Dormancy in stochastic interacting systems

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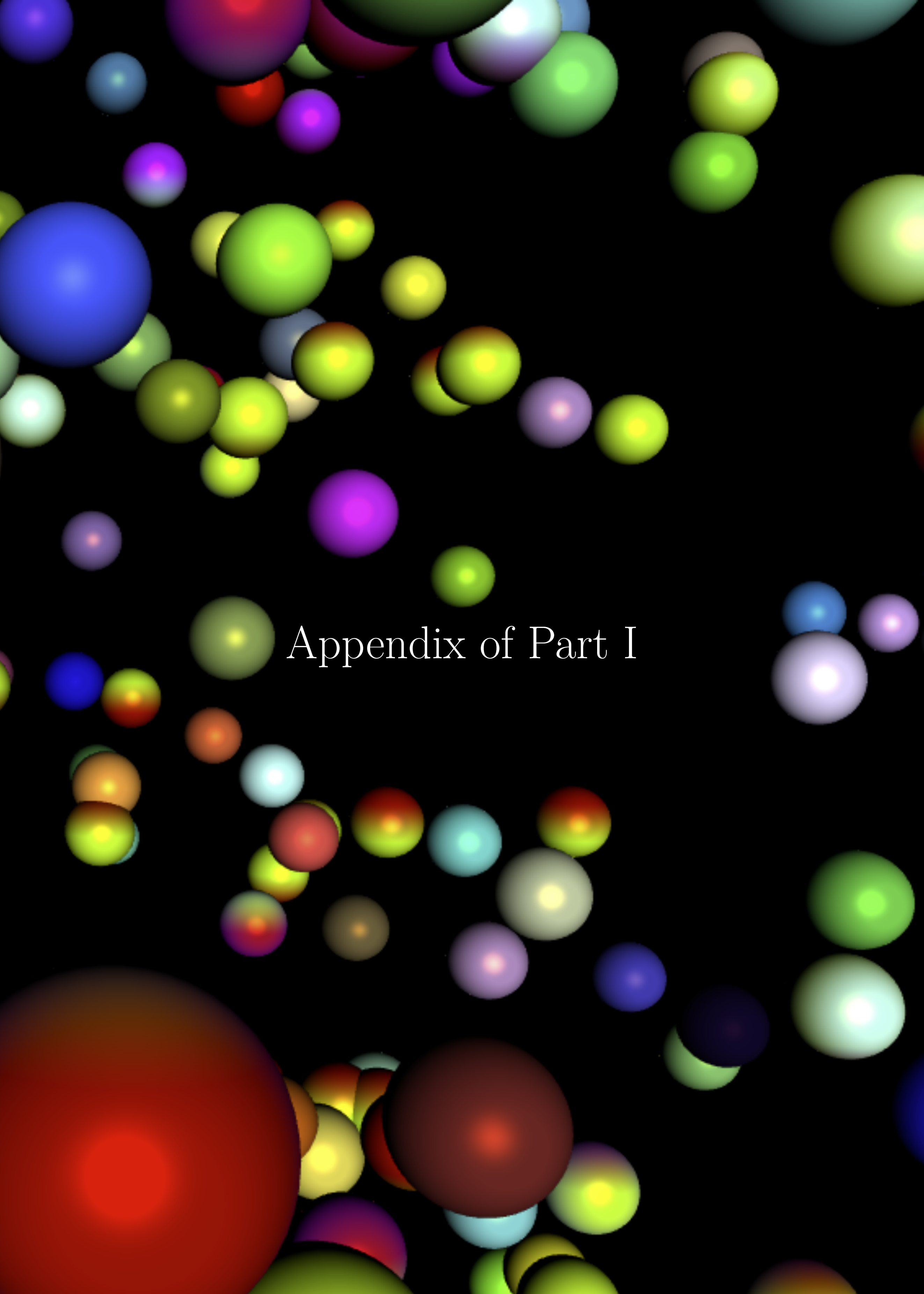

Appendix of Part I

APPENDIX A

Appendix: Chapter 3

§A.1 Two-particle dual and alternative representation

In this appendix, we give a short description of the original dual process $\tilde{Z}$ started with two particles, which was introduced in full generality as a configuration process Z_* in Section 2.4.2 of Chapter 2. Further, we briefly outline the derivation of the *interacting RW1* process ξ defined in Definition 3.3.1 from the configuration process $\tilde{Z}$, and show that the absorption of ξ and coalescence of the two particles in $\tilde{Z}$ are basically equivalent.

Definition A.1.1 (Two-particle dual). The two-particle dual process

$$\tilde{Z} := (\tilde{Z}(t))_{t\geq 0}, \qquad \tilde{Z}(t) := (\tilde{n}_i(t), \tilde{m}_i(t))_{i\in\mathbb{Z}^d}, \tag{A.1}$$

is the continuous-time Markov chain with state space

$$\tilde{\mathcal{X}} := \Big\{(\tilde{n}_i, \tilde{m}_i)_{i\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i]\times[M_i] \colon \sum_{i\in\mathbb{Z}^d} (\tilde{n}_i + \tilde{m}_i) \leq 2\Big\} \tag{A.2}$$

and with transition rates

$$(n_k, m_k)_{k\in\mathbb{Z}^d} \to$$
$$\begin{cases} (n_k,m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} & \text{at rate } \frac{2a(i,i)}{N_i}\binom{n_i}{2}\mathbb{1}_{\{n_i\geq 2\}} \\ & \qquad + \sum\limits_{j\in\mathbb{Z}^d\setminus\{i\}} \frac{n_i a(i,j) n_j}{N_j} \quad \text{for } i\in\mathbb{Z}^d, \\ (n_k,m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} + \vec{\delta}_{i,D} & \text{at rate } \frac{\lambda n_i (M_i - m_i)}{M_i} \quad \text{for } i\in\mathbb{Z}^d, \\ (n_k,m_k)_{k\in\mathbb{Z}^d} + \vec{\delta}_{i,A} - \vec{\delta}_{i,D} & \text{at rate } \frac{\lambda (N_i - n_i) m_i}{M_i} \quad \text{for } i\in\mathbb{Z}^d, \\ (n_k,m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} + \vec{\delta}_{j,A} & \text{at rate } \frac{n_i a(i,j)(N_j - n_j)}{N_j} \quad \text{for } i\neq j\in\mathbb{Z}^d, \end{cases} \tag{A.3}$$

where, for $i\in\mathbb{Z}^d$, the configurations $\vec{\delta}_{i,A}$, $\vec{\delta}_{i,D}$ are defined as in (2.19), and addition (subtraction) of configurations are defined component-wise by (2.20). The support of the distribution of $\tilde{Z}(0)$ is contained in

$$\tilde{\mathcal{X}}_0 := \Big\{(\tilde{n}_i, \tilde{m}_i)_{i\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i]\times[M_i] \colon \sum_{i\in\mathbb{Z}^d} (\tilde{n}_i + \tilde{m}_i) = 2\Big\}. \tag{A.4}$$

■

Here, $\tilde{n_i}(t)$ and $\tilde{m_i}(t)$ are the number of active and dormant particles at site $i \in \mathbb{Z}^d$ at time t. The first transition describes the coalescence of an active particle at site i with active particles at other sites. The second and third transitions describe the switching between the active and the dormant state of the particles at site i. The fourth transition describes the migration of an active particle from site i to site j.

