Translational symmetry breaking in holographic strange metals

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2. AdS/CFT: The Holographic Duality

2.1. History of AdS/CFT

The AdS/CFT correspondence (also often called the holographic duality) has its roots in string theory and the search for a theory of quantum gravity. The holographic principle, formulated by ‘t Hooft and refined by Susskind, made the identification between the information[1] contained in a volume of space-time, and their encoding on its boundary.[57, 58] The big discovery that turned everybody’s heads was the discovery of the AdS/CFT correspondence by Maldacena [59]. The correspondence was hinted at by the similarity in symmetry structure between certain string theory models and supersymmetric Yang-Mills (SYM) theories in one lower dimension. To be precise, Maldacena initially formulated the correspondence between a type IIB string theory living on an $AdS_5 \times S^5$ space-time and an $\mathcal{N} = 4$, $U(N)$ SYM theory living in one less dimension. That explains the reason for the name ‘AdS-CFT’: it is a duality between on one side string theory, which can have classical gravity as its low-energy limit, on a space-time with negative curvature, so-called Anti-De Sitter space, and on the other side a conformal field theory. I will not go into details of either string theory or conformal field theory in this work, but there are many resources out there that discuss both in considerable depth, ranging from pedagogical introductions to reference works.[60–64]

The discovery of the correspondence triggered a major response in the string theory community, and it is commonly included as part of the second revolution in the field of string theory. It unearthed a deep connection between gravitational theories and conformal field theory. Soon after the discovery, it was found that there were other examples of theories where there exists such a duality.[65, 66] These all have their own specific details, such as the geometry and type of string theory, as well as the type of conformal field theory, but they all follow a very similar spirit of duality.

2.1.1. A More General Statement of the Duality

The statement of the duality for a general holographic correspondence is deceptively simple, and was made precise by Gubser, Polyakov, Klebanov and Witten [67]. Imagine a pair of theories between which the duality applies. One is the ‘gravitational’ side, where the string theory lives on some form of space-time with negative curvature. A technical aspect of those space-times is that they all have a boundary. The other theory is a (conformal) quantum field theory with one less

[1] This uses the very abstract notion of ‘information’ as used in the field of information theory. It can be thought of as essentially ‘degrees of freedom’ of some kind.

dimension than the gravitational theory. The conformal field theory, like all quantum field theories, has all of its information contained in its generating functional. In this generating functional, we have the operators $\mathcal{O}_i$ that are present in the theory, coupled to their respective sources j_i. As such, the generating functional contains enough information to compute all n-point correlators in the theory by taking functional field derivatives with respect to the sources. We will denote this functional as $Z_{QFT}[j_i]$. On the other hand, we have a gravitational theory which is coupled to extra fields ϕ_i, with boundary values j_i, and an associated partition function $\mathcal{Z}[\phi_i]\big|_{\phi_i \to j_i}$. Then the duality states that

$$Z_{QFT}[j_i] = \mathcal{Z}[\phi_i]\big|_{\phi_i \to j_i}. \qquad (2.1)$$

In other words, using the duality one way can encode for and compute operators in the quantum field theory by having fields in the gravitational theory with the correct boundary behaviour. Applying the duality the other way, which says we can understand the gravitational side by doing computations in conformal field theory, is less relevant for the purposes of this thesis. This statement is incredibly deep, and allows for investigation both ways. It is possible to learn more about the QFT side by doing computations in the string theory side, but also the other way around. However, we are now struck with the problem that the theories specified on either side are far from easy to work with. A good example is the fields in the field theory: in the canonical example of $\mathcal{N} = 4$ SYM, the number of fields in the theory N is very large. Making any computation can be difficult, and how do we know to couple fields for these into the gravitational theory? The duality luckily has a way to simplify this in a way that is also useful for the purposes of our condensed matter aspirations.

2.1.2. Limits of the Duality

Crucial in the simplification that we desire is the fact that this duality is a weak-strong duality. This is evident when computing exactly which limits can be taken in terms of coupling strengths and numbers of degrees of freedom.[62] Since the string theory on the AdS side is hard to formulate and even harder to compute anything with, we would like to make use of its low energy limit, which is simpler. Let us take here the canonical example of AdS/CFT, to be precise the one where $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and coupling g_{YM}.[59] This is dual to type IIB string theory with string length l_s and coupling g_s, which lives on AdS$_5$ $\times$ S^5 with AdS radius L and N units of $F_(5)$ flux on S^5.[61, 62] There duality maps the parameters g_{YM} and N to the parameters g_s and L/l_s on the string theory side, via[62]

$$g_{YM}^2 = 2\pi g_s \qquad\qquad 2g_{YM}^2 N \equiv 2\lambda = L^4/l_p^4, \qquad (2.2)$$

where we have defined λ as the 't Hooft coupling $g_{YM}^2 N$. The way we want to use the duality is to be able to compute things on the gravity side, which we then want to be able to interpret on the CFT side through the duality. For this, we want to get rid of as much stringiness as possible, as loop corrections and string couplings make our lives difficult here. In order to accomplish this, we will want the string coupling $g_s \to 0$, while having the length of the strings to be inconsequential compared to the AdS radius, such that $L^4/l_p^4 \to \infty$. Equation (2.2) then tells us that here we want to take $g_{YM}^2 \to 0$, while simultaneously $\lambda = g_{YM}^2 N \to \infty$. In other words, for the gravity to be classical[2] we need to be in the limit of infinitely strong 't Hooft coupling. This shows that the duality is of the weak-strong nature: in order to be weakly coupled on the gravitational side, we end up in the limit of matrix large-N in the CFT.[61] This will turn out to be both a blessing and a curse: by going to the classical limit on the gravitational side, we are able to actually perform the computations required to explore extremely strongly coupled systems that are normally far out of reach of ordinary perturbative treatments on classical (though high-power) computers. For condensed matter physics, the presence of large-N spoils a lot of the fun, because this is never physically realised in any real world system and any significant claims will always have to be taken with the grain of salt that there could be significant corrections stemming from an expansion in $1/N$. For the rest of this work, we will stay in the large-N limit. This is in a sense the 'weakest' form of the duality from the sense that we get the easiest to handle gravitational physics, at the cost of only being able access a very restricted parameter space in the conformal field theory.

