



Universiteit
Leiden
The Netherlands

Imperfect information variants of combinatorial games

Bergh, M.J.H. van den

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Chapter 6

Synchronized Cherries

In this chapter, we consider the synchronized version of a variant of the combinatorial game of Cherries, called *Synchronized Stack Cherries*. In this variant, the players may only remove tokens from the front of a segment, instead of from both sides as in regular Cherries. The material in this chapter is based on joint work with Thomas de Mol, started in [18].

We provide a linear-time algorithm to decompose any given Synchronized Stack Cherries position into irreducible stacks and show that these irreducible stacks are fully ordered, allowing for a quick assessment of the winner of a given position and the magnitude of the win. We conjecture that a similar decomposition method can be used to analyze positions in the synchronized version of the regular game of Cherries.

6.1 Introduction

Combinatorial game theory considers the class of two-person, deterministic games with perfect information. Most of the underlying theory can be found in [1], [2] and [3]. Games belonging to this class are, e.g., Nim, Hackenbush, Hex and Domineering. We will focus on variants of Cherries.

In combinatorial games, players move in a sequential fashion, alternating turns until the game's end. In recent years, research has been done on what remains of the theoretical framework if we instead allow the players to move

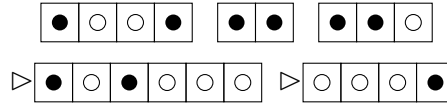


Figure 6.1: Examples of a Cherries position (top) and a Stack Cherries position (bottom).

simultaneously, leading to *synchronized* versions of combinatorial games [13,21]. Every turn, both players communicate their intended move to an impartial referee, who then executes both moves at once.

In this chapter, we study a variant of the combinatorial game of Cherries introduced in Example 2.1.22, which we name Stack Cherries. Recall that a Cherries position consists of ordered connected sequences of tokens, named *cherries*, each colored either black or white. See Figure 6.1 (top) for an example position. In the normal combinatorial game, the two players Left and Right alternate turns, Left removing one black cherry from exactly one sequence on her turn, and Right removing a white cherry on his; only cherries at the beginning or end of a sequence can be removed. A player unable to move on their turn immediately loses, the other player wins. In Stack Cherries, only cherries at the beginning of sequences can be removed, as indicated by a triangle in Figure 6.1 (bottom).

In the synchronized version of the two games, Left and Right pick their cherries simultaneously, after which both are removed at once. Again, if either player is unable to move, the other player immediately wins the game. If all sequences are cleared and therefore neither player can move, the game ends in a draw — which cannot happen for regular Cherries. We call the synchronized version Synchronized Cherries and Synchronized Stack Cherries (or briefly SSC), respectively. Sequences of cherries are usually referred to as *segments* or (in the context of SSC) sometimes as *stacks*; for example, the position in Figure 6.1 (bottom) consists of two stacks.

Though Synchronized Stack Cherries seems simple, it has surprising complexity and expressive power. Our main result, Theorem 6.4.5, is that any SSC segment can be decomposed into a sum of irreducible segments using a linear-time algorithm. Theorem 6.3.9 shows that these irreducible segments are ordered in a very strong sense, allowing us to readily draw conclusions about the winner of the given position, as well as the magnitude of the win.

First, in Section 6.2, we give a full characterization of the combinatorial game

of Stack Cherries and provide the necessities concerning synchronized games. We continue by defining the irreducible elements for SSC in Section 6.3 and provide our main decomposition result in Section 6.4. Finally, in Section 6.5, we conclude with a few words on the relation between Synchronized Cherries and SSC, conjecturing that every position of the former is — in a very precise way — equivalent to a position of the latter.

6.2 Basics

We start by providing the full solution to the combinatorial game of Stack Cherries. In a stack of cherries, we say a *block* is a consecutive series of maximal length with cherries of the same color. A block is *non-trivial* if it consists of more than one cherry.

Theorem 6.2.1. *Let G be a Stack Cherries stack starting with $m \geq 1$ black cherries. If G contains a non-trivial block past the first black block, let $c = 1$ if this block is black, and $c = 0$ if it is white. Otherwise, let $c = 1$ if the last cherry of the stack is black, and $c = 0$ if it is white. Then $G = m - 1 + c$.*

The result is readily proven by induction. As an example, the two stacks in Figure 6.1 (bottom) have values 0 and -2 , respectively. Notice that all Stack Cherries games are integers, just like regular Cherries games.