Let $\tilde{\mathcal{X}}_1$ be the set of configurations containing a single particle, i.e.,

$$\tilde{\mathcal{X}}_1 := \Big\{(\tilde{n_i}, \tilde{m_i})_{i\in\mathbb{Z}^d} \in \tilde{\mathcal{X}} \colon \sum_{i\in\mathbb{Z}^d} (\tilde{n_i} + \tilde{m_i}) = 1\Big\}, \tag{A.5}$$

and let $\tilde{\tau}$ be the first time at which coalescence has occurred, i.e.,

$$\tilde{\tau} = \inf\{t \geq 0\colon\ (\tilde{n_i}(t), \tilde{m_i}(t))_{i\in\mathbb{Z}^d} \in \tilde{\mathcal{X}}_1\}. \tag{A.6}$$

As indicated earlier in Section 3.3.1 of Chapter 3, we are only required to analyse the coalescence probability of two dual particles and thus, it boils down to lumping all the configurations in $\tilde{\mathcal{X}}_1$ into a single state $\circledast$ and consider the resulting lumped process. Note that, on the event $\{\tilde{\tau} < s\}$, the process $(\tilde{Z}(t))_{t\geq s}$ a.s. stays in $\tilde{\mathcal{X}}_1$. Therefore the lumped process is a well-defined continuous-time Markov chain with state space $\tilde{\mathcal{X}}_0 \cup \{\circledast\}$, where $\circledast$ is an absorbing state.

With a little abuse of notation, from here onwards we denote the lumped process by $(\tilde{Z}(t))_{t\geq 0}$. We give the formal description of this process in a definition.

Definition A.1.2 (Lumped two-particle dual). The lumped two-particle dual process

$$\tilde{Z} := (\tilde{Z}(t))_{t\geq 0} \tag{A.7}$$

is the continuous-time Markov chain with state space

$$\tilde{\mathcal{X}} := \Big\{(n_i, m_i)_{i\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i] \times [M_i] \colon \sum_{i\in\mathbb{Z}^d} (n_i + m_i) = 2\Big\} \bigcup \Big\{\circledast\Big\} \tag{A.8}$$

and with transition rates

$$(n_k, m_k)_{k\in\mathbb{Z}^d} \to$$
$$\begin{cases} \circledast, & \text{at rate } \displaystyle\sum_{i\in\mathbb{Z}^d} \Big[\tfrac{2a(0,0)}{N_i}\binom{n_i}{2}\mathbb{1}_{\{n_i\geq 2\}} \\ & \qquad + \displaystyle\sum_{j\in\mathbb{Z}^d\setminus\{i\}} \tfrac{n_i a(i,j) n_j}{N_j}\Big] \ \text{for } i \in \mathbb{Z}^d, \\ (n_k, m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} + \vec{\delta}_{i,D}, & \text{at rate } \tfrac{\lambda n_i (M_i - m_i)}{M_i} \qquad \text{for } i \in \mathbb{Z}^d, \\ (n_k, m_k)_{k\in\mathbb{Z}^d} + \vec{\delta}_{i,A} - \vec{\delta}_{i,D}, & \text{at rate } \tfrac{\lambda (N_i - n_i) m_i}{M_i} \qquad \text{for } i \in \mathbb{Z}^d, \\ (n_k, m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} + \vec{\delta}_{j,A}, & \text{at rate } \tfrac{n_i a(i,j)(N_j - n_j)}{N_j} \qquad \text{for } i \neq j \in \mathbb{Z}^d, \end{cases} \tag{A.9}$$

where, for $i \in \mathbb{Z}^d$, $\vec{\delta}_{i,A}$ and $\vec{\delta}_{i,D}$ are as in (2.19). ■

We write $\tilde{\mathbb{P}}^\eta$ to denote the law of the process $\tilde{Z}$ started from $\eta \in \mathcal{X}$. Note that, by construction, the coalescence time $\tilde{\tau}$ is now same as the absorption time of the process $\tilde{Z}$. In the following proposition, we show that the configuration process $\tilde{Z}$ is an alternative representation of the coordinate process ξ defined in Definition 3.3.1.

Proposition A.1.3 (Equivalence between $\tilde{Z}$ and ξ). *Let $\xi = (\xi(t))_{t\geq 0}$ be the process defined in Definition* 3.3.1 *with initial distribution μ. Let $\phi\colon \mathcal{S} \to \tilde{\mathcal{X}}$ be the map defined by*

$$\phi(\eta) := \begin{cases} (\alpha\delta_{k,i} + \beta\delta_{k,j}, (1-\alpha)\delta_{k,i} + (1-\beta)\delta_{k,j})_{k\in\mathbb{Z}^d}, & \text{if } \eta = [(i,\alpha),(j,\beta)] \neq \circledast, \\ \circledast, & \text{otherwise.} \end{cases} \tag{A.10}$$

For $t \geq 0$, let $\tilde{Z}(t) := \phi(\xi(t))$. Then the process $(\tilde{Z}(t))_{t\geq 0}$ is the lumped dual process defined in Definition A.1.2*, and its initial distribution is the push-forward of μ under the map ϕ. Furthermore, $\tilde{Z}$ is absorbed to $\circledast$ if and only if ξ is.*

Proof. Due to the Assumption 3.A, we see that $\phi(\eta) \in \tilde{\mathcal{X}}$, and so $\tilde{Z}(t) \in \mathcal{X}$ for all $t \geq 0$, and ϕ is onto. For $\eta \in \mathcal{S}$, define

$$\bar{\eta} := \begin{cases} [(j,\beta),(i,\alpha)], & \text{if } \eta = [(i,\alpha),(j,\beta)] \neq \circledast, \\ \circledast, & \text{otherwise.} \end{cases} \tag{A.11}$$

Note that $\phi^{-1}(\phi(\eta)) = \{\eta, \bar{\eta}\}$. Let $Q(\eta_1, \eta_2)$ denote the transition rate from η_1 to η_2 for the process ξ, where $\eta_1 \neq \eta_2 \in \mathcal{S}$. Furthermore, let $z_1 \neq z_2 \in \tilde{\mathcal{X}}$ be fixed and $\eta_1 \in \mathcal{S}$ be such that $\phi(\eta_1) = z_1$. Since $Q(\eta_1, \eta_2) = Q(\bar{\eta}_1, \bar{\eta}_2)$ for any $\eta_1 \neq \eta_2 \in \mathcal{S}$, we have

$$\sum_{\eta\in\phi^{-1}(z_2)} Q(\eta_1, \eta) = \sum_{\eta\in\phi^{-1}(z_2)} Q(\bar{\eta}_1, \eta). \tag{A.12}$$

Hence the Dynkin criterion for lumpability is satisfied, and ϕ preserves the Markov property. So $\tilde{Z}$ is a Markov process on $\tilde{X}$. We can easily verify that the sum in (A.12) is indeed the transition rate from z_1 to z_2 defined in (A.9). Thus, $\tilde{Z}$ is the lumped dual process defined in Definition A.1.2. Clearly, the distribution of $\tilde{Z}(0)$ is $\mu \circ \phi^{-1}$. The second claim trivially follows, since $\phi(\eta) = \circledast$ if and only if $\eta = \circledast$. □

Appendix A

§A.2 Completion of the proof of a theorem on the clustering regime

Let us restate the theorem before we proceed with the completion of the proof.

Theorem A.2.1 (Clustering regime). *Suppose that $K^{-1} = \sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$ and that Assumption* 3.B *holds. If the process ξ^* defined in Definition* 3.3.4 *is absorbed to $\circledast$ with probability 1, then it is necessary that the symmetrised kernel $\hat{a}(\cdot\,,\cdot)$ is recurrent, i.e.,*

$$\int_0^\infty \hat{a}_t(0,0)\,\mathrm{d}t = \infty. \tag{A.13}$$

Furthermore, if $(N_i)_{i\in\mathbb{Z}^d}$ satisfies the non-clumping condition in (3.6) *and $a(\cdot\,,\cdot)$ is symmetric, then* (A.13) *is also sufficient.*

The above theorem states that, under some suitable conditions, two particles evolving according to the process ξ^* defined in Definition 3.3.4 coalesce with probability 1 if and only if the symmetrised migration kernel $\hat{a}(\cdot\,,\cdot)$ is recurrent, where

$$\hat{a}(i,j) = \tfrac{1}{2}[a(i,j) + a(j,i)], \quad i,j \in \mathbb{Z}^d. \tag{A.14}$$

