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Unextendible product operator basis

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ABSTRACT

Quantum nonlocality is associated with the local indistinguishability of orthogonal states. Unextendible product basis (UPB), a widely used tool in quantum information, exhibits nonlocality, which is the powerful resource for quantum information processing. In this work, we extend the definitions of nonlocality and genuine nonlocality from states to operators. We also extend UPB to the notions of unextendible product operator basis, unextendible product unitary operator basis (UPUOB), and strongly UPUOB. We construct their examples and show the nonlocality of some strongly UPUOBs under local operations and classical communications. We study the phenomenon of these operators acting on quantum states. As an application, we distinguish the two-dimensional strongly UPUOB, which only consumes three ebits of entanglement. Our results imply that such UPUOBs exhibit nonlocality as UPBs, and the distinguishability of them requires entanglement resources.

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I. INTRODUCTION

If a set of orthogonal quantum states cannot be distinguished by local operations and classical communications (LOCC), it is locally indistinguishable. Such a set exhibits the phenomenon of quantum nonlocality.^{1,2} In many previous studies, authors considered the problem of distinguishing a given set of states by LOCC.^{3–6} For example, unextendible product bases (UPBs) are locally indistinguishable.^{7,8} As far as we know, little is known about the local indistinguishability of unitary operators.^{9–11} The locality of n -partite unitary operators means that they are the tensor products of unitary operators locally acting on subsystems, that is, $U_{A_1} \otimes \cdots \otimes U_{A_j} \otimes \cdots \otimes U_{A_n}$, where the j th system performs the operator U_{A_j} .^{12,13} Hence, there is no entanglement between distributed parties. The local unitary operators can be perfectly implemented by LOCC. However, there remains whether the discrimination of n -partite ($n \geq 2$) product unitary operators can be done locally or in some m -partitions ($m \in \{2, \dots, n-1\}$) unknown. Our work builds towards filling this gap.

Quantum nonlocality is a non-additive resource that can be activated.¹⁴ Recently, quantum nonlocality was extended to genuine quantum nonlocality^{15,16} and strong nonlocality.² They play a key role in state discrimination and multipartite secret sharing.¹⁷ It has been proved that any discrete finite-dimensional unitary operators can be constructed in the laboratory using optical devices.¹⁸ A series of them are the bases of

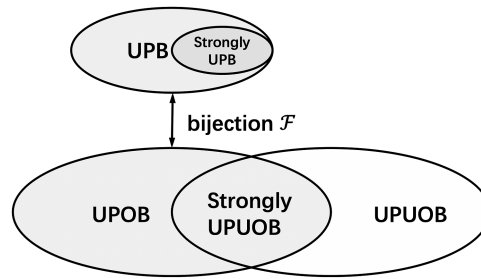


FIG. 1. Using the bijection \mathcal{F} from vectors to matrices in (2), one can obtain a UPOB from a UPB and a strongly UPUOB from strongly UPB, respectively. Furthermore, the strongly UPUOBs are the intersections of UPUOBs and UPOBs.

quantum computation or algorithms.^{19–21} The product unitary operators, such as product Pauli operators, are useful in quantum information masking under noise.²² For maximally entangled states (MESs), the local distinguishability of them has been well studied.^{23–25} It has been proved that any l mutually orthogonal generalized $d \times d$ Bell states are locally distinguishable if d is a prime and $l(l-1) < 2d$.²⁵ We extend the distinguishability of orthogonal MESs to the nonlocality of unitary operators in terms of MESs. That is, if some orthogonal MESs can be distinguished by LOCC, are the tensor powers of these orthogonal MESs distinguishable by LOCC? Our work gives the answer for the tensor of bipartite two-level MESs.

In this work, motivated by the nonlocality and genuine nonlocality of orthogonal product (OP) states in Definition 2, we define the counter part of unitary operators in Definition 3. In Definition 5, extended from UPBs, we propose the notion of unextendible product operator basis (UPOB), unextendible product unitary operator basis (UPUOB), and strongly UPUOB. The connection among them is shown in Fig. 1. Then, we give the properties of UPOB and UPUOB in Lemmas 6–8. By using Lemma 9, we prove that arbitrary tensor powers of bipartite UPOBs are again UPOBs. Next, a bijection from vectors to matrices is proposed to construct UPOBs from UPBs. In Proposition 12, we show the existence of UPOBs. We also show that there exists an n -qubit strongly UPUOB for $n \geq 2$ in Theorem 13. In Example 15 and Theorem 16, we prove that some bipartite UPUOBs are not strongly UPUOBs. In Theorem 17, we show that the two-qubit strongly UPUOB is genuinely nonlocal in terms of the two-level MESs. Finally, in Theorem 18, we prove that the strongly UPUOB can be distinguished by using three ebits of entanglement. It can be used in multiparty secret sharing and unitary gates discrimination.

The remaining of this paper is organized as follows: In Sec. II, we introduce the preliminary knowledge used in this paper, such as the nonlocality, UPBs, and UPOBs. In Sec. III, we construct examples of UPOBs, UPUOBs, and strongly UPUOBs. Then, we study the nonlocality of UPUOBs and apply our results in Sec. IV. We conclude in Sec. V.

II. PRELIMINARIES

In this section, we introduce the notations and preliminary facts used in this paper. We may not normalize states and operators for simplicity. Let \mathbb{C}^d be the d -dimensional Hilbert space. Assume that $\{|i\rangle_A\}_{i=0}^{d_A-1}$ and $\{|j\rangle_B\}_{j=0}^{d_B-1}$ are computational bases in \mathcal{H}_A and \mathcal{H}_B , respectively, where $\dim(\mathcal{H}_A) = d_A$, $\dim(\mathcal{H}_B) = d_B$. For any linear operator \mathcal{M} from \mathcal{H}_A to \mathcal{H}_B , we assume that the $d_B \times d_A$ matrix M is the matrix representation of the operator \mathcal{M} under the computational bases $\{|i\rangle_A\}_{i=0}^{d_A-1}$ and $\{|j\rangle_B\}_{j=0}^{d_B-1}$. In general, we do not distinguish the operator \mathcal{M} and the matrix M . We shall use the notation $\mathbb{O}(\mathcal{H}_A, \mathcal{H}_B)$ for the space of linear operators from \mathcal{H}_A to \mathcal{H}_B and \mathbb{M}_{d_B, d_A} for the space of $d_B \times d_A$ matrices. Obviously, $\mathbb{O}(\mathcal{H}_A, \mathcal{H}_B) \cong \mathbb{M}_{d_B, d_A}$. We also do not distinguish the space $\mathbb{O}(\mathcal{H}_A, \mathcal{H}_B)$ and the space \mathbb{M}_{d_B, d_A} . In particular, we denote $\mathbb{O}(\mathcal{H}) = \mathbb{O}(\mathcal{H}, \mathcal{H})$. For any two matrices $M_1, M_2 \in \mathbb{O}(\mathcal{H}_A, \mathcal{H}_B)$, the inner product of M_1 and M_2 is $\text{Tr}(M_1^\dagger M_2)$. Furthermore, two subspaces $\mathcal{O}, \mathcal{O}^\perp$ of $\mathbb{O}(\mathcal{H}_1, \mathcal{H}_2)$ are said to be complementary if their direct sum gives the entire space, and every matrix in \mathcal{O} is orthogonal to every matrix in \mathcal{O}^\perp . For a product matrix $M = M_A \otimes M_B \in \mathbb{O}(\mathcal{H}_A) \otimes \mathbb{O}(\mathcal{H}_B)$, we shall refer to M_A as the A partition of M and M_B as the B partition of M . For a set \mathbb{U} , we denote the cardinality of it by $|\mathbb{U}|$. We say that the linearly independent operator set $\{U_i, i = 1, \dots, d^2\}$ is complete if it spans $\mathbb{O}(\mathcal{H})$ with $\dim(\mathcal{H}) = d$. Hence, it is a basis. Let $\omega_k = e^{\frac{2\pi i}{k}}$ be a primitive k th root of unity. We say that M and M' are equivalent modulo a global phase if and only if there is a real θ such that the matrix $M' = e^{i\theta}M$. The global phase means there is a real θ such that the matrix M becomes $e^{i\theta}M$. We denote the Pauli matrices as σ_α , where $\alpha = x, y, z$.