We continue by delving slightly into synchronized games. In Section 2.3.1, we defined eight outcome classes for synchronized games. For the remainder of this chapter, we gather all games in the classes \mathcal{D} , \mathcal{LD} , \mathcal{RD} , \mathcal{LR} and \mathcal{LRD} into a single class of games \mathcal{U} for which the outcome of the game is a priori *uncertain*. We order the classes $\mathcal{R} < \mathcal{U} < \mathcal{L}$. In this framework, the following general lemma holds.

Lemma 6.2.2. *Let G be a synchronized game for which $\mathcal{G}^L \neq \emptyset$ and $\mathcal{G}^R = \emptyset$. Then $G > 0$.*

Proof. Let X be an arbitrary synchronized game. We will show that $o(G + X) \geq o(X)$ by induction on the birthday of X .

First, if X has no options at all, then $o(G + X) = o(G) = \mathcal{L} > \mathcal{U} = o(0) = o(X)$. If X has left options, but no right options, then $o(X) = \mathcal{L}$. Moreover, $o(G + X) = \mathcal{L}$, as G also has no right options, so indeed $o(G + X) \geq o(X)$. Conversely, if X only has right options, $o(X) = \mathcal{R}$, so certainly $o(G + X) \geq o(X)$.

Next, suppose X has both left and right options, and assume first that $o(G + X) = \mathcal{R}$. Then, G having no right options, there is some right option X_j^R of X such that $G + X_{ij}^S$ is a Right-player win for all i . By induction, then also $o(G + X_{ij}^S) \geq o(X_{ij}^S)$, so X_{ij}^S is a win for Right for all i , and thus $o(X) = \mathcal{R}$.

Second, suppose $o(X) = \mathcal{L}$. Left winning X , there must be some i such that $o(X_{ij}^S) = \mathcal{L}$ for all j . Then Left can also win $G + X$ by playing to $G + X_{ij}^S$ and continuing locally on X_{ij}^S until Right's options have run out. Hence, $o(G + X) = \mathcal{L} \geq o(X)$.

We conclude that $o(G + X) \geq o(X)$. Now, note that $o(G + 0) = o(G) = \mathcal{L} > \mathcal{U} = o(0)$, so $G \neq 0$. Hence $G > 0$. \square

The technique demonstrated in this proof will be used more often throughout this chapter: to show that $G \geq H$ for any two synchronized games, it suffices to show that $o(G + X) = \mathcal{R} \Rightarrow o(H + X) = \mathcal{R}$ and $o(H + X) = \mathcal{L} \Rightarrow o(G + X) = \mathcal{L}$ for all synchronized games X . This type of proof is necessary as, unfortunately, even having introduced the class \mathcal{U} , for general synchronized games, $G \geq H$ need not imply $G - H \geq 0$.

Example 6.2.3. Define $G = \boxed{\bullet}$, and consider the somewhat unnatural game $X = \left\{ \boxed{\circ\circ} \mid \boxed{\bullet\bullet} \mid \boxed{\circ\circ} \right\}$. It is clear that $o(X) = \mathcal{L}$. Now, consider $G - G + X$. If Left moves on G , Right can move on X to $\boxed{\circ} + \boxed{\circ\circ}$, which is won by Right. If Left moves on X , Right can respond on $-G$, playing to $\boxed{\bullet} + \boxed{\circ\circ}$, which is also won by Right. Hence, Left has no move guaranteeing a win, so $o(G - G + X) \neq \mathcal{L}$. Hence $o(G - G + X) \not\geq o(X)$, so $G - G \not\geq 0$, while obviously $G \geq G$. \triangleleft

6.3 Irreducible segments

We now narrow our focus to the game of Synchronized Stack Cherries, setting out to prove Theorem 6.3.9, that orders the basic building blocks: the irreducible segments. We start by introducing some notation. For any Stack Cherries segment G of length $n \geq 1$, we denote by $G_{i:j}$ the part of the segment only consisting of the cherries $i, i + 1, \dots, j - 1, j$; here $1 \leq i \leq j \leq n$. Moreover, we denote by G_i the suffix of the segment consisting of the cherries

$i, i + 1, \dots, n$; here $1 \leq i \leq n$. If $i > n$, G_i denotes the empty segment. Note that $G = G_{1:n} = G_{1:}$.

