Imperfect information variants of combinatorial games
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Chapter 3

Hackenbush variants

In this chapter, we explore two combinatorial variants of Red-Blue Hackenbush. In Section 3.1, we consider Childish Hackenbush, briefly introduced in [1]. The contents are largely based on joint work with Nienke Burgers [14]. In Section 3.2, we consider the new variant Uncolored Hackenbush.

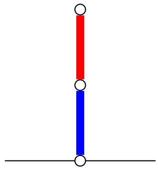
3.1 Childish Hackenbush

In this variant of the game, described in [1], moves that disconnect a part of the configuration from the ground are not allowed. The results in the following sections are largely adapted from [14].

3.1.1 Rules

A position of Childish Hackenbush is the same as one in regular Hackenbush. However, both players are only allowed to cut an edge if this does not result into a part of the graph becoming disconnected from the floor. We focus only on the Red-Blue version of the game.

Example 3.1.1. Consider the following position:

$G =$ 

In the regular version of Hackenbush, we know that $G = \frac{1}{2}$. However, for the Childish rules, we find that $G = \{ \mid 1\} = 0$, as Left is not allowed to cut her blue edge on the first turn, as this would result in the red edge being disconnected from the ground. ◁

3.1.2 Stalks

In this section, we give an exhaustive characterization of Childish Hackenbush stalks, that is, positions essentially consisting of a line graph. We make extensive use of what we will call the *sign expansion* of a Hackenbush stalk, which also lies at the base of Thea van Roode's method to compute the number value of a Red-Blue Hackenbush stalk under regular rules [29].

Definition 3.1.2. Let G be a Hackenbush stalk. Its unique *sign expansion*, denoted by $G_\pm$, is a string of $+$'s and $-$'s: a $+$ for every blue edge, and a $-$ for every red edge, reading from the ground.

Example 3.1.3. For the game G in Example 3.1.1, we have $G_\pm = +-$. ◁

We call every sequence of ≥ 1 consecutive copies of the same symbol ($+$ or $-$) a *block*, and the number of identical symbols in a block its *length*. A block of length at least 2 is called a *series*. A series not being the last block in the sign expansion is called a *non-terminal* series.

Theorem 3.1.4. *Let G be a Childish Hackenbush stalk and let $G_\pm = x_1x_2\ldots x_n$ be its sign expansion. Let a be the length of the last block of $G_\pm$, and let x be the symbol occurring in the last non-terminal series, if any. If no non-terminal series exist, $x = x_1$. Then, concatenating signs and numbers into a string expression, we find*

$$G = \begin{cases} x_n a & \text{if } x = x_n, \\ x_n a x 1 & \text{if } x \neq x_n. \end{cases}$$

Example 3.1.5. Let $G_\pm = + - + - - + - + - + +-$. In the terminology of Theorem 3.1.4, we have $a = 1$, $x = +$ (occurring in the second to last block, being a non-terminal series of length 2) and $x_n = -$. Hence, $G = -1+1 = 0$. ◁

Proof of Theorem 3.1.4. Assume first that $x_n = +$. We proceed by induction on the size of the last block in $G_\pm$, the base case being size 1.

First, suppose $x = -$. If Right starts, he immediately loses. If Left starts, the players alternate turns until the last edge defining x is taken, leaving Left to move on either a string ending in a red edge, or the empty game, losing regardless. Hence, the starting player loses, so $G = 0$, in accordance with the theorem.

Next, suppose $x = +$ and consider $G - 1$. If Right starts, he must move on -1 in the first turn, after which the game proceeds on G as above, resulting in a loss for Right. If Left starts, after having played on G, Right can respond by playing on -1 and win. Hence, $G - 1 = 0$, so $G = 1$.

Now let the size of the last block be $k > 1$. Again, first suppose that $x = -$. Note that $\mathcal{G}^L = \{G'\}$, where $G'_\pm = x_1 \ldots x_{n-1}$, and $\mathcal{G}^R = \emptyset$. By induction, noting that the configuration of blocks in $G_\pm$ does not change by removing the last edge, $G' = k - 2$, so $G = \{k - 2 \mid \} = k - 1$. If $x = +$, by a similar argument, $G = \{k - 1 \mid \} = k$.

The argument for $x_n = -$ is symmetric. □

3.1.3 Trees

We continue by giving a complete characterization of Childish Hackenbush trees, determining their value in an algorithmic way. For a Childish Hackenbush tree, we call a sequence of edges connecting two nodes of degree larger than 2, or connecting such a node to the ground or a leaf node, a *string*. Note that a stalk consists of a single string. A string originating in a leaf node is called a *branch*.