Next, we review the orthonormality and k -orthonormality of matrices. We say that s $d \times d$ matrices U_1, \dots, U_s are orthonormal if $\text{Tr}(U_j^\dagger U_k) = d\delta_{jk}$ for any j, k . Moreover, if U_1, \dots, U_s are n -partite tensor product unitary matrices of system A_1, \dots, A_n and they are orthonormal, then we refer to them as orthogonal product (OP) unitary operators of system A_1, \dots, A_n . To characterize them, we propose the definition of k -orthonormality.

Definition 1. Two n -partite OP unitary matrices $\otimes_{i=1}^n U_i$ and $\otimes_{i=1}^n U'_i$ are called k -orthonormal if U_k and U'_k are orthonormal for the smallest $k \in \{1, \dots, n\}$.

The remaining of this section is divided into two subsections. In Sec. II A, we introduce the definition of nonlocality and genuine nonlocality of states and operators in Definitions 2 and 3, respectively. In Sec. II B, we introduce the definition of the unextendible product operator basis (UPOB), unextendible product unitary operator basis (UPUOB), and strongly UPUOB in Definition 5. The connection among them is given in Fig. 1. We give some constructions of UPOBs and UPUOBs in Lemmas 6–9 and Theorem 10.

A. Nonlocality

A set of orthogonal quantum states is locally indistinguishable if it cannot be distinguished by LOCC. Nonlocality and genuine nonlocality play a key role in various quantum information tasks, such as quantum data hiding^{26,27} and secret sharing.²⁸ In the following, Definition 2 reviews the nonlocality and genuine nonlocality. They will be extended to operators in Definition 3. This will be used in Theorem 17 to show that the two-qubit strongly UPUOB is genuinely nonlocal in terms of the two-level MES.

Definition 2. Consider the quantum systems A_1, \dots, A_n in the n -partite Hilbert space $\otimes_{i=1}^n \mathbb{C}^{d_i}$.

- (i) A set of n -partite OP states $\mathbb{B}_{\text{nl}} \equiv \{|\psi_j\rangle = \otimes_{i=1}^n |\alpha_j^i\rangle_{A_i} \mid j = 1, 2, \dots, N \leq \prod_{i=1}^n d_i\} \subset \otimes_{i=1}^n \mathbb{C}^{d_i}$ is called nonlocal if the states in \mathbb{B}_{nl} are locally indistinguishable.
- (ii) A set of n -partite OP states $\mathbb{B}_{\text{gnl}} \equiv \{|\psi_j\rangle = \otimes_{i=1}^n |\alpha_j^i\rangle_{A_i} \mid j = 1, 2, \dots, N \leq \prod_{i=1}^n d_i\} \subset \otimes_{i=1}^n \mathbb{C}^{d_i}$ is called genuinely nonlocal if the states in \mathbb{B}_{gnl} are locally indistinguishable for every bipartition of the subsystems.

To extend the definition, we define the nonlocality and genuine nonlocality of unitary operators. Most of the related results are about the nonlocality of quantum states, and little is known about the operators. Hence, we introduce the nonlocality of operators and study the phenomenon of unitary operators in terms of MESs.

Definition 3. For $n \geq 2$, consider the quantum systems A_1, \dots, A_n in the n -partite Hilbert space $\otimes_{i=1}^n \mathbb{C}^{d_i}$ and B_1, \dots, B_n in the same Hilbert space. Let

$$\left\{ |a_k\rangle = U_k \otimes I_{B_1, \dots, B_n} (|\psi\rangle_{A_1 B_1} \otimes \dots \otimes |\psi\rangle_{A_n B_n}) \mid k = 1, \dots, N \leq \prod_{i=1}^n d_i^2 \right\}, \quad (1)$$

where U_k are n -partite unitary operators of system A_1, \dots, A_n . Let $\mathbb{U} \equiv \{U_k \mid k = 1, 2, \dots, N\}$.

- (i) If the set of states $\{|a_k\rangle\}_{k=1}^N$ is nonlocal, then the set of unitary operators \mathbb{U} is called nonlocal in terms of $|\psi\rangle$.
- (ii) If the set of states $\{|a_k\rangle\}_{k=1}^N$ is genuinely nonlocal, then the set of unitary operators \mathbb{U} is called genuinely nonlocal in terms of $|\psi\rangle$.
- (iii) If the states $\{|a_k\rangle\}_{k=1}^N$ satisfy the conditions in (i) and (ii) for any bipartite state $|\psi\rangle$, respectively, then the set of unitary operators \mathbb{U} in (i) and (ii) is called nonlocal and genuinely nonlocal, respectively.

Evidently, the genuine nonlocality of operators ensures the nonlocality of operators. Hence, we have extended the nonlocality from states to operators, which will be used in the discrimination of operators in Theorems 17 and 18.

B. UPOB and UPUOB

An unextendible product basis (UPB) is a basis of orthonormal product states in the subspace $\mathcal{H} \subseteq \otimes_{i=1}^n \mathcal{H}_i$ whose complementary subspace $\mathcal{H}^\perp \subseteq \otimes_{i=1}^n \mathcal{H}_i$ does not contain any product state.²⁹ It has been proven that any UPB is locally indistinguishable.⁸ The following is an example of UPB in $\mathbb{C}^4 \otimes \mathbb{C}^4$.

Example 4. The basis $\{|\psi_j\rangle\}_{j=1}^{11}$ is a UPB in $\mathbb{C}^4 \otimes \mathbb{C}^4$,³⁰ where

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle(|0\rangle - |1\rangle)), & |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |3\rangle)|2\rangle, \\ |\psi_3\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega_3|1\rangle + \omega_3^2|2\rangle)|3\rangle, & |\psi_4\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \omega_3^2|1\rangle + \omega_3|2\rangle)|3\rangle, \\ |\psi_5\rangle &= \frac{1}{\sqrt{3}}(|1\rangle + \omega_3|2\rangle + \omega_3^2|3\rangle)|1\rangle, & |\psi_6\rangle &= \frac{1}{\sqrt{3}}(|1\rangle + \omega_3^2|2\rangle + \omega_3|3\rangle)|1\rangle, \\ |\psi_7\rangle &= \frac{1}{\sqrt{2}}|3\rangle(|0\rangle - |3\rangle), & |\psi_8\rangle &= \frac{1}{2}(|1\rangle + |2\rangle)(|0\rangle - |2\rangle), \\ |\psi_9\rangle &= \frac{1}{2}(|1\rangle - |2\rangle)(|0\rangle + |2\rangle), & |\psi_{10}\rangle &= \frac{1}{2}(|1\rangle - |2\rangle)(|0\rangle - |2\rangle), \\ |\psi_{11}\rangle &= \frac{1}{4}(|0\rangle + |1\rangle + |2\rangle + |3\rangle)(|0\rangle + |1\rangle + |2\rangle + |3\rangle). \end{aligned}$$

Next, we extend the notion of UPBs to UPOBs. We shall omit the trivial case of complete UPOBs, just like the case of UPBs.

Definition 5. Let the subspace $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$. Consider the operator subspace $\mathcal{O} \subset \mathbb{O}(\mathcal{H})$ and its complementary subspace \mathcal{O}^\perp .

- (i) A set of n -partite OP operators is a UPOB in \mathcal{O} whose complementary subspace \mathcal{O}^\perp does not contain any product operator.
- (ii) A set of n -partite OP unitary operators is a UPUOB in \mathcal{O} whose complementary subspace \mathcal{O}^\perp does not contain any product unitary operator.
- (iii) A set of n -partite OP unitary operators is a strongly UPUOB in \mathcal{O} whose complementary subspace \mathcal{O}^\perp does not contain any product operator.

One can see that the strongly UPUOB is a UPUOB, while the converse fails. We refer to authors to Fig. 1 for the connection among UPOBs, UPUOBs, and strongly UPUOBs. Specifically, we give an example of UPUOB on $\mathbb{C}^2 \otimes \mathbb{C}^3$, which is not a strongly UPUOB in Example 15.