Previously, we saw that for two synchronized games G and H in general, it is not always easy to determine whether $G \geq H$. For SSC segments, this turns out to be more straightforward. We introduce a *lexicographical order* on the set of SSC segments, denoted by \preceq . We define a white cherry to be lexicographically smaller than an empty square, which is in turn smaller than a black cherry. To compare two segments consisting of more than one cherry, we align to the left, adding an empty square at the end of the shorter segment if necessary, and then compare.

Example 6.3.1. Consider

$$G_1 = \triangleright \left[\begin{array}{|c|c|c|c|} \hline \bullet & \circ & \circ & \bullet \\ \hline \end{array} \right], \quad G_2 = \triangleright \left[\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} \right] \quad \text{and} \quad G_3 = \triangleright \left[\begin{array}{|c|c|c|} \hline \bullet & \bullet & \circ \\ \hline \end{array} \right].$$

We have $G_1 \preceq G_2$ and $G_1 \preceq G_3$, as the second cherry of G_1 is white, while the second cherries of G_2 and G_3 are black. Also $G_3 \preceq G_2$, as G_2 has no cherry in the third position, whereas the third cherry of G_3 is white. \triangleleft

It turns out the lexicographic order is in fact the synchronized order as we know it.

Theorem 6.3.2. *For any two SSC segments G and H , it holds that $G \leq H$ if and only if $G \preceq H$.*

Proof. First, if either segment is empty, the result immediately follows from Lemma 6.2.2. If, say, G starts with a white cherry and H with a black cherry, then also $G < 0 < H$ by Lemma 6.2.2. Hence, it remains to prove the statement for two segments starting with a cherry of the same color. Assume without loss of generality that both G and H start with a black cherry.

Let X be an arbitrary synchronized game and consider $G + X$ and $H + X$. We proceed by induction on the birthday of X . Note that, if X has no right options, then neither does $G + X$ nor $H + X$, so that $o(G + X) = o(H + X) = \mathcal{L}$. Hence, suppose that X does have at least one right option. Assume that $G \preceq H$.

First, suppose $o(H + X) = \mathcal{R}$. Right, having no moves on H , but winning $H + X$, must have some move to $H + X_j^R$ so that both $H_{2:} + X_j^R$ and $H + X_{ij}^S$ are Right-player wins for all i . By induction, also $o(G + X_{ij}^S) = \mathcal{R}$ for all i . Moreover, as $G_{2:} \preceq H_{2:}$, also $o(G_{2:} + X_j^R) \leq o(H_{2:} + X_j^R) = \mathcal{R}$. Hence, $G + X$ is won by Right playing to $G + X_j^R$.

Next, suppose $o(G + X) = \mathcal{L}$. If Left wins by playing to $G_{2:} + X$, then $o(G_{2:} + X_j^R) = \mathcal{L}$ for all j , so also $\mathcal{L} = o(G_{2:} + X_j^R) \leq o(H_{2:} + X_j^R)$ for all j , and thus Left wins $H + X$ moving on H . If Left wins $G + X$ by playing to $G + X_i^L$ for some i , then $G + X_{ij}^S$ is a win for Left for all j . Therefore, by induction, $H + X_{ij}^S$ is a Left win for all j , as well, so Left wins $H + X$ playing to $H + X_i^L$. So $G \leq H$.

Conversely, if $G \leq H$, we know in particular that $o(G + X) \leq o(H + X)$ where X is the longest common prefix of G and H with black and white cherries toggled, say of length $\ell \geq 1$. Playing in these two games leads to comparison of the $(\ell + 1)$ -st cherries of G and H (perhaps even an empty square), showing that $G \preceq H$. \square

Using the (lexicographic) order on SSC segments, determining the best move for either player is relatively straightforward.

Theorem 6.3.3. *Let G and H be SSC segments that start with a cherry of the same color. If $G \leq H$, then $G_{2:} + H \leq G + H_{2:}$.*

We omit the laborious proof, which can be found in [18]. This theorem gives us a deterministic optimal strategy for both players: Left repeatedly plays on the greatest segment (starting with a black cherry), whereas Right takes a cherry from the smallest segment (starting with a white cherry). Hence, the outcome of a game of Synchronized Stack Cherries is always deterministic under optimal play, and any position in \mathcal{U} always ends in a draw.

However, though this observation does provide us with a way to determine the winner and therewith outcome class of a position, it does not give us an efficient way to compare positions, nor a way to measure the magnitude of a position. We continue by developing these concepts in more detail.

As illustrated previously, many beautiful results for combinatorial games do not readily carry over to synchronized games. However, by somewhat relaxing our definition of equality, we do pave the way for fundamental results more in line with combinatorial game theory.