Algorithm 3.1.6. *Let G be a Childish Hackenbush tree. We compute G as follows:*

(i) *Assign a value to every branch of G using Theorem 3.1.4, acting as if the node of degree larger than 2 in which the branch originates is the ground, if applicable.*

(ii) *If, in a vertex in G of degree k,*

- *$k - 1$ of the outgoing edges are part of a string which has already been assigned a value;*
- *these strings do not contain a non-terminal series in their sign expansions;*
- *the first symbol in the sign expansion of these strings is not equal to the last symbol of the sign expansion of the k-th string,*

then we assign a value the k-th string. If it contains a series, the value is 1 if both the last symbol and the sign of the last series in the sign expansion of the string is $+$, -1 if both are $-$, and 0 otherwise. If not, go to step (iii).

(iii) *Assign a value to the k-th string according to Theorem 3.1.4, again regarding the vertex at the other end as the ground. If this vertex now meets the requirements of step (ii), go to step (ii).*

(iv) *The sum of the computed values is the value of G.*

Theorem 3.1.7. *Algorithm 3.1.6 is correct.*

Proof. To prove the theorem, we start by proving two claims.

Claim 1. Any path from a leaf to some vertex with degree at least 3 in the tree consisting of valued strings has total value 0, 1 or -1. The total value of the path is 0 or 1 if the first non-zero string has value 1, and 0 or -1 if it has value -1.

Proof of Claim 1. Consider such a path, consisting of k strings, say. For all strings to be valued by the above procedure, at least the first $k-1$ strings must not have a series. Moreover, any string of which the sign expansion ends in $+$ must be followed by a string with sign expansion starting with $-$ and vice versa. Hence, by Theorem 3.1.4, after a string of value 1 (starting and ending in $+$), we must encounter at least one string being valued -1 (starting and ending in $-$) before encountering a string of value 1 again. A similar arument holds for the value of the k-th valued string.

Claim 2. Let $w(G)$ be the value assigned to G by the above procedure. Then $w(G^L) < w(G)$ for all $G^L \in \mathcal{G}^L$, and $w(G^R) > w(G)$ for all $G^R \in \mathcal{G}^R$.

Proof of Claim 2. Without loss of generality, consider $\mathcal{G}^L$. Making a move on G, Left has three truly different options.

First, Left may move to G^L by removing an edge from a branch having a non-terminal series, resulting in the branch still having a non-terminal series. Now, in computing the value of G^L, the only difference is the value of the branch in which Left moved. If the sign expansion still ends in $+$, the value is 1 lower. Otherwise, its value becomes 0 (if the last non-terminal series has sign $+$, the branch having value 1 in G) or negative (if the last non-terminal series has sign $-$, the branch having value 0 in G). In any case, the claim holds.

Second, Left may move to G^L by removing an edge from a branch B having a non-terminal series, resulting in B no longer having a non-terminal series. This can only be the case if $B_\pm$ ended in a series of $-$, followed by a single $+$, with no other non-terminal series present. The branch B now goes from having value 0 to having a value at most equal to -1. If in G^L no previously unvalued string is now valued, this proves the claim.

Hence, suppose at least one more string is valued in G^L compared to G, ending in the vertex where B begins. If $B_\pm$ begins with $-$, having value at most -2, the newly valued series ends with $+$, so that the path of all potentially newly valued strings has value at most 1 by Claim 1. Hence, the claim is true. Similarly, if $B_\pm$ begins with $+$, B having value at most -1, the newly valued path adds at most 0 to the value of the tree.

Third, Left may move to G^L by removing an edge from a branch which does not have a non-terminal series. Again, in computing the value of G^L, the only difference is the value of this branch in which Left moved, which always decreases by exactly 1.

Proof of the theorem. Induction on the number of edges in the tree, the base case being the empty game having value 0. Consider a tree G and suppose first that $w(G) > 0$. If there is a branch with a positive terminal series in its sign expansion, Left may play on this branch to $w(G) - 1$. If there is a branch with a blue end of which the last non-terminal series is positive, or which does not have a non-terminal series and starts with a blue edge, Left may also play on this branch to $w(G) - 1$.

If there are no such branches, note that all branches have value 0 or lower. Hence, for $w(G) > 0$ to hold, there must be some path from a leaf to an internal vertex of degree at least 3 having value 1. The first non-zero string encountered starting at the leaf must have value 1. Hence, the edge connected to the leaf must be blue. By playing on this edge, Left can play to $w(G) - 1$.