When $n = 1$, the UPUOB reduces to the unextendible unitary operator (UUO) set $\{U_i\}_{i=1}^N$ on \mathbb{C}^d .³¹ The conditions of a UUO set can be expressed as (i) $U_i \in \mathbb{M}_{d,d}$'s are unitary; (ii) $\text{Tr}(U_i^\dagger U_j) = d\delta_{ij}$, $i, j = 1, \dots, N$; and (iii) if $\text{Tr}(U_i^\dagger U) = 0$ for all $i = 1, \dots, N$, then U is not a unitary matrix. The UUO set is equivalent to the unextendible maximally entangled bases (UMEBs).³¹ It is a set of orthonormal maximally entangled states $\{|\psi_i\rangle = \frac{1}{\sqrt{d}}(I \otimes U_i) \sum_{j=1}^d |j, j\rangle |i = 1, \dots, N\}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$ that consists of $N < d^2$ vectors, which have no additional maximally entangled state orthogonal to all of them. The following lemma has been investigated in Theorem 1 of Refs. 32 and 33. We restate it using UUO sets.

Lemma 6. If there is a UUO set of cardinality N on the space \mathbb{C}^d , then for any $q = 1, \dots, n$, there exists a UUO set of cardinality $q^2 d^2 - qd^2 + qN$ on the space \mathbb{C}^{qd} .

This lemma gives a method to construct a high dimensional UUO set from a lower dimensional UUO set. Furthermore, we can obtain the UPUOB by using the tensor product of the UUO sets and the complete unitary operator sets. In the following, we propose some properties of the tensor product of UPUOBs.

Lemma 7. (i) The tensor product of UUO sets is always a UPUOB.

- (ii) The tensor product of UPUOBs is still a UPUOB.
- (iii) The tensor product of strongly UPUOBs is still a strongly UPUOB.

Proof. (i) We prove it by contradiction. Let $\{U_i\}_{i=1}^{n_1}, \{U'_j\}_{j=1}^{n_2}$ be UUO sets on the spaces \mathbb{C}^{d_1} and \mathbb{C}^{d_2} , respectively, where $n_1 \leq d_1^2$, $n_2 \leq d_2^2$. Suppose there is a product unitary operator $U_{n_1+1} \otimes U'_{n_2+1} \in \mathbb{M}_{d_1, d_1} \otimes \mathbb{M}_{d_2, d_2}$ orthogonal to the set $\{U_i \otimes U'_j\}_{i,j=1}^{n_1, n_2}$. That is, $\text{Tr}((U_{n_1+1} \otimes U'_{n_2+1})^\dagger (U_i \otimes U'_j)) = 0$. Hence, we have $\text{Tr}(U_{n_1+1}^\dagger U_i) \text{Tr}((U'_{n_2+1})^\dagger U'_j) = 0$ for any i, j . It is a contradiction with the fact that $\{U_i\}_{i=1}^{n_1}, \{U'_j\}_{j=1}^{n_2}$ are UUO sets on the space \mathbb{C}^{d_1} and \mathbb{C}^{d_2} , respectively.

The above proof can be straightforwardly extended to multipartite UPUOBs. We have proven assertion (i).

- (ii) It suffices to prove that the tensor product of two bipartite UPUOBs is still a UPUOB. Let $\{A_{i_1}^{(1)} \otimes A_{i_2}^{(2)}\}_{i_1, i_2=1}^{n_1, m_2}$ and $\{B_{j_1}^{(1)} \otimes B_{j_2}^{(2)}\}_{j_1, j_2=1}^{m_1, m_2}$ be two bipartite UPUOBs on the space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and $\mathbb{C}^{d_3} \otimes \mathbb{C}^{d_4}$, respectively. Next, we prove by contradiction that $\{A_{i_1}^{(1)} \otimes A_{i_2}^{(2)} \otimes B_{j_1}^{(1)} \otimes B_{j_2}^{(2)}\}$ is still a UPUOB on $\otimes_{k=1}^4 \mathbb{C}^{d_k}$. Suppose there exists a product unitary operator $A_{n_1+1}^{(1)} \otimes A_{n_2+1}^{(2)} \otimes B_{m_1+1}^{(1)} \otimes B_{m_2+1}^{(2)} \in \otimes_{k=1}^4 \mathbb{M}_{d_k, d_k}$ orthogonal to the set $\{A_{i_1}^{(1)} \otimes A_{i_2}^{(2)} \otimes B_{j_1}^{(1)} \otimes B_{j_2}^{(2)}\}$. Since $\{A_{i_1}^{(1)} \otimes A_{i_2}^{(2)}\}_{i_1, i_2=1}^{n_1, m_2}$ is a bipartite UPUOB, we have $\text{Tr}((A_{n_1+1}^{(1)} \otimes A_{n_2+1}^{(2)})^\dagger (A_{i_1}^{(1)} \otimes A_{i_2}^{(2)})) \neq 0$. Then, $B_{m_1+1}^{(1)} \otimes B_{m_2+1}^{(2)}$ must be orthonormal to $B_{j_1}^{(1)} \otimes B_{j_2}^{(2)}$. It is a contradiction with the fact that $\{B_{j_1}^{(1)} \otimes B_{j_2}^{(2)}\}_{j_1, j_2=1}^{m_1, m_2}$ is a bipartite UPUOB. Thus, $\{A_{i_1}^{(1)} \otimes A_{i_2}^{(2)} \otimes B_{j_1}^{(1)} \otimes B_{j_2}^{(2)}\}$ is a UPUOB on the space $\otimes_{k=1}^4 \mathbb{C}^{d_k}$.
- (iii) One can prove the claim using the idea of (ii). This completes the proof. □

We can construct UPUOBs based on Lemma 7 by using UUO sets^{32,34} and complete unitary operators sets. Using Lemmas 6 and 7 (i), one can straightforwardly obtain the following lemma:

Lemma 8. If there are UUO sets \mathbb{U}_i of cardinality N_i on the space \mathbb{C}^{d_i} , $i = 1, \dots, k$, then for any $q_i = 1, \dots, n_i$, there exists a UPUOB of cardinality $\prod_{i=1}^k (q_i^2 d_i^2 - q_i d_i^2 + q_i N_i)$, which is the tensor product of \mathbb{U}_i on the space $\otimes_{i=1}^k \mathbb{C}^{q_i d_i}$.

Next, we prove that arbitrary tensor products of bipartite UPOBs are still bipartite UPOBs in Theorem 10. Before Theorem 10, we propose a preliminary fact on local dimensions extended from Lemma 1 in Ref. 29. The proof of the fact uses the idea of the same lemma, and we present it here for completeness.

Lemma 9. Let $\mathbb{S} = \{A_j = \otimes_{i=1}^m A_{i,j} : j = 1, \dots, n\}$ be an orthogonal product operator basis spanning a subspace of $\otimes_{i=1}^m \mathbb{O}(\mathcal{H}_i)$ with $\dim \mathcal{H}_i = d_i$. Assume that P is a partition of \mathbb{S} into m disjoint subsets: $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \cdots \cup \mathbb{S}_m$. Let $\mathcal{A}_i = \text{span}\{A_{i,j} : A_j \in \mathbb{S}_i\}$ and $r_i = \dim \mathcal{A}_i$ be the local dimension of subset \mathbb{S}_i of the i -th party. Then, \mathbb{S} is extendible if and only if there exists a partition P such that for all $i = 1, \dots, m$, the local dimension r_i of the i -th subset is less than d_i .