Definition 6.3.4. For two SSC positions G and H , we say $G =_{SC} H$ if $o(G + X) = o(H + X)$ for all SSC positions X . Similarly, we use \leq_{SC} , $<_{SC}$, \geq_{SC} and $>_{SC}$.

Lemma 6.3.5. *Let G be a SSC segment. Then $G - G =_{SC} 0$.*

Proof. Assume without loss of generality that G starts with a black cherry. Let X be an arbitrary SSC position, and proceed by induction on the birthdays of

G and X . First, suppose $o(G - G + X) = \mathcal{L}$. If Left can win by moving on G , then in particular $o(G_{2:} - G_{2:} + X) = \mathcal{L}$, so by induction also $o(X) = \mathcal{L}$. If Left wins by moving on X , then there is some i such that $o(G - G + X_{ij}^S) = \mathcal{L}$ for all j . Hence, by induction, also $o(X_{ij}^S) = \mathcal{L}$ for all j , so X is also won by Left.

Next, suppose $o(X) = \mathcal{L}$, and let X_i^L be a winning move for Left. Consider $G - G + X$. If the (lexicographically) smallest segment starting with a white cherry in this game is located in X , then moving on this segment to X_j^R is the dominating move for Right by Theorem 6.3.3. The result for Left also moving on X is $G - G + X_{ij}^S$, which is a win for Left by induction — hence, $o(G - G + X) = \mathcal{L}$. If the (lexicographically) smallest segment starting with a white cherry in $G - G + X$ is $-G$, then this is the dominating move for Right. Left can win by playing on G to $G_{2:} - G_{2:} + X$ by induction, so also in this case $G - G + X$ is a win for Left. Note that the latter case includes the case of G being part of X .

Hence, $o(G - G + X) = \mathcal{L}$ if and only if $o(X) = \mathcal{L}$. An analogous argument works for the case in which X is a Right-player win, from which the conclusion follows. □

Through the (lexicographical) order, all SSC segments are comparable. However, some segments turn out to be much smaller than others, similar to the infinitesimals in combinatorial game theory and the value ε described in [21]. To show this, we need the notion of irreducible elements.

First, note that, by Lemma 6.2.2, for all non-empty SSC segments G , we find either $G > 0$ or $G < 0$, depending on whether G starts with a black or white cherry. Hence, it makes sense to define

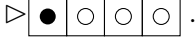
$$|G| = \begin{cases} G & \text{if } G \geq 0, \\ -G & \text{if } G < 0. \end{cases}$$

If $|G| > |H|$ for two SSC segments G and H , we say G is (lexicographically) stronger than H , and H is weaker than G .

Definition 6.3.6. Let G be a SSC segment of length $n \geq 1$. We call G an *irreducible element* if $|G| < |G_i|$ for all $1 < i \leq n$.

In words, an irreducible G is weaker than all its suffixes, apart from itself.

Example 6.3.7. The five positive irreducible elements of length at most 4 are, in decreasing order, $\triangleright \blacksquare$, $\triangleright \blacksquare \square$, $\triangleright \blacksquare \square \square$, $\triangleright \blacksquare \square \square \blacksquare$, $\triangleright \blacksquare \square \square \square$ and



It turns out that all different irreducible elements are infinitesimal with respect to each other. We need the following lemma, the proof of which is based on a direct comparison of the segments in question and can be found in [18].

Lemma 6.3.8. *Let G be an irreducible element of length n . Then $|G_i| \geq |G_{1:i-1}|$ for all $1 < i \leq n$.*

Theorem 6.3.9. *Let G and H be irreducible elements such that $G > H > 0$. Then $G >_{\text{SC}} k \cdot H$ for all $k \in \mathbb{N}$.*

Proof. By Lemma 6.3.5, it suffices to prove $G - k \cdot H >_{\text{SC}} 0$. We proceed by induction on k , the base case being true by assumption. Let X be an arbitrary SSC position and consider $G - k \cdot H + X$.

First, suppose that $-H$ is the lexicographically strongest segment starting with a white cherry, which Right will remove. Left plays on G , resulting in $G_2: -H_2: - (k-1) \cdot H + X$. Note that, as $G - H > 0$, at least one of $G_2:$ and $-H_2:$ starts with a black cherry. If the other starts with a white cherry, by the assumption of G and H being irreducible elements, this is the lexicographically strongest segment starting with a white cherry. Hence, optimal play for Right dictates the removal of this white cherry, and Left may respond on the other of the two segments. Play continues in this fashion until we arrive at $G_i: -H_i: - (k-1) \cdot H + X$ for some i , where both $G_i:$ and $-H_i:$ start with a black cherry, or one of them is empty.