2.2. Renormalization Group and Geometry

Perhaps of greatest interest from a condensed matter perspective is the way that the Renormalization Group (RG) flow manifests itself. Dating back to the 1950's, the idea of the RG and RG flow was formulated from the desire to look at the physics of systems at different energy scales.[68, 69] The canonical way of thinking about renormalization in field theory is through the idea of a beta function. One considers a coupling constant g and looks at the properties of that coupling constant at different energy scales μ. When the energy scale is decreased, and more and more of the higher-energy modes of the theory are thrown out, the effective coupling strength of the theory changes. This is what the beta function of the theory encodes, and this can be stated as

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g(\mu)). \tag{2.3}$$

When $\beta(\mu) = 0$, the scale transformation has no effect on the coupling strength and the theory is said to be scale-invariant. The magic now happens when we take a certain theory, and consider a series of copies of this theory, each evaluated at a slightly different, ever lower energy scale μ. The parameter μ then tells us at each point along its evolution how the physics of the theory behaves. We can imagine this now as an extra dimension of the theory: in essence we are adding a new 'energy scale' coordinate to the space-time of our theory,[3] and we can track how it evolves. The realisation was made that this maps neatly to holographic physics, where the boundary represents the original theory. The theory living at the boundary is commonly called the ultraviolet (UV) theory: no degrees of freedom have been integrated out yet, and it contains all the microscopics. Going deeper into the interior of this space-time, renormalization effects start to kick in, and the deeper you enter into the 'bulk' of the geometry, the more you let the RG flow continue into the infra-red (IR) of the theory. Taking the example of the CFT, we know that in a CFT we must have $\beta(\mu) = 0$, as conformal field theories by have scale invariance as one of their defining

[2] Here the low-energy limit is technically supergravity, but in the setups we use we only have classical Einstein gravity, without supersymmetry.

[3] This is not an exact 'energy' dimension, but thinking about it proves to be enlightening in many respects, e.g. in section 6.5.

properties. The bulk geometry must therefore reflect the same scale invariance, which turns out to be manifestly true for empty AdS space-time. The mathematics are of course more involved, but these statements together with the requirement of having a 'boundary' for the space-time quite naturally leads to an AdS-like space-time. This is schematically represented in figure 2.1.

This consideration of RG flow brings to light the important aspect that, in a sense, what the AdS/CFT correspondence is doing is geometrizing the RG flow of the quantum field theory into a dynamical gravitational bulk, which is sometimes abbreviated in the statement [61]

$$\mathrm{RG} = \mathrm{GR}. \tag{2.4}$$

This provides a compelling reason to use the holographic duality: in physics, we are often interested in low-energy excitations that come from some microscopically detailed theory. The details of how to map microscopics to low-energy excitations is highly non-trivial. RG is one of the canonical ways to get the low-energy physics. These can prove to be very complicated, and there are a lot of limitating factors to their applicability, such as strong coupling and dense entanglement in general.[70] Holography turns the game on its head here. The RG scale is a fundamental dimension of the space-time, and once a black hole solution has been found, we can interpret what the RG is doing in the field theory simply by looking to a different radial slice of the bulk space-time, overall a much more straightforward affair.

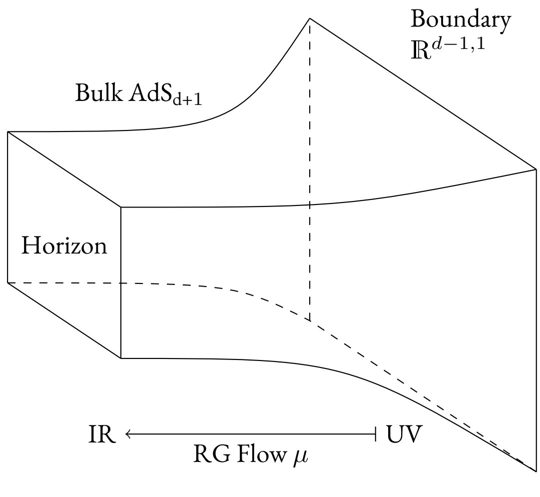

Figure 2.1.: The AdS space-time can be constructed by an RG flow, where each value of μ is a successive point in the flow.

2.2.1. Finite Temperature

Any real condensed matter system exists at finite temperature, simply by the third law of thermodynamics. This has to be taken into account when doing computations and one must be concerned with finite temperature field theory in order to make accurate predictions. This is in general quite tricky again, and once more holography offers a convenient way out. In the bulk space-time, we can encode for a finite temperature field theory by placing a non-extremal black hole in the centre of the space time. This modifies things. For example, the bulk scale invariance that was present in the otherwise empty AdS space-time now no longer applies, as it is definitely possible to tell how close to the black hole you are. This results in some more non-trivial RG flow. The resulting flow is also rather pleasing: the black hole horizon turns out to correspond to a new infrared fixed point in the RG flow, where the physics of the original theory have indeed been changed from the ultraviolet scale invariant physics.[61] What is even more impressive is that these black holes can encode not only for temperature, but they can also carry different charges, such as electric and magnetic charge, which are crucial to condensed matter physics as they give rise to finite density and magnetic fields in the boundary theory. This thesis always operates at finite temperature, and therefore there will always be a non-extremal black hole of some type present in the centre of the bulk space-time that is being considered.

2.3. GPKW Dictionary

Now let us look at some specific examples, as the summary so far sounds intriguing, but it is not yet a useful apparatus that can be used for performing computations. The interpretation and use of many of the ingredients is expressed in what is commonly referred to as the dictionary, which translates quantities between the boundary field theory and gravity sides of the duality. This dictionary is named after Gubser, Polyakov, Klebanov and Witten.[67] Rather than just stating the results, some of the results deserve a bit more attention.

2.3.1. Fields and Scaling Dimensions

Suppose we are in the large-N limit, where we should be able to use classical gravity in the bulk geometry without quantum corrections as a dual to the field theory. If we have a single field ϕ that we want to have include in the holographic dual, we have to find the solution to the Einstein equations of motion of that field, under the condition that its boundary value

$$\phi|_{\text{Bdy}} \to j_i, \tag{2.5}$$

as described above in section 2.1.1. To set things up, let us first look at the space-time itself.[71, 72] We want to have a solution to Einstein's equations, which can be derived by doing a functional variation with respect to the curved metric from Einstein-Hilbert action

$$S = \int d^{d+2}x\sqrt{-g}\,(R - 2\Lambda)\,, \quad 2\Lambda = -\frac{d(d+1)}{L^2} \tag{2.6}$$

where L is the AdS radius. Since we have $d+2$ dimensions in the bulk, the dual field theory will live in $d+1$ dimensions. A metric which is a solution to the Einstein equations with negative cosmological constant is the aforementioned anti-De Sitter space-time, which can be parameterised as

$$ds^2_{\text{AdS}} = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + d\vec{x}^2\right), \tag{2.7}$$

where $d\vec{x}^2 = dx_1^2 + dx_2^2 + \ldots$ runs over all purely spatial dimensions of the theory. The isometries of this AdS space-time turn out to be identical to those of a $d+1$ dimensional conformal field theory – exactly what we expect from the way the duality was described in section 2.2.[60]

Let us assume now that the scalar field has a mass m, and that it is perturbatively coupled into the system, i.e., we assume a fixed background metric and we do not consider the back-reaction of the scalar onto the metric. The contribution to the action of the scalar field is given by

$$S_\phi = \int d^{d+2}x\sqrt{-g}\,\frac{1}{2}\left((\nabla\phi)^2 + m^2\phi^2\right), \tag{2.8}$$

from which we can deduce that the scalar then obeys the equation of motion

$$\left(\nabla^2 - m^2\right)\phi(x) = 0. \tag{2.9}$$

We can use separation of variables to propose a plane-wave solution in the non-holographic direction, but leave the holographic direction as a general function of r. Under this assumption the equation of motion expands to

$$\partial_r^2\phi - \frac{d}{r}\partial_r\phi + \left(\omega^2 - k^2 - \frac{(mL)^2}{r^2}\right)\phi = 0. \tag{2.10}$$

