 1)H + (2 - 3) + (n_2 - 1)H_2^L = (n_1 - 1)H + (n_2 - 1)H_2^L$  with value  $v_{n_1-1}$ .

Finally, if both players play on the same copy of  $H_2^L$ , the result is  $n_1H + (2 - 2) + (n_2 - 1)H_2^L$ , with value  $v_{n_1}$ . If they manage to play on different copies of  $H_2^L$ , the result is  $n_1H + (2 - 3) + H + (n_2 - 2)H_2^L$ , having value  $v_{n_1+1} - 1$  by induction. Hence, both parties playing on  $H_2^L$  yields an average result of  $\frac{1}{n_2}v_{n_1} + \frac{n_2-1}{n_2}(v_{n_1+1} - 1)$ . We thus find that  $n_1H + n_2H_2^L$  boils down to

$$\begin{pmatrix} v_{n_1} & v_{n_1} \\ v_{n_1-1} & \frac{1}{n_2}v_{n_1} + \frac{n_2-1}{n_2}(v_{n_1+1} - 1) \end{pmatrix}.$$

By Theorem 8.2.2,  $v_{n_1} \geq v_{n_1-1}$  and  $v_{n_1+1} - 1 \leq v_{n_1}$ . Hence, the first row dominates, yielding a value of  $v_{n_1} = v(n_1H)$ .  $\square$

**Lemma 8.3.2.** *For  $m, n \in \mathbb{N}$ , let  $v_{m,n} = v(mH_2 + nH)$ . Then*

- (i)  $v_{m+1,n} \leq v_{m,n} + 1$ ;
- (ii)  $v_{m,n} \leq v_{m,n+1} \leq v_{m,n} + 1$ ;
- (iii)  $v_{m+1,n} \leq v_{m,n+1}$ .

*Proof.* We prove all statements through Theorem 2.3.32.

- (i) Consider  $G = (m + 1)H_2 + nH$ , from which Right can play to  $G^{RR} = mH_2 + (n - 1)H + 1$  by moving on the same copy of  $H_2^L$  twice.

- (ii) For the first inequality, consider  $G = mH_2 + (n + 1)H$ , from which Left can move to  $G^L = mH_2 + nH$ . For the second inequality, consider  $G = mH_2 + (n + 1)H$ , from which Right plays to  $G^R = mH_2 + nH + 1$ .
- (iii) Consider  $G = (m + 1)H_2 + nH$  and note that Right can move to  $G^R = mH_2 + (n + 1)H$ .

□

**Lemma 8.3.3.** For all  $n_1, n_2, n_3 \in \mathbb{N}$ , we find  $v(n_1H_2 + n_2H + n_3H_2^L) = v(n_1H_2 + n_2H)$ .

*Proof.* Let  $G = n_1H_2 + n_2H + n_3H_2^L$ . By induction on the birthday, using the previous lemmas, we compute  $G$  being equal to the zero-sum game given by the matrix

$$\begin{pmatrix} \frac{1}{n_1}v_{n_1-1, n_2+1} & v_{n_1-1, n_2-1} + 1 & v_{n_1-1, n_2+1} \\ + \frac{n_1-1}{n_1}v_{n_1-2, n_2+1} & & \\ v_{n_1-1, n_2} & \frac{1}{n_2}(v_{n_1, n_2-1} + 1) & v_{n_1, n_2} \\ + \frac{n_2-1}{n_2}(v_{n_1, n_2-2} + 1) & & \\ v_{n_1-1, n_2+1} - 1 & v_{n_1, n_2-1} & \frac{1}{n_3}v_{n_1, n_2} \\ & & + \frac{n_3-1}{n_3}(v_{n_1, n_2+1} - 1) \end{pmatrix}$$