Hence, we see that $w(G) - 1 \in \mathcal{G}^L$ regardless. Moreover, note that $w(G)$ and therewith $w(G) - 1$ is an integer by construction, and, $w(G^L) < w(G)$ holding by the second claim, moving to $w(G) - 1$ is dominating for Left. Now, if $\mathcal{G}^R = \emptyset$, we find that $G = \{w(G) - 1 \mid \} = w(G)$ by induction. Otherwise, by the second claim, we have that $w(G^R) > w(G)$ for all $G^R \in \mathcal{G}^R$. Now, G^R again being an integer by construction, we have that $G = w(G)$ by induction.

The case $w(G) < 0$ is symmetrical. Finally, if $w(G) = 0$, by induction and the second claim, it immediately follows that $G = 0 = w(G)$. □

Example 3.1.8. Consider the Childish Hackenbush position G as depicted in Figure 3.1.

We first value the branches using Theorem 3.1.4, finding values of 1, -2, 3, 0 and 0, going from left to right. Now, both the bottommost edge of the second

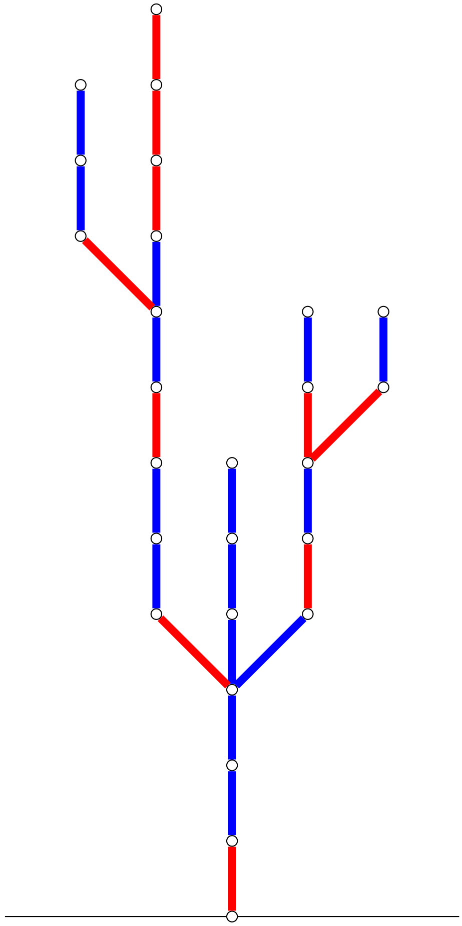


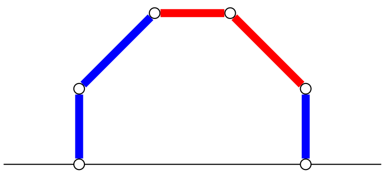
Figure 3.1: A large Childish Hackenbush position.

string and the topmost edge of the string below that are blue, so by the third bullet of step 2, the lower string is not assigned a value. The rightmost vertex rooting the last two strings does meet the requirements of step 2. It does not contain a series, so we proceed to step 3 and value it using Theorem 3.1.4, finding a value of 1. Summing the computed values yields a result of 3, hence $G = 3$. ◁

3.1.4 Cycles

By the results in the previous sections, in Childish Hackenbush, stalks and trees are all integer-valued. This turns out not to be the case if we allow cycles in the graph.

Example 3.1.9. Consider the following position G:

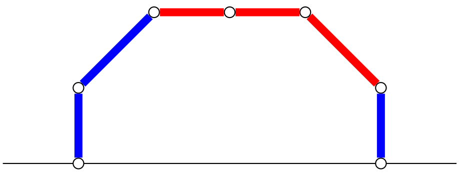


It is readily verified that

$$G = \{0, 0, -1 \mid 2, 1\} = \{0 \mid 1\} = \tfrac{1}{2}.$$

◁

Example 3.1.10. Consider the following position G:



We compute

$$G = \{0, -1, -2 \mid 1, 0, 0\} = \{0 \mid 0\} = *.$$

◁

Unlike in regular Red-Blue Hackenbush, we thus see that Childish Hackenbush allows for non-numeric values.

In [1], it is shown that Red-Blue Hackenbush is NP-hard by a reduction from the Steiner tree problem to that of determining the value of a *Redwood bed*. Such a Redwood bed is a Hackenbush position of the form as shown in Figure 3.2, where G is any graph consisting of solely red edges.

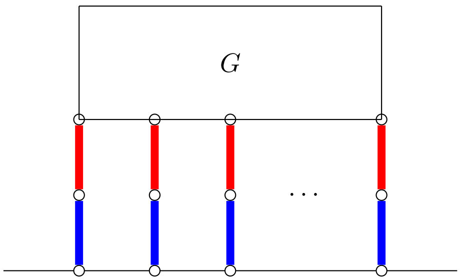


Figure 3.2: The form of a redwood bed. G is a graph consisting only of red edges.

However, determining the value of these beds turns out to be simple in the childish version of the game.

