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Random walk in cooling random environment: Recurrence versus transience and mixed fluctuations

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Abstract. This is the third in a series of papers in which we consider one-dimensional Random Walk in Cooling Random Environment (RWCRE). The latter is obtained by starting from one-dimensional Random Walk in Random Environment (RWRE) and resampling the environment along a sequence of deterministic times, called *refreshing times*. In the present paper we explore two questions for general refreshing times. First, we investigate how the recurrence versus transience criterion known for RWRE changes for RWCRE. Second, we explore the fluctuations for RWCRE when RWRE is either recurrent or satisfies a classical central limit theorem. We show that the answer depends in a delicate way on the choice of the refreshing times. An overarching goal of our paper is to investigate how the behaviour of a random process with a rich correlation structure can be affected by resettings.

Résumé. Ceci est le troisième d'une série d'articles dans lesquels nous considérons une marche aléatoire unidimensionnelle dans un milieu aléatoire refroidissant (MAMAR). Ce processus est obtenu en partant d'une marche aléatoire unidimensionnelle dans un milieu aléatoire (MAMA) et en rafraîchissant l'environnement le long d'une séquence de temps déterministes, appelée *temps de rafraîchissement*. Dans le présent article, nous explorons deux questions pour des moments de rafraîchissement généraux. Tout d'abord, nous examinons comment le critère de récurrence connu pour MAMA change pour MAMAR. Deuxièmement, nous explorons les fluctuations de MAMAR lorsque MAMA est récurrent ou satisfait un théorème central limite classique. Nous montrons que la réponse dépend de manière subtile du choix des moments de rafraîchissement. Un objectif primordial de notre article est d'étudier comment le comportement d'un processus aléatoire avec une riche structure de corrélation peut être affecté par des rafraîchissements.

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Keywords: Random walk; Dynamic random environment; Refreshing times; Cooling regimes; Recurrence versus transience; Mixed fluctuations

1. Introduction, main results and discussion

1.1. *Background and outline*

Random Walk in Random Environment (RWRE) is a classical model for a particle moving in a non-homogeneous medium, consisting of a random walk with random transition probabilities, sampled at time zero from a given law. *Random Walk in Cooling Random Environment* (RWCRE) is a dynamic version of RWRE in which the environment is *fully resampled* along a sequence of deterministic times, called *refreshing times*.

RWRE exhibits *anomalous behaviour* due to the occurrence of *trapping* (i.e., the random walk spends a long time in local niches of the environment). One-dimensional RWRE is well understood (see [19]). Much less is known in higher dimensions, because the geometry of random walk paths is more complicated. Random walks in dynamic random environments (RWDRE) can be even more challenging. Over the last few decades, there has been significant progress in this area (see [1] for a survey). Often work concentrates on specific types of dynamics with good mixing properties that allow for the identification of scaling limits.

RWCRE is a version of RWDRE that aims to capture the *crossover* between homogeneous RW and static RWRE. If the increments between consecutive refreshing times remain bounded, then correlations decay rapidly over time and we expect to see a behaviour close to that of a homogeneous RW. Conversely, if these increments diverge, then we expect to see a behaviour close to that of RWRE. In particular, the faster the divergence, the more RWCRE resembles RWRE. Importantly, RWCRE allows for different resampling regimes, which are determined by the incremental structure of the refreshing times. We will see that different regimes give rise to interesting *new phenomena*.

In order to understand RWCRE, we need certain *concentration properties* of RWRE. Some of these are available from the literature, but others are not. A few preliminary results were obtained in Avena and den Hollander [3] under the *annealed* measure and subject to certain regularity conditions on the refreshing times. In the present paper we find conditions for recurrence versus transience and we identify fluctuations for *general* cooling schemes with non-standard limit laws. In Section 1.2 we define one-dimensional RWRE and recall some basic facts that are needed throughout the paper. In Section 1.3 we define RWCRE. Both these sections are largely copied from [2], but are needed to set the stage and fix the notation. In Section 1.4 we state our main theorems. In Section 1.6 we place these theorems in their proper context and state a number of open problems. Proofs are provided in Sections 2–4. Along the way we need a few refined properties of RWRE that are of independent interest. These properties are stated in Section 1.5 and are proved in Appendices A–C.

1.2. *RWRE*: *Basic facts*

Throughout the paper we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $\mathbb{N} = \{1, 2, \dots\}$. The classical one-dimensional static model is defined as follows. Let $\omega = \{\omega(x) \colon x \in \mathbb{Z}\}$ be an i.i.d. sequence with probability distribution

$$\mu := \alpha^{\mathbb{Z}} \tag{1.1}$$

for some probability distribution α on $(0, 1)$. We assume that α is *non-degenerate* and write $\langle\cdot\rangle$ the corresponding expectation. We also assume that α is *uniformly elliptic*, i.e.,

$$\exists \mathfrak{c} > 0\colon \quad \alpha\big(\mathfrak{c} \le \omega(0) \le 1 - \mathfrak{c}\big) = 1. \tag{1.2}$$

Definition 1 (RWRE). Let ω be an environment sampled from μ. We call *Random Walk in Random Environment* the Markov chain $Z = (Z_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{Z}$ and transition probabilities

$$P^\omega(Z_{n+1} = x + e \mid Z_n = x) = \begin{cases} \omega(x) & \text{if } e = 1, \\ 1 - \omega(x) & \text{if } e = -1, \end{cases} \tag{1.3}$$

for $x \in \mathbb{Z}, n \in \mathbb{N}_0$. We denote by $P_x^\omega(\cdot)$ the *quenched* measure of Z starting from $Z_0 = x \in \mathbb{Z}$, and by $P_x^\mu(\cdot) := \int_{(0,1)^{\mathbb{Z}}} P_x^\omega(\cdot)\mu(\mathrm{d}\omega)$, the *annealed* measure. The corresponding expectations are denoted by E_x^ω and E_x^μ.

The understanding of one-dimensional RWRE is well developed, both under the quenched and the annealed measure. For a general overview, we refer the reader to the lecture notes by Zeitouni [19]. Below we collect some basic facts and definitions.

The average displacement is $E_0^\mu[Z_1] = \langle\frac{1-\rho}{1+\rho}\rangle$, where $\rho := \frac{1-\omega(0)}{\omega(0)}$. The following proposition due to Solomon [16] characterises recurrence versus transience and limiting speed. Without loss of generality we may assume that

$$\langle\log\rho\rangle \le 0, \tag{1.4}$$

because the reverse can be included via a reflection argument. Indeed, if $\widetilde{\omega}$ is defined by $\widetilde{\omega}(x) = 1 - \omega(-x)$, for $x \in \mathbb{Z}$, then $P_0^\omega(-Z \in \cdot) = P_0^{\widetilde{\omega}}(Z \in \cdot)$.

Proposition 1 (Recurrence, transience, speed of RWRE [16]). *Suppose that* (1.4) *holds*. *Then*:

- *Z is recurrent when* $\langle\log\rho\rangle = 0$.
- *Z is transient to the right when* $\langle\log\rho\rangle < 0$, *and for μ-a.e.ω*,

$$\lim_{n\to\infty} \frac{Z_n}{n} =: v_\mu = \begin{cases} 0 & \text{if } \langle\rho\rangle \ge 1, \\ \frac{1-\langle\rho\rangle}{1+\langle\rho\rangle} > 0 & \text{if } \langle\rho\rangle < 1. \end{cases} \tag{1.5}$$

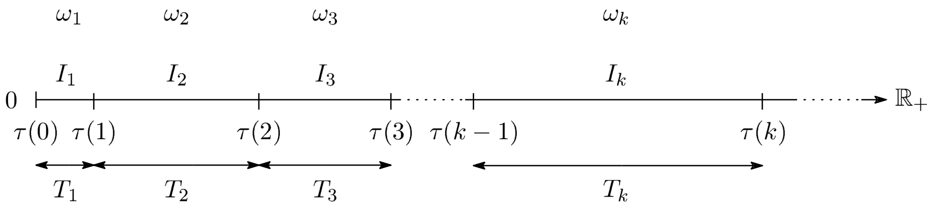

Fig. 1. Structure of the cooling random environment $(\overline{\omega}, \tau)$.

The above proposition shows that the speed of RWRE is a deterministic function of μ (or α; recall (1.1)).

In the recurrent case the scaling was identified by Sinai [15] and the limit law by Kesten [9]. The next proposition summarises their results. We write $\xrightarrow{(d)}$ to denote convergence in distribution and $\xrightarrow{L^p}$ to denote convergence in L^p. We say that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in L^p to a random variable X_* if there is a coupling under which difference converges to 0 in L^p.

Proposition 2 (Scaling limit: recurrent RWRE [9,15]). *Let α be such that $\langle \log \rho \rangle = 0$ and $\sigma_0^2 := \langle \log^2 \rho \rangle \in (0, \infty)$. Then, under the annealed measure P_0^μ,*

$$\frac{Z_n}{\sigma_0^2 \log^2 n} \xrightarrow{(d)} V, \tag{1.6}$$

where the Sinai–Kesten *random variable V is defined by $P(V \in A) := \int_A v(x)\,dx$ with*

$$v(x) := \frac{2}{\pi} \sum_{k\in\mathbb{N}_0} \frac{(-1)^k}{2k+1} \exp\left[-\frac{(2k+1)^2\pi^2}{8}|x|\right], \quad x \in \mathbb{R}. \tag{1.7}$$

Note that the law of V is symmetric with finite variance $\sigma_V^2 \in (0, \infty)$. It was shown in [3] that for α satisfying (1.2), under the annealed measure P_0^μ,

$$\frac{Z_n}{\sigma_0^2 \log^2 n} \xrightarrow{L^p} V \quad \forall p > 0, \tag{1.8}$$

In the transient case the scaling and the limit law were identified by Kesten, Kozlov and Spitzer [10]. The next proposition recalls their result only for the case where the scaling and the limit law are classical. We say that α is *s-transient* when $\langle \log \rho \rangle < 0$, $\langle \rho^s \rangle = 1$ and $\langle \rho (\log \rho)_+ \rangle < \infty$.

Proposition 3 (Scaling limit: transient RWRE [10]). *Let α be s-transient with $s \in (2, \infty)$. Then there exists a $\sigma_s \in (0, \infty)$ such that, under the annealed measure P_0^μ,*

$$\frac{Z_n - v_\mu n}{\sigma_s \sqrt{n}} \xrightarrow{(d)} \Phi, \tag{1.9}$$

where Φ stands for a standard normal random variable.

1.3. *RWCRE: Cooling*

The cooling random environment is the *space–time* random environment built by partitioning $\mathbb{N}_0$, and assigning independently to each piece an environment sampled from μ in (1.1) (see Figure 1). Formally, let $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ be a strictly increasing function with $\tau(0) = 0$, referred to as the *cooling map*. The cooling map determines a sequence of *refreshing times* $(\tau(k))_{k\in\mathbb{N}_0}$.

Definition 2 (Cooling Random Environment). Given a cooling map τ and an i.i.d. sequence of random environments $\overline{\omega} = (\omega_k)_{k\in\mathbb{N}}$ with law $\mu^{\mathbb{N}}$, the *cooling random environment* is built from the pair $(\overline{\omega}, \tau)$ by assigning, for each $k \in \mathbb{N}$, the environment ω_k to the kth interval I_k defined by $I_k := [\tau(k-1), \tau(k))$, which has size $T_k := \tau(k) - \tau(k-1)$ for $k \in \mathbb{N}$.

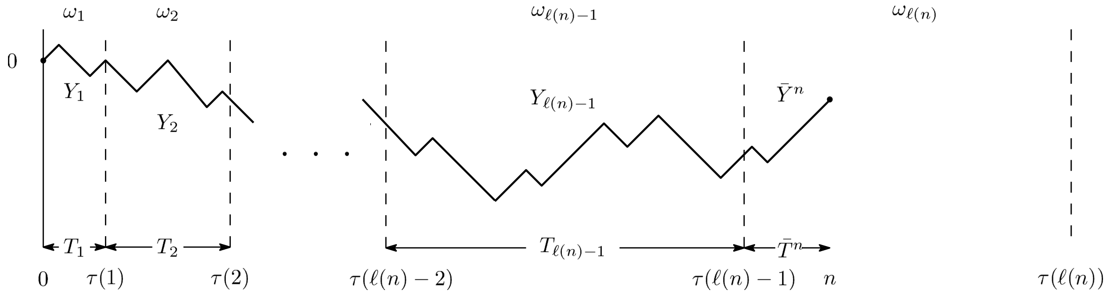

Fig. 2. The decomposition of RWCRE into pieces of RWRE as in (1.14). The random variables $(Y_k)_{1\le k<\ell(n)}$ $(Y_{\ell(n)})$ defined in (1.12) measure the spatial displacement (vertical axis) on each time interval $[\tau(k)\wedge n, \tau(k+1)\wedge n]$ (horizontal axis).

Definition 3 (RWCRE). Let τ be a cooling map and $\overline{\omega}$ an environment sequence sampled from $\mu^{\mathbb{N}}$. We call *Random Walk in Cooling Random Environment* (RWCRE) the Markov chain $X = (X_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{Z}$ and transition probabilities

$$P^{\overline{\omega},\tau}(X_{n+1} = x + e \mid X_n = x) = \begin{cases} \omega_{\ell(n)}(x), & e = 1, \\ 1 - \omega_{\ell(n)}(x), & e = -1, \end{cases} \tag{1.10}$$

for $x \in \mathbb{Z}, n \in \mathbb{N}_0$, where $\ell(n) := \inf\{k \in \mathbb{N} \colon \tau(k) > n\}$, is the index of the interval that n belongs to. Similarly to Definition 1, we denote by

$$P_x^{\overline{\omega},\tau}(\cdot), \quad P_x^{\mu,\tau}(\cdot) := \int_{[(0,1)^{\mathbb{Z}}]^{\mathbb{N}}} P_x^{\overline{\omega},\tau}(\cdot)\mu^{\mathbb{N}}(\mathrm{d}\overline{\omega}), \tag{1.11}$$

the corresponding *quenched* and *annealed* measures, respectively.

The position X_n admits a decomposition into independent pieces. For $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, define the *refreshed increments* and *boundary increment* as

$$Y_k := X_{\tau(k)} - X_{\tau(k-1)}, \qquad \bar{Y}^n := X_n - X_{\tau(\ell(n)-1)} \tag{1.12}$$

and the *running time at the boundary* as $\bar{T}^n := n - \tau(\ell(n) - 1)$. Note that

$$\sum_{k=1}^{\ell(n)-1} T_k + \bar{T}^n = n. \tag{1.13}$$

By construction, we can write X_n as the sum

$$X_n = \sum_{k=1}^{\ell(n)-1} Y_k + \bar{Y}^n, \quad n \in \mathbb{N}_0. \tag{1.14}$$

This decomposition shows that, in order to analyse X_n, we need to analyse the vector $(Y_1, \dots, Y_{\ell(n)-1}, \bar{Y}^n)$, which consists of independent components, each distributed as an increment of Z (defined in Section 1.2) in a given environment over a given length of time determined by $\overline{\omega}$, τ and n. Figure 2 illustrates the decomposition of X_n.

To ease the notation, when n is explicit we will sometimes write

$$T_0 := \bar{T}^n \quad \text{and} \quad Y_0 := \bar{Y}^n. \tag{1.15}$$

1.4. *Main results for RWCRE*

In what follows, we write $\mathbb{P}$ for the annealed measure in (1.11) when the random walk starts at the origin, suppressing $\mu, \tau, 0$ from the notation. We will denote by $\mathbb{E}$ and $\mathbb{V}\mathrm{ar}$ the corresponding expectation and variance. We will further denote by $\mathfrak{X}_n$ the variance-scaled displacement at time $n \in \mathbb{N}_0$,

$$\mathfrak{X}_0 := 0, \qquad \mathfrak{X}_n := \frac{X_n}{\sqrt{\mathbb{V}\mathrm{ar}(X_n)}}, \quad n \in \mathbb{N}. \tag{1.16}$$

1.4.1. *Recurrence versus transience*

We start by exploring how the cooling map affects the recurrence versus transience criterion for RWRE (see Proposition 1). A few remarks are in place. Since for any event A,

$$\mathbb{P}(A)=1 \quad\Longleftrightarrow\quad P_0^{\overline{\omega},\tau}(A)=1, \quad \mu^{\mathbb{N}}\text{-a.s.}, \tag{1.17}$$

we do not distinguish between quenched and annealed statements when it comes to zero-one laws. Moreover, due to the resampling, RWCRE is *tail-trivial*, i.e., all events in the tail sigma-algebra have probability zero or one. We know from Proposition 1 that RWRE is recurrent if and only if $\langle\log\rho\rangle=0$. We say that α is *recurrent* or *right-transient* when $\langle\log\rho\rangle=0$, respectively, $\langle\log\rho\rangle<0$. For RWCRE the classification of recurrence versus transience is more delicate, because it also depends on the cooling map τ. In what follows we say that (α,τ) is *recurrent* or *transient* when

$$\mathbb{P}(X_n=0\text{ i.o.})=1 \quad\text{or}\quad \mathbb{P}(X_n=0\text{ i.o.})=0. \tag{1.18}$$

We say that (α,τ) is *right transient* or *left transient* when

$$\mathbb{P}\Big(\lim_{n\to\infty}X_n=\infty\Big)=1 \quad\text{or}\quad \mathbb{P}\Big(\lim_{n\to\infty}X_n=-\infty\Big)=1. \tag{1.19}$$

By tail triviality, $\{0,1\}$ are the only possible values for the above events.

Our first theorem gives two conditions on the cooling map under which recurrence and transience are not affected by the resampling.

Theorem 1 (Stability of recurrence versus transience).

(a) *If α is right-transient, then (α,τ) is right-transient for all τ such that*

$$\lim_{k\to\infty}T_k=\infty. \tag{1.20}$$

(b) *If α is recurrent, then (α,τ) is recurrent when*

$$\liminf_{n\to\infty}\big|\mathbb{E}[\mathfrak{X}_n]\big|=0. \tag{1.21}$$

The latter holds for all symmetric α and all τ, and also for all non-symmetric α when τ is such that

$$\liminf_{k\to\infty}\frac{1}{k^\gamma}\log T_k>0 \quad \textit{for some } \gamma>\frac{3}{4}. \tag{1.22}$$

Non-symmetric α means that the laws of ω and $\widetilde{\omega}$ are different (see below (1.4)). Note that (1.22) is much more stringent than (1.20).

