Imperfect information variants of combinatorial games

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Chapter 1

Introduction

In this thesis, we consider combinatorial games and all sorts of variants of these games. We start off with a preliminary exploration of these topics.

1.1 Combinatorial games

A combinatorial game is a game with perfect information and no random elements in which two players take turns to compete for the win. Every turn, the current player has a set of moves to choose from, arriving in the next game state. Usually, once a player no longer has any possible moves to do, that player loses.

Example 1.1.1. The game *Hackenbush*, or *Red-Blue-Hackenbush*, is played on a graph, one node being designated as the "ground", and each of the edges colored either blue or red. On their turn, the one player may cut any one of the blue edges, while the other player may cut a single red edge. If, after a move, a part of the graph is disconnected from the ground node, it disappears.

Traditionally, the ground is not drawn as a single node, but as a line, making no technical difference. An example position is shown in Figure 1.1. From here, the blue player could, for example, cut the right arm, removing the whole balloon in the process, as well. ◁

Formally, a game is defined by the states to which both players can move in a single turn, also called the *options* of the players.

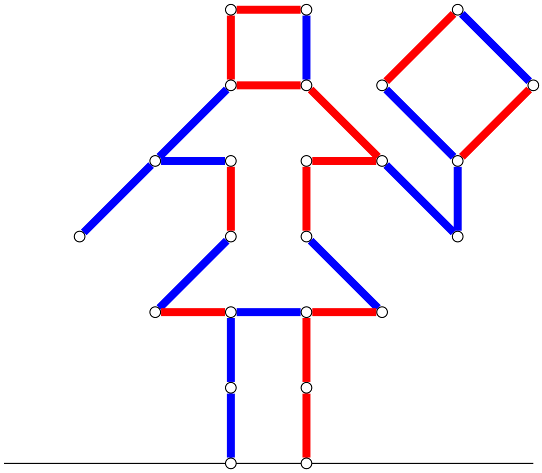


Figure 1.1: A position of Red-Blue Hackenbush.

Definition 1.1.2. We define a *game* G by its set of Left options, $\mathcal{G}^L$, and its set of Right options, $\mathcal{G}^R$, both consisting of games. Notation: $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$.

Hence, from a game G, Left may play to any $G^L \in \mathcal{G}^L$, and Right may play to any $G^R \in \mathcal{G}^R$. The "first" game in some sense, from which the recursive definition is started, is $\{\emptyset \mid \emptyset\}$, which is also denoted by $\{ \mid \}$. We assume both $\mathcal{G}^L$ and $\mathcal{G}^R$ to be finite, and we denote by $\mathbb{G}$ the set of all such games.

Standard works on combinatorial game theory include Winning Ways for Your Mathematical Plays [1] and Lessons in Play [2], in which the theory is introduced in a lighthearted fashion, as well as the comprehensive Combinatorial Game Theory [3]. On Numbers and Games [4] takes a somewhat more formal approach, focusing more on the algebraic structure of the games.

The distinction in colors between the players, such as used in Hackenbush, is commonplace. The one player, using the blue or black pieces, is often called Left, being addressed as female, while the other, playing the red or white pieces, is called Right and uses male pronouns. By the deterministic nature of the games, under perfect play of both players, a game can have any of four outcomes: the Left player wins, the Right player wins, the starting player wins, or the second player to move wins. By this distinction, one may divide combinatorial games in four *outcome classes*, summarized in Table 1.1.

		Right moves first	
		Left wins	Right wins
Left moves first	Left wins	$\mathcal{L}$	$\mathcal{N}$
	Right wins	$\mathcal{P}$	$\mathcal{R}$

Table 1.1: The possible outcome classes of a game in $\mathbb{G}$.

A key insight in combinatorial game theory is that games may often be deconstructed into smaller, mostly independent parts, which are more easy to analyze. For positional games such as Go, Chess or Tic Tac Toe, which do not traditionally end upon a player no longer having a legal move, this can prove very useful, modelling the games as a combinatorial game and using the available theory. In doing so, steps have been made in the endgame analysis of these games [5,6].