Proof. By definition, \mathbb{S} is extendible if and only if there exists a product operator W orthogonal to \mathbb{S} . Equivalently, a partition can be found such that W is orthogonal to all the operators in \mathbb{S}_1 for party 1, all the operators in \mathbb{S}_2 for party 2, and so on through \mathbb{S}_m . This can be done if each of the sets \mathbb{S}_i has local dimension r_i less than the dimension d_i of the i th party's Hilbert space. Conversely, if there exists at least one of the sets \mathbb{S}_i having full local dimension r_i equal to d_i , then there is no way to choose W orthogonal to \mathbb{S} . Thus, the original set \mathbb{S} is unextendible. This completes the proof. \square

If the A_j 's are vectors, then one can see that Lemma 9 reduces to Lemma 1 in Ref. 29. Due to the bijection $|i\rangle\langle j| \leftrightarrow |i, j\rangle$ between matrices and vectors, every UPOB in $\mathbb{M}_{a,b} \otimes \mathbb{M}_{c,d}$ one-to-one corresponds to a bipartite UPB in $\mathbb{C}^{ab} \otimes \mathbb{C}^{cd}$. Hence, the following observation can be obtained from Theorem 8 of Ref. 7 on the version of UPBs, though a straightforward proof would require the use of Lemma 9.

Theorem 10. Let two bipartite UPOBs be $S_1 = \{A_i\}_{i=1}^{l_1}$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and $S_2 = \{A_i\}_{i=1}^{l_2}$ on $\mathbb{C}^{d_3} \otimes \mathbb{C}^{d_4}$. The tensor product operator basis $\{A_i \otimes A_j\}_{i,j=1}^{l_1, l_2}$ is a bipartite UPOB on $\mathbb{C}^{d_1 d_3} \otimes \mathbb{C}^{d_2 d_4}$.

This theorem has the consequence that arbitrary tensor powers of bipartite UPOBs are again UPOBs. Furthermore, a generalization of this theorem for multipartite UPOBs still holds.

III. CONSTRUCTION OF UPOBs, UPUOBs, AND STRONGLY UPUOBs

In this section, we construct UPOBs and strongly UPUOBs. A bijection from vectors to matrices is proposed to construct UPOBs from UPBs. We give an example of UPOB on $\mathbb{C}^2 \otimes \mathbb{C}^2$ constructed from a 4×4 UPB using this bijection in Example 11. Furthermore, Proposition 12 shows the existence of some size of UPOBs. We also show that there exists a strongly UPUOB on $(\mathbb{C}^2)^{\otimes n}$ for $n \geq 2$ in Theorem 13. In Theorem 14, we give a construction of the $(n + 2)$ -partite UPUOB on the space $(\mathbb{C}^2)^{\otimes 2} \otimes (\otimes_{i=1}^n \mathbb{C}^{d_i})$ with $n \geq 0$. In Example 15 and Theorem 16, we show that a set of bipartite UPUOB is not strongly UPUOBs.

To begin with, we define a bijection \mathcal{F} from the set of d -dimensional vectors $|a^{(j)}\rangle := (a_1^{(j)}, a_2^{(j)}, \dots, a_d^{(j)})^T$ to the matrices $[c_{p,q}^{(j)}] \in \mathbb{M}_{m,n}$, $j = 1, \dots, N$ as follows:

$$\mathcal{F} : |a^{(j)}\rangle \mapsto [c_{p,q}^{(j)}]. \tag{2}$$

We choose the set

$$\mathcal{A} = \{f_1, f_2, \dots, f_d\} \subseteq \mathcal{S} = \{(p, q), \quad p = 1, \dots, m, \quad q = 1, \dots, n\}. \tag{3}$$

Let $a_1^{(j)} = c_{f_1}^{(j)}, a_2^{(j)} = c_{f_2}^{(j)}, \dots, a_d^{(j)} = c_{f_d}^{(j)}$, and other $c_{p,q}^{(j)}$ be zero entries, where $\max\{m, n\} \leq d \leq mn$. With the bijection \mathcal{F} in (2), we can obtain the matrices $M^{(1)}, M^{(2)} \in \mathbb{M}_{m,n}$ from the d -dimensional vectors $|a^{(1)}\rangle, |a^{(2)}\rangle$, respectively. One can verify that $M^{(1)}, M^{(2)}$ are orthonormal if and only if $|a^{(1)}\rangle, |a^{(2)}\rangle$ are orthonormal.

One can construct a UPOB $\{\otimes_{i=1}^k M_{m_i, n_i}^{(j)}\}$ from a k -partite UPB $\{\otimes_{i=1}^k |a_i^{(j)}\rangle\}$ in $\otimes_{i=1}^k \mathbb{C}^{d_i}$ with cardinality N using the bijection \mathcal{F} for each party, where $j = 1, \dots, N$ and $d_i = m_i n_i$ for $m_i, n_i > 1$. For the k -partite UPB $\{\otimes_{i=1}^k |a_i^{(j)}\rangle\}$ in $\otimes_{i=1}^k \mathbb{C}^{d_i}$, it has been proved that $N \leq \prod_{i=1}^k d_i - 4$ in Ref. 35. Thus, we also know that the bound of the cardinality of multipartite UPOB is $\prod_{i=1}^k d_i - 4$ using this bijection \mathcal{F} . There are also some results on the bound of bipartite UPB in terms of Ramsey number. For example, the cardinality of a UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is smaller than the Ramsey number $R(3, 3) = 6$.⁷ However, the relationship between the cardinality of multipartite UPOB and Ramsey number remains unknown. Furthermore, we say a UPB $\{\otimes_{i=1}^k |a_i^{(j)}\rangle\}$ in $\otimes_{i=1}^k \mathbb{C}^{d_i}$ is a strongly UPB if $\sum_{x=0}^{d_i-1} |a_i^{(j)}\rangle_{i, d_i+x+t} \overline{\langle a_i^{(j)} |}_{i, x+s} = \delta_{s,t}$, where $\delta_{s,t} = 1$ if $s = t$ and 0 otherwise, $s, t = 0, 1, \dots, d_i - 1$. The strongly UPUOB can be obtained from the strongly UPB using the bijection \mathcal{F} for each party.

In the following, we give an example of UPOB on $\mathbb{C}^2 \otimes \mathbb{C}^2$ from a 4×4 UPB.

Example 11. Consider the UPB in Example 4. Since $d_i = 4 = 2 \times 2$, we can construct UPOB on $\mathbb{C}^2 \otimes \mathbb{C}^2$ using the bijection \mathcal{F} . For both parties, using the set $\mathcal{A} = \{f_1 = (1, 1), f_2 = (1, 2), f_3 = (2, 1), f_4 = (2, 2)\}$ in (3), we can obtain the UPOB $\{M_j, j = 1, \dots, 11\}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$, where

$$\begin{aligned}
 M_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, & M_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
 M_3 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega_3 \\ \omega_3^2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, & M_4 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega_3 \\ \omega_3^2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
 M_5 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 \\ \omega_3 & \omega_3^2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & M_6 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 \\ \omega_3^2 & \omega_3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
 M_7 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & M_8 &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \\
 M_9 &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, & M_{10} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \\
 M_{11} &= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
 \end{aligned}$$

Using the above idea, we extend Example 11 to the UPOB on $\mathbb{C}^m \otimes \mathbb{C}^n$ from a UPB in $\mathbb{C}^{m^2} \otimes \mathbb{C}^{n^2}$.

Proposition 12. (i) There exists a UPOB of size $d_1 d_2 - 4 \lfloor \frac{d_1 - 1}{2} \rfloor$ in $\mathbb{M}_{m_1, n_1} \otimes \mathbb{M}_{m_2, n_2}$, where $3 \leq d_1 \leq d_2$ and $d_1 = m_1 n_1, d_2 = m_2 n_2$.

(ii) There exists a UPOB of size $d_1 d_2 - k$ in $\mathbb{M}_{m_1, n_1} \otimes \mathbb{M}_{m_2, n_2}$ for $4 \leq k \leq 2d_1 - 1$, where $4 \leq d_1 \leq d_2$ and $d_1 = m_1 n_1, d_2 = m_2 n_2$.

Proof. (i) Since there exists a UPB of size $d_1 d_2 - 4 \lfloor \frac{d_1 - 1}{2} \rfloor$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ for $3 \leq d_1 \leq d_2$,³⁰ one can obtain a UPOB in $\mathbb{M}_{m_1, n_1} \otimes \mathbb{M}_{m_2, n_2}$ of size $d_1 d_2 - 4 \lfloor \frac{d_1 - 1}{2} \rfloor$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ by using the bijection \mathcal{F} in (2) in every partition, where $d_1 = m_1 n_1, d_2 = m_2 n_2$.