Remark. If the refreshing increments stay bounded (a regime that in [3] was referred to as 'no cooling'), then RWCRE has little relation to RWRE and no resemblance is to expected.

A recurrence criterion for general cooling maps is lacking and is presumably delicate, as shown by the following examples for which a weaker form of divergence of the increments is still in force. To weaken (1.20) we consider refreshing time increments that *Cesaro diverge*, i.e., increments satisfying

$$\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{k=1}^{\ell}T_k=\infty. \tag{1.23}$$

Counterexamples to stability.

(Ex.1) *Right-transient can turn into left-transient or recurrent*: There exist a right-transient α and two cooling maps $\tau'=\tau'(\alpha)$ and $\tau''=\tau''(\alpha)$ satisfying (1.23) such that (α,τ') and (α,τ'') are left-transient and recurrent, respectively.

(Ex.2) *Recurrent can turn into transient*: There exist a recurrent α and a cooling map $\tau=\tau(\alpha)$ satisfying (1.20) such that (α,τ) is transient.

In Section 2 we prove Theorem 1 and show (Ex.1) and (Ex.2).

1.4.2. *Fluctuations in the Sinai regime*

The following statements identify the scaling limits of RWCRE for recurrent α. They show that the scaling depends in a delicate way on the cooling map. In particular, Theorem 2 below gives a characterisation of the possible limit points as mixtures of Sinai–Kesten and Gaussian random variables, while Corollary 1 and (Ex.3)–(Ex.6) below give a further characterisation of the various possible regimes.

To state our results we need the following definitions. Set

$$\lambda_{\tau,n}(k) := \frac{\sqrt{\mathbb{V}\mathrm{ar}(Y_k)}}{\sqrt{\mathbb{V}\mathrm{ar}(X_n)}} \mathbb{1}_{\{0\le k<\ell(n)\}}, \quad n\in\mathbb{N}, k\in\mathbb{N}_0, \tag{1.24}$$

and $\lambda_{\tau,0}(k) := \delta_{k0}$, $k\in\mathbb{N}_0$. Note that, by (1.14), $\lambda_{\tau,n}$ is a vector of real numbers with unit $\ell_2(\mathbb{N}_0)$-norm, i.e., $\|\lambda_{\tau,n}\|_2^2 := \sum_{k\in\mathbb{N}_0}\lambda_{\tau,n}(k)^2 = 1$, and recall that Y_0, the boundary value defined in (1.15), is determined by τ and n, the indices in $\lambda_{\tau,n}$. With this notation, we can write

$$\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n] = \sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k)\frac{Y_k-\mathbb{E}[Y_k]}{\sqrt{\mathbb{V}\mathrm{ar}(Y_k)}}. \tag{1.25}$$

Let $(V_j)_{j\in\mathbb{N}_0}$ be a family of i.i.d. Sinai–Kesten random variables (see (1.7)). Define for $\lambda = (\lambda(j))_{j\in\mathbb{N}_0}\in\ell_2(\mathbb{N}_0)$, the λ-mixture of normalised Sinai–Kesten random variables by

$$V^{\otimes\lambda} := \sum_{j\in\mathbb{N}_0}\lambda(j)\big(\sigma_V^{-1}V_j\big) = \lim_{n\to\infty}\sum_{j=0}^{n}\lambda(j)\big(\sigma_V^{-1}V_j\big), \tag{1.26}$$

where the above limit is well defined from the convergence in L^2 of the series.

For $\lambda\in\ell_2(\mathbb{N}_0)$, let $\lambda^{\downarrow}$ be the vector obtained from λ by reordering the entries of λ in decreasing order. Consider the equivalence relation $\lambda\sim\lambda'$ when $\lambda^{\downarrow}=\lambda'^{\downarrow}$ and put $[\lambda] := \{\lambda'\in\ell_2(\mathbb{N}_0)\colon \lambda'\sim\lambda\}$. The following lemma, which is proven in Section 3.2, guarantees that up to reordering $V^{\otimes\lambda}$ corresponds to a unique vector λ.

Lemma 1 (Characterisation of Sinai–Kesten mixtures). *$V^{\otimes\lambda}$ and $V^{\otimes\lambda'}$ have different distributions if and only if $[\lambda]\neq[\lambda']$.*

Define by $\lambda^{0\downarrow}$ the vector obtained from λ by putting $\lambda(0)$ as the first entry and reordering the other entries in decreasing order. This notation is needed in order to isolate the boundary increment. In what follows, $(n_i)_{i\in\mathbb{N}_0}$ denotes a strictly increasing sequence of integers with $n_0=0$.

Theorem 2 (Limit distributions in the Sinai regime). *Let α be recurrent with $\sigma_0\in(0,\infty)$ and let τ be a cooling map. Under the annealed measure $\mathbb{P}$, the sequence of centred random variables $(\mathfrak{X}_n-\mathbb{E}[\mathfrak{X}_n])_{n\in\mathbb{N}_0}$ is tight in the weak topology and its limit points are characterised as follows. If $(n_i)_{i\in\mathbb{N}_0}$ is such that*

$$\lim_{i\to\infty}\lambda^{0\downarrow}_{\tau,n_i}(k) =: \lambda_*(k) \quad \forall k\in\mathbb{N}_0, \tag{1.27}$$

then $\lambda_* = (\lambda_*(k))_{k\in\mathbb{N}_0}\in\ell_2(\mathbb{N}_0)$ and

$$\mathfrak{X}_{n_i}-\mathbb{E}[\mathfrak{X}_{n_i}] \xrightarrow{L^p} V^{\otimes\lambda_*} + a(\lambda_*)\Phi \quad \forall p>0, \tag{1.28}$$

where $a(\lambda_*) := (1-\|\lambda_*\|_2^2)^{\frac{1}{2}}$, Φ is a standard normal random variable, and $V^{\otimes\lambda_*}$ is as in (1.26).

Remark 1. We note that if one is allowed to take subsequences, then condition (1.27) is not restrictive. Indeed, since for any $k\in\mathbb{N}_0$ and $n\in\mathbb{N}_0$ the value $\lambda^{0\downarrow}_{\tau,n}(k)$ belongs to $[0,1]$, if we take a (diagonal) subsequence $(n_i)_{i\in\mathbb{N}_0}$, then condition (1.27) will be satisfied for some vector λ_*. By Fatou's lemma we get

$$\sum_{k\in\mathbb{N}_0}\lambda_*^2(k) = \sum_{k\in\mathbb{N}_0}\liminf_{i\in\mathbb{N}_0}\big(\lambda^{0\downarrow}_{\tau,n_i}(k)\big)^2 \le \liminf_{i\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\big(\lambda^{0\downarrow}_{\tau,n_i}(k)\big)^2 = 1, \tag{1.29}$$

which guarantees that $\lambda_*\in\ell_2(\mathbb{N}_0)$ and so $V^{\otimes\lambda_*}$ in (1.28) is well defined. With this it follows that Theorem 2 characterizes *all limit points* of $(\mathfrak{X}_n-\mathbb{E}[\mathfrak{X}_n])_{n\in\mathbb{N}_0}$.

It is possible to distinguish between the different scaling limits by looking at the asymptotic behavior of $(\lambda_{\tau,\tau(k)}(k))_{k\in\mathbb{N}_0}$, the sequence of relative weights of the refreshed increments.

Corollary 1 (Limit distributions for regular cooling maps). *For any* $p > 0$, *under the annealed measure* $\mathbb{P}$

(a) $\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]$ *converges in* L^p *if and only if* $\lambda_{\tau,\tau(k)}(k) \to 0$, *in which case*

$$\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n] \xrightarrow{L^p} \Phi. \tag{1.30}$$

(b) *If* $\lambda_{\tau,\tau(k)}(k) \to q \in (0, 1]$, *then*

$$\mathfrak{X}_{\tau(k)} \xrightarrow{L^p} V^{\otimes\lambda_q}, \tag{1.31}$$

where $\lambda_q(0) := 0$, *and for* $j \in \mathbb{N}$, $\lambda_q^2(j) := q^2(1-q^2)^{j-1}$. *Moreover, if for a subsequence* $(n_i)_{i\in\mathbb{N}_0}$ *the limit* $w := \lim_{i\to\infty} \lambda_{\tau,n_i}(0)$ *exists, then*

$$\mathfrak{X}_{n_i} \xrightarrow{L^p} w\sigma_V^{-1} V_0 + \left(1-w^2\right)^{\frac{1}{2}} V^{\otimes\lambda_q}. \tag{1.32}$$

The proofs of Theorem 2 and Corollary 1 are given in Section 3.

Examples of subsequential limits. We illustrate Corollary 1 by considering examples of cooling maps that diverge at different rates. In examples (Ex.3)– (Ex.6) below all convergence statements are under the annealed measure $\mathbb{P}$.

(Ex.3) *Polynomial cooling* : If $k^{-\beta} T_k \to B$ for some $B, \beta \in (0,\infty)$, then

$$\frac{X_n - \mathbb{E}[X_n]}{\sigma_0^2 n^{\frac{1}{2(\beta+1)}} \log^2 n} \xrightarrow{L^p} \sigma_V \left(\frac{\beta}{\beta+1}\right)^2 \left(\frac{\beta+1}{B}\right)^{\frac{1}{2(\beta+1)}} \Phi. \tag{1.33}$$

(Ex.4) *Exponential cooling*: If $k^{-1}\log T_k \to c \in (0,\infty)$, then

$$\frac{X_n}{\sigma_0^2 \log^{\frac{5}{2}} n} \xrightarrow{L^p} \frac{1}{\sqrt{5c^5}} \sigma_V \Phi. \tag{1.34}$$

(Ex.5) *Double exponential cooling*: If $k^{-1}\log\log T_k \to c \in (0,\infty)$, then

$$\frac{X_{\tau(\ell)}}{\sigma_0^2 \log^2 \tau(\ell)} \xrightarrow{L^p} q_c^{-1}\sigma_V V^{\otimes\lambda_{q_c}} \quad \text{with } q_c^2 = \frac{e^{4c}-1}{e^{4c}} \in (0,1). \tag{1.35}$$

(Ex.6) *Faster than double exponential cooling*: If $k^{-1}\log\log T_k \to \infty$, then

$$\frac{X_{\tau(\ell)}}{\sigma_0^2 \log^2 \tau(\ell)} \xrightarrow{L^p} V. \tag{1.36}$$

In (Ex.5) and (Ex.6) we can even characterise *all* the limit points. Indeed, if a subsequence $(n_i)_{i\in\mathbb{N}_0}$ is such that

$$\lim_{i\to\infty} \frac{\log \bar{T}^{n_i}}{\log \tau(\ell(n_i)-1)} =: b \in [0,\infty], \tag{1.37}$$

then

$$\frac{X_{n_i}}{\sigma_0^2 \log^2 n_i} \xrightarrow{L^p} \begin{cases} q_c^{-1}\sigma_V V^{\otimes\lambda_{q_c}} + b^2 V_0 & \text{if } b \le 1, \\ b^{-2} q_c^{-1}\sigma_V V^{\otimes\lambda_{q_c}} + V_0 & \text{if } b > 1, \end{cases} \tag{1.38}$$

with $b^{-1} = 0$ when $b = \infty$.

The claims in (Ex.3)–(Ex.6) are proven in Section 3.4.

1.4.3. *Fluctuations in the Gaussian regime*

We next examine the scaling limit when α is s-transient with $s \in (2, \infty)$, i.e., when RWRE satisfies a classical CLT (recall Proposition 3).

Theorem 3 (Scaling limit in the Gaussian regime). *Let α be s-transient with $s \in (2, \infty)$ and let τ be any cooling map. Then, under the annealed measure $\mathbb{P}$,*

$$\mathfrak{X}_{\mathfrak{n}} - \mathbb{E}[\mathfrak{X}_{\mathfrak{n}}] \xrightarrow{L^2} \Phi. \tag{1.39}$$

Theorem 3 says that in the Gaussian regime also RWCRE converges to a Gaussian. However, the scaling of the variance as a function of the cooling map is subtle, as we show next.

Corollary 2 (Gaussian limits and stability of the variance). *Fix $s \in (2, \infty)$. The sequence $((X_n - \mathbb{E}[X_n])/\sqrt{n})_{n\in\mathbb{N}_0}$ is tight in the weak topology and its limit points correspond to the limit points of $\sigma^2_{s,\tau}(n) := \mathbb{V}\mathrm{ar}(X_n)/n$. Namely, given a subsequence $(n_i)_{i\in\mathbb{N}_0}$, if $\sigma_{s,\tau}(n_i) \to \sigma$, then, under the annealed measure $\mathbb{P}$,*

$$\frac{X_{n_i} - \mathbb{E}[X_{n_i}]}{\sigma\sqrt{n_i}} \xrightarrow{L^2} \Phi. \tag{1.40}$$

Moreover, if $T_k \to \infty$, then

$$\sigma_{s,\tau}(n) \to \sigma_s, \tag{1.41}$$

with σ_s the standard deviation from Proposition 3.

We conclude our analysis of the Gaussian regime by looking into the centering term in (1.39).

Centering and correction in the law of large numbers. In general the centering term $\mathbb{E}[X_n]$ in Theorem 3 (recall (1.16)) cannot be replaced by the limiting speed of X. In (Ex.7) below we provide a class of rapidly diverging cooling maps for which such a replacement causes no harm. In (Ex.8) below we indicate that there exist slowly diverging cooling maps for which it does.

(Ex.7) *Stable centering for rapidly diverging cooling maps*: For $s \in (2, \infty)$, if $T_k \to \infty$ and

$$\sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k) < \infty, \tag{1.42}$$

then

$$\frac{\mathbb{E}[X_n] - v_\mu n}{\sqrt{n}} \to 0, \tag{1.43}$$

with v_μ the RWRE speed in (1.5), from which it follows via (1.39) and (1.41) that

$$\frac{X_n - v_\mu n}{\sigma_s\sqrt{n}} \xrightarrow{L^2} \Phi. \tag{1.44}$$

Moreover, (1.42) holds when $\liminf_{k\to\infty} k^{-1}\log T_k > 0$.

(Ex.8) *Counterexample with slowly diverging cooling maps*: For any $s \in (2, \infty)$ there exist an s-transient α and a cooling map τ with $T_k \to \infty$, such that (1.44) fails. In particular, there exist "extreme" examples for which

$$\limsup_{n\to\infty} \frac{\mathbb{E}[X_n] - v_\mu n}{\sqrt{n}} = \infty. \tag{1.45}$$

In such cases, the sequence $((X_n - v_\mu n)/\sqrt{n})_{n\in\mathbb{N}_0}$ is not even tight (compare with Corollary 2).

Condition (1.42) in (Ex.7) imposes a growth condition on T_k. (Ex.8) shows that the convergence in (1.44) may fail even when $T_k \to \infty$.

The proofs of Theorem 3 and Corollary 2 are given in Section 4.

1.5. *Auxiliary properties of RWRE*

In our analysis of RWCRE we need a few results about RWRE. The first states that in Proposition 3 the convergence can be extended to L^p for $p < s$.

Theorem 4 (L^p-convergence in the Gaussian regime). *Suppose that the assumptions in Proposition 3 are in force. Then*

$$\frac{Z_n - v_\mu n}{\sigma_s \sqrt{n}} \overset{L^p}{\longrightarrow} \Phi \quad \forall p < s. \tag{1.46}$$

The second result concerns various forms of oscillation of the mean of RWRE.

Theorem 5 (Oscillations of the mean).

(I) *There is a recurrent α such that $E_0^\mu[Z_n] \neq 0$ for infinitely many $n \in \mathbb{N}$.*
(II) *For every $s \in (2, \infty)$, there is an s-transient α such that $E_0^\mu[Z_n] \neq v_\mu n$ for infinitely many $n \in \mathbb{N}$.*
(III) *If α is recurrent with $\sigma_0 \in (0, \infty)$, then for every $0 < \gamma < \frac{2}{3}$ there is a $C = C(\alpha, \gamma) \in (0, \infty)$ such that*

$$\left| E_0^\mu \left[\frac{Z_n}{\sigma_0^2 \log^2 n} \right] \right| \leq \frac{C}{\log^\gamma n}, \quad n \in \mathbb{N}. \tag{1.47}$$

The proofs of Theorems 4–5 are given in Appendices A–C. The line of proof of Theorem 5(III) was suggested by Zhan Shi.

1.6. *Discussion and open problems*

Ellipticity. The uniform ellipticity assumption in (1.2) is needed in the proof of Theorem 5(III) only. Once this would be extended, all our results would carry over. In the proof of Theorem 1(a) we need a concentration property for which it suffices to have a very mild form of ellipticity. In the proof of Theorem 2 and Corollary 1 we use (1.8), which was proved in [3] under (1.2) only, but should be true more generally.

Stability of recurrence and transience. While RWRE asymptotics are non-local due to space–time correlations, for RWCRE, resampling *adds extra noise* and weakens space–time dependencies. From this perspective, we can view RWCRE as a *perturbation* of RWRE. Theorem 1 describes how this perturbation affects the recurrence versus transience criterion known for RWRE. Theorem 1(a) shows that transience is preserved as soon as the increments of the refreshing times diverge, while Theorem 1(b) says that the situation is more delicate for recurrence, unless α is symmetric. In fact, as shown in (Ex.2), for non-symmetric α, resampling is capable of destroying recurrence. We will see in Section 2 that this happens because there are increments of the refreshing times during which the average displacement of RWRE is strictly positive. By repeating such increments often enough, we are able to pull the random walk away from the origin. The increments of the refreshing times in such cooling maps are diverging, but *slowly enough* so that RWCRE is qualitatively different from RWRE. As shown in (Ex.1), cooling can even turn right-transience into left-transience.

Mixed fluctuations in Sinai regime. It is well-known that trapping phenomena are predominant when RWRE is recurrent (see Proposition 2). The underlying correlation structure gives rise to subdiffusive scaling with a non-Gaussian limit law. Theorem 2 and Corollary 1 show that this scenario is affected by the extra noise introduced by the cooling. Indeed RWCRE is less localised, although convergence in distribution of the full sequence is not guaranteed in general. Theorem 2 shows that regular subsequential limits are characterised by mixtures of Gaussian laws and properly weighted Sinai–Kesten laws. Corollary 1(a) is stated in terms of the last increment in the sum (1.14) and is equivalent to the statement that the boundary increment is negligible. It provides a necessary and sufficient condition under which all subsequential limits coincide, in which case a standard Gaussian law emerges after a scaling that is gauged by the divergence in the cooling map. Corollary 1(b), instead, says that if the boundary increment is not negligible, then the full sequence *does not converge*. Indeed, as illustrated by (Ex.3)–(Ex.6), we see that properly chosen subsequences lead to different mixed limit laws. These subsequences can be further characterised depending on whether the boundary term dominates or competes with the other terms, as illustrated in (1.38). Such results yield the answer to a conjecture put forward in [3], where the analysis of the fluctuations in the Sinai regime was carried out for cooling maps for which Lindeberg–Feller type conditions are satisfied, essentially corresponding to the condition in Corollary 1(a).