We first show that the first column dominates the third. Note that, from the game  $(n_1 - 1)H_2 + (n_2 + 1)H + n_3H_2^L$ , Left can move to  $(n_1 - 2)H_2 + (n_2 + 1)H + (n_3 + 1)H_2^L$ , which has value  $v_{n_1-2, n_2+1}$  by induction. Hence,  $v_{n_1-2, n_2+1} \leq v_{n_1-1, n_2+1}$ , so also  $\frac{1}{n_1}v_{n_1-1, n_2+1} + \frac{n_1-1}{n_1}v_{n_1-2, n_2+1} \leq v_{n_1-1, n_2+1}$ . By similar reasoning, also  $v_{n_1-1, n_2} \leq v_{n_1, n_2}$  and  $v_{n_1-1, n_2+1} \leq v_{n_1, n_2+1}$ . By Lemma 8.3.2,  $v_{n_1-1, n_2+1} - 1 \leq v_{n_1-1, n_2}$ , so that  $v_{n_1-1, n_2+1} - 1 \leq \frac{1}{n_3}v_{n_1, n_2} + \frac{n_3-1}{n_3}(v_{n_1, n_2+1} - 1)$ . Hence, the first column indeed dominates the third one.

Next, we show that the first row dominates the third. By Lemma 8.3.2, we find that  $v_{n_1-2, n_2+1} \geq v_{n_1-1, n_2+1} - 1$ , so also  $\frac{1}{n_1}v_{n_1-1, n_2+1} + \frac{n_1-1}{n_1}v_{n_1-2, n_2+1} \geq v_{n_1-1, n_2+1} - 1$ . Moreover,  $v_{n_1-1, n_2-1} + 1 \geq v_{n_1, n_2-1}$ . Hence, the first row indeed dominates the third.

By induction, play continues by making moves only on copies of  $H_2$  and  $H$  until only terminal games and copies of  $H_2^L$  are left, which do not contribute to the value of the game by Lemma 8.3.1. We conclude that  $v(n_1H_2 + n_2H + n_3H_2^L) = v(n_1H_2 + n_2H)$ . □

Note that, by the reasoning in the proof, we may conclude that  $v_{m, n} \leq v_{m+1, n}$  for all  $m, n \in \mathbb{N}$ , extending Lemma 8.3.2. Moreover, recall that, for Blue-Red-

Hackenbush, a Nash equilibrium exists in which both players first play on copies of  $H_2$ , only turning to copies of  $H$  after the quarters have been exhausted. For Push, this is not the case, which makes the argument much less neat.

**Lemma 8.3.4.** *Let  $G = H_2 + nH$  for  $n \geq 4$ . Then Left moving on  $H_2$  and Right moving on any copy of  $H$  uniformly at random is a Nash equilibrium, with value  $v(G) = v_{n-1} + 1$ .*

*Proof.* We proceed by induction on  $n$ . The base cases  $n = 4, 5$  can be checked using computational software to find the Nash equilibria of the games. Hence, suppose  $n \geq 6$ , in which case the game boils down to the zero-sum game given by

$$\begin{pmatrix} v_{n+1} & v_{n-1} + 1 \\ v_n & \frac{1}{n}(v_{1,n-1} + 1) + \frac{n-1}{n}(v_{2,n-2} + 1) \end{pmatrix},$$

where the last entry evaluates to  $1 + \frac{1}{n}v_{1,n-1} + \frac{n-1}{n}v_{2,n-2}$ . First, note that  $v_{n+1} \geq v_n + \frac{1}{2} \geq v_{n-1} + 1$  by Theorem 8.2.2. Next, note that, by induction,  $v_{1,n-1} = v_{n-2} + 1$  and  $v_{2,n-1} = v_{n-3} + 1$ . Hence,

$$\begin{aligned} 1 + \frac{1}{n}v_{1,n-1} + \frac{n-1}{n}v_{2,n-2} &= 1 + \frac{1}{n}(v_{n-2} + 1) + \frac{n-1}{n}(v_{n-3} + 1) \\ &= 2 + \frac{1}{n}v_{n-2} + \frac{n-1}{n}v_{n-3} \\ &\leq 2 + \frac{1}{n-1}v_{n-2} + \frac{n-2}{n-1}v_{n-3} \\ &= 1 + v_{n-1}, \end{aligned}$$

where the inequality follows from Theorem 8.2.2 and the last equality from Theorem 8.2.1. Hence,  $v_{n-1} + 1$  is a saddle point.  $\square$