This equation has a solution in the form of Bessel functions.[73] When looking at the $r \to 0$ behaviour, which is the direction of the boundary, the series expansion at small r has two distinct sectors, namely

$$\phi \sim \phi_A r^{d+1-\Delta}\left(1+\ldots\right) + \phi_B r^{\Delta}\left(1+\ldots\right), \tag{2.11}$$

where $\ldots$ are higher powers of r and $\phi_{A,B}$ are integration constants. Δ are the solutions to

$$\Delta\left(\Delta - d - 1\right) = (mL)^2\,. \tag{2.12}$$

Depending on the value of Δ, one of these fields will be the dominant contribution in the region near the boundary, and the other one will be subdominant. The accepted terminology in the field for these terms is leading and subleading, respectively. Typically, the dominant term actually diverges near the boundary, and is called the non-normalisable mode.[4]

Let us now assume that ϕ_A is the leading component, and ϕ_B is the subleading one. It is exactly the leading component that is the one dual to the source of the operator in the side of the field theory. When evaluating the boundary action coming from equation 2.8, this will turn out to reduce to [74]

$$S_{\phi,r\to 0} \sim \int d^{d+1}x\sqrt{-\tilde{g}}\left(\phi_A\phi_B\right). \tag{2.13}$$

[4] I will not cover the subtleties such as the appearance of logarithmic terms in these expansions.

Making the identification of ϕ_A with its source j, we are still left wondering what the operator ϕ_B is.[5] If we now remember the generating functional and the partition function, we know that we can use it in order to compute expectation values:

$$\langle O \rangle \sim \frac{\delta}{\delta j} Z[j] \big|_{j=0}. \tag{2.14}$$

Comparing this to equation (2.13), this gives us the identification with the vacuum expectation value (VEV)

$$\langle O \rangle_{CFT} \sim \phi_B. \tag{2.15}$$

This example glosses over some of the more subtle points here. One is that in the case of for example the stress tensor, the expectation values might be naively divergent. This can be addressed by instead putting these sources at some $\epsilon > 0$ away from the boundary, and then adding counterterms which kill any divergent behaviour, after which we can safely send $\epsilon \to 0$ and recover the renormalised boundary values.[75]. Another is what might when both modes are normalisable. This turns out to allow for an admixture of boundary conditions into what we call the source and the response. This actually turned out to have important consequences in several parts of this thesis, for probe fermions in chapter 4 and for background space-time geometries and the thermodynamic interpretation thereof in chapter 5.

Subtleties aside, we can identify two-point functions in momentum space by[74]

$$G(k) = \frac{\phi_B(k)}{\phi_A(k)}. \tag{2.16}$$

From the point of view of condensed matter physics, we always prefer to use the retarded Green's function to study the two-point properties of our system. This can be achieved by imposing infalling boundary conditions on the scalar field, meaning that towards late times, the wave in the field theory will be travelling towards and eventually (partially) falling into the black hole horizon.

All this together gives us an entry in the dictionary, where we look to compute the solutions to the gravitational equations of motion, and then pull this through the duality to the boundary field theory by properly identifying source and response components. In this simple example we only looked at two-point functions, but one can look at any n-point function in general.

There are many more entries in the dictionary. Another interesting quality is how gauge fields and symmetries interact. The general identification here is that global symmetries in the boundary will correspond to local symmetries in the boundary. This is for example made explicit when considering a $U(1)$ gauge field in the boundary. The corresponding local $U(1)$ symmetry is dual to a global $U(1)$ symmetry in the boundary. In general, a gauge field will correspond to the conserved current that is associate with the related global symmetry. A more comprehensive, but by no means complete, overview can be found in table 2.1 below.[61]

[5] There are some exact factors of r to be taken into account, but these are not essential to the schematic discussion here.

Field Theory	Gravity
Large N	Classical gravity
Operator $\mathcal{O}$ with dimension Δ	Field ϕ with mass m
Source of operator $\mathcal{O}$	Leading behaviour of ϕ
Vacuum expectation value $\langle\mathcal{O}\rangle$	Subleading behaviour of ϕ
Global symmetry	Local symmetry
Stress tensor	Gravitons
2-form current	2-form field
Finite temperature	Non-extremal black hole
Finite chemical potential	Charged black hole
Free energy	Value of Euclidean on-shell action
$\vdots$	$\vdots$

Table 2.1.: Some basic dictionary entries which feature in this thesis.

2.3.2. Black Hole Thermodynamics

Mentioned before was that black holes in the interior of the space-time are able to encode for finite temperature in the boundary CFT. This is actually tightly linked to discoveries by Bekenstein and Hawking in the 1970's that black holes carry entropy and can emit thermal radiation.[76, 77] The laws we are familiar with in our normal everyday thermodynamics have their own parallels in the form of the laws of black hole thermodynamics. Like the first, second, and third law of ordinary thermodynamics, there are laws of black hole thermodynamics where we can identify black hole quantities like surface area and surface gravity with entropy and temperature, respectively.

The black holes in these curved space-times allow for a wide variety of interesting phenomena, unlike their closely related cousins that live in flat space. The applicability of the 'no-hair' theorem is an example of this: in flat space, it is not possible for black holes to have any kind of structure on the horizon.[78] Instead, they are uniquely determined by the parameters charge, mass and angular momentum, which are just global quantum numbers, without any more structure. This no-go theorem does not hold in negatively curved space-time and therefore we can have all kinds of 'hair' on our black holes, such as scalars that can even acquire some spatial modulation.[79] Another example is black hole evaporation. Astrophysical black holes in flat space evaporate over time.[6] A peculiar property of AdS space-time is that it takes only a finite time for massless objects, for example photons from Hawking radiation, to go from the centre of AdS space all the way to the boundary. This is in stark constrast to geodesics of massive particles, which will never reach the boundary in finite proper time. If one then assumes that any energy that these photons could carry is reflected back from the boundary into the interior of the space-time, eventually the reflected radiation will end up in thermal equilibrium with the radiation emitted by the black hole. As such the black holes can form an equilibrium state.[80] Therefore unlike in flat space, stationary

[6] For astrophysical black holes this happens extremely slowly, even on cosmological timescales.

black holes in AdS can exist and we can understand their dual to be some very late-time field theory in full local thermal equilibrium.[61]

2.3.3. Top-Down vs Bottom-Up

One of the open problems in the field of holography is that while it is conjectured that the AdS/CFT correspondence holds quite generally, we only have a limited number of examples where the duality is known to be exact. In order to understand fully what system we are describing and that the duality remains valid, we can start to play mathematical tricks with these known exact correspondences, such as dimensional Kaluza-Klein reductions. Together with convenient limits we can have a good idea what the conformal field theory is on the field theory side of the duality, while ending up in a gravitational theory that is more manageable than full-blown type IIB string theory, just to give an example. A side effect of this is that almost invariably, Kaluza-Klein reductions which get rid of some dimensions give rise to extra dilatonic scalar fields that get coupled into the gravitational side.[61] This general framework is known as the top-down approach to AdS/CFT. It is not always the most convenient approach to take, as the actions can either be difficult to deal with or in our case uninteresting from a condensed matter point of view. This is for example due to those extra scalars. The holographic dictionary in table 2.1 has some relevant entries such as the free energy and chemical potential, but these scalars have no good interpretation and they certainly do not turn up explicitly in real-world condensed matter systems. From the top-down approach, it can be very hard or impossible to get the exact ingredients in terms of fields and operators that you would like to describe on the condensed matter side.