Proposition 3.1.11. *Let G be a Childish Redwood bed with m blue edges and n red edges. Then $G = m - n$.*

Proof. Note that $m \leq n$. We proceed by induction on m and n. For $m = n = 1$, we have $G = \{ \mid 1\} = 0$, as shown in Example 3.1.1. For $n > 1$ fixed, we find $G = \{ \mid G^R\}$, where G^R is a bed with 1 blue and $n - 1$ red edges. Hence $G = \{ \mid m - (n-1)\} = m - n$.

Now, let G be some bed with $m > 1$ blue edges. If $m = n$, we find $G = 0$, as it is the disjunctive sum of m copies of the bed with $m = n = 1$.

Next, let the bed have $n > m$ red edges. Left can only cut an edge which does not disconnect a part of the red mattress from the floor, resulting in a bed having $m - 1$ blue and n red edges. Hence, $\mathcal{G}^L = \emptyset$ or $\mathcal{G}^L = \{m - n - 1\}$.

A move of red can either remove a single red edge, leading to a bed with value $m-(n-1)$, or it can split the bed into two. In the latter case, say the move results in one bed with x blue and y red edges, and the other in a bed with $m - x$ blue and $n-y-1$ red edges, so that the total game has m blue and n red edges. These two beds together then have value $x - y + (m - x) - (n - y - 1) = m - (n - 1)$. Hence $G = \{m - n - 1 \mid m - n + 1\} = m - n$. □

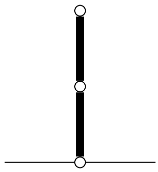
3.2 Uncolored Hackenbush

In this variant of the game, a given graph first needs to be colored before commencing normal play. We start out with an explanation of the rules of the game, followed by results on increasingly complex classes of graphs.

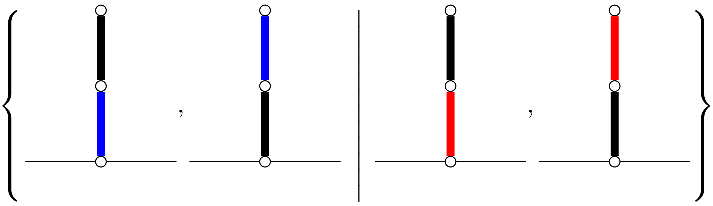
3.2.1 Rules

An initial game position for Uncolored Hackenbush is a Hackenbush position in which none of the edges is colored — or, in other words, each edge is colored black. As long as there is still at least one black edge left, a legal move of Left is to color a black edge blue, and a legal move of Right is to color a black edge red. Once all the edges in the graph have been assigned a color (blue or red), game continues as in the regular combinatorial version of Hackenbush.

Example 3.2.1. Consider the following starting configuration G of a game of Uncolored Hackenbush:

$G =$ 

Writing out the options, we find

$G =$  .

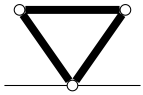
Numbering these options G_1 through G_4, by inspection, we find that $G_1 = \{2 \mid \frac{1}{2}\}$, $G_2 = \{2 \mid -\frac{1}{2}\}$, $G_3 = \{-\frac{1}{2} \mid -2\}$ and $G_4 = \{\frac{1}{2} \mid -2\}$. Removing dominated options, we conclude that

$$G = \{G_1 \mid G_3\} = \left\{2 \mid \tfrac{1}{2} \parallel -\tfrac{1}{2} \mid -2\right\} = \pm\left\{2 \mid \tfrac{1}{2}\right\},$$

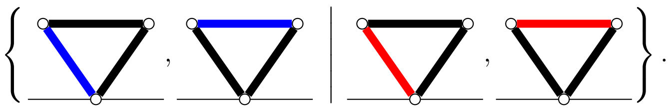
using the nested notation $\left\{2 \mid \frac{1}{2} \parallel -\frac{1}{2} \mid -2\right\} = \left\{\left\{2 \mid \frac{1}{2}\right\} \mid \left\{-\frac{1}{2} \mid -2\right\}\right\}$. ◁

We see that coloring the edge closest to the ground is an optimal move for both players. In the following sections, we will find that this is always the case in some sense.

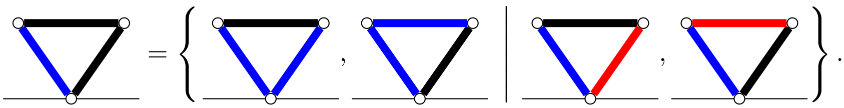
Example 3.2.2. Consider

$G =$ 

Writing out the options, using symmetry to exclude two, we find

$G =$ 

We further analyze the options.