**Theorem 8.3.5.** *Let  $G = mH_2 + nH$  for  $m \geq 11$  and  $n \in \mathbb{N}$ . Then Left and Right both moving on a copy of  $H_2$  uniformly at random is a Nash equilibrium.*

*Proof.* Consider the game  $G = mH_2 + nH$ . We prove that the statement holds for all pairs  $(m, n)$  for which  $n \geq \max\{1, 5 - \lfloor \frac{m-1}{2} \rfloor\}$  by induction on  $m$ . For the base cases, we prove the validity by using computational software to compute the Nash equilibria for the pairs  $(m, n) \in \{(1, 5), (2, 5), (2, 4), (3, 4), (4, 4), (4, 3), (5, 3), (6, 3), (6, 2), (7, 2), (8, 2), (8, 1)\}$ .

Now, let  $(m, n)$  be no base case, and such that  $n \geq \max\{1, 5 - \lfloor \frac{m-1}{2} \rfloor\}$ . Then  $G = mH_2 + nH$  equals the zero-sum game

$$\begin{pmatrix} \frac{1}{m}v_{m-1,n+1} + \frac{m-1}{m}v_{m-2,n+1} & v_{m-1,n-1} + 1 \\ v_{m-1,n} & \frac{1}{n}(v_{m,n-1} + 1) + \frac{n-1}{n}(v_{m,n-2} + 1) \end{pmatrix}$$

by induction, where the last entry evaluates to  $1 + \frac{1}{n}v_{m,n-1} + \frac{n-1}{n}v_{m,n-2}$ . We set out to show that the first entry is a saddle point. By the extended Lemma 8.3.2, we have  $v_{m-1,n} \leq v_{m-2,n+1} \leq v_{m-1,n+1}$ , so that  $v_{m-1,n} \leq \frac{1}{m}v_{m-1,n+1} + \frac{m-1}{m}v_{m-2,n+1}$ . Remains to show that  $\frac{1}{m}v_{m-1,n+1} + \frac{m-1}{m}v_{m-2,n+1} \leq v_{m-1,n-1} + 1$ .

Consider

$$\begin{aligned} G_1 &= (m-1)H_2 + (n-1)H + 1, \\ G_2 &= (m-2)H_2 + (n+1)H, \\ G_3 &= (m-1)H_2 + (n+1)H. \end{aligned}$$

By induction, the Nash equilibrium strategy for both players is to play on copies of  $H_2$  uniformly at random until none are left. We provide a coupling of the three games. For  $2 \leq k \leq m$ , define

$$Z_k = \begin{cases} 1 & \text{with probability } \frac{1}{k}, \\ 0 & \text{with probability } \frac{k-1}{k}, \end{cases}$$

and

$$X(k) = \begin{cases} 1 + X(k-1) & \text{with probability } \frac{1}{k}, \\ 1 + X(k-2) & \text{with probability } \frac{k-1}{k}. \end{cases}$$

Set  $X(0) = 0$  and  $X(1) = -1$ . Then  $X(k)$  models the number of copies of  $H$  obtained when starting play from  $kH_2$  for  $k$  sufficiently large; note that if we end up with only one copy of  $H_2$ , this indeed costs a copy of  $H$  by Lemma 8.3.4. Now, for  $i = 1, 2, 3$ , set

$$X_i(k) = 1 + Z_k X_i(k-1) + (1 - Z_k) X_i(k-2).$$

We consider  $X_1(m-1)$ ,  $X_2(m-2)$  and  $X_3(m-1)$  and let  $A_i$  be the event that  $X_i$  lands on 1. Then  $G_1$ ,  $G_2$  and  $G_3$  will result in

$$\begin{aligned} G'_1 &= (n-1 + X_1(m-1))H + 1 + \mathbb{1}(A_1), \\ G'_2 &= (n+1 + X_2(m-2))H + \mathbb{1}(A_2), \\ G'_3 &= (n+1 + X_3(m-1))H + \mathbb{1}(A_3), \end{aligned}$$

and we have  $v(G_i) = \mathbb{E}_{Z_2, \dots, Z_{m-1}}[v(G'_i)]$  for  $i = 1, 2, 3$ . Note that we may indeed omit copies of  $H_2^L$  if so created by Lemma 8.3.3. Define

$$\ell = \begin{cases} \max\{k \mid 2 \leq k \leq m-1, Z_k = 1\} & \text{if } \prod_{k=2}^{m-1} (1 - Z_k) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $\ell$  tells us the first time that both players play on the same copy of  $H_2$ , with  $\ell = 0$  signifying that this never happens. We distinguish between four cases.