More formally, we look at the deconstruction of games from a bottom-up point of view, facilitating the construction of larger games from smaller components. For two games $G, H \in \mathbb{G}$, we can define a new game by putting the games next to each other, a legal move in the new game now being a move in either component.

Definition 1.1.3. Let $G, H \in \mathbb{G}$. Then the (*disjunctive*) *sum* of G and H is

$$G + H = \{\mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R\},$$

where we write $\mathcal{G}^L + H = \{G^L + H : G^L \in \mathcal{G}^L\}$.

Using this concept of sums of games and the outcome classes in Table 1.1, we may define a notion of equality of games. Two games are called equal if they "behave" the same in any context, that is, adding the games to any other context of games cannot produce a different outcome class. Note that this combinatorial definition of equality defines an equivalence relation on $\mathbb{G}$.

Definition 1.1.4. Let $G, H \in \mathbb{G}$. We define $G = H$ if $o(G + X) = o(H + X)$ for all $X \in \mathbb{G}$.

Even if games are unequal, we may often compare them, showing that one or the other is more beneficial to either of the players. These comparisons hinge on the order of the outcome classes depicted in Figure 1.2, associating greater games with being more favorable for Left.

Definition 1.1.5. Let $G, H \in \mathbb{G}$. We define $G \geq H$ if $o(G + X) \geq o(H + X)$ for all $X \in \mathbb{G}$.

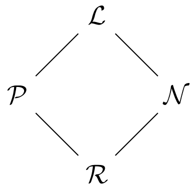


Figure 1.2: The partial order on the outcome classes.

The theory built from the definitions of the outcome classes, sums of games, and their equality and order provides a beautiful framework and myriad of tools for analyzing combinatorial games. We further explore this theory in Chapter 2, as well as introducing other concepts from, e.g., algorithmic game theory. Then, after looking at some actual combinatorial and positional games in Chapters 3 and 4, we ask the question of what remains of the theoretical framework if we drop one of the core assumptions for combinatorial games.

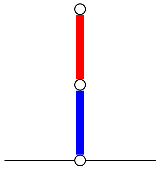
1.2 Variants

By definition, combinatorial games are games for two players taking turns, with perfect information and no chance. Naturally, each of these assumptions can be dropped, fundamentally altering the type of game encountered. Work has, for example, been done on deterministic perfect information games for more than two players [7,8], or on perfect information games for two players involving randomness [9].

In this thesis, we will focus on the removal of the perfect information component. On the one hand, this can be done by not revealing all details of a move to the opponent, such as in the game of Kriegspiel [10,11]. By doing so, a player at any time may be unsure in which state the game is precisely, and more information may be obtained through attempting to move. The game becomes a non-cooperative game in extensive form, detailed in [12]. We look more closely at this type of game in Chapter 5.

Another possibility of introducing imperfect information is to allow the players to move simultaneously, instead of on a turn-by-turn basis [13]. Under this regime, the outcome classes $\mathcal{N}$ and $\mathcal{P}$ no longer exist, as the concept of order between the players is lost. Instead, games may now end in a draw.

Example 1.2.1. Consider the Red-Blue Hackenbush game

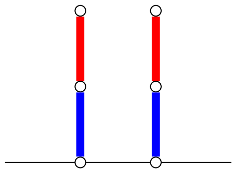
$G =$ 

Playing simultaneously, both players remove their only edge at once, resulting in the empty game. Subsequently, neither player having any options left, we declare the game a draw. ◁

One option of analyzing these games is to extend the definitions of equality and order for combinatorial games to this new class of synchronized games. This is explored for the synchronized game of Cherries in Chapter 6.

A second option is to view the synchronized games as nested zero-sum games, ultimately resulting in some payoff for either player. For Hackenbush, for example, one might argue that a game consisting of only n blue edges should be assigned value n, representing n "free" moves for Left. A game consisting of both blue and red edges is then valued equal to the value of its Nash equilibria.