The bottom-up approach attempts to remedy this problem by taking a different point of view of the duality. It treats the duality as a more phenomenological tool by simply choosing some elements that have in top-down models been shown to have a certain dual interpretation and building an action from that and trusting that the correspondence will still apply. The downside of this is that while we can compute quantities like retarded Green's functions and vacuum expectation values this way, there is no way of finding out what the exact Hamiltonian is of the theory dual to the gravitational action we have posed. As a result, the duality can now not be used to find the physics of a specific boundary theory, but it can give generalities about the physics of strongly coupled field theories. In this work, most of the models presented are of the bottom-up variety, as the translational symmetry breaking that we employ does not follow naturally from top-down constructions.

2.4. Holographic Applications to Condensed Matter Physics

The stage is now set, we have a way of posing our problem and a large number of questions that we would like to address. Let us see how the holographic duality can aid us in this. While the statement of the duality is clear, and the weak-strong nature of it is appealing for the purposes of performing computations, there is some more machinery to discuss.

2.4.1. Hydrodynamics

It turns out that the duality can encode for real-world physics in some surprising ways. What we are often interested in, as mentioned earlier, is the long-wavelength, low energy limit of the physics of the boundary theory. This can be achieved by doing what boils down to a gradient expansion, assuming that the only fluctuations that play a role are small and slow. Degrees of freedom are always slow when they correspond to conserved or nearly conserved quantities. Quantities that are not conserved for any perturbation in them will dissipate and thermally equilibrate quickly, without much of a chance to propagate over wavelengths. On the other hand, conserved quantities obey many continuity equations, which prevents them from doing exactly this, and instead perturbations will propagate for a long time or undergo some diffusion process in order to return to local equilibrium.[81] The gradient expansion of the theory into these slow modes is often synonymous with the hydrodynamics of the theory if they are the only slow modes in the theory. This is a more general consideration than what we would typically call hydrodynamics, after all it does not concern the flow of a real-world physical fluid like water flowing around in our field theory. Instead, the hydrodynamics we are concerned with would describe a relativistic fluid.[7] This is a powerful machinery, and it is able to include relevant effects such as (perturbative) ionic lattices and external electric and magnetic fields.

The applications to holography are quite easy to state and is related to a by now somewhat famous result. For a relativistic fluid, the stress tensor is given by

$$T^{\mu\nu} = (E+P)u^\mu u^\nu + Pg^{\mu\nu} + \Pi^{\mu\nu} \tag{2.17}$$

where E, P are the internal energy and pressure and u^μ is the fluid velocity.[81] The first two terms describe ideal hydrodynamics. $\Pi^{\mu\nu}$ is the interesting part, the part where the derivative expansion comes into play. This can be a complicated term, but crucially it contains a set of constants that are known as transport coefficients such as the shear viscosity and bulk modulus of the fluid we are considering. However, these are a priori just a set of coefficients without a particular value: they represent the microscopic behaviour of the theory, and while their presence may be universal, their values are anything but that.[81]

This is where we can again use holography. In the gravitational dual, it is possible to compute some of these transport coefficients directly. This is one of the ways in which we can learn a lot about the field theory by looking at the gravitational side. While we will be considering only linear fluctuations in hydrodynamics, the mapping between gravity and hydrodynamics goes much deeper, showing that the Navier-Stokes equations can actually be found from the Einstein equations.[83, 84]

A particularly famous result that is related to this is the computation of the shear viscosity.[85, 86] The first-order terms in the gradient expansion of equation (2.17) will for example contain the shear viscosity, η. For example, to first order, the hydrodynamic constitutive relations in the

[7] There are other ways of constructing hydrodynamics, for example in a non-boost invariant setting, which do not feature in this work.[82]

Landau frame are given by [81]

$$T^{\mu\nu} = Eu^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_\lambda u^\lambda + O(\partial^2), \tag{2.18}$$

where Δ is a projector with the flat space metric

$$\Delta = \eta^{\mu\nu} + u^\mu u^\nu, \tag{2.19}$$

ζ is the bulk viscosity and $\sigma^{\mu\nu}$ is the transverse traceless symmetric tensor [81]

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\mu u^\mu\right). \tag{2.20}$$

For AdS black holes at zero density, it actually turns out that this shear viscosity is related to the zero-frequency scattering cross-section of the black hole. This in turn can be expressed in terms of the area and volume of the black hole and therefore also the entropy of the black hole by the laws of black hole thermodynamics. When including geometric factors, this yields the ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\hbar}{k_B}. \tag{2.21}$$

This ratio of viscosity to entropy is known as the minimal viscosity, as it is conjectured to be a lower bound on the viscosity of a strongly interacting field theory.[85] Remarkably, this bound also seems to hold at finite density. Most strongly interacting materials found in nature are nowhere near this bound though. Only the quark-gluon plasma, which was studied in detail around the same time as the minimal viscosity was discovered, has a surprisingly small value for this ratio, even though it does not have ingredients like the large-N of holography.[87] A charitable interpretation can see this as a hint that holography can indeed tell us useful things about the real world.

2.4.2. Conductivities from Holography: Real-time Information

Holography is not only limited to computing hydrodynamic transport coefficients. One of the more technically useful aspects of holography is that it gives access to real-time information about the dual system. This is in stark contrast to typical field theoretical results, where it is often necessary to work in imaginary time to make computations feasible. Finite temperature is then encoded in the radius of the time circle in imaginary time. That in itself is not a problem, but the difficulties come when trying to translate back into real time, as that is what we observe. The Wick rotation that has to take place is technically very challenging, and often means that it is impossible to get anything but the most general scaling dimensions out.[88] Holography is in this aspect very different. Since temperature is already encoded in the thermodynamics of the black hole[8], it turns out that computing real-time properties of field theory involve driving the holographic system out of

[8] The temperature equals radius in imaginary time still shows up in black hole physics too. For the Schwarzschild solution in flat space for example, one can rotate to a Euclidean space-time, where the temperature is found from the radius of the time circle is fixed in order to avoid a conical singulartiy in this space-time. This is a more ad-hoc argument than the original derivation of black hole temperatures, but it is an interesting connection.

equilibrium, and looking at the real-time evolution. This gets us real-time field theory observables without having to resort to any Wick rotations.[71]