(ii) It has been proved that there exists a UPB of size $d_1 d_2 - k$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ for $4 \leq k \leq 2d_1 - 1$ and $4 \leq d_1 \leq d_2$.³⁰ Thus, one can construct a UPOB in $\mathbb{M}_{m_1, n_1} \otimes \mathbb{M}_{m_2, n_2}$ of size $d_1 d_2 - 4 \lfloor \frac{d_1 - 1}{2} \rfloor$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ by using the bijection \mathcal{F} in (2) in every partition, where $d_1 = m_1 n_1, d_2 = m_2 n_2$. This completes the proof. \square

Since there do not exist qubit UO sets,³¹ we cannot obtain UPUOB on the space $(\mathbb{C}^2)^{\otimes n}$ using tensor products of qubit UO sets according to Lemma 7. In the following, we give another method to construct strongly UPUOBs on the space $(\mathbb{C}^2)^{\otimes n}$.

Theorem 13. There exists a strongly UPUOB on the space $(\mathbb{C}^2)^{\otimes n}$ for any $n \geq 2$.

Proof. It has been proved that $\mathbb{U}_2 := \{U_i\}_{i=1}^{12}$ is a UO set on the space \mathbb{C}^4 ,³¹ where

$$\begin{aligned}
 U_1 &= \frac{1}{\sqrt{2}} \sigma_x \otimes (\sigma_x - \sigma_y), \\
 U_2 &= \frac{1}{\sqrt{2}} (\sigma_x - \sigma_y) \otimes \sigma_z, \\
 U_3 &= \frac{1}{\sqrt{2}} \sigma_z \otimes (-\sigma_y + \sigma_z), \\
 U_4 &= \frac{1}{\sqrt{2}} (-\sigma_y + \sigma_z) \otimes \sigma_x, \\
 U_5 &= \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \otimes (\sigma_x + \sigma_y + \sigma_z),
 \end{aligned} \tag{4}$$

and

$$U_6, \dots, U_{12} = I \otimes I, I \otimes \sigma_\alpha, \sigma_\beta \otimes I, \tag{5}$$

and $\alpha, \beta = x, y, z$. We prove that \mathbb{U}_2 is a strongly UPUOB of systems A, B on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Assume that there exists a two-qubit product operator $W \otimes X$ on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$, which is orthonormal to the set \mathbb{U}_2 . Since $W \otimes X$ is orthonormal to the operators in (5), we obtain that W, X must be the linear combination of σ_α and $\alpha = x, y, z$. Furthermore, because $W \otimes X$ is orthonormal to the operators in (4), one can obtain that $W \otimes X$ is k -orthonormal to three of U_1, \dots, U_5 in (4), $k = 1$ or 2 . Note that the A -partition of any three of U_1, \dots, U_5 are linearly independent, and they are the linear combination of $\sigma_\alpha, \alpha = x, y, z$, and the B -partition of U_1, \dots, U_5 has the same property. Hence, we have $W = 0$ or $X = 0$. It is a contradiction with the definition of strongly UPUOBs. Thus, \mathbb{U}_2 is a two-qubit strongly UPUOB.

Next, we construct a strongly UPUOB on the space $(\mathbb{C}^2)^{\otimes n}$ by using \mathbb{U}_2 . Let $\mathbb{S} = \{\sigma_{\alpha_1} \otimes \dots \otimes \sigma_{\alpha_{n-2}} \otimes U_i\}$, where $\alpha_1, \dots, \alpha_{n-2} = 0, x, y, z$, the matrix $\sigma_0 = I$, and $U_i, i = 1, \dots, 12$ are UOs in \mathbb{U}_2 . Using Lemma 7 (iii), one can verify that \mathbb{S} is a strongly UPUOB with $|\mathbb{S}| = 3 \cdot 4^{n-1}$ on the space $(\mathbb{C}^2)^{\otimes n}$. We have proven the assertion. \square

Similar to the last paragraph of the Proof of Theorem 13, one can construct a UPUOB on the space $(\mathbb{C}^2)^{\otimes 2} \otimes (\otimes_{i=1}^n \mathbb{C}^{d_i})$.

Theorem 14. *There exists an $(n + 2)$ -partite UPUOB on the space $(\mathbb{C}^2)^{\otimes 2} \otimes (\otimes_{i=1}^n \mathbb{C}^{d_i})$, $n \geq 0$.*

Proof. If there does not exist a UO set on \mathbb{C}^{d_i} , then we can choose a complete operator basis for the $i + 2$ parties. Using the strongly UPUOB \mathbb{U}_2 in (4) and (5), Lemmas 6–8, one can prove this assertion. \square

The strongly UPUOBs must be UPUOBs on the space $(\mathbb{C}^2)^{\otimes n}$ according to the definitions. In the following, we give an example of UPUOB on $\mathbb{C}^2 \otimes \mathbb{C}^3$, which is not a strongly UPUOB.

Example 15. Let

$$\begin{aligned} U_{n,m}^\pm &= \xi_\pm \otimes U_{n,m} = \xi_\pm \otimes \sum_{k=0}^2 (\omega_3)^{kn} |k \oplus m\rangle \langle k|, n, m = 1, 2, 3, \\ U_s^\pm &= \eta_\pm \otimes U_s = \eta_\pm \otimes (I - (1 - e^{i\theta})|\psi_s\rangle \langle \psi_s|), \quad s = 1, \dots, 6, \end{aligned} \tag{6}$$

where $\omega_3 = e^{\frac{2\pi i}{3}}$, $\xi_\pm = \begin{bmatrix} 0 & 1 \\ \pm 1 & 0 \end{bmatrix}$, $\eta_\pm = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$, $k \oplus m$ denotes $(k + m) \bmod 3$, $\cos \theta = -\frac{7}{8}$, and

$$\begin{aligned} |\psi_{1,2}\rangle &= \frac{1}{\sqrt{1 + \phi^2}} (|0\rangle \pm \phi|1\rangle), \\ |\psi_{3,4}\rangle &= \frac{1}{\sqrt{1 + \phi^2}} (|1\rangle \pm \phi|2\rangle), \\ |\psi_{5,6}\rangle &= \frac{1}{\sqrt{1 + \phi^2}} (|2\rangle \pm \phi|0\rangle), \end{aligned}$$

$\phi = \frac{1 + \sqrt{5}}{2}$. This set $\{U_{n,m}^\pm, U_s^\pm\}_{n,m,s=1}^{3,3,6}$ is a UO set on the space \mathbb{C}^6 .³² It is also a UPUOB on the space $\mathbb{C}^2 \otimes \mathbb{C}^3$ because there do not exist a unitary operator in its complementary space orthogonal to the set. However, it is not a strongly UPUOB because of the claim that one can find a bipartite product operator $\tilde{U} = W \otimes X$ on the space $\mathbb{C}^2 \otimes \mathbb{C}^3$ orthogonal to the set $\{U_{n,m}^\pm, U_s^\pm\}_{n,m,s=1}^{3,3,6}$. The proof of this claim is as below.

Proof. Suppose that the operator $\tilde{U} = W \otimes X$ satisfies $\text{Tr}((\tilde{U})^\dagger U_{n,m}^\pm) = 0$ and $\text{Tr}((\tilde{U})^\dagger U_s^\pm) = 0$. Let the three matrix subspaces $\mathcal{V}_1 = \text{span}\{U_{n,m}^\pm\}$, $\mathcal{V}_2 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid A, B \in \mathbb{M}_{3,3} \right\}$, and $\mathcal{V}_3 = \text{span}\{U_{n,m}^\pm, U_s^\pm\}$ in (6). Then, we have $\dim(\mathcal{V}_1) = \dim(\mathcal{V}_2) = 18$ and $\dim(\mathcal{V}_3) = 30$.

Since $\text{Tr}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^\dagger \begin{bmatrix} 0 & U_{n,m} \\ \pm U_{n,m} & 0 \end{bmatrix}\right) = 0$, we can obtain $\mathcal{V}_1^\perp = \mathcal{V}_2$, where \mathcal{V}_1^\perp is the complementary space of \mathcal{V}_1 . Hence, we have $\mathcal{V}_3^\perp \subset \mathcal{V}_1^\perp = \mathcal{V}_2$.