Recall that in the process ξ^* the two particles evolve independently on the state space $G = \mathbb{Z}^d \times \{0,1\}$ according to the motion described in Section 3.4.2, until they coalesce with each other to form an absorbed state $\circledast$. Under the conditions mentioned in the theorem, every time the two particles share the same location there is a positive probability of absorption within the next unit time interval and this probability is bounded away from zero uniformly over the location of meeting. Therefore we argue, as in the proof of the theorem, that the two independent particles coalesce with probability 1 if and only if in the process obtained from ξ^* after suppressing the coalescent events the total accumulated time spent by the two particles at the same location is infinite with probability 1. This accumulated time $\tilde{I}$ is the random variable given by

$$\tilde{I} = \int_0^\infty \mathbb{1}_{\mathcal{E}(t)}\,\mathrm{d}t, \tag{A.15}$$

where $\mathcal{E}(t)$ is the event that both particles are at the same location at time $t \geq 0$. Let $\eta \in G \times G$ be any state where both particles are at the same location, i.e., $\eta = [(i,\alpha),(i,\beta)]$ for some $i \in \mathbb{Z}^d$ and $\alpha,\beta \in \{0,1\}$. The average accumulated time, written as I when the process starts at state η, can be expressed as

$$I = \mathbb{E}^*_\eta[\tilde{I}] = \int_0^\infty \mathbb{P}^*_\eta(\mathcal{E}(t))\,\mathrm{d}t. \tag{A.16}$$

In the proof of Theorem 3.4.4, we exploit the fact that $\tilde{I}$ is infinite with probability 1 iff $I = \infty$. The forward direction of this fact is obvious. However, the converse is non-trivial and is stated without justification. In the following lemma we provide the proof.

Lemma A.2.2. *Let $\tilde{I}$ be as in* (A.15) *and $\eta = [(i,\alpha),(i,\beta)] \in G \times G$ for some $i \in \mathbb{Z}^d$. Assume that the migration kernel $a(\cdot\,,\cdot)$ is symmetric. If $\mathbb{P}^*_\eta$ is the law of the process ξ^* in Definition* 3.3.4 *after coalescence is turned off and $\mathbb{E}^*_\eta[\tilde{I}] = \infty$, then $\mathbb{P}^*_\eta(\tilde{I} = \infty) = 1$.*

Proof. First observe that the event $\{\tilde{I} = \infty\} \in \mathcal{I}$ where $\mathcal{I}$ is the time-shift invariant σ-field of the process ξ^*. Note that $\mathcal{I}$ is trivial because all bounded harmonic functions of the process ξ^* are constant. The later follows from [93, Theorem 26.11], [112, Corollary 3.7, Chapter II], and the fact that bounded harmonic functions for the corresponding process with a single particle are constant. Therefore, $\mathbb{P}^*_\zeta(\tilde{I} = \infty) \in \{0,1\}$ for any $\zeta \in G \times G$, and so it suffices to show that $\mathbb{P}^*_\eta(\tilde{I} = \infty) > 0$ for the choice of initial state $\eta = [(0,1),(0,1)]$ where both particles are at 0 in the active state. To this end, for $T > 0$ let us consider the truncated random variable

$$\tilde{I}_T := \int_0^T \mathbb{1}_{\mathcal{E}(t)}\,\mathrm{d}t. \tag{A.17}$$

We show that there exists a constant $C > 0$ independent of T, such that

$$\mathbb{E}^*_\eta[\tilde{I}_T^2] \leq C\,\mathbb{E}^*_\eta[\tilde{I}_T]^2, \quad T \geq 0. \tag{A.18}$$

By the Paley-Zygmund inequality, this will imply

$$\mathbb{P}^*_\eta(\tilde{I}_T > \tfrac{1}{2}\mathbb{E}^*_\eta[\tilde{I}_T]) \geq \frac{1}{4}\frac{\mathbb{E}^*_\eta[\tilde{I}_T]^2}{\mathbb{E}^*_\eta[\tilde{I}_T^2]} \geq \frac{1}{4C}, \quad T \geq 0. \tag{A.19}$$

Since $\tilde{I}_T \uparrow \tilde{I}$ almost surely as $T \to \infty$ and $\mathbb{E}^*_\eta[\tilde{I}] = \infty$ by assumption, it will follow that

$$\mathbb{P}^*_\eta(\tilde{I} = \infty) \geq \limsup_{T\to\infty} \mathbb{P}^*_\eta(\tilde{I}_T > \tfrac{1}{2}\mathbb{E}^*_\eta[\tilde{I}_T]) \geq \tfrac{1}{4C} > 0, \tag{A.20}$$

which will complete the proof.

To prove (A.18), we use the Markov property of ξ^* and Fubini's theorem to write

$$\begin{aligned}
\mathbb{E}^*_\eta[\tilde{I}_T^2] &= \mathbb{E}^*_\eta\Big[\int_0^T \int_0^T \mathbb{1}_{\mathcal{E}(t)}\, \mathbb{1}_{\mathcal{E}(s)}\, \mathrm{d}t\, \mathrm{d}s\Big] = \mathbb{E}^*_\eta\Big[2\int_0^T \mathrm{d}t\, \mathbb{1}_{\mathcal{E}(t)} \int_0^t \mathrm{d}s\, \mathbb{1}_{\mathcal{E}(s)}\Big] \\
&= 2\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \mathbb{E}^*_\eta[\mathbb{1}_{\mathcal{E}(t)}\mathbb{1}_{\mathcal{E}(s)}] = 2\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \mathbb{E}^*_\eta\big[\mathbb{1}_{\mathcal{E}(s)}\mathbb{E}^*_\eta[\mathbb{1}_{\mathcal{E}(t)} \mid \mathcal{F}_s]\big] \\
&= 2\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \mathbb{E}^*_\eta\big[\mathbb{1}_{\mathcal{E}(s)}\mathbb{P}^*_{\xi^*(s)}(\mathcal{E}(t-s))\big].
\end{aligned} \tag{A.21}$$

Now, if we show that, for any $t \geq 0$ and some constant C not depending on either t or ζ,

$$\mathbb{P}^*_\zeta(\mathcal{E}(t)) \leq C\, \mathbb{P}^*_\eta(\mathcal{E}(t)), \tag{A.22}$$

where $\zeta \in G \times G$ is any state such that both particles are at the same location, i.e., $\zeta = [(i,\alpha),(i,\beta)]$ for some $i \in \mathbb{Z}^d$ and $\alpha, \beta \in \{0,1\}$, then (A.21) implies

$$\begin{aligned}
\mathbb{E}^*_\eta[\tilde{I}_T^2] &\leq 2C\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \mathbb{E}^*_\eta\big[\mathbb{1}_{\mathcal{E}(s)}\mathbb{P}^*_\eta(\mathcal{E}(t-s))\big] \\
&= 2C\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \mathbb{E}^*_\eta\big[\mathbb{1}_{\mathcal{E}(s)}\big]\mathbb{E}^*_\eta\big[\mathbb{1}_{\mathcal{E}(t-s)}\big] \\
&= 2C\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, g(s)g(t-s) = 2C\int_0^T \mathrm{d}t\, g*g(t) \\
&\leq 2C\|g*g\|_{L^1(\mathbb{R})},
\end{aligned} \tag{A.23}$$

where $g : \mathbb{R} \to [0,1]$ is the map $t \mapsto \mathbb{E}^*_\eta[\mathbb{1}_{\mathcal{E}(t)}]\mathbb{1}_{[0,T]}(t)$ and $*$ denotes the usual convolution operation. Finally, applying Young's inequality for convolution on $g*g$, we get

$$\mathbb{E}^*_\eta[\tilde{I}_T^2] \leq 2C\|g\|^2_{L^1(\mathbb{R})} = 2C\,\mathbb{E}^*_\eta[\tilde{I}_T]^2. \tag{A.24}$$

Thus, it only remains to show (A.22).