For the sake of illustration, let us consider the electrical AC conductivity of the field theory. We can choose to take a static black hole in equilibrium and apply a time-dependent electric field $E(t)$ to it. This will in general drive the black hole out of equilibrium. Let us assume that this field is oscillating at some finite real frequency ω. This electric field is assumed to be a small perturbation, which does not back-react on the initial black hole background. The electric field E is then induces a current response J, and their proportionality is the conductivity σ:

$$J(\omega) = \sigma(\omega)E(\omega). \tag{2.22}$$

In a conformal field theory at zero density, on dimensional grounds this conductivity takes the form

$$\sigma(\omega) \sim \omega^{d-2}, \tag{2.23}$$

where d are the number of transverse spatial dimensions. Since we typically consider AdS_4, our boundary will have $d = 2$ transverse dimensions and hence σ becomes exactly frequency independent.[61]

That is a fairly trivial result, and we do not need to rely on any holographic computation to find this out. At finite temperature, one can no longer rely on dimensional analysis. Nevertheless, the holographic computation of this result at finite temperature is rather straightforward at zero chemical potential and it is a good starting point to show the power of holography. In order to compute the conductivity, we will make use of linear response theory.[88] In linear response, we know that we can find the conductivity from the retarded Green's function via

$$\sigma(\omega) = \frac{1}{i\omega} G^R_{JJ}(\omega). \tag{2.24}$$

The retarded Green's function is that of the current operator, which is defined by

$$G^R_{JJ}(\omega) = -i \int dtdx e^{i\omega t} \Theta(t) \left\langle \left[J(t,x), J(0,x)\right] \right\rangle. \tag{2.25}$$

$\Theta(t)$ is the step function that takes care of the time ordering here such that $t > 0$. In a previous chapter, we encountered exactly how to compute Green's functions by considering leading and subleading behaviours of a field theory. In this case, we now need to consider a black hole in order to do the computation at finite temperature. As we are at zero density, this will be the AdS-Schwarzschild black hole, which can be parameterised by only its horizon radius which sets its temperature after scaling out some other parameters such as the AdS radius and Newton's constant. In order to find the conductivity, we need the bulk dual of the current. What we now need to do is find the Green's function that is related to turning on a perturbative electric field $E_x = F_{tx}$. For a spatially homogeneous electric field, in terms of the gauge field this will correspond to turning on a component $A_x dx$ which in Fourier space will have behaviour like $A_x = a_x(r)e^{-i\omega t}$. The exact boundary behaviour can be extracted by carefully looking at the near-boundary expansion,[9] which will find that the expansion for A_x goes as [61]

$$A_x \propto a_x^0 + \frac{a_x^1}{r} + \cdots \,, \tag{2.26}$$

where $a_x^{0,1}$ are the leading and subleading behaviours near the boundary, respectively. One can solve for the equations of motion, and the condition for the retarded Green's function is fixed by choosing the infalling boundary conditions.[71] The solution[10] will have

$$a_x^0 = c, \quad a_x^1 = i\omega c \tag{2.27}$$

for some constant c. Using the dictionary to extract the Green's function from holography, we can find that for any temperature

$$\sigma(\omega) = \frac{1}{i\omega} G^R_{JJ}(\omega) = \frac{1}{i\omega}\frac{a_x^1}{a_x^0} = 1. \tag{2.28}$$

This is exactly the result is found for the zero-density CFT, since it has no dependence on frequency.[64] More importantly though, there is no dependence on temperature either, which one might not have expected. At zero temperature, there is no scale in the CFT, so there cannot be any identifiable features at some ω, as this would indeed be an indication of some scale. However, temperature is a scale in the theory, as it can be combined with k_B for example to create an energy scale. It is then surprising that the introduction of this scale does not appear to have any effects. [61]

2.5. Finite Density: The Reissner-Nordström Black Hole

Zero density is not the most interesting system though. Apart from some very fine-tuned systems, such as graphene at exact charge neutrality, this is not a system we are likely to see appear in experiments.[89] Luckily going to finite chemical potential in holography is a rather simple affair, and it does not require the development of a lot of new machinery. The star player in respect has long been the Reissner-Nordström black hole. This black hole can be constructed as a compactification of some 11-dimensional M-theory, which we will not go into the details of.[90, 91] For our purposes, the starting point will be the Einstein-Maxwell action

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}\left[R - 2\frac{\Lambda}{L^2}\right] - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.29}$$

where $F = dA$ is the field strength for the $U(1)$ gauge field A. We will always use the units where $2\kappa^2 = 16\pi G = 1, e = 1, \Lambda = -3, L = 1$. Note that here, in contrast to the scalar example that was discussed in the section above, this gauge field is coupled in to gravitation action in the

[9] For the gauge field no holographic renormalisation is necessary/
[10] See box 7.3 [61]

full action, as such the back-reaction of the gauge field on the metric is taken into account. The equations of motion that arise can with some manipulation be written as

$$\begin{aligned} R_{\mu\nu} + 3g_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \\ \nabla_{\mu}F^{\mu\nu} &= 0. \end{aligned} \tag{2.30}$$

This set of equations admits a black hole solution with a finite charge with AdS_4 asymptotics. The black hole can be written using a metric

$$\begin{aligned} ds^2_{RN} &= \frac{z_+^2}{z^2}\left(-f(z)dt^2 + \frac{1}{f(z)}dz^2 + dx^2 + dy^2\right) \\ A = A_t dt &= \mu\left(1 - \frac{z}{z_+}\right)dt \end{aligned} \tag{2.31}$$

The emblackening factor $f(z)$ for Reissner-Nordström is given by [90, 92]

$$f(z) = \left(1 - \frac{z}{z_h}\right)\left(1 + \frac{z}{z_h} + \frac{z}{z_h}^2 - \frac{\mu^2 z^3}{4z_h^3}\right). \tag{2.32}$$

The emblackening factor has two roots at z_+, z_-, which are the locations of the outer and inner horizon at finite temperature respectively. At zero temperature these collapse to a double root. We will only be considering finite temperature here, and therefore we will treat our space-time as if it ends at $z_h \equiv z_+$. We will always be able to rescale the horizon radius such that $z_h = 1$, for both ease of notation and later numerical convenience.

However, there are some scenarios where keeping the horizon radius explicit is useful, for example when computing thermodynamic susceptibilities, as was done in section 6.7. Most observables that are directly computed are typically given or found in units of horizon radius as that is often a free parameter in the solutions. This has a deeper root in the diffeomorphism invariance that is encountered in the gravity side. For practical purposes, that means that all the parameters we referring to, whether they are radii, expectation values, or other objects, have to be phrased with reference to some scale. In the dual boundary, z_h has no natural meaning, but there we can express all dimensionfull quantities in terms of the chemical potential.[11] When we state for example that we take the parameter $\mu = 2$, that is shorthand for $\mu = 2z_h$. The boundary of the AdS space-time is located at $z = 0$ in this parametrization.