Thus, $\tilde{U} = W \otimes X \in \mathcal{V}_3^\perp = \mathcal{V}_2$ and must be the form $\tilde{U} = W \otimes X = \begin{bmatrix} w_1X & w_2X \\ w_3X & w_4X \end{bmatrix} = \begin{bmatrix} w_1X & 0 \\ 0 & w_4X \end{bmatrix}$. As $\tilde{U} = W \otimes X$ is orthogonal to U_s^\pm in (6), we have

$\text{Tr}\left(\begin{bmatrix} w_1X & 0 \\ 0 & w_4X \end{bmatrix}^\dagger \begin{bmatrix} U_s & 0 \\ 0 & \pm U_s \end{bmatrix}\right) = 0$. We can obtain that $\text{Tr}(w_1^* X^\dagger U_s) \pm \text{Tr}(w_4^* X^\dagger U_s) = 0$. Hence, $\text{Tr}(w_1^* X^\dagger U_s) = \text{Tr}(w_4^* X^\dagger U_s) = 0$. Since U_s in (6) are symmetric matrices in the computational basis $|0\rangle, |1\rangle, |2\rangle$, we obtain that w_1X and w_4X are in the antisymmetric subspace represented by the computational basis $|0\rangle, |1\rangle, |2\rangle$. Thus, $(w_1X)^T = -w_1X, (w_4X)^T = -w_4X$. As $w_1w_4 \neq 0$, we have $X^T = -X$. That is X is an anti-symmetric matrix. Hence, we have constructed a bipartite product operator $\tilde{U} = W \otimes X$ orthogonal to the set $\{U_{n,m}^\pm, U_s^\pm\}_{n,m,s=1}^{3,3,6}$. This completes the proof. \square

In the following, we re-derive the construction by using the idea in Example 15. It has been proved that one can construct a UUO set on \mathbb{C}^{qd} from the UUO set on \mathbb{C}^d for any $q = 1, 2, \dots$.³² Let the order- q matrix

$$P_q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad W_q = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_q & \omega_q^2 & \cdots & \omega_q^{q-1} \\ 1 & \omega_q^2 & \omega_q^4 & \cdots & \omega_q^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_q^{q-1} & \omega_q^{2(q-1)} & \cdots & \omega_q^{(q-1)^2} \end{bmatrix},$$

where $\omega_q = e^{\frac{2\pi i}{q}}$. We denote $U_{n,m} = \sum_{k=0}^{d-1} (\omega_d)^{kn} |k \oplus m\rangle\langle k|$, $m, n = 0, 1, \dots, d-1$, and $k \oplus m = (k+m) \bmod d$. Let $\{U_t\}_{t=1}^{N < d^2}$ be a UUO set on \mathbb{C}^d . Then, we set

$$\begin{aligned} U_{n,m}^{(s,j)} &= (W_q^s P_q^j) \otimes U_{n,m}, \\ U_t^{(s)} &= W_q^s \otimes U_t, \end{aligned} \tag{7}$$

where $s = 0, 1, \dots, q-1, j = 1, 2, \dots, q-1, m, n = 0, 1, \dots, d-1$, and $t = 1, 2, \dots, N < d^2$. It has been proven that $\{U_{n,m}^{(s,j)}, U_t^{(s)}\}$ is a UUO set on \mathbb{C}^{qd} with cardinality $q^2 d^2 - qd^2 + qN$ constructed from the UUO set $\{U_t\}$ on \mathbb{C}^d .³² Similar to Example 15, we present the following observation:

Theorem 16. *If there exists a UUO set on \mathbb{C}^d , then the set $\{U_{n,m}^{(s,j)}, U_t^{(s)}\}$ in (7) is a UPUOB but not a strongly UPUOB on $\mathbb{C}^q \otimes \mathbb{C}^d$.*

Proof. Let $\{U_t\}_{t=1}^{N < d^2}$ be the UUO set on \mathbb{C}^d . Then, one can construct the UUO set on the space \mathbb{C}^{qd} ³² as shown in (7). This UUO set $\{U_{n,m}^{(s,j)}, U_t^{(s)}\}$ is a UPUOB on the space $\mathbb{C}^q \otimes \mathbb{C}^d$ because there does not exist a unitary operator in its complementary space, which is orthogonal to the set.³² However, it is not a strongly UPUOB because there exists a bipartite product operator $\hat{U}' = W'_q \otimes X'_d$ on the space $\mathbb{C}^q \otimes \mathbb{C}^d$, which is orthogonal to the set $\{U_{n,m}^{(s,j)}, U_t^{(s)}\}$. The proof is similar to the claim of Example 15. In particular, W'_q must be of the form $W'_q = \text{diag}(w'_1, \dots, w'_q)$ and X'_d must be an anti-symmetric matrix. \square

Thus, from the above Example 15 and Theorem 16, we can see the connection between UPUOBs and strongly UPUOBs. Similar to the application of UPBs, we ask whether we can distinguish UPOBs or strongly UPUOBs by LOCC. This will be discussed in Sec. IV.

IV. APPLICATION: NONLOCALITY OF UPUOB

In this section, we prove that the strongly UPUOB \mathbb{U}_2 in (4) and (5) is genuinely nonlocal in terms of the two-level maximally entangled state (MES). This is presented in Theorem 17. Furthermore, Theorem 18 shows that the strongly UPUOB \mathbb{U}_2 in (4) and (5) can be locally distinguished by using three ebits of entanglement. As shown in Fig. 2, it has an important application in multipartite secret sharing and the distinguishability of product unitary gates.

Let the d -level MES $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i, i\rangle$. We define $|a'_k\rangle$ as follows:

$$|a'_k\rangle = U_k \otimes I_{B_1, \dots, B_n} (|\psi_d\rangle_{A_1 B_1} \otimes \cdots \otimes |\psi_d\rangle_{A_n B_n}), \tag{8}$$

where U_k are n -partite UPUOs of system A_1, \dots, A_n in $\otimes_{i=1}^n \mathbb{C}^d$, $k = 1, \dots, N$, and N is the cardinality of the UPUOB $\{U_k\}$. One can verify that the states $|a'_k\rangle$'s are n -partite OP states. Next, we show that these states are indistinguishable under LOCC when U_k in (8) are the genuinely UPUO in (4) and (5).

Theorem 17. *The strongly UPUOB \mathbb{U}_2 in (4) and (5) is genuinely nonlocal in terms of the two-level MES.*

Proof. Let U_k in (8) be the genuinely UPUO in (4) and (5). According to Definition 3 (i), to show that the strongly UPUOB \mathbb{U}_2 is genuinely nonlocal in terms of the two-level MES $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we need to prove that the bipartite state $|a'_k\rangle$ in (8) are nonlocal in terms of the MES. Furthermore, it is impossible to distinguish more than d' MESs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ by LOCC.³⁶ Note that except for the $A_1 B_1 |A_2 B_2$ bipartition, other bipartitions are all MESs in $\mathbb{C}^2 \otimes \mathbb{C}^8$, such as $A_1 |B_1 A_2 B_2$ partition, or MESs in $\mathbb{C}^4 \otimes \mathbb{C}^4$, such as $A_1 A_2 |B_1 B_2$ partition. Since the number of UPUO in (4) and (5) is 12, which is larger than 8 and 4, we only need to consider the $A_1 B_1 |A_2 B_2$ bipartition for nonlocality. Next, we show that $|a'_k\rangle$ are locally indistinguishable in $A_1 B_1 |A_2 B_2$ bipartition, where