To this aim, let $S(t)$ and $S'(t)$ denote the respective accumulated activity times in the time interval $[0,t]$ for the two particles evolving according to the process ξ^* with coalescence switched off. For $i \in \mathbb{Z}^d$, let $\zeta = [(i,\alpha),(i,\beta)] \in G \times G$ be fixed arbitrarily. Observe that, by the independence of the two particles and the symmetry assumption of $a(\cdot\,,\cdot)$,

$$\begin{aligned}\mathbb{P}^*_\zeta(\mathcal{E}(t)) &= \sum_{j\in\mathbb{Z}^d} \mathbb{E}^*_\zeta[a_{S(t)}(i,j)]\mathbb{E}^*_\zeta[a_{S'(t)}(i,j)] = \mathbb{E}^*_\zeta[a_{S(t)+S'(t)}(i,i)] \\ &= \mathbb{E}^*_\zeta[a_{S(t)+S'(t)}(0,0)] \\ &\le \mathbb{E}_{(i,\alpha)}[a_{S(t)}(0,0)],\end{aligned} \tag{A.25}$$

where we use that $s\mapsto a_s(0,0)$ is non-increasing in s when $a(\cdot\,,\cdot)$ is symmetric (which follows from standard Fourier analysis), and the last expectation on the right-hand side is taken w.r.t. the law of a single particle process started at $(i,\alpha)\in G$. Fix $\epsilon\in\big(0,\frac{1}{1+K^{-1}}\big)$ arbitrarily, where $K^{-1} := \sup_{j\in\mathbb{Z}^d} K_j^{-1} < \infty$. Once more, applying the non-increasing property of $s\mapsto a_s(0,0)$, we get from (A.25) that

$$\begin{aligned}\mathbb{P}^*_\zeta(\mathcal{E}(t)) &\le a_{\epsilon t}(0,0)\mathbb{P}_{(i,\alpha)}(S(t)>\epsilon t) + \mathbb{E}_{(i,\alpha)}[a_{S(t)}(0,0);\, S(t)\le\epsilon t] \\ &\le a_{\epsilon t}(0,0) + \mathbb{P}_{(i,\alpha)}(S(t)\le\epsilon t).\end{aligned} \tag{A.26}$$

Using the same arguments as in the proof of Lemma 3.4.1, we can construct an alternating renewal process $(R_s)_{s\ge0}$, with successive exponential waiting times $(V_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$, on the probability space of the single particle such that

$$D(t) \le \int_0^t \mathbb{1}_{\{R_s=1\}}\,\mathrm{d}s, \tag{A.27}$$

where $D(t)=t-S(t)$ is the accumulated dormant period of the particle in the time interval $[0,t]$, the process $s\mapsto R_s$ takes value 0 (resp., 1) during the periods $(V_n)_{n\in\mathbb{N}}$ (resp., $(U_n)_{n\in\mathbb{N}}$) and

$$V_n \overset{\text{i.i.d.}}{=} \exp(\lambda), \quad U_n \overset{\text{i.i.d.}}{=} \exp(\lambda K), \quad n\in\mathbb{N}. \tag{A.28}$$

By the renewal reward theorem, we get

$$\lim_{l\to\infty}\frac{1}{l}\int_0^l \mathbb{1}_{\{R_s=1\}}\,\mathrm{d}s = \tfrac{K^{-1}}{1+K^{-1}} \quad \mathbb{P}_{(i,\alpha)} \text{ a.s.}, \tag{A.29}$$

and, by Cramer's theorem, deviations of the left-hand quantity away from the limiting constant are exponentially costly in l. Because $\epsilon\in\big(0,\frac{1}{1+K^{-1}}\big)$, using (A.27) and the above we can find constants $A_\epsilon, A'_\epsilon, C_\epsilon>0$ such that

$$\begin{aligned}\mathbb{P}_{(i,\alpha)}(S(t)\le\epsilon t) &= \mathbb{P}_{(i,\alpha)}\big(\tfrac{D(t)}{t} > (1-\epsilon)\big) \\ &\le \mathbb{P}_{(i,\alpha)}\Big[\tfrac{1}{t}\int_0^t \mathbb{1}_{\{R_s=1\}}\,\mathrm{d}s > 1-\epsilon\Big] \\ &\le A_\epsilon\, e^{-A'_\epsilon t} \\ &\le C_\epsilon\, a_{\epsilon t}(0,0).\end{aligned} \tag{A.30}$$

Inserting (A.30) into (A.26), we get that

$$\mathbb{P}^*_\zeta(\mathcal{E}(t)) \le [1+C_\epsilon]a_{\epsilon t}(0,0) \tag{A.31}$$

for any $t \geq 0$. Finally, noting that $\mathbb{P}^*_\eta(\mathcal{E}(t)) = \mathbb{E}^*_\eta\big[a_{S(t)+S'(t)}(0,0)\big] \geq a_{2t}(0,0)$, we get

$$\frac{\mathbb{P}^*_\zeta(\mathcal{E}(t))}{\mathbb{P}^*_\eta(\mathcal{E}(t))} \leq [1 + C_\epsilon]\frac{a_{\epsilon t}(0,0)}{a_{2t}(0,0)} \quad \leq C(\epsilon), \tag{A.32}$$

where $C(\epsilon) := [1 + C_\epsilon] \sup_{s \geq 0} \frac{a_{\epsilon s}(0,0)}{a_{2s}(0,0)} < \infty$ where we use the symmetry of $a(\cdot\,,\cdot)$ and Assumption 3.B. The proof of (A.22) is therefore complete. $\square$

APPENDIX A

APPENDIX B

Appendix: Chapter 4

§B.1 Proof of stationarity of environment process and law of large numbers

In this section we prove Theorem 4.3.6 stated in Section 4.3 of Chapter 4. As an application, we also prove a strong law of large numbers which is stated later in Theorem B.1.1.

§B.1.1 Stationary distribution of environment process

Proof of Theorem 4.3.6. We first prove part (1) of the theorem. To prove stationarity of W under $\mathbb{Q}$, it suffices to show that, for any bounded measurable $f \in \mathcal{F}_b(\Omega_{\mathfrak{K}})$,

$$\int_{\Omega_{\mathfrak{K}}} \mathfrak{R} f(\mathfrak{e}, \alpha)\, \mathrm{d}\mathbb{Q}(\mathfrak{e}, \alpha) = \int_{\Omega_{\mathfrak{K}}} f(\mathfrak{e}, \alpha)\, \mathrm{d}\mathbb{Q}(\mathfrak{e}, \alpha), \tag{B.1}$$

where $\mathfrak{R}$ is the Markov kernel operator given in (4.38). Let $\theta := \frac{1}{1+\bar{\mathbb{E}}[M_0/N_0]}$ and q_s, $\hat{p}(\cdot)$, $\omega = (\omega(k))_{k\in\mathbb{Z}^d}$ be as in (4.28), where ω is the only parameter that depends on the realisation of the environment $\mathfrak{e}$. In terms of these parameters, from (4.39) we get that

$$\int_{\Omega_{\mathfrak{K}}} g(\mathfrak{e}, \alpha)\, \mathrm{d}\mathbb{Q}(\mathfrak{e}, \alpha) = \theta \int_{\Omega_{\mathfrak{K}}} \left[g(\mathfrak{e}, 1) + \tfrac{q_s}{\omega(0)}\, g(\mathfrak{e}, 0)\right] \mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \tag{B.2}$$

for any $g \in \mathcal{F}_b(\Omega_{\mathfrak{K}})$. Thus, taking $g = \mathfrak{R} f$ in the above equation, we have

$$\int_{\Omega_{\mathfrak{K}}} \mathfrak{R} f(\mathfrak{e}, \alpha)\, \mathrm{d}\mathbb{Q}(\mathfrak{e}, \alpha) = \theta \int_{\mathcal{E}_{\mathfrak{K}}} \left[\mathfrak{R} f(\mathfrak{e}, 1) + \tfrac{q_s}{\omega(0)}\, \mathfrak{R} f(\mathfrak{e}, 0)\right] \mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) = \theta(I_1 + I_2), \tag{B.3}$$

where $I_1 := \int_{\mathcal{E}_{\mathfrak{K}}} \mathfrak{R} f(\mathfrak{e}, 1)\, \mathrm{d}\bar{\mathbb{P}}(\mathfrak{e})$ and $I_2 := \int_{\mathcal{E}_{\mathfrak{K}}} \frac{q_s}{\omega(0)}\, \mathfrak{R} f(\mathfrak{e}, 0)\, \mathrm{d}\bar{\mathbb{P}}(\mathfrak{e})$.