CLT in the Gaussian regime and centering issues. RWCRE can be seen as an interpolation between RWRE and a homogeneous random walk. Thus, not surprisingly, Theorem 3 shows that if RWRE satisfies a CLT (i.e., when $s \in$

$(2, \infty)$), then the same is true for RWCRE. Yet, as is clear from Corollary 2, the cooling can make the variance oscillate on scale n, but not under (1.20). (Ex.7) and (Ex.8) shown that, if the cooling map is "sufficiently concentrated" as captured in condition (1.42), it *must* be centered with the average displacement.

Refined properties of RWRE. Section 1.5 collects a few refined properties of RWRE that are not available in the literature but are needed in our proofs. In particular, Theorem 4 extends the mode of convergence in Proposition 3 to L^p, and we use the latter in the proof of Theorem 3. Concerning Theorem 5, items (I) and (II) are similar in spirit, and say that in the recurrent and transient regime, respectively, the limiting speeds are not achieved after a finite time. These statements may sound plausible, but the disorder does not allow for a simple proof, as can be appreciated from Appendix B. We use items (I) and (II) to construct (Ex.2) and (Ex.8), respectively. Item (III) gives some control (possibly not optimal) on the rate of convergence in Proposition 2, which we use in the proof of (1.21).

Extensions and open problems:

- (*Regime with limiting stable laws*). The only regime for which we have not analysed RWCRE fluctuations is when α is s-transient with $s \in (0, 2]$. In this regime, the RWRE fluctuations are more intricate. Under the annealed measure it is known that, after an appropriate scaling, RWRE converges to certain stable laws or inverse-stable laws. Under the quenched measure fluctuations are drastically different and actually have only been partially characterised. In particular, different subsequential limits are possible under the quenched measure. For precise statements we refer the reader to [19] and references therein. The analysis of RWCRE with $s \in (0, 2]$ should lead to interesting cooling-dependent crossover phenomena.
- (*Higher dimensions*). The focus in the present paper and in [2,3] is on *one-dimensional* RWCRE. It is natural to consider RWCRE also in higher dimensions. However, much less is known for RWRE in higher dimensions, and most of the relevant results require additional and often technical assumptions (see [19]). Nonetheless, some of our arguments and results may be adapted to higher dimensions, in particular, those concerning the stability of directional transience and directional speed.
- (*Recurrence criterion for arbitrary cooling*). We partially solved the problem of recurrence versus transience in Theorem 1. The following problem is left open: If α is recurrent and non-symmetric, then what is a *necessary and sufficient* condition on τ such that RWCRE is recurrent?
- (*RWRE oscillations*). Some of the statements in Theorem 5 are non-optimal. For example, in part (2) we should be able to show that $E_0^\mu[Z_n] \neq v_\mu n$ for infinitely many $n \in \mathbb{N}$ for every s-transient α with $s \in (2, \infty)$. Such an improvement would allow us to strengthen the statement in (Ex.8) by saying that for every s-transient α with $s \in (2, \infty)$ there exists a τ such that (1.45) is satisfied.

2. Proofs: Recurrence versus transience

2.1. *Transience is preserved for any cooling with diverging increments*

Proof of Theorem 1(a). We assume that $\langle \log \rho \rangle < 0$.

Basic coupling. Let us consider a probability space $(S, \mathcal{S}, \mathcal{P})$ on which random variables $(X_n)_{n\in\mathbb{N}_0}$ and $(Z_n^{(k)})_{k\in\mathbb{N},n\in\mathbb{N}_0}$ are defined such that

$$\begin{aligned}
&Y_k = Z^{(k)}_{T_k}, \quad k \in \mathbb{N}, \qquad \bar{Y}^n = Z^{(\ell(n))}_{\bar{T}^n}, \quad n \in \mathbb{N}_0,\\
&\mathcal{P}\big((X_n)_{n\in\mathbb{N}_0} \in \cdot\big) = \mathbb{P}\big((X_n)_{n\in\mathbb{N}_0} \in \cdot\big),\\
&\big(Z_n^{(k)}\big)_{k\in\mathbb{N},n\in\mathbb{N}_0} \text{ are independent in } k \in \mathbb{N},\\
&\mathcal{P}\big(\big(Z_n^{(k)}\big)_{n\in\mathbb{N}_0} \in \cdot\big) = P_0^\mu\big((Z_n)_{n\in\mathbb{N}_0} \in \cdot\big), \quad k \in \mathbb{N}.
\end{aligned} \tag{2.1}$$

This constitutes a *coupling* of RWRE and RWCRE. We write $\mathcal{E}$ to denote expectation with respect to $\mathcal{P}$.

Leftmost record. Set $W := \inf\{Z_n : n \in \mathbb{N}_0\}$ and $W^{(k)} := \inf\{Z_n^{(k)} : n \in \mathbb{N}_0\}$, $k \in \mathbb{N}$. By (1.14), for any $a > 0$ and $\ell \in \mathbb{N}$,

$$X_{\tau(\ell)} = \sum_{k=1}^{\ell} Y_k = \sum_{k=1}^{\ell} Z^{(k)}_{T_k} \mathbb{1}_{\{Z^{(k)}_{T_k} > a\}} + \sum_{k=1}^{\ell} Z^{(k)}_{T_k} \mathbb{1}_{\{Z^{(k)}_{T_k} \le a\}}$$

$$\geq \sum_{k=1}^{\ell} Z_{T_k}^{(k)} \mathbb{1}_{\{Z_{T_k}^{(k)}>a\}} + \sum_{k=1}^{\ell} W^{(k)}. \tag{2.2}$$

The following lemma tells us that the expectation of $-W$ is finite.

Lemma 2. *Suppose that $\langle \log \rho \rangle < 0$. Then $E_0^{\mu}[-W] < \infty$.*

Proof. Write

$$E_0^{\omega}[-W] = \sum_{m \in \mathbb{N}} P_0^{\omega}(W \leq -m). \tag{2.3}$$

For $j \in \mathbb{Z}$, let $\rho_j := \frac{1-\omega(j)}{\omega(j)}$ and for $m \in \mathbb{N}$, $\varepsilon > 0$, define

$$\Omega(m,\varepsilon) := \left\{\omega\colon \sup_{i \in \mathbb{N}} \left| \frac{\sum_{j=-m+1}^{i-1} \log \rho_j}{m+i-1} - \langle \log \rho \rangle \right| < \varepsilon \right\}. \tag{2.4}$$

For $0 < \varepsilon < -\frac{1}{2}\langle \log \rho \rangle$ and $\omega \in \Omega(m,\varepsilon)$,

$$\prod_{j=-m+1}^{i-1} \rho_j \leq e^{\frac{1}{2}\langle \log \rho \rangle (m+i-1)}. \tag{2.5}$$

Therefore there is a $c > 0$ such that, for all $m \in \mathbb{N}$ and $\omega \in \Omega(m,\varepsilon)$,

$$P_0^{\omega}(W \leq -m) \leq \sum_{i=1}^{\infty} \prod_{j=-m+1}^{i-1} \rho_j \leq e^{-cm}, \tag{2.6}$$

where the first inequality follows from a standard computation for RWRE (see [19, p. 196 (2.1.4)]), and the inequality uses (2.5).

Next we note that there is a $c' > 0$ such that, for all $m \in \mathbb{N}$, using

$$\mu\left(\omega\colon \sup_{i \in \mathbb{N}} \left| \frac{\sum_{j=-m+1}^{i-1} \log \rho_j}{m+i-1} - \langle \log \rho \rangle \right| \geq \varepsilon \right) \leq e^{-c'm}, \tag{2.7}$$

where the inequality follows from the union bound in combination with the large deviation principle for the i.i.d. random variables $(\log \rho_j)_{j \in \mathbb{Z}}$. (For the latter the uniform ellipticity assumption in (1.2) amply suffices, but can be substantially weakened.). Combining (2.6) and (2.7) we see that, for all $m \in \mathbb{N}$,

$$P_0^{\mu}(W \leq -m) \leq 2e^{-(c \wedge c')m}. \tag{2.8}$$

The result follows from (2.3) and (2.8). □

Transience along subsequences via the leftmost record. We continue the proof of Theorem 1(a). Pick $a := 4E_0^{\mu}[-W] < \infty$. Since α is right-transient and $T_k \to \infty$, we have $\mathbb{P}(Y_k > a) = P_0^{\mu}(Z_{T_k} > a) \to 1$. From stochastic domination together with the independence of $Z_{T_k}^{(k)}$, $k \in \mathbb{N}$, we get that

$$\liminf_{\ell \to \infty} \frac{1}{\ell} \sum_{k=1}^{\ell} Z_{T_k}^{(k)} \mathbb{1}_{\{Z_{T_k}^{(k)}>a\}} \geq a \liminf_{\ell \to \infty} \frac{1}{\ell} \sum_{k=1}^{\ell} \mathbb{1}_{\{Z_{T_k}^{(k)}>a\}} \geq a, \quad \mathcal{P}\text{-a.s.} \tag{2.9}$$

Now, applying the law of large numbers and (2.9) into (2.2) we get

$$\liminf_{\ell \to \infty} \frac{X_{\tau(\ell)}}{\ell} \geq \frac{3}{4} a, \quad \mathbb{P}\text{-a.s.}, \tag{2.10}$$

which settles right-transience along the sequences of refreshing times.

Transience of the full sequence. Let $\overline{\Omega}(k) := \{\overline{\omega}\colon \frac{X_{\tau(k)}}{k} \geq \frac{3}{4}a\}$. On $\overline{\Omega}(k)$, for $\ell > k$ and $n \in [\tau(\ell), \tau(\ell+1))$ we have $X_n \geq \frac{1}{2}a\ell + W^{(\ell)}$. Let $\mathbb{P}^k$ be $\mathbb{P}$ conditioned on $\overline{\Omega}(k)$. It follows that for $\ell > k$,

$$\mathbb{P}^k\Big(\inf_{n\in[\tau(\ell),\tau(\ell+1))} X_n \leq 0\Big) \leq \mathcal{P}\bigg(W^{(\ell)} \leq -\frac{1}{2}a\ell\bigg). \tag{2.11}$$

By (2.8), $\sum_{\ell\in\mathbb{N}} \mathcal{P}(W^{(\ell)} \leq -\frac{1}{2}a\ell) < \infty$ and hence, by the first Borel–Cantelli lemma,

$$\mathbb{P}^k\Big(\inf_{n\in[\tau(\ell),\tau(\ell+1))} X_n \leq 0 \text{ i.o.}\Big) = 0. \tag{2.12}$$

This implies that $\{n \in \mathbb{N}_0\colon X_n \leq 0\}$ is $\mathbb{P}^k$-a.s. finite. Since $\mathbb{P}(\overline{\Omega}(k)) \to 1$ as $k \to \infty$ it follows that $\{n \in \mathbb{N}_0\colon X_n \leq 0\}$ is $\mathbb{P}$-a.s. finite, which by the irreducibility of RWCRE implies the right-transience of the sequence. □

2.2. *Recurrence is preserved for fast enough cooling*

Proof of Theorem 1(b). The sequence $(\lambda_{\tau,n})_{n\in\mathbb{N}_0}$ of $\ell_2(\mathbb{N}_0)$-unit vectors in (1.24) admits an increasing subsequence $(n_i)_{i\in\mathbb{N}_0} \in \{\tau(k)\colon k \in \mathbb{N}\}^{\mathbb{N}}$ for which there is a vector λ_* with $\|\lambda_*\|_2 \leq 1$ such that, for every $k \in \mathbb{N}_0$, $\lambda^{\downarrow}_{\tau,n_i}(k) \to \lambda_*(k)$. By Theorem 2 (to be proved in Section 3.1), and condition (1.21)

$$\frac{X_{n_i}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} \xrightarrow{(d)} V^{\otimes\lambda_*} + a\Phi. \tag{2.13}$$

Since $\mathbb{V}\mathrm{ar}(X_n) \to \infty$, $(n_i)_{i\in\mathbb{N}_0}$ can be chosen such that

$$\frac{n_{i-1}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} < \frac{1}{2}, \quad i \in \mathbb{N}. \tag{2.14}$$

Now, because $V^{\otimes\lambda_*} + a\Phi$ has full support on $\mathbb{R}$, there is an $\varepsilon > 0$ for which

$$\begin{aligned}
&\mathbb{P}\bigg(\frac{X_{n_i} - X_{n_{i-1}}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} > \frac{1}{2}\bigg) \geq \mathbb{P}\bigg(\frac{X_{n_i}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} > 1\bigg) > \varepsilon,\\
&\mathbb{P}\bigg(\frac{X_{n_i} - X_{n_{i-1}}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} < -\frac{1}{2}\bigg) \geq \mathbb{P}\bigg(\frac{X_{n_i}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} < -1\bigg) > \varepsilon.
\end{aligned} \tag{2.15}$$

Note that because $n_i \in \{\tau(k)\colon k \in \mathbb{N}\}$ for every $i \in \mathbb{N}$, the increments $(X_{n_i} - X_{n_{i-1}})_{i\in\mathbb{N}}$ are independent and therefore, by the Borel–Cantelli lemma, we have that

$$\mathbb{P}\bigg(\frac{X_{n_i} - X_{n_{i-1}}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} > \frac{1}{2} \text{ i.o.}\bigg) = 1, \qquad \mathbb{P}\bigg(\frac{X_{n_i} - X_{n_{i-1}}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{n_i})}} < -\frac{1}{2} \text{ i.o.}\bigg) = 1. \tag{2.16}$$

Since X makes steps of size 1 only, we get from (2.14) that $\mathbb{P}(X_{n_i} > 0 \text{ i.o.}) = 1$ and $\mathbb{P}(X_{n_i} < 0 \text{ i.o.}) = 1$, which proves the first claim in Theorem 1(b).

It remains to show that (1.22) implies (1.21). In the remainder of the proof, c, C denote constants that may change from line to line, but do not depend on n. First note that (1.8) and (1.14) imply

$$c\log^4 T_k \leq \mathbb{V}\mathrm{ar}(Y_k) \leq C\log^4 T_k, \qquad \mathbb{V}\mathrm{ar}(X_{\tau(\ell)}) \geq c\sum_{k=1}^{\ell} \log^4 T_k. \tag{2.17}$$

For any fixed $\varepsilon > 0$, Theorem 5(III), (2.17) yield

$$\big|\mathbb{E}[\mathfrak{X}_{\tau(\ell)}]\big| \leq \sum_{k=1}^{\ell} \frac{\sqrt{\mathbb{V}\mathrm{ar}(Y_k)}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}}\bigg|\mathbb{E}\bigg[\frac{Y_k}{\sqrt{\mathbb{V}\mathrm{ar}(Y_k)}}\bigg]\bigg| \leq C\frac{\sum_{k=1}^{\ell} \log^{(\frac{4}{3}+\varepsilon)} T_k}{\sqrt{\sum_{k=1}^{\ell} \log^4 T_k}}. \tag{2.18}$$

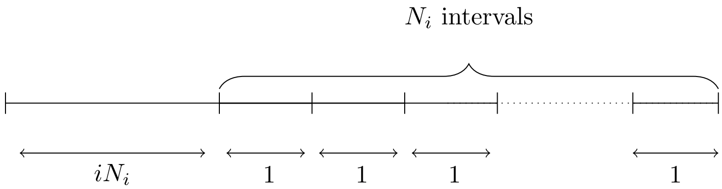

Fig. 3. The ith environment.

By Hölder's inequality it follows that

$$\sum_{k=1}^{\ell} \log^{(\frac{4}{3}+\varepsilon)} T_k \le \left(\sum_{k=1}^{\ell} \log^4 T_k\right)^{\frac{4/3+\varepsilon}{4}} \ell^{\frac{8/3-\varepsilon}{4}}, \tag{2.19}$$

which leads to

$$\left|\mathbb{E}[\mathfrak{X}_{\tau(\ell)}]\right| \le C \frac{(\sum_{k=1}^{\ell} \log^4 T_k)^{\frac{4/3+\varepsilon}{4}} \ell^{\frac{8/3-\varepsilon}{4}}}{\sqrt{\sum_{k=1}^{\ell} \log^4 T_k}} \le C \frac{\ell^{\frac{8/3-\varepsilon}{4}}}{(\sum_{k=1}^{\ell} \log^4 T_k)^{\frac{2/3-\varepsilon}{4}}}. \tag{2.20}$$

By (1.22) it follows that $\sum_{k=1}^{\ell} \log^4 T_k \ge c\ell^{4\gamma+1}$, and so

$$\left|\mathbb{E}[\mathfrak{X}_{\tau(\ell)}]\right| \le C \frac{\ell^{\frac{8/3-\varepsilon}{4}}}{(\ell^{4\gamma+1})^{\frac{2/3-\varepsilon}{4}}} = C\ell^{\frac{1}{2}-\gamma(\frac{2}{3}-\varepsilon)}. \tag{2.21}$$

when $\gamma > \frac{3}{4-6\varepsilon}$, $|\mathbb{E}[\mathfrak{X}_{\tau(\ell)}]| \to 0$.