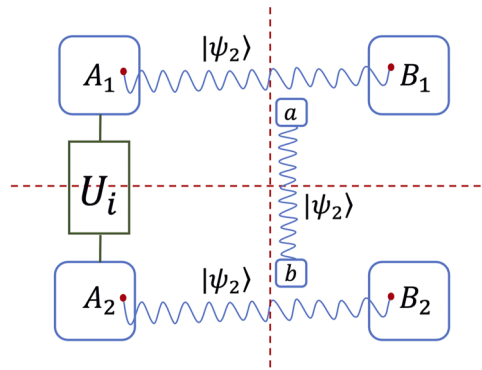


FIG. 2. A_1 and B_1 share a two-level MES $|\psi_2\rangle$ and A_2 and B_2 also share a two-level MES $|\psi_2\rangle$. The operators $\{U_i\}_{i=1}^{12}$ are the strongly UPUOB in (4) and (5). The availability of additional entanglement resources across the vertical and/or the horizontal dotted lines give rise to different sets of allowed operations performed by A_i and B_i , $i = 1, 2$. When we perform the operation U_i on A_1 and A_2 , then as shown in Theorem 18, it only requires a two-level MES $|\psi_2\rangle$ between the ancillary systems a and b to distinguish U_i , $i = 1, \dots, 12$. Thus, it requires three ebits of entanglement resources to distinguish U_i , $i = 1, \dots, 12$.

$$\begin{aligned}
 |a'_1\rangle &= (U_1 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) \\
 &= \frac{1}{2\sqrt{2}}(\sigma_x \otimes (\sigma_x - \sigma_y) \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_2\rangle &= (U_2 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) \\
 &= \frac{1}{2\sqrt{2}}((\sigma_x - \sigma_y) \otimes \sigma_z \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_3\rangle &= (U_3 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) \\
 &= \frac{1}{2\sqrt{2}}(\sigma_z \otimes (-\sigma_y + \sigma_z) \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_4\rangle &= (U_4 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) \\
 &= \frac{1}{2\sqrt{2}}((-\sigma_y + \sigma_z) \otimes \sigma_x \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_5\rangle &= (U_5 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) \\
 &= \frac{1}{6}((\sigma_x + \sigma_y + \sigma_z) \otimes (\sigma_x + \sigma_y + \sigma_z) \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_6\rangle &= (U_6 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(I_4 \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_7\rangle &= (U_7 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(I_2 \otimes \sigma_x \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_8\rangle &= (U_8 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(I_2 \otimes \sigma_y \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_9\rangle &= (U_9 \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(I_2 \otimes \sigma_z \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_{10}\rangle &= (U_{10} \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(\sigma_x \otimes I_2 \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_{11}\rangle &= (U_{11} \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(\sigma_y \otimes I_2 \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \\
 |a'_{12}\rangle &= (U_{12} \otimes I_4)(|\psi_2\rangle \otimes |\psi_2\rangle) = \frac{1}{2}(\sigma_z \otimes I_2 \otimes I_4)(|0, 0\rangle + |1, 1\rangle) \otimes (|0, 0\rangle + |1, 1\rangle), \tag{9}
 \end{aligned}$$

and U_1, U_2, \dots, U_{12} are the UPUO in (4) and (5).

Define a bijection from the basis $\{|p, q\rangle\}_{p,q=0}^1$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ to the basis in \mathbb{C}^4 as follows: $|0, 0\rangle \rightarrow |0\rangle, |0, 1\rangle \rightarrow |1\rangle, |1, 0\rangle \rightarrow |2\rangle, |1, 1\rangle \rightarrow |3\rangle$. Then, we can rewrite the set of states $\{|a'_k\rangle, k = 1, \dots, 12\}$ in $(\mathbb{C}^2)^{\otimes 4}$ as the set of bipartite product states $\{|b_k\rangle, k = 1, \dots, 12\}$ in $\mathbb{C}^4 \otimes \mathbb{C}^4$ in Eq. (12), that is,

$$\begin{aligned} |b_1\rangle &= \frac{1}{2\sqrt{2}}(|1\rangle + |2\rangle)((1+i)|1\rangle + (1-i)|2\rangle), \\ |b_2\rangle &= \frac{1}{2\sqrt{2}}((1+i)|1\rangle + (1-i)|2\rangle)(|0\rangle - |3\rangle), \\ |b_3\rangle &= \frac{1}{2\sqrt{2}}(|0\rangle - |3\rangle)(|0\rangle - i|2\rangle + i|1\rangle - |3\rangle), \\ |b_4\rangle &= \frac{1}{2\sqrt{2}}(|0\rangle + i|1\rangle - i|2\rangle - |3\rangle)(|1\rangle + |2\rangle), \\ |b_5\rangle &= \frac{1}{6}(|0\rangle + (1-i)|1\rangle + (1+i)|2\rangle - |3\rangle)(|0\rangle \\ &\quad + (1-i)|1\rangle + (1+i)|2\rangle - |3\rangle), \\ |b_6\rangle &= \frac{1}{2}(|0\rangle + |3\rangle)(|0\rangle + |3\rangle), \\ |b_7\rangle &= \frac{1}{2}(|0\rangle + |3\rangle)(|1\rangle + |2\rangle), \\ |b_8\rangle &= \frac{i}{2}(|0\rangle + |3\rangle)(|2\rangle - |1\rangle), \\ |b_9\rangle &= \frac{1}{2}(|0\rangle + |3\rangle)(|0\rangle - |3\rangle), \\ |b_{10}\rangle &= \frac{1}{2}(|1\rangle + |2\rangle)(|0\rangle + |3\rangle), \\ |b_{11}\rangle &= \frac{i}{2}(-|1\rangle + |2\rangle)(|0\rangle + |3\rangle), \\ |b_{12}\rangle &= \frac{1}{2}(|0\rangle - |3\rangle)(|0\rangle + |3\rangle). \end{aligned} \tag{10}$$

The set of orthonormal states, $\{\frac{1}{\sqrt{2}}(|0\rangle - |3\rangle), |1\rangle, |2\rangle\}$, spans a three-dimensional Hilbert space \mathcal{H}_3 . Consider a bijection from \mathcal{H}_3 to \mathbb{C}^3 as follows:

$$\begin{aligned} \frac{1}{\sqrt{2}}(|0\rangle - |3\rangle) \in \mathcal{H}_3 &\rightarrow |0\rangle \in \mathbb{C}^3, \\ |1\rangle \in \mathcal{H}_3 &\rightarrow |1\rangle \in \mathbb{C}^3, \\ |2\rangle \in \mathcal{H}_3 &\rightarrow |2\rangle \in \mathbb{C}^3. \end{aligned}$$

Then, $|b_1\rangle, |b_2\rangle, \dots, |b_5\rangle \in \mathcal{H}_3 \otimes \mathcal{H}_3$ can be transformed into $|c_1\rangle, |c_2\rangle, \dots, |c_5\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$, where

$$\begin{aligned} |c_1\rangle &= \frac{1}{2\sqrt{2}}(|1\rangle + |2\rangle)((1+i)|1\rangle + (1-i)|2\rangle), \\ |c_2\rangle &= \frac{1}{2}((1+i)|1\rangle + (1-i)|2\rangle)|0\rangle, \\ |c_3\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - i|2\rangle + i|1\rangle), \\ |c_4\rangle &= \frac{1}{2}(|0\rangle + i|1\rangle - i|2\rangle)(|1\rangle + |2\rangle), \\ |c_5\rangle &= \frac{1}{3}(|0\rangle + (1-i)|1\rangle + (1+i)|2\rangle)(|0\rangle + (1-i)|1\rangle + (1+i)|2\rangle). \end{aligned} \tag{11}$$

Evidently, each partition of $|c_1\rangle, |c_2\rangle, \dots, |c_5\rangle$ are composed by the vectors $|1\rangle + |2\rangle, (1+i)|1\rangle + (1-i)|2\rangle, |0\rangle, |0\rangle + i|1\rangle - i|2\rangle$ and $|0\rangle + (1-i)|1\rangle + (1+i)|2\rangle$. One can show that any three of them are linearly independent. Thus, using Lemma 1 in Ref. 7, we know that these five states $|c_1\rangle, |c_2\rangle, \dots, |c_5\rangle$ form a two-qutrit UPB. Thus, the set $\{|b_k\rangle, k = 1, \dots, 12\}$ in (12) is locally indistinguishable. Hence, the bipartite state $|a'_k\rangle$ in (8) are nonlocal in terms of the MES. Then, from Definition 3, one can obtain that the strongly UPUOB \mathbb{U}_2 is genuinely nonlocal in terms of the two level MES. This completes the proof. \square

Theorem 17 implies that one cannot distinguish the strongly UPUOB \mathbb{U}_2 in (4) and (5) in every bipartition using LOCC. Inspired by Theorem 17, we ask whether a strongly UPUOB or UPUOB on higher dimensional Hilbert space is nonlocal or genuinely nonlocal. In the following theorem, we show that the strongly UPUOB \mathbb{U}_2 in (4) and (5) can be distinguished by the protocol, which only consumes three ebits of entanglement resource.