First, suppose  $\ell = 0$  and  $m$  is even. Then  $X_1(m-1) = X_3(m-1) = \frac{m-2}{2} + X(1) = \frac{m-2}{2} - 1$  and  $X_2(m-2) = \frac{m-2}{2} + X(0) = \frac{m-2}{2}$ . Setting  $n' = n + \frac{m-2}{2} - 1$ , we thus find that, in this case,  $v(G'_1) = v_{n'-1} + 2$ ,  $v(G'_2) = v_{n'+2}$  and  $v(G'_3) = v_{n'+1} + 1$ . By Lemma 8.2.2, we find

$$\begin{aligned} \frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) &= \frac{1}{m}(v_{n'+1} + 1) + \frac{m-1}{m}v_{n'+2} \\ &\leq \frac{1}{2}(v_{n'+1} + 1) + \frac{1}{2}v_{n'+2} \\ &= \frac{1}{2}(v_{n'-1} + d_{n'} + d_{n'+1} + 1) + \frac{1}{2}(v_{n'-1} + d_{n'} + d_{n'+1} + d_{n'+2}) \\ &= v_{n'-1} + d_{n'} + d_{n'+1} + \frac{1}{2}d_{n'+2} + \frac{1}{2}. \end{aligned}$$

For  $n' = 2$ , we have

$$d_2 + d_3 + \frac{1}{2}d_4 + \frac{1}{2} = \frac{1}{2} + \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} = \frac{23}{12} < 2,$$

and  $n' = 3$  yields

$$d_2 + d_3 + \frac{1}{2}d_4 + \frac{1}{2} = \frac{2}{3} + \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} = \frac{59}{30} < 2.$$

Noting that  $d_{2j} = \frac{1}{2}$  and  $d_{2j+1}$  is decreasing in  $j$ , we conclude that  $d_{n'} + d_{n'+1} + \frac{1}{2}d_{n'+2} + \frac{1}{2} \leq 2$  for all  $n'$ , so that

$$\frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) \leq v_{n'-1} + 2 = v(G'_1).$$

Second, suppose  $\ell = 0$  and  $m$  is odd. Then  $X_1(m-1) = X_3(m-1) = \frac{m-1}{2} + X(0) = \frac{m-1}{2}$  and  $X_2(m-2) = \frac{m-1}{2} + X(1) - 1 = \frac{m-1}{2} - 2$ . Hence, now setting  $n' = n + \frac{m-1}{2}$ , we find  $v(G'_1) = v_{n'-1} + 1 = v(G'_2)$  and  $v(G'_3) = v_{n'+1}$ . Again by Theorem 8.2.2,

$$\begin{aligned} \frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) &= \frac{1}{m}v_{n'+1} + \frac{m-1}{m}(v_{n'-1} + 1) \\ &\leq \frac{1}{3}v_{n'+1} + \frac{2}{3}(v_{n'-1} + 1) \\ &= \frac{1}{3}(v_{n'-1} + d_{n'} + d_{n'+1} + \frac{2}{3}(v_{n'-1} + 1)) \\ &= v_{n'-1} + \frac{1}{3}d_{n'} + \frac{1}{3}d_{n'+1} + \frac{2}{3}. \end{aligned}$$

By similar reasoning as above, for  $n' \geq 2$ , we find

$$\frac{1}{3}d_{n'} + \frac{1}{3}d_{n'+1} + \frac{2}{3} \leq \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} = \frac{19}{18} < \frac{11}{10},$$

so that

$$\frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) \leq v_{n'-1} + \frac{11}{10} = v(G'_1) + \frac{1}{10}.$$