If $G_i:$ starts with a black cherry, then $G_i: -H_i: \geq G_i: > G$, since G is an irreducible element. If not, then $G_i:$ is the empty segment, and, by the mirrored play so far, we find $G = H_{1:i-1}$. By Lemma 6.3.8, $|H_i| \geq |H_{1:i-1}|$, so $G_i: -H_i: = 0 + |H_i| \geq |H_{1:i-1}| = |G| = G$. Hence, $G_i: -H_i: \geq G$ in any case, so

$$G_i: -H_i: - (k-1) \cdot H + X \geq G - (k-1) \cdot H + X > X$$

by induction. Right having played optimally, we find $o(G - k \cdot H + X) \geq o(X)$.

Second, suppose that the lexicographically strongest segment starting with a white cherry is contained in X , and that removing this cherry leads to X_j^R . First, assume that $o(X) = \mathcal{L}$, Left winning by moving to X_i^L . By induction on the birthday of X , we conclude $o(G - k \cdot H + X_{ij}^S) \geq o(X_{ij}^S) = \mathcal{L}$. Hence, Right moving optimally, $G - k \cdot H + X$ is a Left-player win. Next, suppose $o(G - k \cdot H + X) = \mathcal{R}$, the winning move for Right being to $G - k \cdot H + X_j^R$. It

follows that $\mathcal{R} = o(G - k \cdot H + X_{ij}^S) \geq o(X_{ij}^S)$ for all i by induction, so $o(X) = \mathcal{R}$. Hence $o(G - k \cdot H + X) \geq o(X)$.

We conclude that $G - k \cdot H \geq_{\text{SC}} 0$. Now, noting that $o(G - k \cdot H) \geq o(G - (k-1) \cdot H)$ by the reasoning above, and that $o(G) = \mathcal{L}$, we conclude by induction that $o(G - k \cdot H) = \mathcal{L}$, whereas $o(0) = \mathcal{U}$. Hence, strict equality does not hold, and the result follows. \square

Example 6.3.10. Consider the irreducible elements G_1 and G_2 defined by $G_1 = \triangleright \boxed{\bullet}$ and $G_2 = \triangleright \boxed{\bullet \mid \circ}$. Then $G_1 > G_2$, and by Theorem 6.3.9, $G_1 >_{\text{SC}} k \cdot G_2$ for any $k \in \mathbb{N}$. \triangleleft

Before continuing with the decomposition of SSC segments into irreducible elements, we make a small detour, providing a method to count the number of unique irreducible elements of a given length. To do so, we consider words over the alphabet $\{0, 1\}$. We use the lexicographical order \preceq as defined before for Stack Cherries, writing $0 \prec \varepsilon \prec 1$, denoting the empty word by ε . We denote the canonical lexicographical order by \triangleleft , writing $\varepsilon \triangleleft 0 \triangleleft 1$. We define $\bar{0} = 1$ and $\bar{1} = 0$, and naturally extend this definition to all words in $\{0, 1\}^*$. For a word $w = w_1 w_2 \dots w_n$, we write $\rho_k(w) = w_{k+1} w_{k+2} \dots w_n w_1 \dots w_k$ for the counterclockwise rotation over k positions. We say that the *fundamental period* of w equals p if $\rho_p(w) = w$, and $\rho_q(w) \neq w$ for all $1 \leq q < p$. Finally, we write $|w| = w$ if $w_1 = 1$, and $|w| = \bar{w}$ otherwise.

Theorem 6.3.11. *Let G be a SSC segment starting with a black cherry and represent G by a word $w \in \{0, 1\}^*$, writing a 1 for a black cherry and 0 for a white cherry. Then G is irreducible if and only if w has fundamental period n and $\rho_1(w)$ is the lexicographically smallest element of $S(w) = \bigcup_{k=1}^n \{\rho_k(w), \rho_k(\bar{w})\}$ under the canonical lexicographical order \triangleleft .*

Proof. Suppose G is irreducible. First, suppose w has fundamental period $k < n$. Note that $k \mid n$; hence, $G = HH \dots H$ (n/k times) for some suffix H of length k which starts with a black cherry. But then $|H| = H < G = |G|$, which contradicts the fact that G is irreducible. Hence, w indeed has fundamental period n .