Numbering these options G_1 through G_4, we find that $G_1 = G_2 = \{3 \mid 1\}$ and $G_3 = G_4 = \{1 \mid -1\}$. Hence, the position above is $\{3 \mid 1 \parallel 1 \mid -1\} = 1$. The same holds for the position in which the bottom two edges of the cycle are black and the top edge is blue.

The same holds for the other side, with the colors reversed, so that $G = \pm 1$. ◁

In contrast with games of regular Red-Blue Hackenbush, which are all numbers, we see that the games depicted above are both switches. We note that this is the case for every fully Uncolored Hackenbush position.

Theorem 3.2.3. *Let G be an Uncolored Hackenbush position containing only black edges. Then $G = \pm H$ for some game H.*

The statement follows from the following lemma, which is also useful in its own right.

Lemma 3.2.4. *Let G be an Uncolored Hackenbush position. Then $-G$ is G with all red edges colored blue and all blue edges colored red.*

Proof. Trivial. □

3.2.2 Stalks

In this section, we compute the value of an arbitrary Uncolored Hackenbush stalk. We start by proving that the optimal strategy for both players is to color the black edge closest to the ground.

Proposition 3.2.5. *Let G be an Uncolored Hackenbush stalk with n black edges, $n > 0$. For $k = 0, \ldots, 2^n$, denote by G_k the position in which the i-th black edge counting from the ground is colored blue if the binary expansion of k has a 0 as i-th digit and red if it has a 1 as i-th digit. Then*

$$G = \{G_0 \mid G_1 \parallel G_2 \mid \ldots \mid G_{2^n-1}\}, \tag{3.1}$$

where the $| \ldots |$ symbols are nested so that the games are successively paired.

Example 3.2.6. Let G be the Uncolored Hackenbush stalk consisting of three black edges. Proposition 3.2.5 gives that

$$\begin{aligned} G &= \{G_0 \mid G_1 \mid\mid G_2 \mid G_3 \mid\mid\mid G_4 \mid G_5 \mid\mid G_6 \mid G_7\} \\ &= \{\{\{G_0 \mid G_1\} \mid \{G_2 \mid G_3\}\} \mid \{\{G_4 \mid G_5\} \mid \{G_6 \mid G_7\}\}\}, \end{aligned}$$

where, for example, G_6 represents the stalk in which the bottom two edges are red and the top edge is blue, as $6 = 110$ in binary notation. Filling in the values we know from regular Red-Blue Hackenbush, we obtain

$$G = \{3 \mid \tfrac{3}{2} \mid\mid \tfrac{3}{4} \mid \tfrac{1}{4} \mid\mid\mid -\tfrac{1}{4} \mid -\tfrac{3}{4} \mid\mid -\tfrac{3}{2} \mid -3\} = \pm\{3 \mid \tfrac{3}{2} \mid\mid \tfrac{3}{4} \mid \tfrac{1}{4}\}.$$

◁

Proof of Proposition 3.2.5. We show by induction on n that coloring the edge closest to the ground is the dominating option for both players, the base case $n = 1$ being trivial. Note first that G indeed represents the game in which moving on the lowest edge is the only possible option for both players, which is most easily seen by picturing the possible moves as a binary tree. We label every left child of a node 0, corresponding to a move by Left coloring the lowest black edge blue, and every right child of a node 1, corresponding to a move by Right coloring the lowest black edge red.

Remains to show that coloring a different edge is not beneficial for both players. We only give the argument for Left. Suppose Left chooses to color another edge than the lowest edge blue on the first move, say edge i, playing to G'. We need to show that

$$G' \leq \{G_0 \mid G_1 \mid \ldots \mid G_{2^{n-1}-1}\},$$

i.e, that $G' - \{G_0 \mid G_1 \mid \ldots \mid G_{2^{n-1}-1}\}$ is won by Right playing second. Whenever Left colors some edge other than 1 or i blue in G', Right responds on $-\{G_0 \mid G_1 \mid \ldots \mid G_{2^{n-1}-1}\}$ by coloring the corresponding edge blue, and vice versa. When Left colors edge 1 in G' blue, Right responds by coloring edge i in $-\{G_0 \mid G_1 \mid \ldots \mid G_{2^{n-1}-1}\}$ blue. The result will then eventually be $G'' - H$, where G'' and H are identical; hence, Left loses playing first. When Left colors edge i in $-\{G_0 \mid G_1 \mid \ldots \mid G_{2^{n-1}-1}\}$ red, Right responds by coloring edge 1 in G' red. The result will then be $G'' - H$ in which the only difference between G'' and H is that in G'', edge 1 is colored red and i blue, and vice versa in $-H$. Now, Left starting, Right can always respond by mirroring, except when Left cuts edge i in either game, in which case Left responds by cutting edge 1 in the same game. The result is a win for Right. □

We can, in fact, be more specific. Recall the definition of the ordinal sum from Definition 2.1.37.