Third, suppose  $\ell \geq 2$  and  $\ell \equiv m - 1 \pmod{2}$ . Then  $X_1(m - 1) = X_3(m - 1) = X_2(m - 2) + 1$ , as the processes meet. Moreover,  $A_1 = A_2 = A_3$ . Hence, setting  $A = A_1$  and  $N = n + X(m - 1)$ , we find

$$\begin{aligned} v(G'_1) &= v_{N-1} + 1 + \mathbb{1}(A), \\ v(G'_2) &= v_N + \mathbb{1}(A), \\ v(G'_3) &= v_{N+1} + \mathbb{1}(A). \end{aligned}$$

We compute

$$\begin{aligned} \frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) &= \frac{1}{m}(v_{N+1} + \mathbb{1}(A)) + \frac{m-1}{m}(v_N + \mathbb{1}(A)) \\ &\leq \frac{1}{2}(v_{N+1} + \mathbb{1}(A)) + \frac{1}{2}(v_N + \mathbb{1}(A)) \\ &= \frac{1}{2}(v_{N-1} + d_N + d_{N+1} + \mathbb{1}(A)) + \frac{1}{2}(v_{N-1} + d_N + \mathbb{1}(A)) \\ &= v_{N-1} + d_N + \frac{1}{2}d_{N+1} + \mathbb{1}(A). \end{aligned}$$

We compute

$$d_4 + \frac{1}{2}d_5 = \frac{4}{5} \quad \text{and} \quad d_5 + \frac{1}{2}d_6 = \frac{17}{20},$$

so that, noting that  $d_{2J} + \frac{1}{2}d_{2J+1}$  and  $d_{2J+1} + \frac{1}{2}d_{2J+2}$  are decreasing in  $J$ ,

$$\begin{aligned} \frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) &= v_{N-1} + d_N + \frac{1}{2}d_{N+1} + \mathbb{1}(A) \\ &\leq v_{N-1} + \frac{9}{10} + \mathbb{1}(A) \\ &= v(G'_1) - \frac{1}{10}, \end{aligned}$$

for  $N \geq 4$ .

Fourth, and finally, suppose  $\ell \geq 2$  and  $\ell \equiv m \pmod{2}$ . Defining  $A$  and  $N$  as above, we find  $v(G'_1)$  and  $v(G'_3)$  arrive at the same values, the only difference being that in this case  $v(G'_2) = v_{N+1} + \mathbb{1}(A)$ . We thus compute

$$\begin{aligned} \frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) &= \frac{1}{m}(v_{N+1} + \mathbb{1}(A)) + \frac{m-1}{m}(v_{N+1} + \mathbb{1}(A)) \\ &= v_{N+1} + \mathbb{1}(A) \\ &= v_{N-1} + d_N + d_{N+1} + \mathbb{1}(A). \end{aligned}$$

We compute  $d_4 + d_5 = \frac{11}{10}$ , so that

$$\begin{aligned} \frac{1}{m}v(G'_3) + \frac{m-1}{m}v(G'_2) &= v_{N-1} + d_N + d_{N+1} + \mathbb{1}(A) \\ &\leq v_{N-1} + \frac{11}{10} + \mathbb{1}(A) \\ &= v(G'_1) + \frac{1}{10}. \end{aligned}$$

In the cases  $\ell \geq 2$ , the statements only hold for  $N = n + X_1(m-1) \geq 4$ , i.e.,  $n \geq 4 - X_1(m-1)$ . Noting that  $X_1(m-1) \geq \lfloor \frac{m-1}{2} \rfloor - 1$ , we find that the statements hold for  $n \geq 5 - \max\{1, 5 - \lfloor \frac{m-1}{2} \rfloor\}$ , which was the assumption.