2.5.1. Scaling Properties of Reissner-Nordström

Another key property of the RN black hole is its particular near-horizon geometry. If we depart from the metric as presented in equation (2.31), we can take the horizon to be located at $z = 1$. In

[11] In relevant condensed matter systems, such as the cuprates, the typical size of μ is about 1 eV. In natural units, this corresponds to a temperature of about $10^4 K$.Room temperature would be on the order of $T = 3 \times 10^{-2}\mu$, to give a sense of scale.

a series expansion around $z = 1$, the emblackening factor will expand as

$$f(z) \approx f'(1)(z-1) + \frac{1}{2}f''(1)(z-1)^2 + \ldots, \tag{2.33}$$

where black hole thermodynamics tells us (see equation (2.40) below) that $f'(1) \propto T$. At $T = 0$ then, we can find after a rescaling of the radial coordinate that the near-horizon geometry up to some constant factors can be written as

$$ds^2 = \frac{1}{\xi^2}\left(-dt^2 + d\xi^2\right) + dx^2 + dy^2. \tag{2.34}$$

This geometry is now no longer AdS_4, but rather $AdS_2 \times \mathbb{R}^2$ in coordinates $(t, \xi) \times (x, y)$.[12] A similar metric can still be found by making the right choices of coordinate substitutions in a series expansion in T for $T/\mu \ll 1$.[71, 93] What is interesting here is the symmetries of this $AdS_2 \times \mathbb{R}^2$ space-time. In empty AdS_4, there is a global scaling symmetry

$$(t, x, y, z) \rightarrow \lambda(t, x, y, z). \tag{2.35}$$

In this $AdS_2 \times \mathbb{R}^2$ case there is a symmetry that goes as

$$(t, \xi) \rightarrow \lambda(t, \xi), \quad (x, y) \rightarrow (x, y). \tag{2.36}$$

When writing this in terms of a dynamical critical exponent z, which is associated with a scaling

$$t \rightarrow \lambda t, \quad x \rightarrow \lambda^{\frac{1}{z}} x, \tag{2.37}$$

it can be seen that this $AdS_2 \times \mathbb{R}^2$ near-horizon geometry corresponds to a $z \rightarrow \infty$ system. In other words, the spatial dimensions completely decouple from the temporal and radial dynamics. This is exactly the local quantum criticality that is seen in the strange metal.[50, 90] This local quantum criticality is a feature of the near-horizon geometry, and therefore only of the lowest energy scales that enter in the problem, near the end of the RG flow. In the UV of the theory this symmetry is not present, and therefore this is an emergent property of the system. This peculiar fact, that the RN black hole turns out to have an emergent quantum critical sector, is one of the reasons that the string theory community has been looking to it as a point of departure for looking at interesting condensed matter systems such as the strange metal. This emergent quantum critical sector is unique and not found in any conventional condensed matter theory, however it turns out to be one of the most natural and simple to find things when doing holography.[61]

2.5.2. Thermodynamics of Reissner-Nordström

The RN black hole is dual to a quantum field theory at both finite temperature and chemical potential. For this solution, it is possible to evaluate the Euclidean on-shell action I_E, which according to the dictionary in table 2.1 yields the thermodynamic potential

$$\Omega = T I_E. \tag{2.38}$$

[12] Or, in general, $AdS_2 \times \mathbb{R}^d$ when starting from AdS_{d+2}.

From this it is possible to compute all thermodynamical quantities such as entropy and compressibilities in terms of derivatives of this free energy.[94–96] As luck would have it, in the Reissner-Nordström black hole some of these quantities can actually be evaluated directly from the near boundary behaviour or horizon integrals through standard dictionary entries without having to compute the entire free energy and compute derivatives. This is especially useful in numerical settings, where computing and finding the derivatives of the free energy by integrating the on-shell action can be very challenging. Rescaling all observables to the chemical potential μ[13] and one can find out that there is also a finite charge density

$$\frac{\rho}{\mu^2} = -\partial_z A_t \Big|_{z=0} = \frac{1}{\mu}. \tag{2.39}$$

The temperature of the black hole can be evaluated by evaluating the surface gravity of the black hole and is given by [76, 77]

$$\frac{T}{\mu} = \frac{|f'(1)|}{4\pi} = \frac{12-\mu^2}{16\pi\mu}. \tag{2.40}$$

The black hole entropy, which via black hole thermodynamics can be related to black hole surface area, is given by

$$s = \frac{S}{\mu^2} = \frac{4\pi}{\mu^2}. \tag{2.41}$$

This formula for the entropy may seem innocuous, but there is some important physics in it. As can be deduced from equation (2.40), the zero-temperature limit corresponds to $\mu \to \sqrt{12}$. One of the characteristics of the RN black hole is that the zero temperature limit has a finite horizon area.[92] Since the entropy is proportional to the horizon area, this means that the entropy is still finite even at zero temperature. For condensed matter systems, which we are interested in, this is a large complication, as they have no entropy at $T = 0$.[14] Naturally, one should feel sceptic of the results coming out of RN for this very reason. This does not mean though that everything coming out of this model is useless. In reality, what it means is that we should be keeping an eye open to places where this might become an important factor and either adapt our interpretations accordingly, or alternatively switch to other types of black hole solutions where this issue does not appear. The more cynical view is that this is a fundamental sickness of the AdS-Reissner-Nordström black hole for the purposeses of condensed matter physics and a strong argument for the use of other black hole solutions such as the Gubser-Rocha black hole of section 2.6, which is why it has been used for the majority of chapters 5 and 6.[98, 99]

Regarding the energy density and pressure, we can compute these via the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$ of the field theory by standard holographic renormalization techniques. In this particular case, that is done by constructing

$$\tilde{T}_{\mu\nu} = 2\left(K_{\mu\nu} - K h_{\mu\nu}\right) - 4h_{\mu\nu}, \tag{2.42}$$

[13] For the sake of simplicity, I have not written down the black hole in terms of its charge Q, which is the more usual starting point, but rather already in terms of μ. These two quantities are of course related, and can be expressed in terms of each other.

[14] Zero-temperature entropy does arise sometimes, for example in frustrated systems, but these are not in the scope of this thesis.[97]

where $h_{\mu\nu}$ is the induced metric at a slice at a constant radial $z = \epsilon$ away from the boundary, $K_{\mu\nu}$ and K are extrinsic curvature and its trace, respectively, and μ, ν run over the non-radial indices (t, x, y). The constant -4 arises from the Gibbons-Hawking-York counterterm.[75, 100] When appropriate counterterms are taken into consideration, in the limit $\epsilon \to 0$ the following expression becomes finite, yielding us the stress tensor

$$\langle T_{\mu\nu} \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \tilde{T}_{\mu\nu}. \tag{2.43}$$

Evaluating this in the homogeneous Reissner-Nordström black hole this gives

$$-T^t{}_t = 2T^x{}_x = 2T^y{}_y = 2 + \frac{\mu^2}{2}. \tag{2.44}$$

This means that the corresponding internal energy and pressure are given by

$$\frac{E}{\mu^3} = \frac{2P}{\mu^3} = \frac{1}{\mu^3}\left(2 + \frac{\mu^2}{2}\right). \tag{2.45}$$

Note from this that this is a conformal system as the trace of the stress tensor vanishes

$$\left\langle T^\mu{}_\mu \right\rangle = E - 2P = 0. \tag{2.46}$$

2.6. Einstein-Maxwell-Dilaton Theory

The Reissner-Norström black holes that have been discussed so far are ubiquitous in AdS/CFT, as they are just about the simplest system that can provide a finite chemical potential. The ease of computations in the system has been the main driver for its popularity. The thermodynamics of the RN states are however rather problematic, as there is no condensed matter system we could want to describe that has the property of finite ground-state entropy. In this sense, the RN black hole is only one of a much larger family of black holes which can have different types of near-IR scaling behaviours.