Theorem 3.2.7. *Let S_n be an Uncolored Hackenbush stalk consisting of n black edges. Then S_n is determined by the following recurrence relation:*

$$\begin{cases} S_0 = 0, \\ S_n = \pm(1 : S_{n-1}), \quad n \geq 1. \end{cases}$$

The solution is (3.1), which is in canonical form. It may be rewritten explicitly as

$$S_n = \pm\{n \mid \tfrac{2(n-1)-1}{2} \mid \tfrac{4(n-2)-1}{4} \mid \tfrac{4(n-2)-3}{4} \mid \tfrac{8(n-3)-1}{8} \mid \ldots$$
$$\ldots \mid \tfrac{8(n-3)-7}{8} \mid \ldots \mid \tfrac{2^k(n-k)-i}{2^k} \mid \ldots\}.$$

Proof. It is clear that $S_0 = 0$. By Proposition 3.2.5, the dominating move for both players in S_n, $n > 0$, is to color the edge closest to the floor. Hence, $S_n = \{S_n^L \mid S_n^R\}$, where in S_n^L the lowest edge is colored blue, and likewise in S_n^R the lowest edge is colored red. By the theory on ordinal sums in [4], we find that

$$S_n = \{1 : S_{n-1} \mid -1 : S_{n-1}\}.$$

Using Lemma 3.2.4, and the properties of ordinal sums described in [2], we find that

$$\begin{aligned} S_n &= \{1 : S_{n-1} \mid -1 : S_{n-1}\} \\ &= \{1 : S_{n-1} \mid -(1 : (-S_{n-1}))\} \\ &= \{1 : S_{n-1} \mid -(1 : S_{n-1})\} \\ &= \pm(1 : S_{n-1}). \end{aligned}$$

The fact that (3.1) is a solution, is Proposition 3.2.5. To show that G is in canonical form, it suffices to show that neither the move of Left nor the move of Right is reversible. For the former, we need to prove that

$$\{G_{2^{n-2}} \mid G_{2^{n-2}+1} \mid \ldots \mid G_{2^{n-1}-1}\} > G,$$

i.e., that $G - \{G_{2^{n-2}} \mid G_{2^{n-2}+1} \mid \ldots \mid G_{2^{n-1}-1}\}$ is a win for Right, regardless of who makes the first move. Suppose first that Left starts. If Left plays on G, Right responds on G^L, playing to

$$\{G_{2^{n-2}} \mid G_{2^{n-2}+1} \mid \ldots \mid G_{2^{n-1}-1}\} - \{G_{2^{n-2}} \mid G_{2^{n-2}+1} \mid \ldots \mid G_{2^{n-1}-1}\} = 0$$

with Left moving and losing. If Left plays on $-\{G_{2^{n-2}} \mid G_{2^{n-2}+1} \mid \ldots \mid G_{2^{n-1}-1}\}$, note that all options in this game are negative, as all indices start with a 0, and all represented games are thus blue-based. By responding on G to $\{G_{2^{n-1}} \mid G_{2^{n-1}+1} \mid \ldots \mid G_{2^n-1}\}$, Right guarantees that also this component becomes negative, winning the game. To show that Right's move on G is not reversible follows a similar argument.

Finally, to show that the explicit solution is correct, we note that G_i is the sign expansion obtained by replacing every 0 in the binary expansion of i by $+$ and every 1 by $-$. Indeed, note that the G_i are precisely the number values encountered in the game tree of G. To compute $1 : G$, we replace every node in which G_i is encountered by $1 : G_i$, which is equivalent to adding a $+$ at the left side of the sign expansion represented by G_i. Similarly, to compute $-1 : G$, we replace every G_i by $-1 : G_i$, which amounts to adding a $-$ at the left side of the sign expansion. $\square$

3.2.3 Trees

We continue by assessing trees. We start out by giving two helpful lemmas, in which we abuse the notation of $G : H$ to mean H connected to some leaf node of G.

Lemma 3.2.8. *Let $G = H : S$ be an Uncolored Hackenbush position, where S is a stalk. Coloring the edge in S closest to H dominates all other moves in S.*

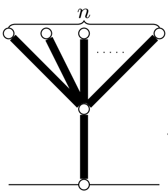
Proof. Analogous to the proof of Proposition 3.2.5. □

Lemma 3.2.9. *Let $G = S : H$ be an Uncolored Hackenbush position, where S is a stalk. Coloring the edge in S closest to the ground dominates all other moves.*

Proof. Analogous to the proof of Proposition 3.2.5. □

We first consider, in a sense, the simplest trees around.