Next, we compute the relevant probabilities. As the  $Z_k$  are independent, for  $k \geq 2$  we compute

$$\mathbb{P}(\ell = k) = \mathbb{P}(Z_k = 1) \prod_{j=k+1}^{m-1} \mathbb{P}(Z_j = 0) = \frac{1}{k} \prod_{j=k+1}^{m-1} \frac{j}{j+1} = \frac{1}{m-1}.$$

For  $\ell = 0$ , we compute

$$\mathbb{P}(\ell = 0) = 1 - \mathbb{P}(\ell \geq 2) = 1 - \sum_{k=2}^{m-1} \mathbb{P}(\ell = k) = 1 - \frac{m-2}{m-1} = \frac{1}{m-1}.$$

Moreover, using these probabilities, we compute

$$\mathbb{P}(\ell \geq 2, \ell \equiv m-1 \pmod{2}) = \begin{cases} \frac{m-2}{2} \cdot \frac{1}{m-1}, & \text{if } m \text{ is even,} \\ \frac{m-1}{2} \cdot \frac{1}{m-1}, & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\mathbb{P}(\ell \geq 2, \ell \equiv m \pmod{2}) = \begin{cases} \frac{m-2}{2} \cdot \frac{1}{m-1}, & \text{if } m \text{ is even,} \\ (\frac{m-1}{2} - 1) \cdot \frac{1}{m-1}, & \text{if } m \text{ is odd,} \end{cases}$$

Hence, for  $m$  even, we compute

$$\begin{aligned} & \mathbb{E}_{Z_2, \dots, Z_{m-1}} \left[ \frac{1}{m} v(G'_3) + \frac{m-1}{m} v(G'_2) \right] \\ & \leq \mathbb{P}(\ell = 0) v(G'_1) + \mathbb{P}(\ell \geq 2, \ell \equiv m-1 \pmod{2}) (v(G'_1) - \frac{1}{10}) \\ & \quad + \mathbb{P}(\ell \geq 2, \ell \equiv m \pmod{2}) (v(G'_1) + \frac{1}{10}) \\ & = \frac{1}{m-1} v(G'_1) + \frac{m-2}{2} \cdot \frac{1}{m-1} (v(G'_1) - \frac{1}{10}) \\ & \quad + \frac{m-2}{2} \cdot \frac{1}{m-1} (v(G'_1) + \frac{1}{10}) \\ & = (\frac{1}{m-1} + \frac{m-2}{m-1}) v(G'_1) = v(G'_1). \end{aligned}$$

For  $m$  odd, we compute

$$\begin{aligned}
 & \mathbb{E}_{Z_2, \dots, Z_{m-1}} \left[ \frac{1}{m} v(G'_3) + \frac{m-1}{m} v(G'_2) \right] \\
 & \leq \mathbb{P}(\ell = 0) \left( v(G'_1) + \frac{1}{10} \right) + \mathbb{P}(\ell \geq 2, \ell \equiv m-1 \pmod{2}) \left( v(G'_1) - \frac{1}{10} \right) \\
 & \quad + \mathbb{P}(\ell \geq 2, \ell \equiv m \pmod{2}) \left( v(G'_1) + \frac{1}{10} \right) \\
 & = \frac{1}{m-1} \left( v(G'_1) + \frac{1}{10} \right) + \frac{m-1}{2} \cdot \frac{1}{m-1} \left( v(G'_1) - \frac{1}{10} \right) \\
 & \quad + \left( \frac{m-1}{2} - 1 \right) \cdot \frac{1}{m-1} \left( v(G'_1) + \frac{1}{10} \right) \\
 & = v(G'_1).
 \end{aligned}$$

Hence, indeed  $\frac{1}{m} v_{m-1, n+1} + \frac{m-1}{m} v_{m-2, n+1} \leq v_{m-1, n-1} + 1$ , so the top-left entry of the matrix is a saddle point.  $\square$

From the proof of Theorem 8.3.5, it follows that, for  $m$  and  $n$  large enough,

$$v_{m, n} = \frac{1}{m} v_{m-1, n+1} + \frac{m-1}{m} v_{m-2, n+1}.$$

Defining  $d_{m, n} = v_{m, n} - v_{m-1, n}$ , and using the base cases written in the aforementioned proof, one may readily compute values for  $v_{m, 0} = v(mH_2)$  and  $d_{m, 0} = v(mH_2) - v((m-1)H_2)$ . Some results are shown in Figure 8.2.