Next, suppose $\rho_1(w)$ is not the lexicographically smallest element of $S(w)$. First, suppose that $w' = \rho_k(w)$ is the lexicographically smallest element of $S(w)$ for some $k \neq 1$. Naturally, $w'_1 = 0$, as the word consisting of only 1's is not irreducible. Moreover, $w'_n = 1$, as otherwise $\rho^{-1}(w')$ is lexicographically smaller than w' . Now, consider the suffix $z = w_k w_{k+1} \dots w_n$ of w , of which all

but w_k is a prefix of w' . Note that $w_k = w'_n = 1$. By the assumption of $w' \leq \rho_1(w)$, at the first element at which the words differ, the element of w' must be smaller than the element of $\rho_1(w)$. If this element is one of the first $n - k$ of w' , then $z < w$, which is a contradiction. Otherwise, if the first $n - k$ elements match, we find the $n - k + 1$ -st element of w' to be $w_1 = 1$. For $w' \leq \rho_1(w)$ to hold, then also $w_{n-k+2} = 1$. But then $z < w$. Hence, $w' \triangleleft \rho_1(w)$ cannot hold.

Next, suppose that $w' = \rho_k(\bar{w})$ is the lexicographically smallest element of $S(w)$ for some k . By the reasoning above, $w'_1 = 0$ and $w'_n = 1$. Now, first suppose that the first $n - k$ elements of w' are not all 0's. Let ℓ be the position of the first 1 in w' and consider the prefix $w'_1 \dots w'_{\ell-1}$ of w' . Then the same argument as above leads to a contradiction, taking as suffix $z = w_{n-\ell+1} \dots w_n$ of w . If the first $n - k$ elements of w' are 0's, then the last $n - k$ elements of $\rho_1(w)$ are 1's. Note that $n - k > \frac{n}{2}$ must hold; otherwise, a rotation of w would be smaller than this rotation of \bar{w} . But then for the suffix $z = w_{k+1} \dots w_n$ of w we find $|z| < |w| = w$. Hence, $\rho_1(w)$ must indeed be the lexicographically smallest element of $S(w)$.

Now, suppose w has fundamental period n and $\rho_1(w)$ is the lexicographically smallest element of $S(w)$. For the sake of contradiction, suppose that $w > |z|$ for some suffix z of w of length k (note that equality cannot occur for words of different length). First, suppose $|z| = z$. If the first ℓ elements of w and z are equal, and $w_{\ell+1} > z_{\ell+1}$, then we may rotate w such that it starts with $z_2 \dots z_{\ell+1}$, finding a smaller rotation than $\rho_1(w)$. Hence, all elements of z and w must be equal, forcing $w_{k+1} = 1$. But then we can rotate w such that it starts with w_{k+1} , which provides a smaller rotation, as the first $k - 1$ elements remain the same, and the k -th is now a 0 instead of a 1.

Next, suppose $|z| = \bar{z}$. We repeat the above argument, now swapping the colors of w before rotating, again leading to smaller elements of $S(w)$. Hence, w is indeed irreducible. \square

The words over $\{0, 1\}$ as described before are usually called *necklaces* in the context of equivalence under rotation [43].

Corollary 6.3.12. *For a fixed n , the number of irreducible elements of length n starting with a black cherry equals the number of binary necklaces of length n having fundamental period n , in which the colors may be swapped.*

The number of such necklaces, and therewith the number of positive irreducible elements for any fixed length, may be found as A000048 in the OEIS [44].

6.4 Decomposition

If a SSC position consists of only irreducible elements, it is straightforward to find the winner and its value. Hence, we develop a method to decompose any given SSC position into irreducible elements. Again, the omitted proof of Lemma 6.4.1 can be found in [18].

Lemma 6.4.1. *Let G be a SSC segment of length n that is not an irreducible element. Let $G_{i+1:}$ be the weakest suffix of G , and let $0 \leq j < i$ be such that $|G_{j+1:i}| \leq |G_{k+1:i}|$ for all $0 \leq k < i$. Then $|G_{i+1:}| \leq |G_{j+1:i}|$. If the $(i+1)$ -st and $(j+1)$ -st cherries of G have a different color, the inequality is strict.*

Theorem 6.4.2. *Let G be a SSC segment of length n and let $0 \leq i < n$ be such that $|G_{i+1:}| \leq |G_{j+1:}|$ for all $0 \leq j < n$. Then $G = G_{1:i} + G_{i+1:}$.*

Proof. Assume without loss of generality that the first cherry of G is black. For $i = 0$, there is nothing to show, so let $i > 0$, and consider $G_{i+1:}$.