Proposition 3.2.10. *Let*

$G =$  .

Then

$$G = \begin{cases} \pm 1, & n \text{ even}, \\ \pm\{2 \mid \frac{1}{2}\}, & n \text{ odd}. \end{cases}$$

Proof. First, let n be even and consider $G + \pm 1$, which we will show to be losing for the starting player. We only consider the case in which Left starts. If Left plays on G, the optimal move is to color the base of the tree blue. Right reacts by playing ± 1 to -1. Consequently, the players alternate turns until both players have colored half of the n branches of the tree, resulting in $(1 : 0) - 1 = 1 - 1 = 0$ with Left moving losing. If Left plays on ± 1 on her first move, Right reacts by claiming the base of the tree, and the result is similar.

Next, suppose $n > 1$ is odd and consider $G \pm \{2 \mid \frac{1}{2}\}$. Again, suppose Left starts. If Left colors the base of the tree G, Right reacts by playing on $\pm\{2 \mid \frac{1}{2}\}$ to $\{-\frac{1}{2} \mid -2\}$. Now, continuing play, the result is either that $\frac{n+1}{2}$ of the branches of G are colored blue, and $\frac{n-1}{2}$ red, and $\{-\frac{1}{2} \mid -2\}$ is played to -2; or that the amount of blue and red branches in G is reversed and $\{-\frac{1}{2} \mid -2\}$ is played to $-\frac{1}{2}$. In both cases, it is Left's turn to move. The colored G in the first option has value $1 : 1 = 2$, so that the game as a whole has value $2 - 2 = 0$ and is thus losing for Left. Similarly, the colored G in the second scenario has value $1 : -1 = \frac{1}{2}$, so the total game is losing for the starting Left. □

Alternative proof. Instead of arguing in terms of playing games, we can also prove the proposition by using Lemma 3.2.9 and the arithmetic of ordinal sums. For n even, we find

$$\begin{aligned} G &= \left\{ 1 : \sum_{k=1}^{n} \pm 1 \mid -1 : \sum_{k=1}^{n} \pm 1 \right\} \\ &= \{1 : 0 \mid -1 : 0\} \\ &= \{\{0 \mid\} \mid \{\mid 0\}\} \\ &= \pm 1. \end{aligned}$$

For n odd, we have

$$\begin{aligned} G &= \left\{ 1 : \sum_{k=1}^{n} \pm 1 \mid -1 : \sum_{k=1}^{n} \pm 1 \right\} \\ &= \{1 : \pm 1 \mid -1 : \pm 1\} \\ &= \{\{0, 1 : 1 \mid 1 : -1\} \mid \{-1 : 1 \mid 0, -1 : 1\}\} \\ &= \{\{0, 2 \mid \tfrac{1}{2}\} \mid \{-\tfrac{1}{2} \mid 0, -2\} \\ &= \{2 \mid \tfrac{1}{2} \parallel -\tfrac{1}{2} \mid 2\} \\ &= \pm\{2 \mid \tfrac{1}{2}\}. \end{aligned}$$

□

For more complicated trees, we may use the following recursion to determine the value.

Theorem 3.2.11. *Let G be an uncolored tree. We have $G = \pm(1 : G')$, where G' is G with the edge starting in the root node contracted.*

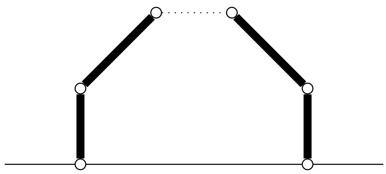
Proof. If the degree of the root of G is $n > 1$, it is straightforward to see that G can be seen as the sum of n trees. Hence, suppose that the root has degree 1. By Lemma 3.2.9, the dominating move for both players is to color the edge originating in the root, from which the statement immediately follows. □

Theorem 3.2.11 provides a linear-time algorithm to determine the value of an arbitrary uncolored tree, albeit not always in canonical form.

3.2.4 Cycles

As for regular Hackenbush, cycles provide more of a challenge. We characterize the value of a single cycle.

Proposition 3.2.12. *Let*

$C_n =$ 

be a cycle with n edges. Then

$$C_n = \begin{cases} 0, & n \text{ even}, \\ \pm 1, & n \text{ odd}. \end{cases}$$

Proof. If n is even, a mirroring strategy yields $C_n = 0$.

Hence, suppose n is odd. We will show that $C_n + \pm 1 = 0$ by constructing an explicit winning strategy for the second player. Throughout, we will represent positions of the game by a string of B's, R's and X's, denoting blue, red and black edges, respectively.