Let us compute I_1 and I_2 using (4.38):

$$
\begin{aligned}
I_1 &= q_s \int_{\mathcal{E}_{\mathfrak{K}}} f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + (1-q_s)\int_{\mathcal{E}_{\mathfrak{K}}} \Big[\sum_{j\in\mathbb{Z}^d} \hat{p}(j) f(T_j\mathfrak{e},1)\Big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \\
&= q_s \int_{\mathcal{E}_{\mathfrak{K}}} f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + (1-q_s)\sum_{j\in\mathbb{Z}^d} \hat{p}(j)\int_{\mathcal{E}_{\mathfrak{K}}} f(T_j\mathfrak{e},1)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \quad \text{(bounded convergence)} \\
&= q_s \int_{\mathcal{E}_{\mathfrak{K}}} f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + (1-q_s)\sum_{j\in\mathbb{Z}^d} \hat{p}(j)\int_{\mathcal{E}_{\mathfrak{K}}} f(\mathfrak{e},1)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \quad \text{(translation-invariance of } \bar{\mathbb{P}}) \\
&= q_s \int_{\mathcal{E}_{\mathfrak{K}}} f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + (1-q_s)\int_{\mathcal{E}_{\mathfrak{K}}} f(\mathfrak{e},1)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}), \quad \Big(\text{using} \sum_{j\in\mathbb{Z}^d} \hat{p}(j) = 1\Big).
\end{aligned}
\tag{B.4}
$$

Similarly,

$$
I_2 = \int_{\mathcal{E}_{\mathfrak{K}}} \tfrac{q_s}{\omega(0)}\,\mathfrak{R} f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) = q_s\int_{\mathcal{E}_{\mathfrak{K}}} [f(\mathfrak{e},1)-f(\mathfrak{e},0)]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + \int_{\mathcal{E}_{\mathfrak{K}}} \tfrac{q_s}{\omega(0)} f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}).
\tag{B.5}
$$

Finally, adding (B.4)–(B.5) and using (B.2)–(B.3), we get

$$
\begin{aligned}
\int_{\Omega_{\mathfrak{K}}} \mathfrak{R} f(\mathfrak{e},\alpha)\,\mathrm{d}\mathbb{Q}(\mathfrak{e},\alpha) = \theta(I_1+I_2) &= \theta\int_{\Omega_{\mathfrak{K}}} \big[f(\mathfrak{e},1) + \tfrac{q_s}{\omega(0)} f(\mathfrak{e},0)\big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \\
&= \int_{\Omega_{\mathfrak{K}}} f(\mathfrak{e},\alpha)\,\mathrm{d}\mathbb{Q}(\mathfrak{e},\alpha),
\end{aligned}
\tag{B.6}
$$

which proves the claim.

Next we proceed to prove ergodicity of W under the stationary law $\mathbb{Q}$. It suffices to show (see e.g. [93]) that if $A \in \Sigma\otimes 2^{\{0,1\}}$ satisfies $\mathfrak{R}\mathbb{1}_A = \mathbb{1}_A$ $\mathbb{Q}$ a.s., then $\mathbb{Q}(A)\in\{0,1\}$. Thus, let us fix a measurable $A\subseteq\Omega_{\mathfrak{K}}$ such that

$$
\mathfrak{R}\mathbb{1}_A(\mathfrak{e},\alpha) = \mathbb{1}_A(\mathfrak{e},\alpha), \qquad \text{for all } (\mathfrak{e},\alpha)\in B,
\tag{B.7}
$$

where $B\subseteq\Omega_{\mathfrak{K}}$ is measurable with $\mathbb{Q}(B)=1$. Define $A_0, A_1\in\Sigma$ as

$$
A_0 := \{\mathfrak{e} : (\mathfrak{e},0)\in A\},\ A_1 := \{\mathfrak{e} : (\mathfrak{e},1)\in A\}.
\tag{B.8}
$$

By Lemma 4.3.8, we can find $B'\in\Sigma$ such that

$$
\bar{\mathbb{P}}(B') = 1,\ B'\times\{0,1\}\subseteq B.
\tag{B.9}
$$

Using (4.38), (B.7) and (B.9), we get that, for all $\mathfrak{e}\in B'$,

$$
\begin{aligned}
&q_s\mathbb{1}_A(\mathfrak{e},0) + (1-q_s)\sum_{j\in\mathbb{Z}^d}\hat{p}(j)\mathbb{1}_A(T_j\mathfrak{e},1) = \mathbb{1}_A(\mathfrak{e},1), \\
&\omega(0)\mathbb{1}_A(\mathfrak{e},0) + (1-\omega(0))\mathbb{1}_A(\mathfrak{e},1) = \mathbb{1}_A(\mathfrak{e},0),
\end{aligned}
\tag{B.10}
$$

where ω is defined in terms of $\mathfrak{e}$ as in (4.28). In terms of A_0, A_1 given in (B.8), for all $\mathfrak{e}\in B'$,

$$
\begin{aligned}
&q_s\mathbb{1}_{A_0}(\mathfrak{e}) + (1-q_s)\sum_{j\in\mathbb{Z}^d}\hat{p}(j)\mathbb{1}_{A_1}(T_j\mathfrak{e}) = \mathbb{1}_{A_1}(\mathfrak{e}), \\
&(1-\omega(0))\mathbb{1}_{A_1}(\mathfrak{e}) = (1-\omega(0))\mathbb{1}_{A_0}(\mathfrak{e}).
\end{aligned}
\tag{B.11}
$$

By ellipticity of $\mathfrak{e} \in B'$, we have $\omega(0) < 1$, and so the second part of the above equation implies that

$$\mathbb{1}_{A_1}(\mathfrak{e}) = \mathbb{1}_{A_0}(\mathfrak{e}), \quad \mathfrak{e} \in B'. \tag{B.12}$$

Integrating the above w.r.t. $\bar{\mathbb{P}}$ over B' and using (B.9), we also have

$$\bar{\mathbb{P}}(A_0) = \bar{\mathbb{P}}(A_1). \tag{B.13}$$

Note that if we show $\bar{\mathbb{P}}(A_1) \in \{0,1\}$, then it follows from (B.13) that $\mathbb{Q}(A) \in \{0,1\}$. Indeed, from (4.39) we see that

$$\mathbb{Q}(A) = \theta\Big[\bar{\mathbb{P}}(A_1) + \int_{A_0} \tfrac{M_0}{N_0}\, \mathrm{d}\bar{\mathbb{P}}\{(N_k, M_k)_{k\in\mathbb{Z}^d}\}\Big], \tag{B.14}$$

where $\theta := \frac{1}{1+\bar{\mathbb{E}}[M_0/N_0]}$. Therefore, if $\bar{\mathbb{P}}(A_1) = \bar{\mathbb{P}}(A_0) = 1$, then

$$\mathbb{Q}(A) = \theta(1 + \bar{\mathbb{E}}[M_0/N_0]) = 1. \tag{B.15}$$

Similarly, if $\bar{\mathbb{P}}(A_1) = \bar{\mathbb{P}}(A_0) = 0$, then by (B.14), trivially $\mathbb{Q}(A) = 0$. We prove $\bar{\mathbb{P}}(A_1) \in \{0,1\}$ by using ergodicity of $\bar{\mathbb{P}}$. To this purpose, let us note that (B.12), combined with the first part of (B.11) and the fact $q_s < 1$, implies

$$\sum_{j\in\mathbb{Z}^d} \hat{p}(j)\mathbb{1}_{A_1}(T_j\mathfrak{e}) = \mathbb{1}_{A_1}(\mathfrak{e}), \quad \mathfrak{e} \in B'. \tag{B.16}$$