Theorem 18. *The strongly UPUOB \mathbb{U}_2 in (4) and (5) can be distinguished by three ebits of entanglement.*

Proof. From Theorem 17, the strongly UPUOB \mathbb{U}_2 in (4) and (5) is genuinely nonlocal in terms of the two-level MES. To distinguish \mathbb{U}_2 , it suffices to distinguish the states $|a'_k\rangle$'s (9). First, a 2-level MES is distributed between A_1 and B_1 , and then, A_1 teleports his subsystem to B_1 by using the teleportation-based protocol.³⁷ Next, a 2-level MES is distributed between A_2 and B_2 , and then, A_2 also teleports his subsystem to B_2 by using the teleportation-based protocol. It means that we obtain the set of bipartite product states $\{|b_k\rangle, k = 1, \dots, 12\}$ in $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^4 \otimes \mathbb{C}^4$, where

$$\begin{aligned} |b_1\rangle &= \frac{1}{2\sqrt{2}}(|1\rangle + |2\rangle)((1+i)|1\rangle + (1-i)|2\rangle), \\ |b_2\rangle &= \frac{1}{2\sqrt{2}}((1+i)|1\rangle + (1-i)|2\rangle)(|0\rangle - |3\rangle), \\ |b_3\rangle &= \frac{1}{2\sqrt{2}}(|0\rangle - |3\rangle)(|0\rangle - i|2\rangle + i|1\rangle - |3\rangle), \\ |b_4\rangle &= \frac{1}{2\sqrt{2}}(|0\rangle + i|1\rangle - i|2\rangle - |3\rangle)(|1\rangle + |2\rangle), \\ |b_5\rangle &= \frac{1}{6}(|0\rangle + (1-i)|1\rangle + (1+i)|2\rangle - |3\rangle)(|0\rangle \\ &\quad + (1-i)|1\rangle + (1+i)|2\rangle - |3\rangle), \\ |b_6\rangle &= \frac{1}{2}(|0\rangle + |3\rangle)(|0\rangle + |3\rangle), \\ |b_7\rangle &= \frac{1}{2}(|0\rangle + |3\rangle)(|1\rangle + |2\rangle), \\ |b_8\rangle &= \frac{i}{2}(|0\rangle + |3\rangle)(|2\rangle - |1\rangle), \\ |b_9\rangle &= \frac{1}{2}(|0\rangle + |3\rangle)(|0\rangle - |3\rangle), \\ |b_{10}\rangle &= \frac{1}{2}(|1\rangle + |2\rangle)(|0\rangle + |3\rangle), \\ |b_{11}\rangle &= \frac{i}{2}(-|1\rangle + |2\rangle)(|0\rangle + |3\rangle), \\ |b_{12}\rangle &= \frac{1}{2}(|0\rangle - |3\rangle)(|0\rangle + |3\rangle), \end{aligned} \tag{12}$$

and $A = A_1B_1$ and $B = A_2B_2$. Here, it consumes two ebits of entanglement resource. Now, the discrimination protocol proceeds as follows:

- Step 1.* Alice performs the measurement $\{M_1 := \frac{1}{2}(|0\rangle + |3\rangle)(\langle 0| + \langle 3|)_A, \bar{M}_1 = I - M_1\}$. If M_1 clicks, Eq. (12) remains $\{|b_k\rangle\}_{k=6}^9$. These four states can be easily distinguished by Bob.³⁸ If \bar{M}_1 clicks, Eq. (12) remains $\{|b_k\rangle\}_{k=1}^5 \cup \{|b_k\rangle\}_{k=10}^{12}$.
- Step 2.* Bob performs the measurement $\{M_2 := \frac{1}{2}(|0\rangle + |3\rangle)(\langle 0| + \langle 3|)_B, \bar{M}_2 = I - M_2\}$. If M_2 clicks, $\{|b_k\rangle\}_{k=1}^5 \cup \{|b_k\rangle\}_{k=10}^{12}$ remains $\{|b_{10}\rangle, |b_{11}\rangle, |b_{12}\rangle\}$. These three states can be easily distinguished by Alice. If \bar{M}_2 clicks, $\{|b_k\rangle\}_{k=1}^5 \cup \{|b_k\rangle\}_{k=10}^{12}$ remains $\{|b_k\rangle\}_{k=1}^5$.
- Step 3.* In the Proof of Theorem 17, we have shown that $|b_1\rangle, \dots, |b_5\rangle$ is isomorphic to a two-qutrit UPB $|c_1\rangle, |c_2\rangle, \dots, |c_5\rangle$ in Eq. (11). It has been proved that one ebit of entanglement is sufficient to distinguish any UPB on $3 \otimes 3$ only by LOCC.⁵ That is, a 2-level MES is distributed between Alice and Bob. The initial state is

$$|\phi_k\rangle = |c_k\rangle_{AB} \otimes (|00\rangle_{ab} + |11\rangle_{ab}), \tag{13}$$

where a, b are the ancillary systems of Alice and Bob, respectively, $k = 1, 2, \dots, 5$. Thus, these five states $\{|b_k\rangle\}_{k=1}^5$ can be locally distinguished.

Hence, it requires three ebits of entanglement resource to distinguish the strongly UPUOB \mathbb{U}_2 in (4) and (5). □

Since a strongly UPUOB cannot be locally distinguished in every bipartition, a perfect entanglement-assisted discrimination with less entanglement resource is desirable. As an application of Theorem 18, Fig. 2 shows that it requires three ebits of entanglement resource to distinguish the strongly UPUOB \mathbb{U}_2 in (4) and (5), which was applied to A_1 and A_2 . Specifically, one two-level MES is distributed between

A_1 and B_1 , and one two-level MES is distributed between A_2 and B_2 . It consumes a two-level MES to distinguish the strongly UPUOB \mathbb{U}_2 in (4) and (5) between systems a and b , where a and b are the ancillary systems of Alice and Bob, respectively. Since the simplest type of product unitary gates are of the general form $U_i = P_i \otimes V_i$ acting on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where P_i 's are orthogonal unitary operators on \mathcal{H}_A and V_i 's are orthogonal unitary operators on \mathcal{H}_B . One can find that the strongly UPUOB \mathbb{U}_2 in (4) and (5) is one of the simplest product unitary operators, which are operational in experiments. Thus, the two-dimensional strongly UPUOB can be used in multipartite secret sharing.^{17,39} For more complicated product unitary gates discrimination and secret sharing scheme, we need to find multipartite strongly UPUOB in higher dimension.

V. CONCLUSIONS

Inspired by the notion of UPB, we have proposed the definitions of UPOB, UPUOB, strongly UPUOB as well as the nonlocality and genuinely nonlocality of unitary operators. We have presented the properties and examples of them. Further, we have investigated the nonlocality of the two-qubit strongly UPUOB in terms of MES. There are many interesting problems left. We do not know any construction of strongly UPUOB in more general cases. Are UPUOBs and strongly UPUOBs nonlocal or genuinely nonlocal in terms of MES? Can we find other bipartite state $|\psi\rangle$ rather than MES such that the states $|a_k\rangle$ in Eq. (1) are locally indistinguishable across any bipartition?

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Mengyao Hu: Investigation (equal); Methodology (equal); Writing – original draft (equal). **Lin Chen:** Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Supervision (equal); Writing – review & editing (equal). **Fei Shi:** Validation (equal); Visualization (equal); Writing – review & editing (equal). **Xiande Zhang:** Funding acquisition (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – review & editing (equal). **Jordi Tura:** Funding acquisition (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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