To conclude the proof we take arbitrary $n \in \mathbb{N}$. We have

$$\left|\mathbb{E}[\mathfrak{X}_n]\right| \le \frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n)-1)})}{\mathbb{V}\mathrm{ar}(X_n)} \left|\mathbb{E}[\mathfrak{X}_{\tau(\ell(n)-1)}]\right| + \frac{\mathbb{V}\mathrm{ar}(\bar{Y}^n)}{\mathbb{V}\mathrm{ar}(X_n)} \frac{|\mathbb{E}[\bar{Y}^n]|}{\mathbb{V}\mathrm{ar}(\bar{Y}^n)}. \tag{2.22}$$

By (2.21), the first term in the right-hand side of (2.22) vanishes as $n \to \infty$. As for second term, it is bounded by $\varepsilon + (K/\mathbb{V}\mathrm{ar}(X_n))$. Indeed, by (1.8) and (2.1), for any $\varepsilon > 0$ there is a $K > 0$ such that $\bar{T}^n > K$ implies $|\mathbb{E}[\bar{Y}^n]|/\mathbb{V}\mathrm{ar}(\bar{Y}^n) < \varepsilon$. As $\mathbb{V}\mathrm{ar}(X_n) \to \infty$ and $\varepsilon > 0$ is arbitrary, $|\mathbb{E}[\mathfrak{X}_n]| \to 0$. □

2.3. *Breaking of transience*

Proof of (Ex.1). We construct the two maps τ' and τ'' in (Ex.1)

The cooling map τ'. There is a measure α on $(0, 1)$ such that (1.2) holds, $\langle\rho\rangle > 1$ and $\langle\log\rho\rangle < 0$. Since $\omega \mapsto \omega^{-1}$ is convex in $\mathbb{R}_+$, by Jensen's inequality we have

$$\langle 1+\rho\rangle = \langle\omega^{-1}\rangle > \frac{1}{\langle\omega\rangle}, \tag{2.23}$$

so $\langle\rho\rangle - 1 > 0$ implies that $1 - 2\langle\omega\rangle > 0$ and therefore $v := E_0^{\mu}[-Z_1] > 0$. By Proposition 1, $P_0^{\mu}(\lim_{n\to\infty} \frac{Z_n}{n} = 0) = 1$. In this case we can build a cooling map satisfying (1.23) for which $\mathbb{P}(\lim_{n\to\infty} X_n = -\infty) = 1$. The construction goes as follows. Using the notation introduced in (2.1), we set $N_0 = 0$ and for each $i \in \mathbb{N}$ we choose N_i such that $N_i \ge 2N_{i-1}$ and

$$E_0^{\mu}[Z_{iN_i}] < vN_i. \tag{2.24}$$

We take the *ith environment piece* to be composed of one increment of size iN_i followed by N_i increments of size 1 (see Figure 3). By (2.24), the increments over the ith piece have negative expectation.

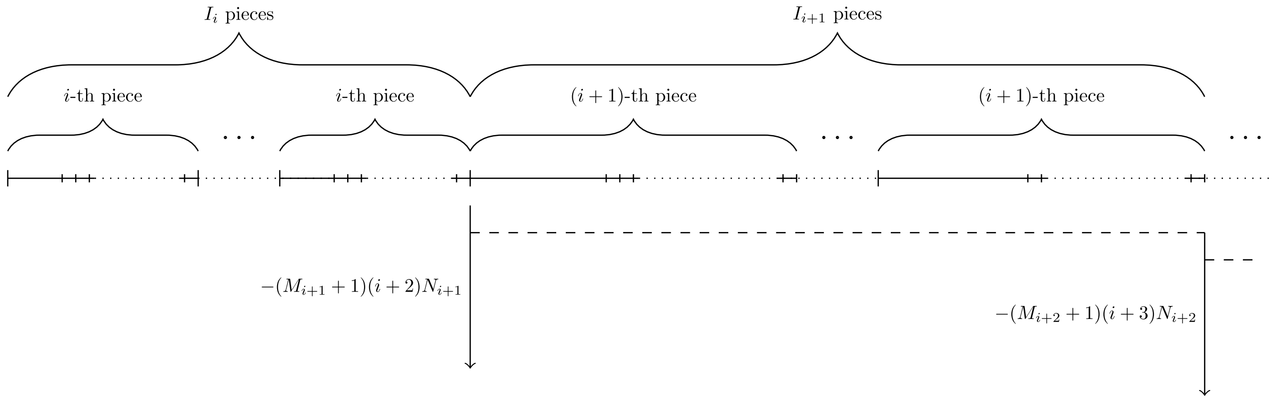

Fig. 4. Picture of the bound encoded in (2.29). The downarrows represent the decrease at the end of the last ith environment piece in comparison with the value at the beginning of the first ith environment piece. The dashed line represents the upper bound on the supremum of the random walk.

The idea to build the cooling map is, for each $i \in \mathbb{N}$, to repeat I_i times the ith environment piece in order to induce left-transience of the random walk. More precisely, for $i \in \mathbb{N}$, $j \in \mathbb{N}_0$, define $s(0) := 0$, $s(i,j) := s(i-1) + j(N_i+1)$, $s(i) := s(i, I_i)$, let $A_i := \{s(i,j)\colon j \le I_i\}$, and define the increments of the map τ', $\{T'_k = \tau'(k) - \tau'(k-1)\}_{k\in\mathbb{N}}$, by

$$T'_k := \begin{cases} iN_i & \text{if } k-1 \in A_i \text{ for some } i \in \mathbb{N}, \\ 1 & \text{else.} \end{cases} \tag{2.25}$$

We note that, irrespective of the choice of $(I_i)_{i\in\mathbb{N}}$, this construction ensures the Cesaro divergence of the increments.

Left transience. Before choosing I_i, we note that the displacement over different ith environment pieces are i.i.d random variables. For $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$, denote such displacements by

$$D_i^{(j)} := Z_{iN_i}^{(s(i,j)+1)} + \sum_{k=s(i,j)+2}^{s(i,j)+N_i+1} Z_1^{(k)}. \tag{2.26}$$

By the strong law of large numbers, there is a sequence of positive integers $(M_i)_{i\in\mathbb{N}}$ satisfying

$$M_{i+1} \ge M_i, \quad \mathcal{P}\left[\sup_{m\ge M_i} \sum_{j=1}^{m} D_i^{(j)} \ge 0\right] < 2^{-i}. \tag{2.27}$$

The sequence $(I_i)_{i\in\mathbb{N}}$ is chosen to satisfy the following condition:

$$\mathcal{P}\left(\sum_{j=1}^{I_i} D_i^{(j)} \ge -(M_{i+1}+1)(i+2)N_{i+1}\right) < 2^{-i}, \tag{2.28}$$

By the Borel–Cantelli Lemma, due to (2.1) (2.27) and (2.28), we have that eventually

$$\begin{aligned} &X_{\tau'(s(i))} - X_{\tau'(s(i-1))} < -(M_{i+1}+1)(i+2)N_{i+1}, \\ &\sup_{n\in[\tau'(s(i)),\tau'(s(i+1))]} X_n - X_{\tau'(s(i))} < (M_{i+1}+1)(i+2)N_{i+1}, \end{aligned} \tag{2.29}$$

where the second line follows from the fact that RWCRE is a nearest-neighbour random walk, so that to go beyond $(M_{i+1}+1)(i+2)N_{i+1}$ we need at least M_{i+1} increments, in which case a positive displacement is bounded by (2.27). The conditions in (2.29) imply left-transience (see Figure 4).

The map τ''. In the same setting as (Ex.1), we construct a recurrent RWCRE by modifying the cooling map τ', inserting large intervals. First note that, since $\langle \log \rho \rangle < 0$, we can define, for any $N \in \mathbb{N}$ and $\varepsilon > 0$,

$$H(N,\varepsilon) := \inf\left\{m \in \mathbb{N}\colon \mathcal{P}\left(\inf_{n>m} Z_n^{(1)} \le N\right) < \varepsilon\right\}. \tag{2.30}$$

Let $T'_k := \tau'(k) - \tau'(k-1)$. Inductively, define the increment sequence $\{T''_k\}_{k\in\mathbb{N}}$ by setting

$$\begin{cases} T''_k = T'_k, & k \in \mathbb{N} \setminus \{s(i) \colon i \in \mathbb{N}\}, \\ T''_{s(i)} = T'_{s(i)} + H(\sum_{i=1}^{s(i-1)} T''_i, 2^{-i}), & i \in \mathbb{N}, \end{cases} \tag{2.31}$$

where $s(0) := 0$ and

$$s(i) := \left\{ \inf_{k > s(i-1)} \colon \mathcal{P}\left(\sum_{i=s(i-1)+1}^{k} Z^{(i)}_{T'_i} \geq - \sum_{i=1}^{s(i-1)} T''_i \right) < 2^{-i} \right\}. \tag{2.32}$$

With these definitions, set $\tau''(k) := \sum_{i=1}^{k} T''_i$ and note that, since $T''_k \geq T'_k$, the increments are Cesaro diverging. We conclude the proof, by noting that (2.1), (2.30), (2.32) and the Borel–Cantelli lemma imply

$$\begin{aligned} &\mathbb{P}\big(X_{\tau''(s(i)-1)} - X_{\tau''(s(i-1))} > -\tau''\big(s(i-1)\big) \text{ i.o.}\big) = 0, \\ &\mathbb{P}\big(X_{\tau''(s(i))} - X_{\tau''(s(i)-1)} < \tau''\big(s(i)-1\big) \text{ i.o.}\big) = 0. \end{aligned} \tag{2.33}$$

□

2.4. *Breaking of recurrence*

Proof of (Ex.2). We show that there exists a recurrent non-symmetric α and a cooling map τ for which (α, τ) is transient. The construction that follows is possible because, by Theorem 5(1), there is a recurrent non-symmetric α for which at least one of the sets

$$\mathcal{N}_+ := \big\{n \in \mathbb{N} \colon E^{\mu}_0[Z_n] > 0\big\}, \qquad \mathcal{N}_- := \big\{n \in \mathbb{N} \colon E^{\mu}_0[Z_n] < 0\big\}, \tag{2.34}$$

is infinite. Assume without loss that $\mathcal{N}_+ = \{n_1 < n_2 < \cdots\}$ is infinite.

Successively choose N_j consecutive increments of size n_j for every $j \in \mathbb{N}$, where the sequence $(N_j)_{j\in\mathbb{N}}$ will be chosen below. More precisely, define $s(0) := 0$, $s(j) := s(j-1) + N_j$, $j \in \mathbb{N}$, and let

$$T_k := \sum_j n_j \mathbb{1}_{(s(j-1), s(j)]}(k), \tag{2.35}$$

where N_j is defined in (2.38), as we explain next. By the strong law of large numbers, for all $j \in \mathbb{N}$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} Z^{(k)}_{n_j} = E^{\mu}_0[Z_{n_j}] > 0, \quad \mathcal{P}\text{-a.s.}, \tag{2.36}$$

from which it follows that there are $(M_j)_{j\in\mathbb{N}}$ satisfying

$$\mathcal{P}\left(\inf_{m \geq M_j} \sum_{k=1}^{m} Z^{(k)}_{n_j} \leq 0 \right) \leq \frac{1}{j^2}. \tag{2.37}$$

Next, pick N_j such that

$$\mathcal{P}\left(\frac{1}{N_j} \sum_{k=1}^{N_j} Z^{(k)}_{n_j} \leq \frac{1}{2} E^{\mu}_0[Z_{n_j}] \right) \leq \frac{1}{j^2}, \tag{2.38}$$

and

$$\frac{1}{2} N_j E^{\mu}_0[Z_{n_j}] \geq (M_{j+1} + 1) n_{j+1}. \tag{2.39}$$

Define $s(0) := 0$, $s(j) := s(j-1) + N_j$, for $j \in \mathbb{N}$. By (2.1), it follows that

$$\mathbb{P}\left(X_{\tau(s(j))} - X_{\tau(s(j-1))} \leq \frac{1}{2} N_j E^{\mu}_0[Z_{n_j}] \right) \leq \frac{1}{j^2}. \tag{2.40}$$

Consequently, by the first Borel–Cantelli lemma, it follows that $\mathbb{P}$-a.s. for j sufficiently large,

$$X_{\tau(s(j))} > X_{\tau(s(j-1))} + \frac{1}{2} N_j E_0^\mu [Z_{n_j}]. \tag{2.41}$$

Now let $A_j = \{\inf_{m \ge M_j} \sum_{k=1}^m Z_{n_j}^{(k)} \le 0\}$, and note that (2.41), (2.39) and (2.1) imply

$$\mathbb{P}(X_n = 0 \text{ i.o.}) \le \mathcal{P}(A_j \text{ i.o.}) = 0, \tag{2.42}$$

where the equality follows from (2.37). □

3. Proofs: Mixed fluctuations

3.1. *Mixed fluctuations in the Sinai-regime*

Proof of Theorem 2. The proof is organised into several steps.

Tightness. Tightness follows from the constant variance scaling in (1.16), because for any $K > 0$, by Chebyshev's inequality,

$$\mathbb{P}\big(\big|\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]\big| > K\big) \le \frac{1}{K^2}. \tag{3.1}$$

We identify the limit points. As noted in Remark 1, the sequence $(\lambda_{\tau,n})_{n\in\mathbb{N}_0}$ of $\ell_2(\mathbb{N}_0)$-unit vectors in (1.24) admits a subsequence $(n_i)_{i\in\mathbb{N}_0}$ for which there is a vector $\lambda_* \in \ell_2(\mathbb{N}_0)$ with $\|\lambda_*\|_2 \le 1$ such that,

$$\lim_{i\to\infty} \lambda_{\tau,n_i}(k) = \lambda_*(k) \quad \forall k \in \mathbb{N}_0. \tag{3.2}$$

We proceed by comparing $\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]$ with $V^{\otimes\lambda_{\tau,n}}$. By (1.8) with $p = 2$,

$$\sigma_0^2(n) := \mathrm{Var}\bigg[\frac{Z_n}{\sigma_0^2 \log^2 n}\bigg] \to \sigma_V^2. \tag{3.3}$$

Coupling with error term. Consider a probability space $(S, \mathcal{S}, \mathcal{P})$ that is rich enough to include the sequence of random variables $(V_k)_{k\in\mathbb{N}_0}$ defined in Section 1.4.2 and an array of random variables $(R_n^{(k)})_{k,n\in\mathbb{N}_0}$ satisfying:

(H1) For any $k, n \in \mathbb{N}_0$ and $x \in \mathbb{R}$,

$$P_0^\mu\bigg(\frac{Z_n - E_0^\mu[Z_n]}{\sigma_0(n)} \le x\bigg) = \mathcal{P}\big(\sigma_V^{-1} V_k + R_n^{(k)} \le x\big). \tag{3.4}$$

(H2) For all $k, n \in \mathbb{N}_0$, $\mathcal{E}[R_n^{(k)}] = 0$, where $\mathcal{E}$ stands for expectation w.r.t. $\mathcal{P}$.
(H3) $(V_k, R_n^{(k)})_{n,k\in\mathbb{N}_0}$ are independent in k under $\mathcal{P}$.
(H4) $R_n^{(k)}$ vanishes in L^2, i.e.,

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}_0} \mathcal{E}\big[\big(R_n^{(k)}\big)^2\big] = 0. \tag{3.5}$$

The construction of the above random variables can be implemented via the Skorohod representation. More concretely, it is based on a family $(U^{(k)})_{k\in\mathbb{N}}$ of independent uniform random variables on $(0, 1)$, which we may assume to be defined in $(S, \mathcal{S}, \mathcal{P})$. For each k, let

$$V_k := \sigma_V F_{\frac{V}{\sigma_V}}^{-1}\big(U^{(k)}\big), \qquad R^{(k)} := F_{\frac{Z_n - E_0^\mu[Z_n]}{\sigma_0(n)}}^{-1}\big(U^{(k)}\big) - \sigma_V^{-1} V_k, \tag{3.6}$$

where, for a random variable X, F_X^{-1} is the generalized inverse function of the distribution of X (see [13, p. 6, Skorohod Theorem]). Properties (H1)–(H3) follow from the construction, while (H4) is a consequence of (1.8). By (1.25) and (3.4), for any bounded continuous function f,

$$\mathbb{E}\big[f\big(\mathfrak{X}_n-\mathbb{E}[\mathfrak{X}_n]\big)\big]=\mathcal{E}\left[f\left(V^{\otimes\lambda_{\tau,n}}+\sum_{k=0}^{\ell(n)-1}\lambda_{\tau,n}(k)R_{T_k}^{(k)}\right)\right], \tag{3.7}$$

i.e., $\mathfrak{X}_n-\mathbb{E}[\mathfrak{X}_n]$ has the same distribution under $\mathbb{P}$ as the $\lambda_{\tau,n}$-mixture of Sinai–Kesten random variables defined in (1.26), up to an error term that is negligible because of (3.5).

The proof proceeds in two parts. First, we remove the error term. Second, we examine the convergence of the main term.

• *asymptotics of the error terms.* As a consequence of (H2)–(H3),

$$\begin{aligned}&\lim_{J\to\infty}\limsup_{n\to\infty}\mathcal{E}\left[\left(\sum_{k=0}^{\ell(n)-1}\lambda_{\tau,n}(k)R_{T_k}^{(k)}\mathbb{1}_{\{T_k>J\}}\right)^2\right]\\&\quad=\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{k=0}^{\ell(n)-1}\lambda_{\tau,n}^2(k)\mathcal{E}\big[\big(R_{T_k}^{(k)}\mathbb{1}_{\{T_k>J\}}\big)^2\big]=0,\end{aligned} \tag{3.8}$$

where the last equality follows from (3.5). For any fixed $J>0$, under $\mathbb{P}$, $(Y_k\mathbb{1}_{\{T_k\le J\}})_{k\in\mathbb{N}_0}$ is a collection of bounded independent random variables. Thus, by the CLT for i.i.d. random variables, for any bounded continuous function $f:\mathbb{R}\to\mathbb{R}$ we have

$$\begin{aligned}\lim_{n\to\infty}\Bigg|&\mathbb{E}\left[f\left(\sum_{k=0}^{\ell(n)-1}\lambda_{\tau,n}(k)\frac{Y_k-\mathbb{E}[Y_k]}{\sigma_0(T_k)}\mathbb{1}_{\{T_k\le J\}}\right)\right]\\&-\mathcal{E}\left[f\left(\left(\sum_{k=0}^{\ell(n)-1}\lambda_{\tau,n}^2(k)\mathbb{1}_{\{T_k\le J\}}\right)^{\frac12}\Phi\right)\right]\Bigg|=0\end{aligned} \tag{3.9}$$

with Φ a standard normal random variable. In view of (3.7)–(3.9), to prove Theorem 2, it suffices to show that

$$V^{\otimes\lambda_{\tau,n}}\overset{(d)}{=}V^{\otimes\lambda_n^\downarrow}\overset{(d)}{\longrightarrow}V^{\otimes\lambda_*}+a(\lambda_*)\Phi, \tag{3.10}$$

where the equality is due to Lemma 1, whose proof is given in the sequel. We note that tightness in combination with (3.10) characterizes all limit points in the weak topology of the sequence $(\mathfrak{X}_n-\mathbb{E}[\mathfrak{X}_n])_{n\in\mathbb{N}_0}$ as mixtures of weighted independent Sinai–Kesten and Gaussian random variables.