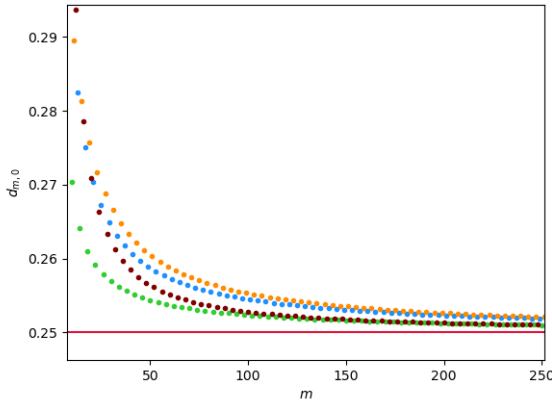


Figure 8.2: Values of  $d_{m,0}$  for  $10 \leq m \leq 250$ . The line  $y = \frac{1}{4}$  is drawn in red, and points are given a color based on their index modulo 4.

We see that, just like the clearly distinguishable subsequences  $d_{2k}$  and  $d_{2k+1}$  when looking at copies of  $H$ , we now find four decreasing subsequences, converging from above to  $\frac{1}{4}$ .

**Theorem 8.3.6.** *We have*

$$\lim_{m \rightarrow \infty} (v(mH_2) - v((m-1)H_2)) = \frac{1}{4}.$$

*Proof.* For  $n \in \mathbb{N}$ , define the stochastic process  $Y(n)$  by  $Y(0) = 0$ ,  $Y(1) = 1$ , and, for  $n \geq 2$ ,

$$Y(n) = \begin{cases} 1 + Y(n-1) & \text{with probability } \frac{1}{n}, \\ 1 + Y(n-2) & \text{with probability } \frac{n-1}{n}. \end{cases}$$

The process  $Y(n)$  models the amount of copies of 1 obtained when starting play from  $nH$ . Hence,  $\mathbb{E}[Y_n] = v(nH)$  and, by Theorem 8.2.2,  $\mathbb{E}[Y(n) - \tilde{Y}(n)] \rightarrow \frac{1}{2}$  for any two copies  $Y, \tilde{Y}$  of the process.

Next, define  $X(m), A, Z_k$  for  $k = 2, \dots, m$ , and  $\ell$  as in the proof of Theorem 8.3.5. Define two copies of each process,  $X$  and  $\tilde{X}$ , and  $Y$  and  $\tilde{Y}$ , coupling them through

$$Y(k) = 1 + Z_k Y(k-1) + (1 - Z_k) Y(k-2)$$

and

$$\tilde{Y}(k) = 1 + Z_k \tilde{Y}(k-1) + (1 - Z_k) \tilde{Y}(k-2),$$

and analogous definitions for  $X$  and  $\tilde{X}$ . We will prove that

$$\lim_{m \rightarrow \infty} \mathbb{E}[Y(X(m)) + \mathbb{1}(A) - \tilde{Y}(\tilde{X}(m)) - \mathbb{1}(\tilde{A})] = \frac{1}{4}.$$

First, note that, by the proof of Theorem 8.3.5,  $\mathbb{P}(\ell = 0) = \frac{1}{m} \rightarrow 0$ , so that also  $\mathbb{E}[\mathbb{1}(A) - \mathbb{1}(\tilde{A})] = 0$ . Next, we consider the difference  $Y(k) - \tilde{Y}(k-1)$ . If  $k \equiv \ell \pmod{2}$ , then  $Y(k) = \frac{k-\ell}{2} + 1 + Y(\ell-1)$  and  $\tilde{Y}(k-1) = \frac{k-\ell-1}{2} + Y(\ell-1)$ . If  $k \equiv \ell-1 \pmod{2}$ , then  $Y(k) = \frac{k-\ell-1}{2} + 1 + Y(\ell-1)$  and  $\tilde{Y}(k-1) = \frac{k-\ell-1}{2} + 1 + Y(\ell-1)$ . Hence, we find that  $Y(k) - \tilde{Y}(k-1) \in \{0, 1\}$  for both cases, so that

$$\mathbb{P}(Y(n) - \tilde{Y}(n-1) = 1) = \mathbb{E}[Y(n) - \tilde{Y}(n-1)] \rightarrow \frac{1}{2}.$$

By similar reasoning, the same holds for  $X$  and  $\tilde{X}$ . Hence, we compute