Example 1.2.2. Consider the Red-Blue Hackenbush game

$G =$ 

The optimal (Nash) strategy for both players is to remove either of their edges with equal probability. Half of the times, this leads to a single blue edge remaining, having value 1; the other half, it leads to the game in Example 1.2.1, having value 0. Hence, the value of the Nash equilibrium of the game, and therewith of the game itself, is $\frac{1}{2}$. ◁

We see that, under this regime, the optimal strategy for both players need no longer be deterministic. We further discuss the analysis of synchronized games using their Nash values in Chapters 7 and 8.

1.3 Structure of the thesis

Though this thesis is designed as a single piece of work, each chapter can be read independently. To this end, some chapters contain some overlap, especially in the introductory paragraphs. We outline the contents of the chapters.

Chapter 2: Background. We start off by discussing the fundamental concepts that are developed further in the rest of the thesis. First, we cover existing material on combinatorial and economic game theory. Next, we introduce the concept of synchronized games. We state and prove some fundamental properties, and show how to construct synchronized versions of existing combinatorial games. Finally, we discuss two methods of evaluating these synchronized games.

Chapter 3: Hackenbush variants. In this chapter, we consider two combinatorial variants of Red-Blue Hackenbush. The first is Childish Hackenbush, introduced in [1]. In this variant of the game, players are not allowed to remove edges that would disconnect a part of the graph from the ground. The presented analysis is based on joint work with Nienke Burgers [14]. In the second variant, Uncolored Hackenbush, the game is started with a graph consisting of uncolored edges. In the first phase of the game, the players take turns coloring the edges blue and red. In the second phase, Hackenbush is played out as usual.

Chapter 4: Order and Chaos. Based on joint work with Sipke Castelein and Daan van Gent [15], in this chapter, we consider the positional game Order versus Chaos, introduced in [16]. In this variant of Tic Tac Toe, both players may place crosses or circles on a board. One player, called Order, attempts to construct a horizontal, vertical or diagonal line of identical symbols, while the other player, named Chaos, tries to prevent this while filling the board. We provide a theoretical analysis of the game on varying board sizes, showing that either player must win under perfect play, utilizing a SAT solver for part of the proofs. Moreover, we use Monte Carlo Tree Search to produce results for the games not covered by the theoretical analysis.

Chapter 5: Nim variants. In this chapter, based on [17], we analyse three turn-based imperfect information variants of the combinatorial game of Nim.

The game is played on heaps of coins. Every turn, the active player may take any number of coins from a single heap. The three variants differ in the amount of information that is provided to the opponent after making a move. One of the variants is inspired by Kriegspiel, a non-perfect information variant of chess. We model the variants as games in extensive form and compute Nash equilibria for some examples.

Chapter 6: Synchronized Cherries. In this chapter, based on joint work with Thomas de Mol [18], we consider the synchronized version of the game of Cherries. The game is played on strips of white and black tokens. Every turn, Left takes a black token from the end of a strip, and Right takes a white token from the end of a strip. We extend the definition of combinatorial equality to this synchronized game, and show that under this definition, every Cherries position is equal to a sum of positions of the game in which both players may only take cherries from one side of every strip. Moreover, we give an algorithm which computes this decomposition.

Chapter 7: Synchronized Hackenbush. We consider the synchronized version of the game of Red-Blue Hackenbush. We model these games as nested matrix games and compute their Nash equilibria. We show that, for some simple games, the Nash value of an increasing amount of copies of the game tends to the combinatorial value. Finally, we shortly consider the variant of the game with green edges, that may be cut by either player.

Chapter 8: Synchronized Push. Finally, we consider the synchronized version of Push, based on joint work with Ronald Takken [19]. Again, we model the games as nested matrix games and compute their Nash equlibria, concluding that the Nash value of copies of a position tends to their number value as a combinatorial game. We conclude with a short analysis of some games of synchronized Shove.