Define the translation invariant set $B_{\text{inv}} := \bigcap_{j\in\mathbb{Z}^d} T_j^{-1}(B')$. By translation invariance of $\bar{\mathbb{P}}$ we see that $\bar{\mathbb{P}}(B_{\text{inv}}) = \bar{\mathbb{P}}(B') = 1$. Also, (B.16) holds for all $\mathfrak{e} \in B_{\text{inv}}$. Let us fix $\mathfrak{e} \in B_{\text{inv}}$. By translation invariance of B_{inv}, we see that $T_i\mathfrak{e} \in B_{\text{inv}}$ for any $i \in \mathbb{Z}^d$ and so, using (B.16), we get

$$\sum_{j\in\mathbb{Z}^d} \hat{p}(j)\mathbb{1}_{A_1}(T_jT_i\mathfrak{e}) = \mathbb{1}_{A_1}(T_i\mathfrak{e}) \implies \sum_{j\in\mathbb{Z}^d} \hat{p}(j-i)\mathbb{1}_{A_1}(T_j\mathfrak{e}) = \mathbb{1}_{A_1}(T_i\mathfrak{e}). \tag{B.17}$$

In particular, the map $i \mapsto \mathbb{1}_{A_1}(T_i\mathfrak{e})$ is harmonic for $\hat{p}(\cdot)$. Finally, because of the irreducibility of the migration kernel $a(\cdot\,,\cdot)$ (see Assumption 2.A), we can apply the Choquet-Deny theorem to the $\hat{p}$-harmonic function $i \mapsto \mathbb{1}_{A_1}(T_i\mathfrak{e})$ to conclude that

$$\mathbb{1}_{A_1}(T_i\mathfrak{e}) = \mathbb{1}_{A_1}(\mathfrak{e}), \quad \forall i \in \mathbb{Z}^d. \tag{B.18}$$

In other words, $B_{\text{inv}} \cap A_1$ is a translation invariant subset of $\mathcal{E}_{\mathfrak{K}}$, and so ergodicity of $\bar{\mathbb{P}}$ implies $\bar{\mathbb{P}}(B_{\text{inv}} \cap A_1) \in \{0,1\}$. But $\bar{\mathbb{P}}(B_{\text{inv}} \cap A_1) = \bar{\mathbb{P}}(A_1)$ because $\bar{\mathbb{P}}(B_{\text{inv}}) = 1$. This concludes the proof of ergodicity of W w.r.t. the law $\mathbb{Q}$.

It remains to prove reversibility of $\mathbb{Q}$ under condition (2) in Assumption 4.A. It is enough to prove that, for $f, g \in \mathcal{F}_b(\Omega_{\mathfrak{K}})$,

$$\int_{\Omega_{\mathfrak{K}}} g\,\mathfrak{R}f\,\mathrm{d}\mathbb{Q} = \int_{\Omega_{\mathfrak{K}}} f\,\mathfrak{R}g\,\mathrm{d}\mathbb{Q}. \tag{B.19}$$

Using (4.38), we get

$$\int_{\Omega_{\mathfrak{K}}} g\,\mathfrak{R}f\,\mathrm{d}\mathbb{Q} = \tfrac{1}{1+\bar{\mathbb{E}}[M_0/N_0]}[I_1(f,g) + I_2(f,g)], \tag{B.20}$$

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where

$$I_1(f,g) = q_s \int_{\mathcal{E}_{\mathfrak{K}}} g(\mathfrak{e},1)f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + (1-q_s)\int_{\mathcal{E}_{\mathfrak{K}}} \Big[\sum_{j\in\mathbb{Z}^d} \hat{p}(j)g(\mathfrak{e},1)f(T_j\mathfrak{e},1)\Big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}),$$
$$I_2(f,g) = q_s \int_{\mathcal{E}_{\mathfrak{K}}} g(\mathfrak{e},0)[f(\mathfrak{e},1)-f(\mathfrak{e},0)]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) + \int_{\mathcal{E}_{\mathfrak{K}}} \tfrac{q_s}{\omega(0)}\, g(\mathfrak{e},0)f(\mathfrak{e},0)\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}). \tag{B.21}$$

Note that, by condition (2) in Assumption 4.A, we have $\hat{p}(k) = \hat{p}(-k)$ for all $k \in \mathbb{Z}^d$, and so by translation invariance of $\bar{\mathbb{P}}$ the second term in $I_1(f,g)$ remains unchanged if we interchange f and g. Indeed,

$$\begin{aligned}\int_{\mathcal{E}_{\mathfrak{K}}} \Big[\sum_{j\in\mathbb{Z}^d}\hat{p}(j)g(\mathfrak{e},1)f(T_j\mathfrak{e},1)\Big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) &= \int_{\mathcal{E}_{\mathfrak{K}}} \Big[\sum_{j\in\mathbb{Z}^d}\hat{p}(j)g(T_{-j}\mathfrak{e},1)f(\mathfrak{e},1)\Big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \\ &= \int_{\mathcal{E}_{\mathfrak{K}}} \Big[\sum_{j\in\mathbb{Z}^d}\hat{p}(-j)g(T_j\mathfrak{e},1)f(\mathfrak{e},1)\Big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}) \\ &= \int_{\mathcal{E}_{\mathfrak{K}}} \Big[\sum_{j\in\mathbb{Z}^d}\hat{p}(j)g(T_j\mathfrak{e},1)f(\mathfrak{e},1)\Big]\,\mathrm{d}\bar{\mathbb{P}}(\mathfrak{e}), \quad \text{(by symmetry of } \hat{p}(\cdot)).\end{aligned} \tag{B.22}$$

Thus, using (B.21) and the above, we see that $I_1(f,g)+I_2(f,g) = I_1(g,f)+I_2(g,f)$, which combined with (B.20) proves the claim in (B.19). □

§B.1.2 An application: strong law of large numbers

As pointed out earlier in Remark 4.3.7, part (1) of Theorem 4.3.6 holds in any dimension $d \geq 1$, even when the migration kernel $\hat{p}(\cdot)$ (see (4.28)) is not symmetric. An interesting application of this theorem is the strong law of large numbers stated below.

Theorem B.1.1 (Strong law of large numbers). *Let $\widehat{\Theta}^{\mathfrak{e}} = (X_n^{\mathfrak{e}}, \alpha_n^{\mathfrak{e}})_{n\in\mathbb{N}_0}$ be the subordinate Markov chain evolving in environment $\mathfrak{e}$ with law $\widehat{P}^{\mathfrak{e}}_{(0,\alpha)}$ (see Definition* 4.3.1*), and let $\bar{\mathbb{P}}$ be the translation-invariant, ergodic field as in Assumption* 4.B. *Assume that the migration kernel $\hat{p}(\cdot)$ (see* (4.28)*) has finite range and mean*

$$v := \sum_{j\in\mathbb{Z}^d} j\,\hat{p}(j). \tag{B.23}$$

Then, for $\bar{\mathbb{P}}$ almost every realisation of $\mathfrak{e}$ and $\alpha \in \{0,1\}$,

$$\lim_{n\to\infty} \frac{X_n^{\mathfrak{e}}}{n} = \frac{1-q_s}{1+\rho}\, v \quad \widehat{P}^{\mathfrak{e}}_{(0,\alpha)} \ a.s., \tag{B.24}$$

where $\rho := \bar{\mathbb{E}}\big[\frac{M_0}{N_0}\big]$ and q_s is as in (4.28).

Recall that $X_n^{\mathfrak{e}}$ denotes the location in $\mathbb{Z}^d$ at time n of a particle that evolves according to the subordinate Markov chain $\widehat{\Theta}^{\mathfrak{e}}$ in environment $\mathfrak{e}$. Therefore, the intuitive meaning of the above result is that the particle on average spends a $\frac{1}{1+\rho}$ fraction of its time in the active state, and since it migrates only while being active with probability $1-q_s$, the overall velocity is scaled by the factor $\frac{1-q_s}{1+\rho}$.