First, suppose $G_{i+1:}$ also starts with a black cherry. By assumption, $G_{i+1:} = |G_{i+1:}| \leq |G_{0+1:}| = G$, so $G - G_{i+1:}$ is a Left-player win, as the segments differ in length. If G and $G_{i+1:}$ do not match for the first i cherries, then play on $G - G_{i+1:}$ will halt with less than i cherries having been taken from G , so $G_{1:i} - G_{i+1:} > 0$ must hold.

If G and $G_{i+1:}$ do match for the first i cherries, play continues to $G_{i+1:} - G_{2i+1} > 0$. Now, if the $i+1$ -st cherry of G and $G_{i+1:}$ would match, being the first cherry of $G_{i+1:}$, these cherries should be black. However, it would follow that $|G_{i+1:}| = G_{i+1:} > G_{2i+1:} = |G_{2i+1:}|$, which is a contradiction. Hence, if G and $G_{i+1:}$ match for the first i cherries, they must mismatch at the $i+1$ -st. This cherry in $G_{i+1:}$ being white or non-existent, we conclude that $G_{1:i} > G_{i+1:}$ in both cases.

Hence, in any context, we find that, in $G_{1:i} + G_{i+1:} + X$, Left's move on $G_{i+1:}$ is dominated. Therefore, by induction,

$$G_{1:i} + G_{i+1:} = \{G_{2:i} + G_{i+1:} \mid \mid\} = \{G_{2:} \mid \mid\} = G.$$

Second, suppose $G_{i+1:}$ starts with a white cherry. By induction, using the statement of Lemma 6.4.1, we can obtain a collection of irreducible elements B_1, \dots, B_k such that $B_1 > 0$, $B_1 \geq \dots \geq B_k$ and $G_{1:i} = B_1 + \dots + B_k$. Let X be an arbitrary SSC position, and consider $B_1 + \dots + B_k + G_{i+1:} + X$. If the best move for Right is on some B_j , then $W \geq B_j$ for all other segments W starting with a

white cherry, as well as $G_{i+1:} \geq B_j$. Furthermore, for all segments B starting with a black cherry, we have $B > 0 > B_j$. Hence, $B_1 + \dots + B_k + G_{i+1:} + X \geq B_1 + \ell \cdot B_i$ for some ℓ . By Lemma 6.4.1, $|B_1| > |B_j|$, as the stacks start with different colors. We conclude that, both being irreducible elements,

$$B_1 + \dots + B_k + G_{i+1:} + X \geq B_1 + \ell \cdot B_i > -\ell \cdot B_j + \ell \cdot B_j = 0.$$

Hence, if Right's best move on $B_1 + \dots + B_k + G_{i+1:} + X$ is to play on some B_j , then the game is won by Left. Hence, leaving out the possible moves for Right on these stacks cannot worsen the outcome for Right, being \mathcal{L} . Furthermore, deleting possible moves for Right never worsens the outcome for Left. Hence, we may delete all moves on the B_j for Right without changing the outcome class in any context. We conclude that

$$\begin{aligned} G_{1:i} + G_{i+1:} &= B_1 + \dots + B_k + G_{i+1:} \\ &= \{(B_1)_{2:} + \dots + B_k + G_{i+1:} \mid \mid\} \\ &= \{G_{2:} \mid \mid\} \\ &= G. \end{aligned}$$

□

By repeatedly applying Theorem 6.4.2, one may decompose any given SSC segment into a sum of irreducible elements, every pair of different elements being infinitesimal with respect to each other by Theorem 6.3.9. Compare this to the optimal notation for dicotic games as described in [2, Chapter 9].