A part of this family of black holes can conveniently be explored by coupling a dilatonic field into the theory.[61, 101, 102] Dilatons are rather general in holography, as mentioned above in section 2.3.3. These dilatonic fields are dynamical and back-react onto the geometry, but more important is the way in which they couple to the Maxwell sector. A generic EMD Lagrangian where a single dilaton field ϕ is coupled in can be written as

$$\mathcal{L}_{\text{EMD}} = R - \frac{Z(\phi)}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - V(\phi). \tag{2.47}$$

There are here two coupling functions, $Z(\phi)$ and $V(\phi)$. V is the potential for the scalar, which also includes the cosmological constant in its series expansion, $V(0) = 2\Lambda$.

Often, $V(\phi)$ can take the form of a sum of exponentials, sometimes balanced to form hyperbolic geometric functions. The coupling function $Z(\phi)$ presents a modification of the coupling constant of the Maxwell sector in the theory. This changes the effective value of the charge density in the system, as through the holographic dictionary

$$\rho_{CFT} = Z(\phi)F^{tz}\Big|_{z=0}. \tag{2.48}$$

Compare this to the Einstein-Maxwell setup that was used for the RN black holes, where $Z(\phi) = 1$. A crucial difference with the RN black hole is in the low-temperature IR geometry. Where for RN we are left with a finite-size extremal black hole that carries a large amount of zero-temperature entropy, the general EMD black holes can have horizons that keep shrinking all the way to zero temperature, giving the entropy a temperature dependence

$$S \sim T^{\alpha}, \quad \alpha > 0. \tag{2.49}$$

The form of the potentials V, Z is generically what determines the scaling properties of the solutions.[101, 102] The dilaton takes on a large value at the horizon, and therefore in order to understand the horizon scaling we only need to consider what the leading near-horizon behaviours of the potentials are.[102] In typical top-down cases, the exponentials scale near the horizon like

$$Z \sim e^{\phi}, \qquad V \sim e^{-\phi}. \tag{2.50}$$

But, as we have seen before, in holography we are not bound by only exact top-down constructions, and we are able to take other values for the parameters. The potentials are typically parametrized by the factors γ, δ as

$$Z = e^{\gamma\phi}, \qquad V = V_0 e^{-\delta\phi}. \tag{2.51}$$

The scaling analysis in these theories is a bit more complicated than in the Reissner-Nordström case. In standard coordinates, the metric of the deep IR at zero temperature can be described as

$$ds^2 \sim r^{-2\theta/d}\left(-r^{2z}dt^2 + r^2 d\vec{x}^2 + \frac{1}{r^2}dr^2\right). \tag{2.52}$$

Here the coordinate r ranges from the interior $r = 0$ to the boundary $r = \infty$. The parameter z is one we have encountered above – this is the same dynamical critical exponent as before. The parameter θ is the hyperscaling violation exponent.[103–105] In a very rough description, a theory with parameters (z, θ) will display scaling of the thermodynamical observables as if the theory has dynamical critical exponent z but lives in $d - \theta$ dimensions.[104] The choices made for the coupling exponents γ, δ can be mapped onto resulting values of z, θ.[102] It is important that in the presence of $\theta \neq 0$, the near-IR theory is no longer scale-invariant, but rather it is scale-covariant, with under the scale transformation[104]

$$t \to \lambda^z t, \quad \vec{x} \to \lambda\vec{x}, \quad r \to \lambda r. \tag{2.53}$$

The metric then transforms as

$$ds^2 \to \lambda^{2\theta/d} ds^2. \tag{2.54}$$

There are several interesting ideas that we can consider for θ, z. For example, $\theta = d - 1$ can be thought of as describing the physics of a system near its Fermi surface, which is always has a lower dimensionality than the dimensionality of the theory.[104] For reasons that will become clear soon, the most interesting value in relation to the strange metal is the rather peculiar choice of $z, -\theta \to \infty$.

2.6.1. The Gubser-Rocha Conformal-to-AdS2 Metal

When considering the thermodynamics of these EMD systems, we can find that the entropy will scale with temperature as

$$S \sim T^{(d-\theta)/z}. \tag{2.55}$$

If we wish to describe a strange metal, we would ideally like to consider an entropy that scales as temperature like

$$S \sim T^{1}. \tag{2.56}$$

This is the well-known Sommerfeld entropy, also experimentally observed in strange metals.[52] To take full advantage of the power of holography we would want to do this in a setting where we are locally quantum critical.[50] In other words, that means $z \to \infty$. These two ideas can only both be satisfied in the limit

$$z, -\theta \to \infty, \text{ such that } \frac{-\theta}{z} = 1. \tag{2.57}$$

A hyperscaling violation exponent of minus infinity is rather strange, and definitely is hard to think about in terms of an effective dimensionality of a problem. Nevertheless, it is a well-defined limit.[101, 102] It has been shown that these models can reproduce the linear-in-T resistivity of strange metals as well.[106] To be more exact, these systems correspond to the choice $\gamma = -\delta = 1/\sqrt{3}$ in term of the coupling exponents. That means we have that the following couplings

$$Z(\phi) = e^{\phi/\sqrt{3}}, \qquad V(\phi) = 2\Lambda \cosh\left(\frac{\phi}{\sqrt{3}}\right). \tag{2.58}$$

This choice of potential is consistent with what was discussed earlier, as it is only the leading exponential behaviour that is relevant in the IR, so only one of the exponential terms in the hyperbolic cosine will be of relevance there. This potential has the added benefit of having the property that

$$V'(\phi)\Big|_{\phi=0} = 0, \tag{2.59}$$

which allows for proper AdS asymptotics.[102] This system turns out to have an analytical solution in the form of a black-brane solution.[98, 107] The metric, scalar and gauge profiles of this solution can be parametrized in terms of the charge Q of the black hole for a horizon radius of $z_H = 1$

as

$$\begin{aligned} ds^2 &= \frac{1}{z^2}\left(-f(z)dt^2 + \frac{1}{f(z)}dz^2 + g(z)(dx^2+dy^2)\right), \\ \phi &= \frac{\sqrt{3}}{2}\log\left(1+Qz\right), \\ A &= \sqrt{3Q(1+Q)}\frac{1-z}{1+Qz}dt. \end{aligned} \tag{2.60}$$