Convergence of mixtures and removal of the error term. We explain why (3.10) suffices. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $\|f\|_\infty<\infty$ and $\|f'\|_\infty<\infty$. Abbreviate

$$\begin{aligned}&\tilde{\mathfrak{X}}_n:=\mathfrak{X}_n-\mathbb{E}[\mathfrak{X}_n],\qquad\tilde{\mathfrak{Y}}_k:=\frac{Y_k-\mathbb{E}[Y_k]}{\sqrt{\mathbb{V}\mathrm{ar}(Y_k)}},\\&\overline{\lambda_n}^{0,J}(k):=\lambda_{\tau,n}(k)\mathbb{1}_{\{T_k<J\}},\qquad\overline{\lambda_n}^{J,\infty}(k):=\lambda_{\tau,n}(k)-\overline{\lambda_n}^{0,J}(k),\\&\overline{\tilde{\mathfrak{X}}_n}^{0,J}:=\sum_{k=0}^{\ell(n)-1}\overline{\lambda_n}^{0,J}(k)\tilde{\mathfrak{Y}}_k,\qquad\overline{\tilde{\mathfrak{X}}_n}^{J,\infty}:=\tilde{\mathfrak{X}}_n-\overline{\tilde{\mathfrak{X}}_n}^{0,J},\\&\overline{R_n}^{0,J}:=\sum_{k=0}^{\ell(n)-1}\overline{\lambda_n}^{J,\infty}(k)R_{T_k}^{(k)},\qquad\overline{R_n}^{J,\infty}:=R_n-\overline{R_n}^{0,J},\end{aligned} \tag{3.11}$$

and note that from (H1) and (H3) we have

$$\mathbb{E}\big[f(\tilde{\mathfrak{X}}_n)\big] = \mathbb{E}\big[f\big(\overline{\tilde{\mathfrak{X}}_n}^{0,J} + \overline{\tilde{\mathfrak{X}}_n}^{J,\infty}\big)\big]$$
$$= \mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}} + \overline{R_n}^{0,J} + V^{\otimes\overline{\lambda_n}^{J,\infty}} + \overline{R_n}^{J,\infty}\big)\big]. \tag{3.12}$$

For fixed $J > 0$, $\sup_{k\in\mathbb{N}_0} \overline{\lambda_n}^{0,J}(k) \to 0$, because if $T_k < J$, then the numerator in $\lambda_{\tau,n}(k)$ remains bounded while the denominator diverges (recall (1.24)). Hence, by the Lindeberg–Feller theorem for triangular arrays [5, Theorem 2.4.5],

$$\lim_{n\to\infty}\big|\mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}}\big)\big] - \mathcal{E}\big[f\big(\|\overline{\lambda_n}^{0,J}\|_2\Phi\big)\big]\big| = 0. \tag{3.13}$$

Via (H1) and (H3), (3.9) translates into

$$\lim_{n\to\infty}\big|\mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}} + \overline{R_n}^{0,J}\big)\big] - \mathcal{E}\big[f\big(\|\overline{\lambda_n}^{0,J}\|_2\Phi\big)\big]\big| = 0. \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\lim_{n\to\infty}\big|\mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}} + \overline{R_n}^{0,J} + V^{\otimes\overline{\lambda_n}^{J,\infty}} + \overline{R_n}^{J,\infty}\big)\big]$$
$$- \mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}} + V^{\otimes\overline{\lambda_n}^{J,\infty}} + \overline{R_n}^{J,\infty}\big)\big]\big| = 0. \tag{3.15}$$

Hence we can estimate

$$\limsup_{n\to\infty}\big|\mathbb{E}\big[f(\tilde{\mathfrak{X}}_n)\big] - \mathcal{E}\big[f\big(V^{\otimes\lambda_*} + a(\lambda_*)\Phi\big)\big]\big|$$
$$= \limsup_{n\to\infty}\big|\mathbb{E}\big[f\big(\overline{\tilde{\mathfrak{X}}_n}^{0,J} + \overline{\tilde{\mathfrak{X}}_n}^{J,\infty}\big)\big] - \mathcal{E}\big[f\big(V^{\otimes\lambda_n}\big)\big]\big|$$
$$= \limsup_{n\to\infty}\big|\mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}} + V^{\otimes\overline{\lambda_n}^{J,\infty}} + \overline{R_n}^{J,\infty}\big)\big]$$
$$- \mathcal{E}\big[f\big(V^{\otimes\overline{\lambda_n}^{0,J}} + V^{\otimes\overline{\lambda_n}^{J,\infty}}\big)\big]\big|$$
$$\le \inf_{\delta,J>0}\limsup_{n\to\infty} C_f\big(\delta + \delta^{-2}\mathcal{E}\big[\big(\overline{R_n}^{J,\infty}\big)^2\big]\big), \tag{3.16}$$

where C_f is a constant that depends on $\|f\|_\infty, \|f'\|_\infty$. The first equality follows from (3.10), the second from (3.12)–(3.15), and the inequality from the following standard bound, which we state for generic random variables X and H:

$$\big|\mathcal{E}\big[f(X+H) - f(X)\big]\big|$$
$$\le \big|\mathcal{E}\big[\big(f(X+H) - f(X)\big)\mathbb{1}_{\{|H|\le\delta\}}\big]\big|$$
$$+ \big|\mathcal{E}\big[\big(f(X+H) - f(X)\big)\mathbb{1}_{\{|H|>\delta\}}\big]\big|$$
$$\le C_f\delta + C_f\mathcal{P}\big(|H|>\delta\big) \le C_f\delta + C_f\delta^{-2}\mathcal{E}\big[|H|^2\big]. \tag{3.17}$$

From (3.8),

$$\lim_{J\to\infty}\sup_{n\in\mathbb{N}}\mathcal{E}\big[\big(\overline{R_n}^{J,\infty}\big)^2\big] = 0, \tag{3.18}$$

and hence (3.16) yields

$$\tilde{\mathfrak{X}}_n = \mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n] \xrightarrow{(d)} V^{\otimes\lambda_*} + a(\lambda_*)\Phi, \tag{3.19}$$

which is the claim in (1.28) with convergence in distribution. We note that the role of the truncation by J in (3.11) is to capture the contribution of the small increments to the Gaussian random variable that appears in the limit.

L^p convergence. We recall that convergence in L^p is understood as the existence of a coupling $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of the random variables and the limit point such that their difference converges to 0 in L^p. Given the convergence in distribution $\mathfrak{X}_n \to \mathfrak{X}_*$, via the Skorohod representation theorem we may consider a coupling for which $\tilde{\mathfrak{X}}_n \overset{d}{=} \mathfrak{X}_n$, $\tilde{\mathfrak{X}}_* \overset{d}{=} \mathfrak{X}_*$, and $\tilde{\mathfrak{X}}_n - \tilde{\mathfrak{X}}_* \to 0$ almost surely. Therefore, to prove the convergence in L^p it suffices to note that for any $r \in \mathbb{N}$,

$$\mathbb{E}\big[\tilde{\mathfrak{X}}_n^{2r}\big] = \mathcal{E}\left[\left(\sum_{k=0}^{\ell(n)} \lambda_{\tau,n}(k)\tilde{\mathcal{Y}}_k\right)^{2r}\right] \le C_{2r} < \infty, \tag{3.20}$$

where we use that $\|\lambda_{\tau,n}\|_2^2 = 1$, $\mathcal{E}[\tilde{\mathcal{Y}}_k] = 0$ for all $k \in \mathbb{N}_0$, and $\sup_k \mathcal{E}[(\tilde{\mathcal{Y}}_k)^{2r}] < C$ by (1.8). The convergence in distribution in (3.19), combined with the uniform bound in (3.20), implies that (1.28) holds. Indeed, let $\tilde{\mathbb{E}}$ denote expectation with respect to $\tilde{\mathbb{P}}$. Since $2r > p$, by Hölder's inequality we have for any $\varepsilon > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \tilde{\mathbb{E}}\big[|X_n - X_*|^p\big] &\le \limsup_n \tilde{\mathbb{E}}\big[|X_n - X_*|^p \mathbb{1}_{|X_n - X_*| > \varepsilon}\big] + \varepsilon^p \\ &\le \limsup_n \tilde{\mathbb{E}}\big[|X_n - X_*|^{2r}\big]^{\frac{p}{2r}} \tilde{\mathbb{P}}\big(|X_n - X_*| > \varepsilon\big)^{1-\frac{p}{2r}} + \varepsilon^p = \varepsilon^p.\end{aligned} \tag{3.21}$$

Limit of Sinai–Kesten mixtures. In order to prove Theorem 2, it remains to show (3.10). We divide this part of the proof into steps.

A triangle inequality. To simplify notation, let us drop the index i from the subsequence $(n_i)_{i\in\mathbb{N}}$ satisfying (3.2). Also, let $\lambda_n := \lambda^{\downarrow}_{\tau,n}$. Because (3.2) holds, it follows that

$$\sum_{j\in\mathbb{N}_0} \lambda_*^2(j) = 1 - a^2 \quad \text{for some } a \ge 0, \tag{3.22}$$

$$\lim_{K\to\infty} \sum_{j>K} \lambda_*^2(j) = 0, \tag{3.23}$$

$$\lim_{n\to\infty} \sum_{j=1}^{K} \big|\lambda_n(j) - \lambda_*(j)\big| = 0, \quad K \in \mathbb{N}, \tag{3.24}$$

$$j \ge K \quad \Longrightarrow \quad \lambda_n(j) \le \frac{1}{\sqrt{K}}, \quad K \in \mathbb{N}, \tag{3.25}$$

where (3.25) follows from $1 \ge \sum_{j=0}^{K-1} \lambda_n^2(j) \ge K\lambda_n^2(K)$. Let $(\Phi_j)_{j\in\mathbb{N}_0}$ be a family of i.i.d. standard normal random variables defined on the same probability space $(S, \mathcal{S}, \mathcal{P})$. Set $\Phi^{\otimes\lambda} := \sum_{j\in\mathbb{N}_0} \lambda(j)\Phi_j$ for a given vector $\lambda \in \ell_2(\mathbb{N}_0)$, and note that the following isometry is in force (recall (1.26)):

$$\mathcal{E}\big[\big|V^{\otimes\lambda_n}\big|^2\big] = \mathcal{E}\big[\big|\Phi^{\otimes\lambda_n}\big|^2\big] = \|\lambda_n\|_{\ell_2}. \tag{3.26}$$

To prove (3.10), we will show via a truncation that, for any $f : \mathbb{R} \to \mathbb{R}$ with bounded derivatives up to order three, $\max\{\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty\} < \infty$,

$$\mathcal{E}\big[f\big(V^{\otimes\lambda_n}\big) - f\big(V^{\otimes\lambda_*} + a\Phi_0\big)\big] \to 0. \tag{3.27}$$

Indeed, for $\lambda \in \ell_2(\mathbb{N}_0)$ and $k, K \in \mathbb{N}_0$ with $k < K$, set

$$\lambda^{k,K}(j) := \begin{cases} 0 & \text{if } 0 \le j < k, \\ \lambda(j) & \text{if } k \le j < K, \\ 0 & \text{if } j \ge K, \end{cases} \tag{3.28}$$

and $\lambda^{K,\infty}(j) := \lambda(j) - \lambda^{0,K}(j)$. By the triangle inequality, for all $K \in \mathbb{N}$,

$$
\begin{aligned}
&\big|\mathcal{E}\big[f\big(V^{\otimes\lambda_n}\big) - f\big(V^{\otimes\lambda_*} + a\Phi_0\big)\big]\big| \\
&\quad\le \big|\mathcal{E}\big[f\big(V^{\otimes\lambda_n}\big) - f\big(V^{\otimes\lambda_n^{0,K}} + \Phi^{\otimes\lambda_n^{K,\infty}}\big)\big]\big| \\
&\qquad + \big|\mathcal{E}\big[f\big(V^{\otimes\lambda_n^{0,K}} + \Phi^{\otimes\lambda_n^{K,\infty}}\big) - f\big(V^{\otimes\lambda_*^{0,K}} + a\Phi_0\big)\big]\big| \\
&\qquad + \big|\mathcal{E}\big[f\big(V^{\otimes\lambda_*^{0,K}} + a\Phi_0\big) - f\big(V^{\otimes\lambda_*} + a\Phi_0\big)\big]\big|.
\end{aligned}
\tag{3.29}
$$

To conclude the proof, we will argue that the three terms in the right-hand side of (3.29) can be made arbitrarily small.

Asymptotic negligibility of the last terms in the triangle inequality. The last two terms in (3.29) can be treated via (3.17), by using (3.22)–(3.26). Indeed, the third term in the right-hand side of (3.29) tends to zero as $K \to \infty$ due to (3.23). For the second term, note that, by (3.22), (3.24) and $\sum_{i\in\mathbb{N}_0} \lambda_n^2(i) = 1$,

$$
\lim_{K\to\infty}\lim_{n\to\infty}\big\|\lambda_n^{K,\infty}\big\|_2 = \lim_{K\to\infty}\lim_{n\to\infty}\sum_{i>K}\lambda_n^2(i) = a^2.
\tag{3.30}
$$

By (3.17), we get for any $\delta > 0$,

$$
\begin{aligned}
&\big|\mathcal{E}\big[f\big(V^{\otimes\lambda_n^{0,K}} + \Phi^{\otimes\lambda_n^{K,\infty}}\big)\big] - \mathcal{E}\big[f\big(V^{\otimes\lambda_*^{0,K}} + a\Phi_0\big)\big]\big| \\
&\quad= \big|\mathcal{E}\big[f\big(V^{\otimes\lambda_n^{0,K}} + \big\|\lambda_n^{K,\infty}\big\|_2^{\frac{1}{2}}\Phi_0\big)\big] - \mathcal{E}\big[f\big(V^{\otimes\lambda_*^{0,K}} + a\Phi_0\big)\big]\big| \\
&\quad\le C_f\delta + C_f\delta^{-2}\mathcal{E}\big[\big|\big(a - \big\|\lambda_n^{K,\infty}\big\|_2^2\big)\Phi + V^{\lambda_n^{0,K}} - V^{\lambda_*^{0,K}}\big|\big]
\end{aligned}
\tag{3.31}
$$

and therefore, by (3.24) and (3.30), the second term vanishes as one takes $K \to \infty$ and then $n \to \infty$. To show that the first term in the right-hand side of (3.29) vanishes as well, we prove a bound that is independent of n by using a classical argument in the spirit of the Lindeberg–Feller theorem (see [5, Theorem 2.4.5]).

Interpolation of random variables. We consider

$$
W_{K,n}(M) := V^{\otimes\lambda_n^{0,K\wedge M}} + \Phi^{\otimes\lambda_n^{K,M}} + V^{\otimes\lambda_n^{M,\infty}}
\tag{3.32}
$$

obtained from $V^{\otimes\lambda_n}$ after replacing $\sigma_V^{-1}V_j$ by Φ_j for $K < j \le M$ in (1.26). Note that, by (3.28), $W_{K,n}(M) = V^{\otimes\lambda_n}$ for $M \le K$, and also that, for fixed $K, n \in \mathbb{N}_0$, $W_{K,n}(M) \xrightarrow{L^2} W_{K,n}(\infty) := V^{\otimes\lambda_n^{0,K}} + \Phi^{\otimes\lambda_n^{K,\infty}}$. With these auxiliary random variables, we see that in order to show that the first term in the right-hand side of (3.29) vanishes, we must prove that

$$
\limsup_{K\to\infty}\limsup_{n\to\infty}\big|\mathcal{E}\big[f\big(W_{K,n}(K)\big) - f\big(W_{K,n}(\infty)\big)\big]\big| = 0.
\tag{3.33}
$$

We will show that

$$
\limsup_{K\to\infty}\limsup_{n\to\infty}\sum_{M>K}\big|\mathcal{E}[f\big(W_{K,n}(M)\big) - f\big(W_{K,n}(M+1)\big)\big| = 0,
\tag{3.34}
$$

which in particular implies (3.33).