Example 6.4.3. Consider the SSC segment

$$G = \triangleright \boxed{\bullet \bullet \circ \bullet \bullet \circ \circ}$$

Comparing all non-empty suffixes, we find that $G_{5:} = \triangleright \boxed{\bullet \circ \circ}$ is the weakest. Hence, in particular, $|G_{5:}| \leq |G_{6:}|$ and $|G_{5:}| \leq |G_{7:}|$, so

$$G = \triangleright \boxed{\bullet \bullet \circ \bullet \bullet} + \triangleright \boxed{\bullet \circ \circ}$$

by Theorem 6.4.2. Now, comparing all suffixes of $\triangleright \boxed{\bullet \bullet \circ \bullet}$, we find that

$\triangleright \boxed{\circ \bullet}$ is the weakest, leading to

$$G = \triangleright \boxed{\bullet \bullet} + \triangleright \boxed{\circ \bullet} + \triangleright \boxed{\bullet \circ \circ}$$

Again looking at the suffixes of $\triangleright \boxed{\bullet \bullet}$, we conclude that this segment can be split once more, producing

$$G = \triangleright \boxed{\bullet} + \triangleright \boxed{\bullet} + \triangleright \boxed{\circ \bullet} + \triangleright \boxed{\bullet \circ \circ}$$

Note that indeed every segment of the decomposition is an irreducible element. \triangleleft

Using Theorem 6.4.2 to decompose a given SSC segment requires continuously comparing the lexicographic strength of suffixes, leading to an $O(n^2)$ algorithm to determine the decomposition of a segment of length n . This can be sped up. We once again omit the proof of Lemma 6.4.4 and refer to [18].

Lemma 6.4.4. *Let G be a SSC segment of length n and let G_{i+1} be its weakest suffix. Then, for all $1 \leq \ell, k \leq i$, it holds that $|G_{\ell}| > |G_k|$ if and only if $|G_{\ell:i}| > |G_{k:i}|$.*

Theorem 6.4.5. *Let G be a non-irreducible SSC segment of length n and let $1 \leq i < n$ be the smallest index for which $|G_{i+1}| \leq |G|$. Then $G = G_{1:i} + G_{i+1}$.*

Proof. Let G_{k+1} be the weakest suffix of G . Suppose first that $G_{1:k}$ is an irreducible element, i.e., $|G_{j:k}| > |G_{1:k}|$ for all $1 \leq j \leq k$. By Lemma 6.4.4, then also $|G_j| > |G_{1:}| = |G|$ for all $1 \leq j \leq k$, so $i \geq k+1$. As G_{k+1} is the weakest suffix of G , also $i \leq k+1$, so we conclude that $i = k+1$. Hence, by Theorem 6.4.2, $G = G_{1:i} + G_{i+1}$.

Next, suppose $G_{1:k}$ is not an irreducible element. It still holds that $i \leq k+1$, as G_{k+1} is the weakest suffix of G . If $i = k+1$, we are done by Theorem 6.4.2, so suppose that $i < k+1$. By Lemma 6.4.4, i is the smallest index for which $|G_{i+1:k}| \leq |G_{1:k}|$. By induction on the birthday, it follows that $G_{1:k} = G_{1:i} + G_{i+1:k}$. As G_{k+1} is the weakest suffix of G , it is also the weakest suffix of G_{i+1} . Hence, by Theorem 6.4.2, $G_{i+1} = G_{i+1:k} + G_{k+1}$. We conclude that

$$G = G_{1:k} + G_{k+1} = G_{1:i} + G_{i+1:k} + G_{k+1} = G_{1:i} + G_{i+1}.$$

□

Theorem 6.4.5 provides us with a faster way of decomposing a given SSC segment into a sum of irreducible elements. By using *suffix arrays* to store information about the suffixes of a given SSC segment, we can develop a linear-time algorithm to decompose the given segment into irreducible elements [45–47].

6.5 Summary and future work

In this chapter, we have considered a variant of the combinatorial game of Cherries, called Synchronized Stack Cherries. In this variant, players are only allowed to take a cherry from the front of a given segment or stack. In the combinatorial version of the game, all encountered positions are integers, readily characterized by an inductive argument.

Subsequently, we analysed the synchronized version of the game, in which the two players make moves simultaneously. For this game of Synchronized Stack Cherries, we showed that any segment can be decomposed into a sum of irreducible elements in linear time using Theorem 6.4.5. These irreducible elements are all infinitesimal with respect to each other by Theorem 6.3.9. Analogous to the optimal notation for dicotic combinatorial games, just looking at the sign of the largest irreducible element in the decomposition immediately tells us the outcome class of a game.

We conjecture that it is possible to decompose a given segment of Synchronized Cherries into a sum of Synchronized Stack Cherries irreducible elements in a similar way. We repeatedly start at the lexicographically strongest end of the segment, taking cherries until the other side becomes stronger. Every group of cherries taken in this way forms a new Stack Cherries segment. More details can be found in [18]. Using this decomposition, it would be possible to determine the outcome class and value of a Synchronized Cherries position in quadratic or perhaps even linear time.