First, we will show that any position of the form

$$G_1 = BBx_1x_2 \ldots x_{n-3}R + R \quad \text{or} \quad G_2 = RRx_1x_2 \ldots x_{n-3}B + B$$

is a second-player win for Right, where the x_i are coupled such that either $x_i = B$ and $x_{n-2-i} = R$ or vice versa for all $i = 1, \ldots, n-3$. Suppose Left moves first on G_1. If Left removes any of the x_i, Right can respond by mirroring and removing x_{n-2-i}. If Left removes the second B edge, Right responds by removing the last R edge, playing to $B + R = 0$. If Right removes the first B edge, Right plays the same response, leading to $R = -1$, a Right win.

Next, suppose Left plays first on G_2. Again, any move on some x_i can be mirrored in x_{n-2-i}. If Left removes the last B edge in the cycle, Right can respond by removing the second R edge, playing to $R + B = 0$. If Left removes the loose B edge, Right responds by playing on x_i with $|i - \frac{n-3}{2}|$ minimal. If Left mirrors this move in x_{n-2-i}, Right continues making the same response until either Left removes the last B in the cycle, at which point Right wins as before, until Left plays on some other x_j, which Right mirrors in x_{n-2-j}, or until the position $RR + B$ is encountered, which is a win for Right playing first.

Next, we will consider the positions

$$G_3 = RRx_1x_2 \ldots x_{n-3}R + B, \quad G_4 = RRx_1x_2 \ldots x_{n-3}B + R,$$
$$G_5 = RBx_1x_2 \ldots x_{n-3}R + R, \quad G_6 = BRx_1x_2 \ldots x_{n-3}R + R$$
$$\text{and } G_7 = RRx_1x_2 \ldots x_{n-3}R + R,$$

where the x_i are again coupled as before, except that now $x_i = x_{n-2-i} = B$ holds for exactly one i. We show that G_3, G_4, G_5, G_6 and G_7 are won by Right playing second.

In G_3 and G_7, regardless of Left's moves, Right may remove the second and last R edge in the cycles on his first two turns, playing to $R + B = 0$. Similarly, in G_5, Right can remove the red ends of the cycle, resulting in R. In G_6, Right removes the first and last R in the cycle, resulting in $B + R = 0$.

Remains to consider G_4. If Left plays on a coupled x_i, then Right mirrors on x_{n-2-i}. If Left removes the last B in the cycle, Right responds by taking the second R, playing to $R + R$, which is certainly a Right win. Next, supppose Left plays on the x_i for which $x_{n-2-i} = B$. If $i < n - 2 - i$, Right responds by playing on x_j with $j > n - 2 - i$ minimal; otherwise, Right responds by playing on x_j with $j < i$ maximal. Now, Right can mirror any move of Left except a move on x_{n-2-j}. If Left makes this move, Right continues replying in the same fashion until $RR + B + R$ is reached with Right moving and winning.

The final step is to prove that the second player can force a game of the form G_1 (if Left goes first) or G_2 (if Right goes first), and this is the best possible. As $G_1 = -G_2 = 0$, we may assume without loss of generality that Left starts. Opening to $BX \ldots X + X$, Right responds to $BX \ldots XR + X$. If Left plays to $BBX \ldots XR + X$ or $BX \ldots XR + B$, Right responds to $BBX \ldots XR + R$ or $BX \ldots XRR + B$, respectively. If Left plays on any other X, Right responds by mirroring, resulting in G_1 or G_2, respectively.

If Left opens to $XBX \ldots X + X$, Right responds to $XBX \ldots XR + X$. Now, if Left plays to $BBX \ldots XR + X$ or $XBX \ldots XR + B$, Right responds to $BBX \ldots XR + R$ or $XBX \ldots XRR + B$, respectively. In the former case, Right can mirror to achieve G_1. In the latter case, Right also mirrors, coupling the first and last X. The result will either be G_2 or G_i for $i = 3, \ldots, 7$, won by Right going second in both cases.

If Left opens to $X \ldots X + B$, Right responds to $RX \ldots X + B$. Next, if Left plays to $RBX \ldots X + B$, Right responds to $RBX \ldots R + B$ and mirrors afterwards. Right then wins the result playing second, identifying the first R with the loose B, and the second B with the last R. If Left plays to $RX \ldots XB + B$, Right responds to $RRX \dots XB + B$ and continues mirroring, leading to G_2. If Left plays on any other X, Right can respond leading to some G_i, $i = 3, \dots, 7$.

Finally, if Left opens by coloring any other edge, Right can always respond to create any G_i, $i = 1, \dots, 7$, winning the game playing second. □

Naturally, further research could focus on determining whether there is some polynomial time algorithm to determine the value of an arbitrary uncolored position, or whether this problem is NP-complete.