Bound by Taylor expansion. For the proof of (3.34) define, for $M \ge K$,

$$
W_{K,n}^*(M) := W_{K,n}(M) - \sigma_V^{-1}\lambda_n(M)V_M.
\tag{3.35}
$$

Note that $W_{K,n}^*(M)$ is independent of Φ_M and V_M, and that

$$
W_{K,n}(M+1) = W_{K,n}^*(M) + \lambda_n(M)\Phi_M
\tag{3.36}
$$

Consider the Taylor expansion of f up to second order,

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + C_f\big(|h|^2 \wedge |h|^3\big). \tag{3.37}$$

Note that, for any $\varepsilon > 0$, $|h|^2 \wedge |h|^3 \le |h|^2 \mathbb{1}_{\{|h|>\varepsilon\}} + |h|^3$, and that for $j \in \mathbb{N}_0$,

$$\mathcal{E}[\Phi_j] = 0, \qquad \mathcal{E}[V_j] = 0, \qquad \mathcal{E}\big[\Phi_j^2\big] = \mathcal{E}\big[\big(\sigma_V^{-1} V_j\big)^2\big] = 1. \tag{3.38}$$

Use (3.35) and (3.36), respectively, to expand $f(W_{K,n}(M)) - f(W_{K,n}(M+1))$ with the help of (3.37), which together with the triangle inequality yield

$$\begin{aligned}
&\big|\mathcal{E}\big[f\big(W_{K,n}(M)\big) - f\big(W_{K,n}(M+1)\big)\big]\big| \\
&\quad\le C_f\big(\mathcal{E}\big[\big|\lambda_n(M)V_M\big|^3 + \big|\lambda_n(M)\Phi_M\big|^3\big] \\
&\qquad + \mathcal{E}\big[\big|\lambda_n(M)\sigma_V^{-1}V_M\big|^2 \mathbb{1}_{\{|\lambda_n(M)\sigma_V^{-1}V_M|^2>\varepsilon\}} \\
&\qquad + \big|\lambda_n(M)\Phi_M\big|^2 \mathbb{1}_{\{|\lambda_n(M)\Phi_M|^2>\varepsilon\}}\big]\big).
\end{aligned} \tag{3.39}$$

Next, note that Hölder's inequality and Markov's inequality imply that

$$\begin{aligned}
&\mathcal{E}\big[\big|\lambda_n(M)\sigma_V^{-1}V_M\big|^2 \mathbb{1}_{\{|\lambda_n(M)\sigma_V^{-1}V_M|^2>\varepsilon\}}\big] \\
&\quad\le \lambda_n(M)^2 \mathcal{E}\big[\big|\sigma_V^{-1}V_M\big|^4\big]^{\frac{1}{2}} \mathcal{P}\big(\big|\lambda(M)\sigma_V^{-1}V_M\big|^2 > \varepsilon\big)^{\frac{1}{2}} \\
&\quad\le \lambda_n(M)^2 \mathcal{E}\big[\big|\sigma_V^{-1}V_1\big|^4\big]^{\frac{1}{2}} \frac{\lambda(M)\mathcal{E}[|\sigma_V^{-1}V_1|^2]^{\frac{1}{2}}}{\sqrt{\varepsilon}} \\
&\quad\le \lambda_n(M)^3 \frac{\mathcal{E}[|\sigma_V^{-1}V_1|^4]^{\frac{3}{4}}}{\sqrt{\varepsilon}}.
\end{aligned} \tag{3.40}$$

Since $\mathcal{E}[|V_1|^4] < \infty$, (3.34) follows from (3.39) and (3.40) via an analogous argument as for the terms involving Φ_M, because, for some $C > 0$ independent of K and n,

$$\begin{aligned}
&\sum_{M>K} \big(\mathcal{E}\big[\big|\lambda_n(M)\sigma_V^{-1}V_M\big|^2 \mathbb{1}_{\{|\lambda_n(M)V_M|^2>\varepsilon\}}\big] \\
&\qquad + \mathcal{E}\big[\big|\lambda_n(M)\sigma_V^{-1}V_M\big|^3\big]\big) \\
&\quad\le C \sum_{M>K} \lambda_n^3(M) \le C \sup_{M>K} \lambda_n(M) \sum_{M>K} \lambda_n^2(M) \le C \frac{1}{\sqrt{K}},
\end{aligned} \tag{3.41}$$

where the last inequality follows from (3.25) and $\sum_{M>K} \lambda_n^2(M) \le 1$. □

3.2. *Characterisation of Sinai–Kesten mixtures*

Proof of Lemma 1. To prove Lemma 1, it is equivalent to prove

$$\lambda \sim \lambda' \quad \Longrightarrow \quad V^{\otimes\lambda} \stackrel{(d)}{=} V^{\otimes\lambda'}, \tag{3.42}$$

$$[\lambda] \neq \big[\lambda'\big] \quad \Longrightarrow \quad V^{\otimes\lambda} \stackrel{(d)}{\neq} V^{\otimes\lambda'}. \tag{3.43}$$

Proof of (3.42). Let λ be a vector with finitely many non-zero entries. In view of the i.i.d. property of the random variables $(V_j)_{j\in\mathbb{N}_0}$, we have that

$$\lambda \sim \lambda' \quad \Longrightarrow \quad V^{\otimes\lambda} \stackrel{(d)}{=} V^{\otimes\lambda'}. \tag{3.44}$$

For general $\lambda \in \ell_2(\mathbb{N}_0)$, let $\sigma, \sigma' \colon \mathbb{N}_0 \to \mathbb{N}_0$ be such that $\lambda\sigma(i) = \lambda^{\downarrow}(i)$, and $\lambda'\sigma'(i) = \lambda^{\downarrow}(i)$. Define

$$\lambda^{\sigma,0,k}(j) = \begin{cases} \lambda(j) & \text{if } j \in \{\sigma(i) \colon i < k\}, \\ 0 & \text{else.} \end{cases} \tag{3.45}$$

As in (3.44), $V^{\otimes \lambda^{\sigma,0,k}} \stackrel{(d)}{=} V^{\otimes \lambda'^{\sigma',0,k}}$. By (3.17), for any $\delta > 0$,

$$\left|\mathcal{E}\left[f\left(V^{\lambda^{\sigma,0,k}}\right)\right] - \mathcal{E}\left[f\left(V^{\lambda}\right)\right]\right| \le C_f \delta + C_f \delta^{-2} \left\|\lambda - \lambda^{\sigma,0,k}\right\|_2^2. \tag{3.46}$$

Since $\|\lambda^{\sigma,0,k}\|_2 \to \|\lambda\|_2$, the claim follows.

Proof of (3.43). We may assume without loss of generality that $\lambda = \lambda^{\downarrow}$, $\lambda' = \lambda'^{\downarrow}$ and that there is a $j_0 \in \mathbb{N}_0$ for which

$$\lambda(j) = \lambda'(j) \quad \forall 0 \le j < j_0, \qquad \lambda(j_0) > \lambda'(j_0). \tag{3.47}$$

Let $t \mapsto \mathcal{L}_X(t) := \mathcal{E}[e^{tX}]$ be the moment generating function of a random variable X. To show that the distributions of $V^{\otimes\lambda}$ and $V^{\otimes\lambda'}$ are different, by [4, Theorem 30.1] we must show that the moment generating function of $V^{\otimes\lambda}$ is finite in a neighbourhood of the origin and

$$\exists t \in \mathbb{R} \colon \mathcal{L}_{V^{\otimes\lambda}}(t) \neq \mathcal{L}_{V^{\otimes\lambda'}}(t). \tag{3.48}$$

The proof proceeds in three steps. First, we analyse the Laplace transform of $V^{\otimes\lambda}$ for general $\lambda \in \ell_2(\mathbb{N}_0)$. Second, we prove (3.48) when $j_0 = 0$ in (3.47). Third, we show (3.48) when $j_0 > 0$ by reducing it to the case $j_0 = 0$.

Laplace transform of $V^{\otimes\lambda}$. Abbreviate $f(t) := \mathcal{L}_{\sigma_V^{-1} V_1}(t)$ and note that

$$\mathcal{L}_{V^{\otimes\lambda}}(t) := \mathcal{E}\left[e^{tV^{\otimes\lambda}}\right] = \prod_{j\in\mathbb{N}_0} \mathcal{E}\left[e^{t\lambda(j)\sigma_V^{-1}V_j}\right] = \prod_{j\in\mathbb{N}_0} f\left(\lambda(j)t\right). \tag{3.49}$$

By (1.7),

$$\begin{aligned} |t| < \frac{1}{8}\pi^2\sigma_V \quad &\Longrightarrow \quad \left|f(t)\right| < \infty, \\ t \to \frac{1}{8}\pi^2\sigma_V \quad &\Longrightarrow \quad f(t) \to \infty. \end{aligned} \tag{3.50}$$

Furthermore, by Morera's theorem [17, Theorem 5.1], $t \mapsto f(t)$ is holomorphic on the open disk

$$B := \left\{t \in \mathbb{C} \colon |t| < \frac{1}{8}\pi^2\sigma_V\right\}. \tag{3.51}$$

Therefore Taylor expansion of f on B around 0 gives that

$$f(t) = 1 + \frac{1}{2}t^2 + t^4 g(t), \tag{3.52}$$

with g a holomorphic function on B. From [17, Proposition 3.2], the finiteness of the ℓ_2-norm of λ, and (3.52), we deduce that $t \mapsto \mathcal{L}_{V^{\otimes\lambda}}(t)$ is holomorphic on the open disk

$$B(\lambda) := \left\{t \in \mathbb{C} \colon |t| < \frac{\pi^2\sigma_V}{8\lambda(0)}\right\}. \tag{3.53}$$

Case $j_0 = 0$. From (3.49) and (3.50), $\mathcal{L}_{V^{\otimes\lambda}}(t) \to \infty$ as $t \to \frac{\pi^2\sigma_V}{8\lambda(0)}$, while, $\lambda(0) > \lambda'(0)$ implies $B(\lambda) \subsetneq B(\lambda')$,

$$\sup_{t\in B(\lambda)} \left|\mathcal{L}_{V^{\otimes\lambda'}}(t)\right| < \infty. \tag{3.54}$$

from which (3.48) follows.

Case $j_0 \in \mathbb{N}$. Recall the notation in (3.28). By (3.47), we have $\lambda^{0,j_0} = \lambda'^{0,j_0}$. Suppose that

$$V^{\otimes\lambda} \overset{(d)}{=} V^{\otimes\lambda'}. \tag{3.55}$$

Since $V^{\otimes\lambda} = V^{\otimes\lambda^{0,j_0}} + V^{\otimes\lambda^{j_0,\infty}}$ and $V^{\otimes\lambda'} = V^{\otimes\lambda^{0,j_0}} + V^{\otimes\lambda'^{j_0,\infty}}$, taking the Laplace transform of both random variables and using the independence, we get that

$$V^{\otimes\lambda^{j_0,\infty}} \overset{(d)}{=} V^{\otimes\lambda'^{j_0,\infty}}, \tag{3.56}$$

which is a contradiction. □

3.3. *Identification of the limit points*

Proof of Corollary 1(a)–(b). (a): To prove necessity of the condition on $\lambda_{\tau,\tau(k)}(k)$, suppose that $\limsup_{k\to\infty} \lambda_{\tau,\tau(k)}(k) = c > 0$, and take a subsequence $(k_i)_{i\in\mathbb{N}}$ such that $\lambda_{\tau,\tau(k_i)}(k_i) \to c$ and $\lambda^{0\downarrow}_{\tau,\tau(k_i)}(j) \to \lambda_*(j)$ for any $j \in \mathbb{N}_0$ and for some $\lambda_* \in \ell_2(\mathbb{N}_0)$. Next, since $\limsup_{n\to\infty} \lambda_{\tau,n}(0) = \limsup_{k\to\infty} \lambda_{\tau,\tau(k)}(k) = c$, we may take a subsequence $(n_i)_{i\in\mathbb{N}}$ for which

$$\lim_{i\to\infty} \lambda_{\tau,n_i}(0) \in \left(\frac{c}{2}, \frac{2c}{3}\right) \quad \text{and} \quad \lim_{i\to\infty} \lambda^{0\downarrow}_{\tau,n_i}(j) = \lambda'_*(j) \quad \forall j \in \mathbb{N}_0 \tag{3.57}$$

for some $\lambda'_* \in \ell_2(\mathbb{N}_0)$. By Theorem 2, $\mathfrak{X}_{\tau(k_i)} - \mathbb{E}[\mathfrak{X}_{\tau(k_i)}] \overset{(d)}{\longrightarrow} V^{\otimes\lambda_*}$, and $\mathfrak{X}_{n_i} - \mathbb{E}[\mathfrak{X}_{n_i}] \overset{(d)}{\longrightarrow} V^{\otimes\lambda'_*}$. To conclude the proof it suffices to show that $V^{\otimes\lambda'_*}$ and $V^{\otimes\lambda_*}$ have different distributions. But this follows from Lemma 1, for which we argue next that $[\lambda'_*] \neq [\lambda_*]$.

Since $\limsup_{n\to\infty} \sup_{j\in\mathbb{N}} \lambda_{\tau,n}(j) \le \limsup_{k\to\infty} \lambda_{\tau,\tau(k)}(k) = c > 0$, it follows that for any $\varepsilon > 0$ there is an $n_\varepsilon > 0$ such that, for $n_i > n_\varepsilon$,

$$\sup_{j\in\mathbb{N}} \lambda^2_{\tau,n_i}(j) < c^2 + \varepsilon. \tag{3.58}$$

Furthermore, by (3.57), for n_i large enough,

$$\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n_i)-1)})}{\mathbb{V}\mathrm{ar}(X_{n_i})} \le 1 - \frac{c^2}{4}. \tag{3.59}$$

Therefore, for $\varepsilon < \frac{c^4}{4-c^2}$,

$$\begin{aligned} \sup_{j\in\mathbb{N}} \lambda'^2_{\tau,n_i}(j) &= \sup_{j\in\mathbb{N}} \frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n_i)-1)})}{\mathbb{V}\mathrm{ar}(X_{n_i})} \lambda^2_{\tau,\tau(\ell(n_i)-1)}(j) \\ &\le \left(1 - \frac{c^2}{4}\right)\left(c^2 + \varepsilon\right) < c^2, \end{aligned} \tag{3.60}$$

and therefore $\sup_{i\in\mathbb{N}_0} \lambda'_*(i) < c \le \sup_{i\in\mathbb{N}_0} \lambda_*(i)$. For the reverse implication, by Theorem 2, it suffices to show that $\lambda_{\tau,\tau(k)}(k) \to 0$ implies that for all $i \in \mathbb{N}_0$, $\lim_{n\to\infty} \lambda^{0\downarrow}_n(i) = 0$. To prove this, we will show that

$$\lim_{n\to\infty} \sup_{i\in\mathbb{N}_0} \lambda_{\tau,n}(i) = 0. \tag{3.61}$$

Note that for $i > 0$,

$$\lim_{n\to\infty} \lambda_{\tau,n}(i) = 0, \qquad \sup_{n\in\mathbb{N}} \lambda_{\tau,n}(i) \le \lambda_{\tau,\tau(i)}(i). \tag{3.62}$$

By (3.62), it follows that

$$\begin{aligned} \lim_{n\to\infty} \sup_{i\in\mathbb{N}} \lambda_{\tau,n}(i) &\le \lim_{J\to\infty} \lim_{n\to\infty} \sup_{i>J} \lambda_{\tau,n}(i) \\ &\le \lim_{J\to\infty} \sup_{k>J} \lambda_{\tau,\tau(k)}(k) = \limsup_{k\to\infty} \lambda_{\tau,\tau(k)}(k) = 0. \end{aligned} \tag{3.63}$$

As to $i=0$, define $s_n = \sigma_0^2(\bar{T}^n)/\sigma_0^2(T_{\ell(n)})$ and note that, by (3.3), there is a constant $C \in (1,\infty)$ such that $s_n < C$. Therefore, since $\bar{T}^n \le T_{\ell(n)}$ and $x \mapsto \frac{x}{x+y}$ is increasing on $\mathbb{R}_+$ for $y>0$, we get

$$\lambda_{\tau,n}^2(0) = \frac{\sigma_0^2(\bar{T}^n)\log^2 \bar{T}^n}{\mathbb{V}\mathrm{ar}(X_{\tau(k)}) + \sigma_0^2(\bar{T}^n)\log^2 \bar{T}^n} \le C\lambda_{\tau,\tau(\ell(n))}\big(\ell(n)\big) \to 0, \tag{3.64}$$

from which (3.61) follows.

(b) As in the proof of (a), we examine the sequence of vectors $(\lambda^{0\downarrow}_{\tau,\tau(k)})_{k\in\mathbb{N}}$ and prove that it converges to λ_q with $q = \lim_{k\to\infty}\lambda_{\tau,\tau(k)}(k)$. Abbreviate $q_k := \lambda_{\tau,\tau(k)}(k)$ and note that $(1-q_k)^2\mathbb{V}\mathrm{ar}(Y_k) = q_k^2\sum_{i=1}^{k-1}\mathbb{V}\mathrm{ar}(Y_i)$. Adding $(1-q_k^2)\sum_{i=1}^{k-1}\mathbb{V}\mathrm{ar}(Y_i)$ on both sides, we get

$$\big(1-q_k^2\big)\sum_{i=1}^{k}\mathbb{V}\mathrm{ar}(Y_i) = \sum_{i=1}^{k-1}\mathbb{V}\mathrm{ar}(Y_i). \tag{3.65}$$

Since, $\mathbb{V}\mathrm{ar}(Y_{k-j}) = q_{k-j}^2\sum_{i=1}^{k-j}\mathbb{V}\mathrm{ar}(Y_i)$, recursively applying (3.65), yields to $\mathbb{V}\mathrm{ar}(Y_{k-j}) = q_{k-j}^2\prod_{i=1}^{j}(1-q_{k-j+i}^2) \times \sum_{i=1}^{k}\mathbb{V}\mathrm{ar}(Y_i)$, which implies that

$$\lambda_{\tau,\tau(k)}^2(k-j) = \frac{\mathbb{V}\mathrm{ar}(Y_{k-j})}{\mathbb{V}\mathrm{ar}(X_{\tau(k)})} = q_{k-j}^2\prod_{i=1}^{j}\big(1-q_{k-j+i}^2\big). \tag{3.66}$$

For any $k\in\mathbb{N}$, $\lambda_{\tau,\tau(k)}(0)=0$. As for $j\in\mathbb{N}$, since $q_k \to q > 0$,

$$\lim_{k\to\infty}\big(\lambda^{0\downarrow}_{\tau,\tau(k)}(j)\big)^2 = \lim_{k\to\infty}\lambda^2_{\tau,\tau(k)}(k-j+1) = q^2\big(1-q^2\big)^{j-1}, \tag{3.67}$$

and since $\|\lambda_q\|_2 = 1$, the first part of (b) follows from a direct application of Theorem 2. As to the second part of (b), if $\lambda_{\tau,n_i}(0) \to w$, then

$$1-w^2 = \lim_{k\to\infty}\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n_k)-1)})}{\mathbb{V}\mathrm{ar}(X_{n_k})}. \tag{3.68}$$

From (3.67) and (3.68), for any $i\in\mathbb{N}$,

$$\begin{aligned}\lim_{i\to\infty}\big(\lambda^{0\downarrow}_{\tau,n_k}(j)\big)^2 &= \lim_{i\to\infty}\lambda^2_{n_i,\tau}\big(\ell(n_i)-j\big)\\ &= \lim_{i\to\infty}\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n_i)-1)})}{\mathbb{V}\mathrm{ar}(X_{n_i})}\lambda^2_{\tau,\tau(\ell(n_i)-1)}\big(\ell(n_i)-j\big)\\ &= \big(1-w^2\big)q^2\big(1-q^2\big)^{i-1}.\end{aligned} \tag{3.69}$$

Then, By Theorem 2, $V^{\otimes\lambda_{n_i,\tau}} \xrightarrow{(d)} w\sigma_V^{-1}V_0 + (1-w^2)^{\frac{1}{2}}V^{\otimes\lambda_q}$. □

3.4. *Divergence of cooling maps and crossovers*

In this section we examine the different classes of cooling maps presented in Section 1.4.2 and identify the corresponding limit laws.

Proof. We treat the different examples one by one.