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \mathbb{E}[Y(X(m)) + \mathbb{1}(A) - \tilde{Y}(\tilde{X}(m-1)) - \mathbb{1}(\tilde{A})] \\
&= \lim_{m \rightarrow \infty} \mathbb{E}[Y(X(m)) - \tilde{Y}(\tilde{X}(m-1))] \\
&= \lim_{m \rightarrow \infty} \left( \mathbb{E}[Y(X(m)) - \tilde{Y}(\tilde{X}(m-1)) \mid X(m) - \tilde{X}(m-1) = 1] \right. \\
&\quad \cdot \mathbb{P}(X(m) - \tilde{X}(m-1) = 1) \\
&\quad \left. + \mathbb{E}[Y(X(m)) - \tilde{Y}(\tilde{X}(m-1)) \mid X(m) - \tilde{X}(m-1) = 0] \right. \\
&\quad \left. \cdot \mathbb{P}(X(m) - \tilde{X}(m-1) = 0) \right) \\
&= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \\
&= \frac{1}{4}.
\end{aligned}$$

The reasoning only holds if  $X(m), Y(m) \rightarrow \infty$  for  $m \rightarrow \infty$ , which is indeed the case.  $\square$

For higher powers of  $\frac{1}{2}$ , the difficulties encountered in the proof of Theorem 8.3.5 become critical. While for Red-Blue Hackenbush, both players always play on the highest power of  $\frac{1}{2}$ , and this is also almost always true if only copies of  $H$  and  $H_2$  are available for Push, such pure Nash equilibria fail to exist for higher powers of  $\frac{1}{2}$  in Push. This makes theoretical analysis difficult. However, experimental results do support the conjecture that a statement along the lines of Theorem 8.3.6 holds in a more general sense.

## 8.4 Shove

We conclude with some preliminary results for the game of Shove. The combinatorial version is fully solved; the following result is from [2].

**Theorem 8.4.1.** *Consider a single strip of Shove  $G$  containing  $n$  pieces. Let  $p(i)$  be the position of the  $i$ -th piece from the left of the strip, and let  $c(i) = 1$  if the  $i$ -th piece is blue and  $-1$  otherwise. Let  $r(i)$  be the number of pieces to the right of the  $i$ -th piece up to and including both pieces of the last color alternation, setting  $r(i) = 0$  if the  $i$ -th piece and all pieces to the right are of the same color. Then*

$$G = \sum_{i=1}^n c(i) \frac{p(i)}{2^{r(i)}}.$$

For the synchronized version, we consider copies of the game

$$K_n = \boxed{\phantom{S}} \boxed{S} + \underbrace{\boxed{S} \boxed{S} \cdots \boxed{S} \boxed{S}}_n$$

which has combinatorial value  $2^{-n}$  by Theorem 8.4.1. For  $n = 1$ , the resulting game is isomorphic to  $H$ , and therefore the same results hold. For  $n = 2$ , some computational results are shown in Figure 8.3.

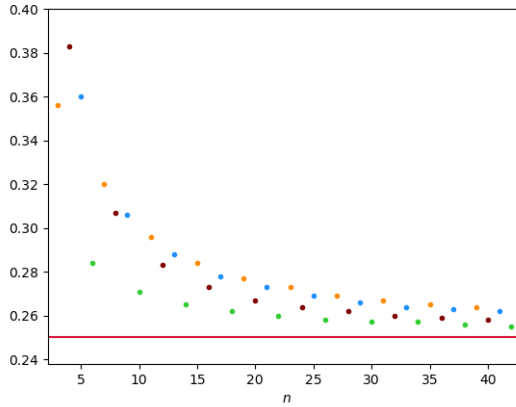


Figure 8.3: The difference  $v(nK_2 - (n-1)K_2)$  for  $k = 3, \dots, 42$ . The line  $y = \frac{1}{4}$  is drawn in red, and points are given a color based on their index modulo 4.

Just like for higher powers of  $\frac{1}{2}$  in Push, copies of  $K_2$  (and also  $K_n$  for  $n > 2$ ) display non-deterministic Nash equilibria, making the analysis difficult. However, Figure 8.3 suggest that also for this game, a result similar to Theorem 8.3.6 holds.