The two functions f, g are given by

$$\begin{aligned} f(z) &= \frac{(1-z)}{g(z)}\left(1+(1+3Q)z+(1+3Q(1+Q))z^2\right) \\ g(z) &= (1+Qz)^{3/2}. \end{aligned} \tag{2.61}$$

The chemical potential is given by $\mu = \sqrt{3Q(1+Q)}$. In contrast to the RN solution, from a simplicity standpoint the preferred parameter for this theory is Q, as the expressions in terms of the chemical potential, though equally valid, are simply horrible to read. Nonetheless, in a similar fashion to RN we will prefer to think of observables and parameters in units of chemical potential, as that is a scale we can make reference to when looking at physical experiment. Black hole charge itself does not have a physically relevant measure. The thermodynamics of this state are given by

$$T = \frac{3\sqrt{1+Q}}{4\pi} \Rightarrow \frac{T}{\mu} = \frac{\sqrt{3}}{4\pi\sqrt{Q}}. \tag{2.62}$$

The entropy is again given by the horizon area

$$S = 4\pi\int\sqrt{h} = 4\pi\left(1+Q\right)^{3/2} \Rightarrow \frac{S}{\mu^2} = 4\pi\frac{\sqrt{1+Q}}{3Q}, \tag{2.63}$$

and the charge density is [15]

$$\rho = \mu\left(1+Q\right) \Rightarrow \rho/\mu^2 = \frac{\sqrt{1+Q}}{\sqrt{3Q}} \tag{2.64}$$

As can be expected, the temperature dependence of the thermodynamical observables is clearly different from those in the Reissner-Nordström black hole. It is easiest to think in the parameter Q. The temperature scales as

$$T/\mu \sim Q^{-1/2}, \tag{2.65}$$

so at low temperature Q is large. The entropy similarly scales as

$$S/\mu^2 \sim Q^{-1/2} \tag{2.66}$$

in the large-Q regime, so indeed we have that

$$\frac{S}{\mu^2} \sim \frac{T}{\mu} \tag{2.67}$$

[15] The factor $Z(\phi)$ mentioned earlier in equation (2.52) is absent, as $Z(\phi(z=0)) = Z(0) = 1$.

at low temperatures $T/\mu \ll 1$. The scaling at large T is different, as for large T we have that the parameter Q is small, and therefore

$$S/\mu^2 = \frac{4\pi\sqrt{1+Q}}{3Q} \sim 1/Q \sim T^2. \tag{2.68}$$

The charge density also shows two different regimes. For large Q the charge density scales as

$$\rho/\mu^2 \sim Q^0 \sim T^0, \tag{2.69}$$

while for small Q the constant term dominates in the numerator, meaning that

$$\rho/\mu^2 \sim Q^{-1/2} \sim T^1. \tag{2.70}$$

The stress tensor in this system requires a bit more consideration. The full discussion of this has been the subject of one of the papers that make up this thesis, as given in chapter 5. Other references that have dealt with this topic have overlooked certain subtle points in relation to near-boundary expansions of the analytical solution when considering quantization choices.[108, 109] The resulting stress tensor including counterterms is now expressed as

$$T_{ij} = 2K_{ij} - 2\left({}^dR_{\gamma,ij}\right) - 2(K+2)\gamma_{ij} + \gamma_{ij}\left[c_\phi \phi N^z \partial_z \phi + \Lambda_\phi \phi^2/2\right], \tag{2.71}$$

where N_μ is the outward-pointing unit normal vector. The coefficients are then fixed to be

$$\Lambda_\phi = 2c_\phi - 1, \quad c_\phi = \frac{1}{3}. \tag{2.72}$$

The first of these is easy to understand, as both terms can contribute to cancelling the lowest-order divergence. The argument for choosing $c_\phi = 1/3$ is more subtle, but essentially boils on choosing the scalar to be a marginal operator in the theory. The resulting thermodynamics are that of a one-charge theory, rather than the generic two-charge that would arise from coupling in a scalar to the boundary theory. The full details can be found in chapter 5. The resulting expressions for the stress energy tensor are rather simple,

$$-T^t_t = 2T^x_x = 2T^y_y = 2(1+Q)^3. \tag{2.73}$$

Hence the energy and pressure can be expressed as

$$\frac{E}{\mu^3} = \frac{2P}{\mu^3} = \frac{2}{\mu^3}(1+Q)^3 = \frac{2}{3\sqrt{3}}\left(1+\frac{1}{Q}\right)^{3/2}. \tag{2.74}$$

where again we recover a conformal theory.

2.6.2. DC Conductivity in the Gubser-Rocha Model

The DC conductivity of this system is formally divergent. The system is translationally invariant, so momentum is conserved and there is no resistance in the system to a DC perturbation. There is a large amount of spectral weight contained in a δ-peak located at the origin. An important result was found in massive gravity variations on the RN and GR model where momentum is no longer conserved.[106, 110–112] Massive gravity is a slightly peculiar construction, where a frame-fixing occurs due to the introduction of a reference metric. This breaks diffeomorphism invariance, and as a result the graviton acquires a mass. Massive modes propagating in the bulk space-time now only have a finite lifetime, so this causes linear momentum to be able to decay in the system. Since momentum is now no longer conserved, this renders the DC conductivity finite. Spectacularly, in the GR model when momentum is indeed not conserved, the resistivity acquires a dependence

$$\rho_{DC} \propto T + O(T^2). \tag{2.75}$$

This is a phenomenal result, and we might expect to be able to find this in other systems where conservation of momentum is removed through more realistic methods, as massive gravity, though holographically a sound theory, has little bearing on the physics in the lab. In particular, one could break translations into an (ionic) lattice, which has indeed been shown to have similar effects in terms of breaking momentum conservation.[110, 111] This is the theme of this thesis.

2.7. Breaking Translational Symmetry

The allusion to massive gravity models just made aside, all results so far have got one thing in common, and that is that all models are perfectly translationally invariant, momentum is still perfectly conserved. The two transverse directions (x, y) do not contain any structure at all. Of course in actual physical systems, this is not the case. For Drude-like transport, which is observed in even the strange metals at low frequency, the source of momentum dissipation is not something as holographic and weird as a 'massive gravity' construction. Instead, we know very well that translational and rotational symmetries of space-time are broken into some crystal lattice.[113] There are several ways to break translational symmetry in holography. The computationally more straightforward ones are models that break translational symmetry in a homogeneous way, which means that the differential equations that describe the systems are only dependent on the radial coordinate and any dependence on the transverse coordinates is engineered to be absent.[114–117] This again is a very particular construction, and if we are to do an honest job using holography to model physical systems in the presence of a crystal lattice we should have an actual lattice present. This requires a periodic modulation that captures both the rotational and translational properties of a crystal lattice. The way to really achieve that is to let go of the simple radial-only models and turn the problem of solving for the bulk geometry into a set of partial differential equations, that now can also depend explicitly on the transverse coordinates as well as the radial coordinate. This type of symmetry breaking is often called inhomogeneous symmetry breaking, and it is of this type that the remainder of this thesis explores the intriguing consequences.