(*Ex.*3) *polynomial cooling*. When $k^{-\beta}T_k \to B$, we have

$$\lim_{n\to\infty}\frac{n}{\frac{B}{\beta+1}\ell(n)^{\beta+1}} = 1. \tag{3.70}$$

Furthermore, by (3.3) we get that

$$\lim_{k\to\infty} \frac{\mathbb{V}\mathrm{ar}(Y_k)}{\sigma_V^2\sigma_0^4\log^4(Bk^\beta)} = 1. \tag{3.71}$$

Since $\frac{\sum_{k=1}^{\ell}\log^4(Bk^\beta)}{\beta^4\ell\log^4\ell} \to 1$, it follows that

$$\lim_{\ell\to\infty} \frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sum_{k=1}^{\ell}\sigma_V^2\sigma_0^4\log^4(Bk^\beta)} = \lim_{\ell\to\infty} \frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sigma_V^2\sigma_0^4\beta^4\ell\log^4\ell} = 1. \tag{3.72}$$

It follows that $\lambda^2_{\tau,\tau(\ell)}(\ell) \to 0$, $\frac{\mathbb{V}\mathrm{ar}(X_n)}{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n)-1)})} \to 1$, and by (3.72)

$$\frac{\mathbb{V}\mathrm{ar}(X_n)}{\sigma_V^2\sigma_0^4(\frac{\beta}{\beta+1})^4(\frac{\beta+1}{B}n)^{\frac{1}{\beta+1}}\log^4 n} \to 1. \tag{3.73}$$

Finally, note that (3.73) implies

$$\frac{X_n - \mathbb{E}[X_n]}{\sigma_0^2 n^{\frac{1}{2(\beta+1)}}\log^2 n} = \alpha_n\big(\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]\big), \quad \text{with } \alpha_n \to \sigma_V\left(\frac{\beta}{\beta+1}\right)^2\left(\frac{\beta+1}{B}\right)^{\frac{1}{2(\beta+1)}}. \tag{3.74}$$

By Corollary 1(a), it follows that

$$\frac{X_n - \mathbb{E}[X_n]}{\sigma_0^2 n^{\frac{1}{2(\beta+1)}}\log^2 n} \xrightarrow{L^p} \sigma_V\left(\frac{\beta}{\beta+1}\right)^2\left(\frac{\beta+1}{B}\right)^{\frac{1}{2(\beta+1)}}\Phi. \tag{3.75}$$

(*Ex*.4) *exponential cooling*. When $k^{-1}\log T_k \to c \in (0,\infty)$,

$$\ell^{-1}\log\tau(\ell) \to c. \tag{3.76}$$

Since $\frac{\sum_{k=1}^{\ell}k^4}{\ell^5} \to \frac{1}{5}$, via (3.3) it follows that

$$\lim_{\ell\to\infty} \frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sum_{k=1}^{\ell}\sigma_V^2\sigma_0^4k^4} = \lim_{\ell\to\infty} \frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sigma_V^2\sigma_0^4 5^{-1}c^{-5}\log^5\tau(\ell)} = 1, \tag{3.77}$$

that $\lambda^2_{\tau,\tau(\ell)}(\ell) \to 0$, and that $\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell(n)-1)})}{\mathbb{V}\mathrm{ar}(X_n)} \to 1$. From (1.22) we obtain that $\mathbb{E}[\mathfrak{X}_n] \to 0$. Finally, note that

$$\frac{X_n}{\frac{1}{\sqrt{5c^5}}\sigma_V\sigma_0^2\log^{\frac{5}{2}} n} = \alpha_n\big(\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]\big) + \beta_n \tag{3.78}$$

with $\alpha_n \to 1$, and $\beta_n \to 0$. By (3.78) and Corollary 1(a),

$$\frac{X_n}{\frac{1}{\sqrt{5c^5}}\sigma_V\sigma_0^2\log^{\frac{5}{2}} n} \xrightarrow{L^p} \Phi. \tag{3.79}$$

(*Ex*.5) *double exponential cooling*. When $k^{-1}\log\log T_k \to c \in (0,\infty)$,

$$\frac{\tau(\ell)}{T_\ell} \to 1, \qquad \frac{\sum_{k=1}^{\ell}\log^4 T_k}{\sum_{k=1}^{\ell}e^{4ck}} \to 1. \tag{3.80}$$

From (3.3) it follows that $\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sigma_V^2\sigma_0^4e^{4c\ell}(1-e^{-4c})^{-1}} \to 1$ and therefore

$$\lambda^2_{\tau,\tau(\ell)}(\ell) \to \frac{e^{4c}-1}{e^{4c}} = q_c^2. \tag{3.81}$$

Note that

$$\lim_{\ell\to\infty}\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sigma_V^2\sigma_0^4\log^4\tau(\ell)}=\lim_{\ell\to\infty}\frac{\sum_{k=1}^{\ell}\log^4 T_k}{\log^4 T_\ell}\frac{\log^4 T_\ell}{\log^4\tau(\ell)}=q_c^{-2}. \tag{3.82}$$

Combining (3.81) and (3.82) with Corollary 1(b), we conclude that

$$\begin{aligned}\frac{X_{\tau(\ell)}}{\sigma_0^2\log^2\tau(\ell)}&=\frac{\sqrt{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}}{\sigma_0^2\log^2\tau(\ell)}\frac{X_{\tau(\ell)}}{\sqrt{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}}\\&=\sigma_V\frac{\sqrt{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}}{\sigma_V\sigma_0^2\log^2\tau(\ell)}\mathfrak{X}_{\tau(\ell)}\xrightarrow{L^p}\sigma_V q_c^{-1}V^{\otimes\lambda_{q_c}}.\end{aligned} \tag{3.83}$$

(*Ex*.6) *faster than double exponential cooling*. In this case

$$\frac{\tau(\ell)}{T_\ell}\to 1,\qquad\frac{\sum_{k=1}^{\ell}\log^4 T_k}{\log^4 T_\ell}\to 1, \tag{3.84}$$

from which, by (3.3), it follows that

$$\lim_{\ell\to\infty}\lambda_{\tau,\tau(\ell)}(\ell)=1,\qquad\lim_{\ell\to\infty}\frac{\mathbb{V}\mathrm{ar}(X_{\tau(\ell)})}{\sigma_0^2\sigma_V^4\log^4 T_\ell}=1, \tag{3.85}$$

and therefore $\lambda^2_{\tau,\tau(\ell)}(\ell)\to 1$. By (3.85) and Corollary 1(b),

$$\frac{X_{\tau(\ell)}}{\sigma_0^2\log^2\tau(\ell)}\xrightarrow[\ell\to\infty]{(d)}V. \tag{3.86}$$

Subsequences. In (Ex.5) and (Ex.6) we need to examine the effect of the boundary. Let $(n_i)_{i\in\mathbb{N}}$ be a subsequence for which (1.37) holds. Then

$$\frac{\log\tau(\ell(n_i)-1)}{\log n_i}\to\begin{cases}1 & \text{if } b\le 1,\\ b^{-1} & \text{if } b>1.\end{cases} \tag{3.87}$$

Decompose $X_{n_i}=X_{\tau(\ell(n_i)-1)}+\bar{Y}^{n_i}$. By conveniently rewriting the scaling factors, we obtain

$$\begin{aligned}\frac{X_{n_i}}{\sigma_V\sigma_0^2\log^2 n_i}&=\frac{\log^2\tau(\ell(n_i)-1)}{\log^2 n_i}\\&\quad\times\left(\frac{X_{\tau(\ell(n_i)-1)}}{\sigma_V\sigma_0^2\log^2\tau(\ell(n_i)-1)}+\frac{\log^2\bar{T}^{n_i}}{\log^2\tau(\ell(n_i)-1)}\frac{\bar{Y}^{n_i}}{\sigma_V\sigma_0^2\log^2\bar{T}^{n_i}}\right).\end{aligned} \tag{3.88}$$

Using (3.83) and (3.86) in combination with (3.87) and (3.88), we conclude that

$$\frac{X_{n_i}}{\sigma_V\sigma_0^2\log^2 n_i}\xrightarrow{L^p}\begin{cases}q_c^{-1}V^{\otimes\lambda_{q_c}}+b^2\sigma_V^{-1}V_0 & \text{if } b\le 1,\\ b^{-2}q_c^{-1}V^{\otimes\lambda_{q_c}}+\sigma_V^{-1}V_0 & \text{if } b>1.\end{cases} \tag{3.89}$$

□

4. Proofs: Gaussian fluctuations

4.1. *Convergence in the Gaussian regime*

Proof of Theorem 3. By Theorem 4,

$$\sigma_s^2(n):=\mathrm{Var}\left[\frac{Z_n}{\sqrt{n}}\right]\to\sigma_s^2. \tag{4.1}$$

Consider a probability space $(S, \mathcal{S}, \mathcal{P})$ that is rich enough to include a sequence of i.i.d. standard normal random variables $(\Phi_k)_{k\in\mathbb{N}_0}$ and a collection of random variables $(\mathcal{R}_n^{(k)})_{k,n\in\mathbb{N}_0}$ satisfying (recall (H1)–(H4) in Section 3.1):

(H1') For any $k, n \in \mathbb{N}_0$ and $x \in \mathbb{R}$,

$$P_0^\mu\left(\frac{Z_n - E_0^\mu[Z_n]}{\sigma_s(n)} \le x\right) = \mathcal{P}\big(\Phi_k + \mathcal{R}_n^{(k)} \le x\big). \tag{4.2}$$

(H2') For all $k, n \in \mathbb{N}_0$, $\mathcal{E}[\mathcal{R}_n^{(k)}] = 0$.
(H3') $(\Phi_k, \mathcal{R}_n^{(k)})_{n,k\in\mathbb{N}_0}$ are independent in k under $\mathcal{P}$.
(H4') $\mathcal{R}_n^{(k)}$ vanishes in L^2, i.e.,

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}_0} \mathcal{E}\big[\big(\mathcal{R}_n^{(k)}\big)^2\big] = 0. \tag{4.3}$$

Note that, by (1.24) $\sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k)\Phi_k \overset{(d)}{=} \Phi$. and by (4.2),

$$\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n] \overset{(d)}{=} \sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k)\big(\Phi_k + \mathcal{R}_{T_k}^{(k)}\big) \overset{(d)}{=} \Phi + \sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k)\mathcal{R}_{T_k}^{(k)}. \tag{4.4}$$

i.e., $\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]$ has the same distribution as a standard normal distribution, up to an error term that is negligible because of (4.3). By (H2') and (H3'), we have

$$\mathcal{E}\left[\left(\sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k) R_{T_k}^{(k)}\right)^2\right] = \sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}^2(k)\mathcal{E}\big[\big(R_{T_k}^{(k)}\big)^2\big]. \tag{4.5}$$

Since $\sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}^2(k) = 1$, it follows from (4.3) that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \mathcal{E}\left[\left(\sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k) R_k \mathbb{1}_{\{T_k > J\}}\right)^2\right] = 0. \tag{4.6}$$

On the other hand, for any fixed $J > 0$, under $\mathbb{P}$, $(Y_k \mathbb{1}_{\{T_k \le J\}})_{k\in\mathbb{N}}$ is a collection of bounded independent random variables. Thus, by the CLT for i.i.d. random variables, we get

$$\begin{aligned} \lim_{n\to\infty} \Bigg| &\mathbb{E}\left[f\left(\sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k) \frac{Y_k - \mathbb{E}[Y_k]}{\sigma_{s,T_k}} \mathbb{1}_{\{T_k \le J\}}\right)\right] \\ &- \mathcal{E}\left[f\left(\left(\sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}^2(k) \mathbb{1}_{\{T_k \le J\}}\right)^{\frac{1}{2}} \Phi\right)\right]\Bigg| = 0 \end{aligned} \tag{4.7}$$

Theorem 3 follows from (4.6) and (4.7) by applying the same arguments that led to (3.16), it follows that

$$\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n] \xrightarrow{(d)} \Phi. \tag{4.8}$$

To prove the convergence in L^2 of $(\tilde{\mathfrak{X}}_n^2)_{n\in\mathbb{N}}$, as in (3.11) we consider the truncated random variables $\overline{\Phi}_n^{0,J}$, $\overline{\Phi}_n^{J,\infty}$, $\overline{\mathcal{R}_n}^{0,J}$, $\overline{\mathcal{R}_n}^{0,J}$. Now,

$$\begin{aligned} &\lim_{M\to\infty} \sup_{n\in\mathbb{N}_0} \mathbb{E}\big[\tilde{\mathfrak{X}}_n^2 \mathbb{1}_{\{\tilde{\mathfrak{X}}_n^2 > M\}}\big] \\ &\quad = \lim_{M\to\infty} \sup_{n\in\mathbb{N}_0} \mathcal{E}\big[\big(\big(\overline{\Phi}_n^{0,\infty} + \overline{\mathcal{R}_n}^{0,\infty}\big)^2 \mathbb{1}_{\{\tilde{\mathfrak{X}}_n^2 > M\}}\big)\big] \\ &\quad \le \inf_J \sup_{n\in\mathbb{N}_0} \mathcal{E}\big[\big(\overline{\mathcal{R}_n}^{J,\infty}\big)^2\big] = 0, \end{aligned} \tag{4.9}$$

where we used the uniform integrability in L^2 of $(\overline{\Phi}_n^{0,J})_{n\in\mathbb{N}_0}$ and $(\overline{\mathcal{R}_n}^{0,J})_{n\in\mathbb{N}_0}$ to obtain the the inequality. □

4.2. *Limit points and stability of the variance*

Proof of Corollary 2. Note first that $(\sigma_{s,\tau}(n))_{n\in\mathbb{N}}$ is a bounded sequence. Indeed, $C := \sup_{T_k\in\mathbb{R}} \sigma_s(T_k) < \infty$ by (4.1), and by independence we obtain that

$$\frac{\mathbb{V}\mathrm{ar}(X_n)}{n} = \sum_{k=0}^{\ell(n)} \frac{T_k}{n} \sigma_s^2(T_k) < C. \tag{4.10}$$

To prove (1.40), note that as the sequence $(\sigma_{s,\tau}(n), n \in \mathbb{N})$ is bounded, it admits a convergent subsequence. Furthermore if $\sigma_{s,\tau}(n_i) \to \sigma$, then

$$\frac{X_n - \mathbb{E}[X_{n_i}]}{\sqrt{n_i}} = \big(\sigma_{s,\tau}(n_i)\big)\big(\mathfrak{X}_{n_i} - \mathbb{E}[\mathfrak{X}_{n_i}]\big) \xrightarrow{L^2} \sigma\Phi. \tag{4.11}$$

Note that (4.10) proves tightness of the sequence $(n^{-1/2}(X_n - \mathbb{E}[X_n])_{n\in\mathbb{N}}$, and (4.11) characterizes its limit points as scalar multiples of a standard Gaussian random variable.

To prove (1.41), use (4.1). Indeed, if $T_k \to \infty$, then $\sigma_s^2(T_k) \to \sigma_s^2$ and

$$\lim_{n\to\infty} \frac{\sigma_{s,\tau}^2(n)}{\sigma_s^2} = \lim_{n\to\infty} \frac{\mathbb{V}\mathrm{ar}(X_n)}{n\sigma_s^2} = \lim_{n\to\infty} \sum_{k=0}^{\ell(n)} \frac{T_k}{n} \frac{\sigma_s^2(T_k)}{\sigma_s^2} = 1, \tag{4.12}$$

where the last equality follows from the Toeplitz Lemma [14, Thm.1.2.3]. □

4.3. *Stable centering and counterexample*

Proof. We first turn to (Ex.7). To prove (1.44), note that the L^2-convergence in (1.46) implies that

$$\mathbb{E}\left[\frac{Z_n - nv_\mu}{\sqrt{n}}\right] \to 0. \tag{4.13}$$

Let $C_n := \sum_{k=0}^{\ell(n)-1} \lambda_{\tau,n}(k)$ and $C := \sup_{n\in\mathbb{N}} C_n$. Condition (1.42) corresponds to $C < \infty$. In this case, by Markov's inequality, Theorem 5(II) and the Toeplitz lemma [14, Thm.1.2.3], we have

$$\left|\mathbb{E}\left[\frac{X_n - nv_\mu}{\sigma_s\sqrt{n}}\right]\right| \le \sum_{k=0}^{\ell(n)-1} \frac{\lambda_{\tau,n}(k)}{C} \left|\mathbb{E}\left[\frac{Y_k - T_k v_\mu}{\sigma_s\sqrt{T_k}}\right]\right| \xrightarrow[n\to\infty]{} 0. \tag{4.14}$$

From (4.12) and (4.14) it follows that

$$\frac{X_n - nv_\mu}{\sigma_s\sqrt{n}} = \alpha_n\big(\mathfrak{X}_n - \mathbb{E}[\mathfrak{X}_n]\big) + \beta_n \tag{4.15}$$

with $\alpha_n \to 1$, $\beta_n \to 0$, and (1.44) follows from (1.39).

We next turn to (Ex.8). To show (1.45), consider the sets

$$\mathcal{N}_+ := \big\{n \in \mathbb{N}\colon E_0^\mu[Z_n] > 0\big\}, \qquad \mathcal{N}_- := \big\{n \in \mathbb{N}\colon E_0^\mu[Z_n] < 0\big\}. \tag{4.16}$$

By Theorem 5(2), there exists an s-transient α for which at least one of these sets is infinite. Assume without loss that $\mathcal{N}_+ = \{n_1 < n_2 < \cdots\}$ is infinite. Define the cooling map by successively picking N_ℓ consecutive increments of size n_ℓ for every $\ell \in \mathbb{N}$, where the values of $(N_\ell)_{\ell\in\mathbb{N}}$ are chosen such that

$$\frac{N_\ell}{\sqrt{\sum_{m=1}^{\ell} N_m n_m}} \big(E_0^\mu[Z_{n_\ell}] - v_\mu\big) > \ell. \tag{4.17}$$

Let $s(0) := 0$, and for $\ell \in \mathbb{N}$, define $s(\ell) := s(\ell-1) + N_\ell n_\ell$. Therefore,

$$\mathbb{E}[X_{n(k)}] - \sqrt{s(k)}v_\mu = \sum_{\ell=1}^{k} \frac{N_\ell}{\sqrt{\sum_{m=1}^{\ell} N_m n_m}} \big(E_0^\mu[Z_{n_\ell}] - v_\mu\big) > k, \tag{4.18}$$

which proves (1.45). □

Appendix A: L^p-convergence in the Gaussian regime

We prove Theorem 4.