Remark B.1.2 (Transference of law of large numbers). Using Theorem B.1.1, Lemma 4.3.4 and the elementary renewal theorem, we can *transfer* the law of large numbers on $\widehat{\Theta}^{\mathfrak{e}}$ to the continuous-time process $\Theta^{\mathfrak{e}} = (x_t^{\mathfrak{e}}, \alpha_t^{\mathfrak{e}})_{t\geq 0}$ (see Definition 4.2.3) and obtain, for $\bar{\mathbb{P}}$-a.s. every realisation of $\mathfrak{e}$ and $\alpha \in \{0,1\}$,

$$\lim_{t\to\infty} \tfrac{x_t^{\mathfrak{e}}}{t} = \tfrac{1}{1+\rho} \sum_{j\in\mathbb{Z}^d} j\, a(0,j), \quad P^{\mathfrak{e}}_{(0,\alpha)}\text{-a.s.} \tag{B.25}$$

We conclude this section with the proof of the above theorem. The proof is based on an application of the classical Birkhoff pointwise ergodic theorem combined with the Azuma inequality for martingales having bounded increments.

Proof of Theorem B.1.1. Following the standard route as taken in [22, Lecture 1], we start by defining a (d-dimensional) Martingale $M^{\mathfrak{e}} := (M_n^{\mathfrak{e}})_{n\in\mathbb{N}}$ constructed from the "local drift" of a particle moving in an environment $\mathfrak{e} \in \mathcal{E}_{\mathfrak{K}}$ according to the subordinate Markov chain $\widehat{\Theta}^{\mathfrak{e}} = (X_n^{\mathfrak{e}}, \alpha_n^{\mathfrak{e}})_{n\in\mathbb{N}_0}$ with law $\widehat{P}^{\mathfrak{e}}_{(0,\alpha)}$. With this aim, let us fix $\mathfrak{e} \in \mathcal{E}_{\mathfrak{K}}$, $\alpha \in \{0,1\}$ and set $M_0^{\mathfrak{e}} := X_0^{\mathfrak{e}}$. For $n \in \mathbb{N}$, define

$$M_n^{\mathfrak{e}} := X_n^{\mathfrak{e}} - (1-q_s)v \sum_{l=0}^{n-1} \alpha_n^{\mathfrak{e}}. \tag{B.26}$$

We show that $M^{\mathfrak{e}}$ is a martingale (viewed component-wise) under the law $\widehat{P}^{\mathfrak{e}}_{(0,\alpha)}$ w.r.t. the natural filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ of the subordinate Markov chain $\widehat{\Theta}^{\mathfrak{e}}$. Indeed,

$$\begin{aligned}
\widehat{E}^{\mathfrak{e}}_{(0,\alpha)}\big[M_{n+1}^{\mathfrak{e}} \mid \mathcal{F}_n\big] &= \widehat{E}^{\mathfrak{e}}_{(0,\alpha)}\Big[X_{n+1}^{\mathfrak{e}} - (1-q_s)v\sum_{l=0}^{n}\alpha_l^{\mathfrak{e}} \,\Big|\, \mathcal{F}_n\Big] \\
&= \widehat{E}^{\mathfrak{e}}_{(0,\alpha)}\big[X_{n+1}^{\mathfrak{e}} \mid \mathcal{F}_n\big] - (1-q_s)v\sum_{l=0}^{n}\alpha_l^{\mathfrak{e}} \\
&= \widehat{E}^{\mathfrak{e}}_{(X_n^{\mathfrak{e}},\alpha_n^{\mathfrak{e}})}\big[X_1^{\mathfrak{e}}\big] - (1-q_s)v\sum_{l=0}^{n}\alpha_l^{\mathfrak{e}} \quad \text{(by Markov property)},
\end{aligned} \tag{B.27}$$

where the last equality holds $\widehat{P}^{\mathfrak{e}}_{(0,\alpha)}$-a.s. Finally, using (4.27), we obtain from the above that

$$\begin{aligned}
\widehat{E}^{\mathfrak{e}}_{(0,\alpha)}\big[M_{n+1}^{\mathfrak{e}} \mid \mathcal{F}_n\big] &= \alpha_n^{\mathfrak{e}}\Big[q_s X_n^{\mathfrak{e}} + (1-q_s)\sum_{j\in\mathbb{Z}^d}\hat{p}(j)(X_n^{\mathfrak{e}}+j)\Big] \\
&\qquad + (1-\alpha_n^{\mathfrak{e}})X_n^{\mathfrak{e}} - (1-q_s)v\sum_{l=0}^{n}\alpha_l^{\mathfrak{e}} \\
&= \alpha_n^{\mathfrak{e}}[q_s\, X_n^{\mathfrak{e}} + (1-q_s)(X_n^{\mathfrak{e}}+v)] + (1-\alpha_n^{\mathfrak{e}})X_n^{\mathfrak{e}} - (1-q_s)v\sum_{l=0}^{n}\alpha_l^{\mathfrak{e}} \\
&= X_n^{\mathfrak{e}} - (1-q_s)v\sum_{l=0}^{n-1}\alpha_n^{\mathfrak{e}} = M_n^{\mathfrak{e}}.
\end{aligned} \tag{B.28}$$

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Thus, $M^{\mathfrak{e}}$ is a martingale under $\widehat{P}^{\mathfrak{e}}_{(0,\alpha)}$, and it has bounded increments by virtue of the finite-range assumption on the migration kernel $\hat{p}(\cdot)$. A standard application of Azuma inequality and the Borel Cantelli lemma yield (see e.g., [22, Lecture 1, page 14])

$$\lim_{n\to\infty} \frac{M^{\mathfrak{e}}_n}{n} = 0 \quad \widehat{P}^{\mathfrak{e}}_{(0,\alpha)}\text{-a.s.} \tag{B.29}$$

Observe from above and (B.26), the proof will be complete if we prove the following: for $\bar{\mathbb{P}}$-a.s. every $\mathfrak{e} \in \mathcal{E}_{\mathfrak{K}}$ and any $\alpha \in \{0,1\}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \alpha^{\mathfrak{e}}_l = \frac{1}{1+\bar{\mathbb{E}}[M_0/N_0]}, \quad \widehat{P}^{\mathfrak{e}}_{(0,\alpha)} \text{ a.s.} \tag{B.30}$$

This is a consequence of Birkhoff's pointwise ergodic theorem. Indeed, let $\tilde{P}_{\mathbb{Q}}$ be the canonical law defined on the path space $\Omega_{\mathfrak{K}}^{\mathbb{N}_0}$ of the auxiliary environment process $W = (\mathfrak{e}_n, \alpha_n)_{n\in\mathbb{N}_0}$ (recall Definition 4.3.5) with initial distribution $\mathbb{Q}$ (see (4.39)). In other words,

$$\tilde{P}_{\mathbb{Q}}(W \in \cdot\,) := \int_{\Omega_{\mathfrak{K}}} \widehat{P}^{\mathfrak{e}}_{(0,\alpha)}(W \in \cdot\,)\, \mathrm{d}\mathbb{Q}(\mathfrak{e},\alpha). \tag{B.31}$$

Let $S : \Omega_{\mathfrak{K}}^{\mathbb{N}_0} \to \Omega_{\mathfrak{K}}^{\mathbb{N}_0}$ be the natural left-shift operator and $f : \Omega_{\mathfrak{K}}^{\mathbb{N}_0} \to \{0,1\}$ be the function

$$(\mathfrak{a}_n, \beta_n)_{n\in\mathbb{N}_0} \mapsto \beta_0, \quad (\mathfrak{a}_n, \beta_n)_{n\in\mathbb{N}_0} \in \Omega_{\mathfrak{K}}^{\mathbb{N}_0}. \tag{B.32}$$

Since, by part (1) of Theorem 4.3.6, $\mathbb{Q}$ is a stationary and ergodic distribution of W, we see that S is a measure-preserving ergodic transformation of the dynamical system $(\Omega_{\mathfrak{K}}^{\mathbb{N}_0}, \tilde{P}_{\mathbb{Q}})$. Applying Birkhoff's pointwise ergodic theorem to the bounded function f, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} f \circ S^l = \int_{\Omega_{\mathfrak{K}}^{\mathbb{N}_0}} f \,\mathrm{d}\tilde{P}_{\mathbb{Q}} = \int_{\Omega_{\mathfrak{K}}} \widehat{E}^{\mathfrak{e}}_{(0,\alpha)}\big[f((\mathfrak{e}_n,\alpha_n)_{n\in\mathbb{N}_0})\big]\, \mathrm{d}\mathbb{Q}(\mathfrak{e},\alpha), \tag{B.33}$$

where the first equality holds $\tilde{P}_{\mathbb{Q}}$ almost everywhere and the second equality follows from (B.31). We compute the left and the right side of (B.33) using the definition of f and (4.39), to obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \beta_l = \int_{\Omega_{\mathfrak{K}}} \alpha \,\mathrm{d}\mathbb{Q}(\mathfrak{e},\alpha) = \frac{1}{1+\bar{\mathbb{E}}[M_0/N_0]} \tag{B.34}$$

for $\tilde{P}_{\mathbb{Q}}$ almost every $(\mathfrak{b}_l, \beta_l)_{l\in\mathbb{N}_0}$. However, (B.31) combined with the above implies that (B.30) holds for all $(\mathfrak{e},\alpha) \in A$ for some $A \in \Sigma$ such that $\mathbb{Q}(A) = 1$. The result now follows from the equivalence of $\mathbb{Q}$ and $\bar{\mathbb{P}}$ stated in Lemma 4.3.8. $\square$

§B.2 Fundamental theorem of Markov chains

In this section we provide the proof of Proposition 4.3.10 stated in Section 4.3.2 of Chapter 4. Let us recall the statement of the proposition for convenience of the reader.

Proposition B.2.1. *Let $(\Omega, \Sigma, \mathbb{Q})$ be a probability space, where the σ-field Σ is countably generated. Let $W := (W_n)_{n\in\mathbb{N}_0}$ be a Markov chain on the state space Ω, and assume that $\mathbb{Q}$ is a reversible and ergodic stationary distribution for W. If -1 is not an eigenvalue of the Markov kernel operator $\mathfrak{R} : L_\infty(\Omega, \mathbb{Q}) \to L_\infty(\Omega, \mathbb{Q})$ associated to W, then for every bounded measurable function $f \in \mathcal{F}_b(\Omega)$ and $\mathbb{Q}$-a.s. every $w \in \Omega$,*

$$\lim_{n\to\infty} \mathbb{E}_w[f(W_n)] = \int_\Omega f \, \mathrm{d}\mathbb{Q}, \tag{B.35}$$

where the expectation on the left is taken w.r.t. the law of W started at w.

Proof. If $Q(\cdot\,,\cdot)$ denotes the transition kernel of W, the action of the Markov operator $\mathfrak{R}$ on $f \in L_\infty(\Omega, \mathbb{Q})$ is well-defined and is given by

$$\mathfrak{R}f(w) := \int_\Omega f(y)\, Q(w, \mathrm{d}y). \tag{B.36}$$

In fact, since $\mathbb{Q}$ is invariant for W, the same definition extends $\mathfrak{R}$ in a canonical way to a positive contraction operator on $L_p(\Omega, \mathbb{Q})$ for any $p \geq 1$. Furthermore, by reversibility of $\mathbb{Q}$, the operator $\mathfrak{R}$ becomes self-adjoint on $L_2(\Omega, \mathbb{Q})$ as well. Let $f \in \mathcal{F}_b(\Omega)$ be fixed. Because f is bounded, $\mathfrak{R}^n f \in \mathcal{F}_b(\Omega)$, and by the Markov property of W it follows that

$$\mathbb{E}_w[f(W_n)] = \mathfrak{R}^n f(w), \quad w \in \Omega,\, n \in \mathbb{N}. \tag{B.37}$$

Because $\mathfrak{R}$ is self-adjoint, we see that $\mathfrak{R}^2$ is a nonnegative-definite operator on the Hilbert space $L_2(\Omega, \mathbb{Q})$ equipped with the natural L_2 inner product, and this allows us to conclude from [147, Corollary 3] (see also [136, Theorem 1]) that there exist $\psi, \widehat{\psi} \in L_2(\Omega, \mathbb{Q})$ satisfying

$$\psi = \lim_{n\to\infty} \mathfrak{R}^{2n} f, \quad \widehat{\psi} = \lim_{n\to\infty} \mathfrak{R}^{2n+1} f, \tag{B.38}$$

where the convergence is in L_2-norm and $\mathbb{Q}$ almost everywhere. It is worth mentioning that the convergence in (B.38), which follows from [147, Corollary 3], essentially uses the classical Banach principle (see e.g., [6]) along with a maximal ergodic inequality. The convergence, in fact, holds for any function in $(L\log^+ L)(\Omega, \mathbb{Q})$. By the almost sure convergence of $\mathfrak{R}^{2n} f$ (resp. $\mathfrak{R}^{2n+1} f$) and the L_∞ contractivity of $\mathfrak{R}$, we see that $\psi, \widehat{\psi} \in L_\infty(\Omega, \mathbb{Q})$ as well. The L_2 contractivity of the linear operator $\mathfrak{R}$ also implies that,

$$\psi = \mathfrak{R}\widehat{\psi}, \quad \widehat{\psi} = \mathfrak{R}\psi \quad \mathbb{Q}\text{-a.s.}, \tag{B.39}$$

from which we get

$$\mathfrak{R}^2\psi = \psi,\ \mathfrak{R}^2\widehat{\psi} = \widehat{\psi}, \quad \mathbb{Q}\text{-a.s.} \tag{B.40}$$

We claim that if -1 is not an eigenvalue of $\mathfrak{R}$ as an operator on $L_\infty(\Omega, \mathbb{Q})$, then we must have

$$\psi = \widehat{\psi} = \int_\Omega f \, \mathrm{d}\mathbb{Q}, \quad \mathbb{Q}\text{-a.s.} \tag{B.41}$$

Note that (B.35) will follow once we prove (B.41). Indeed, (B.41) combined with (B.37)–(B.38) implies that, for $\mathbb{Q}$-a.s. $w \in \Omega$, both the odd and even subsequence of

$(\mathbb{E}_w[f(W_n)])_{n\in\mathbb{N}}$ converge to the same limit $\mathbb{Q}(f) := \int_\Omega f \, \mathrm{d}\mathbb{Q}$, which necessarily forces the convergence of $\mathbb{E}_w[f(W_n)]$ to $\mathbb{Q}(f)$ as $n \to \infty$.

To prove (B.41), it suffices to show that ψ and $\widehat{\psi}$ are constant $\mathbb{Q}$ a.s., because by the invariance of $\mathfrak{R}$ w.r.t. $\mathbb{Q}$ and bounded convergence we have

$$\int_\Omega \psi \, \mathrm{d}\mathbb{Q} = \lim_{n\to\infty} \int_\Omega \mathfrak{R}^{2n} f \, \mathrm{d}\mathbb{Q} = \int_\Omega f \, \mathrm{d}\mathbb{Q} = \lim_{n\to\infty} \int_\Omega \mathfrak{R}^{2n+1} f \, \mathrm{d}\mathbb{Q} = \int_\Omega \widehat{\psi} \, \mathrm{d}\mathbb{Q}. \tag{B.42}$$

We only prove the claim for ψ, as the same argument works for $\widehat{\psi}$. Let us set $g := \mathfrak{R}\psi - \psi$. From (B.40), we have

$$\mathfrak{R} g = -g, \quad \mathbb{Q}\text{-a.s.}, \tag{B.43}$$

and also $\|g\|_\infty \leq 2\|\psi\|_\infty < \infty$. Thus, $g \in L_\infty(\Omega, \mathbb{Q})$ is such that $\mathbb{Q}$-a.s. $\mathfrak{R} g = -g$, and hence by our assumption we must have $g = 0$ a.s. In other words, $\mathbb{Q}$-a.s. $\mathfrak{R}\psi - \psi = 0$ and therefore ergodicity of $\mathfrak{R}$ in $L_2(\Omega, \mathbb{Q})$, which is equivalent to the ergodicity of W under $\mathbb{Q}$, implies that ψ is necessarily a constant $\mathbb{Q}$ a.s. □