Preparation. Recall that $\rho_j = \frac{1-\omega(j)}{\omega(j)}$. Following Zeitouni [19, Section 2.2], we have

$$\Delta(j,\omega) := -1 + v_\mu \Sigma\big(\theta^j \omega\big), \quad j \in \mathbb{Z}, \tag{A.1}$$

where

$$\Sigma(\omega) := \sum_{i=-\infty}^{0} \frac{1}{\omega_i} \prod_{j=i+1}^{0} \rho_j \tag{A.2}$$

and θ denotes the spatial shift operator acting on $(0,1)^{\mathbb{Z}}$ (i.e., $(\theta\omega)(j) = \omega(j+1)$, $j \in \mathbb{Z}$). Now define, for $n \in \mathbb{N}$,

$$M_n := Z_n - v_\mu n + S_n + R_n, \tag{A.3}$$

where ($S_0 = 0$)

$$S_n := \sum_{j=0}^{n v_\mu} \Delta(j,\omega), \qquad R_n := \begin{cases} \sum_{j=Z_n}^{n v_\mu} \Delta(j,\omega) & \text{if } Z_n < n v_\mu, \\ 0 & \text{if } Z_n = n v_\mu, \\ \sum_{j=n v_\mu+1}^{Z_n - 1} \Delta(j,\omega) & \text{if } Z_n > n v_\mu. \end{cases} \tag{A.4}$$

Note that, in this decomposition, S_n depends only on ω. Therefore we will distinguish between the different measures and write E_μ for expectation with respect to μ. Next, by [16, Theorem 1.16 (i)], $v_\mu^{-1} = E_\mu[\Sigma(\omega)]$, and consequently

$$E_\mu\big[\Delta(x,\omega)\big] = 0. \tag{A.5}$$

Therefore, for $s \in (2,\infty)$, (A.3) is a decomposition of $(Z_n - v_\mu n)_{n\in\mathbb{N}_0}$ into a martingale $(M_n)_{n\in\mathbb{N}_0}$ with respect to the natural filtration of the random walk $\mathcal{F}_n = \sigma(Z_i : 0 \le i \le n)$ and the probability measure P_0^ω for any $\omega \in (0,1)^{\mathbb{Z}}$; a mean-zero stationary sequence $(S_n)_{n\in\mathbb{N}_0}$ with respect to the shift operator θ and the measure μ; and a remainder term $(R_n)_{n\in\mathbb{N}_0}$. Furthermore, the assumptions of [19, Theorem 2.2.1] are satisfied and, under the annealed measure P_0^μ,

$$n^{-\frac{1}{2}} R_n \xrightarrow{(d)} 0, \qquad n^{-\frac{1}{2}} M_n \xrightarrow{(d)} \sigma_{1,\mu}\Phi_1, \qquad n^{-\frac{1}{2}} S_n \xrightarrow{(d)} \sigma_{2,\mu}\Phi_2, \tag{A.6}$$

where $\sigma_{1,\mu}, \sigma_{2,\mu}$ will be introduced below and Φ_1, Φ_2 are standard normal random variables. To prove L^p-convergence, it suffices to show that, for any $p \in (2,s)$

$$\sup_{n\in\mathbb{N}} E_0^\mu\big[\big|n^{-\frac{1}{2}} R_n\big|^p\big] < \infty, \tag{A.7}$$

$$\sup_{n\in\mathbb{N}} E_0^\mu\big[\big|n^{-\frac{1}{2}} M_n\big|^p\big] < \infty, \tag{A.8}$$

$$\sup_{n\in\mathbb{N}} E_\mu\big[\big|n^{-\frac{1}{2}} S_n\big|^p\big] < \infty. \tag{A.9}$$

These conditions ensure uniform integrability in L^p for $p < s$ and, combined with (A.6), yield the desired result. The proof of (A.7) is given in Section A.1, and the proofs of (A.8), (A.9) are given in Section A.2.

A.1. Remainder term

For $p \in (2, s)$, note that

$$\sup_{n\in\mathbb{N}} E_0^\mu\big[\big|n^{-\frac{1}{2}} R_n\big|^p\big] = \sup_{n\in\mathbb{N}} \int_0^\infty p\delta^{p-1} P_0^\mu\big(n^{-\frac{1}{2}}|R_n| > \delta\big)\, d\delta. \tag{A.10}$$

As $P_0^\mu(|Z_n - nv_\mu| > 2n) = 0$, by (A.4), we have

$$\begin{aligned}
&P_0^\mu\big(n^{-\frac{1}{2}}|R_n| > \delta\big)\\
&\quad\le \mu\left(\max_{j-,j+\in(v_\mu n-2n, v_\mu n+2n)}\left|\sum_{i=j-}^{j+}\frac{\Delta(i,\omega)}{\sqrt{n}}\right| \ge \delta\right)\\
&\quad= \mu\left(\max_{j-,j+\in(-2n,2n)}\left|\sum_{i=j-}^{j+}\frac{\Delta(i,\omega)}{\sqrt{n}}\right| \ge \delta\right)\\
&\quad\le 2\mu\left(\max_{j\in(0,2n)}\left|\sum_{i=j}^{0}\frac{\Delta(i,\omega)}{\sqrt{n}}\right| \ge \frac{\delta}{2}\right),
\end{aligned} \tag{A.11}$$

where in the first inequality, since the random variable does not depend on the random walk and is a function of the environment ω only, we replace P_0^μ by μ; the equality follows from the stationarity of $\Delta(i,\omega)$ and to obtain the last inequality we estimate the invcrement from $j-$ to $j+$ in terms of the distance to the origin and use symmetry. By Markov's inequality,

$$\mu\left(\max_{j\in(0,2n)}\left|\sum_{i=0}^{j}\frac{\Delta(i,\omega)}{\sqrt{n}}\right| \ge \delta\right) \le \frac{1}{\delta^p} E_\mu\left[\max_{j\in(0,2n)}\left|\sum_{i=0}^{j}\frac{\Delta(i,\omega)}{\sqrt{n}}\right|^p\right]. \tag{A.12}$$

We estimate this expectation with the help of [11, Proposition 7],

$$E_\mu\left(\max_{j+\in(0,n)}\left|\sum_{i=0}^{j+}\frac{\Delta(i,\omega)}{\sqrt{n}}\right|^p\right) \le C_p\left(\frac{\sum_{i=1}^n b_{i,n,p}}{n}\right)^{\frac{p}{2}}, \tag{A.13}$$

where

$$b_{i,n,p} := \max_{i\le\ell\le n}\left\|\Delta(i,\omega)\sum_{k=i}^{\ell}\mu\big[\Delta(k,\omega)|\mathcal{G}_i\big]\right\|_{\frac{p}{2}}, \tag{A.14}$$

$\mathcal{G}_i := \sigma(\omega(j)\colon j \le i)$, and $\|f\|_p = \int_0^1 |f(\omega)|^p\, d\mu(\omega)$. Below we show that

$$\sup_{i,n} b_{i,n,p} =: K < \infty. \tag{A.15}$$

To conclude the proof of (A.7) with the help of (A.15), note that (A.13) is uniformly bounded in $n \in \mathbb{N}$ and therefore by combining it with (A.11)–(A.12) we can bound the right-hand side of (A.10) by

$$\int_0^1 p\delta^{p-1} P_0^\mu\big(n^{-\frac{1}{2}}|R_n| > \delta\big)\, d\delta + C\int_1^\infty \delta^{p-1}\frac{1}{\delta^{p'}}\, d\delta \tag{A.16}$$

for some $C > 0$ and $p' \in (p, s)$. Since for $p' > p$ the second integral above is finite, this conclude the proof of (A.7). It remains to verify (A.15).

Bound on $b_{i,n,p}$. To prove (A.15), the expression (A.2) allows us to bound the conditional expectation in (A.14) by

$$\begin{aligned}
E_\mu\big[\Delta(i+k,\omega)|\mathcal{G}_i\big] &= -1 + \frac{1}{E_\mu[\Sigma(\omega)]}\big(\big\langle\omega(0)^{-1}\big\rangle\big(1 + \langle\rho\rangle + \cdots + \langle\rho\rangle^{k-1}\big) + \langle\rho\rangle^k\Sigma\big(\theta^i\omega\big)\big)\\
&\le \frac{1}{E_\mu[\Sigma(\omega)]}\langle\rho\rangle^k\left(\frac{-\langle\omega(0)^{-1}\rangle}{1-\langle\rho\rangle} + \Sigma\big(\theta^i\omega\big)\right),
\end{aligned} \tag{A.17}$$

where the inequality follows from observing that

$$E_\mu\big[\Sigma(\omega)\big] = \langle\omega^{-1}\rangle\big(1 + \langle\rho\rangle + \langle\rho\rangle^2 + \cdots\big). \tag{A.18}$$

The right-hand side of (A.14) is bounded by

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0} \big\| \Delta(i,\omega) E_\mu\big[\Delta(i+k,\omega)|\mathcal{G}_i\big]\big\|_{\frac{p}{2}} \\
&\quad\le \sum_{k\in\mathbb{N}_0} v_\mu \langle\rho\rangle^k \left\| \big(-1 + v_\mu \Sigma\big(\theta^i\omega\big)\big)\left(\frac{-\langle\omega^{-1}\rangle}{1-\langle\rho\rangle} + \Sigma\big(\theta^i\omega\big)\right)\right\|_{\frac{p}{2}} \\
&\quad\le \sum_{k\in\mathbb{N}_0}^{\infty} v_\mu \langle\rho\rangle^k C\big(1 + \mu\big[\big(\Sigma\big(\theta^i\omega\big)\big)^p\big]\big)^{\frac{2}{p}} \\
&\quad= \frac{1}{1-\langle\rho\rangle} v_\mu C\big(1 + \mu\big[\big(\Sigma(\omega)\big)^p\big]\big)^{\frac{2}{p}} < \infty,
\end{aligned} \tag{A.19}$$

where in the second inequality we used that $ab \le a^2 + b^2$ to separate the constants from the random variable $\Sigma(\theta^i\omega)$, and $C > 0$ is a constant that does not depend on k or i. The last equality follows from $\Sigma(\theta^i\omega) \overset{(d)}{=} \Sigma(\omega)$, and the final bound follows from $E_\mu[(\Sigma(\omega))^p] < \infty$ for $p < s$, as can be seen by applying Minkowsky's inequality on the L^p norm of (A.2).

A.2. L^p-convergence to the normal

To prove (A.8) and (A.9) we will bound the expectations with a bound on the difference of the distribution functions of the each random variables and the standard normal distribution.

Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of random variables, P the underlying probability measure and E its corresponding expectation. Assume that this sequence converges in distribution to the standard normal. To prove that this convergence is also in L^p- we need to show that, for $p < s$,

$$\sup_{n\in\mathbb{N}} E\big[|W_n|^p\big] < \infty. \tag{A.20}$$

We have

$$\begin{aligned}
E\big[|W_n|^p\big] &= \int_0^\infty dx p x^{p-1} P\big(|W_n| > x\big) \\
&= \int_0^\infty dx p x^{p-1} \big[P\big(|W_n| > x\big) - P\big(|\Phi| > x\big)\big] \\
&\quad + \int_0^\infty dx p x^{p-1} P\big(|\Phi| > x\big).
\end{aligned} \tag{A.21}$$

Since $\int_0^\infty dx p x^{p-1} P(|\Phi| > x) < \infty$, if

$$\big|P\big(|W_n| \ge x\big) - P\big(|\Phi| > x\big)\big| \le C a_n f(x), \tag{A.22}$$

where a_n and $f(x)$ satisfy

$$\sup_{n\in\mathbb{N}} a_n < \infty, \qquad \int_0^\infty dx x^{p-1} f(x) < \infty, \tag{A.23}$$

then (A.20) follows.

A.2.1. *Martingale part*

We will use a result in [6] to prove (A.8). Define $M_0 = 0$, and the square-integrable martingale difference sequence $(D_n)_{n\in\mathbb{N}}$ by $D_k := M_k - M_{k-1}$. As shown in [19, p. 211], the quadratic variation of $(M_n)_{n\in\mathbb{N}}$ under P_0^ω is given by

$A_n^\omega := \sum_{k=1}^n E_0^\omega[D_k^2|\mathcal{F}_{k-1}]$, where

$$\begin{aligned} E_0^\omega\big[D_k^2|\mathcal{F}_{k-1}\big] \\ = v_\mu^2\big[\bar\omega_0(k)\big(\Sigma\big(\bar\omega(k)\big)-1\big)^2+\big(1-\bar\omega_0(k)\big)\big(\Sigma\big(\theta^{-1}\bar\omega(k)\big)+1\big)^2\big], \end{aligned}\tag{A.24}$$

As shown in [19, Corollary 2.1.25], the sequence $(\bar\omega(k) := \theta^{Z_k}\omega)_{k\in\mathbb{N}}$ is stationary and ergodic under $Q\otimes P_0^\omega$, where $Q(d\omega) := \Lambda(\omega)P(d\omega)$, and $\Lambda(\omega) := \frac{1}{\omega_0}+\frac{1}{\omega_0}\rho_1+\frac{1}{\omega_0}\rho_1\rho_2+\cdots = \frac{1}{\omega_0}(\sum_{i=0}^\infty\prod_{j=0}^{i-1}\rho_j)$. Therefore, letting E^Q denote the expectation with respect to Q, we see that the following limit exists Q-almost surely:

$$\begin{aligned} \sigma_{\mu,1}^2 &:= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n E_0^\omega\big[D_k^2|\mathcal{F}_{k-1}\big] \\ &= E^Q\big[v_\mu^2\big[\omega_0\big(\Sigma(\omega)-1\big)^2+\big(\Sigma\big(\theta^{-1}\omega\big)+1\big)^2\big]\big]. \end{aligned}\tag{A.25}$$

Fix $\delta>0$ such that $2+2\delta<s$, let $D_{k,n} := (\sigma_{\mu,1}\sqrt{n})^{-1}D_k$, and consider the following two quantities:

$$A_{n,\delta} := \sum_{k=1}^n E_0^\mu\big[|D_{k,n}|^{2+2\delta}\big],\tag{A.26}$$

$$B_{n,\delta} := E_0^\mu\left[\left|1-\sum_{k=1}^n E_0^\mu\big[D_{k,n}^2|\mathcal{F}_{k-1}\big]\right|^{1+\delta}\right].\tag{A.27}$$

Since

$$E_0^\mu\big[|D_{k,n}|^{2+2\delta}\big] = \frac{1}{\sigma_{\mu,1}^{2+2\delta}n^{1+\delta}}E_0^\mu\big[|M_k-M_{k-1}|^{2+2\delta}\big],\tag{A.28}$$

we can bound $A_{n,\delta}$ by:

$$\frac{\sup_{k\in\mathbb{N}}E_0^\mu[|M_k-M_{k-1}|^{2+2\delta}]}{\sigma_{\mu,1}^{2+2\delta}n^\delta}.\tag{A.29}$$

Since $2+2\delta<s$, we have $\sup_{k\in\mathbb{N}}E_0^\mu[|M_k-M_{k-1}|^{2+2\delta}]<\infty$, and therefore $A_{n,\delta}\to 0$.

To estimate $B_{n,\delta}$, we first note that

$$E_0^\mu\big[D_k^2|\mathcal{F}_{k-1}\big] = \int_0^1 E_0^\omega\big[D_k^2|\mathcal{F}_{k-1}\big]\,d\mu(\omega).\tag{A.30}$$

Now note that since $\Lambda(\omega)\ge 1$, for all positive f, $E^Q[f]\ge E_0^\mu[f]$. Next, we apply the von Neumann L^p-ergodic theorem in [18, Corollary 1.14.1] to the ergodic sequence $(E_0^\omega[D_k^2|\mathcal{F}_{k-1}])_{k\in\mathbb{N}}$ in $L^{1+\delta}(Q\otimes P_0^\omega)$, to conclude that

$$\lim_{n\to\infty}E^Q\left[\left|1-\sum_{k=0}^n E_0^\omega\big[D_{k,n}^2|\mathcal{F}_{k-1}\big]\right|^{1+\delta}\right]=0\tag{A.31}$$

and that $\lim_{n\to\infty}B_{n,\delta} = \lim_{n\to\infty}E_0^\mu[|1-\sum_{k=0}^n E_0^\omega[D_{k,n}^2|\mathcal{F}_{k-1}]|^{1+\delta}]=0$. By [6, Theorem 1], whenever $a_{n,\delta} := A_{n,\delta}+B_{n,\delta}<1$, then for any $\delta>0$ there exists a finite constant C_δ such that

$$\left|P_0^\mu\left(\sum_{k=1}^n D_k\le x\right)-P\big(|\Phi|>x\big)\right|\le C_\delta a_{n,\delta}^{\frac{1}{3+2\delta}}\big(1+|x|^{2+2\delta}\big)^{-1}\tag{A.32}$$

for all $x\in\mathbb{R}$. Since $a_{n,\delta}\to 0$, the terms in (A.32) satisfy (A.23). If we replace W_n in (A.20) by $\frac{M_n}{\sigma_{\mu,1}\sqrt{n}}$, then we obtain (A.8).

A.2.2. Stationary part

We show (A.9) with the help of [7, Theorem 2.4]. Indeed, if $\{\Delta(j,\omega)\}_{j\in\mathbb{N}}$ satisfies [7, Assumption 2.1], then for some constants $C_p > 0$ and $b_{n,p} > 0$

$$\left| E_0^{\mu}\left(\sum_{k=1}^{nv_\mu} \Delta(j,\omega) \le \sigma_{\mu,2}\sqrt{n}x\right) - \phi(x)\right| \le C_p b_{n,p}\big(1+|x|^p\big)^{-1}, \tag{A.33}$$

for any $x \in \mathbb{R}$, where

$$\sigma_{\mu,2}^2 := \lim_{n\to\infty} \frac{1}{n} E_0^{\mu}\left[\left(\sum_{k=0}^{n} \Delta(k,\omega)\right)^2\right]. \tag{A.34}$$

To verify the conditions in [7], we need to introduce some notation.

Let $\omega'(0) \in (0,1)$ be an independent random variable selected according to α, and define

$$\omega'(k) := \begin{cases} \omega(k) & \text{if } k \neq 0, \\ \omega'(0) & \text{if } k = 0. \end{cases} \tag{A.35}$$

Recall (A.1) and (A.4). Since the sequence $(\omega_x)_{x\in\mathbb{Z}}$ is stationary with respect to θ under μ, we have

$$\Delta(j,\omega) \text{ is stationary with respect to } \theta \text{ under } \mu. \tag{A.36}$$

In what follows, we verify the remaining conditions [7, Assumption 2.1] and fix $p \in (2,s)$. First note that

$$\big\|\Delta(k,\omega)\big\|_p \le 1 + v_\mu \big\|\Sigma(\omega)\big\|_p < \infty, \qquad E_0^{\mu}\big[\Delta(j,\omega)\big] = 0. \tag{A.37}$$

Next note that, since $\|\Delta(k,\omega) - \Delta(k,\omega')\|_p \le C_p\|\rho\|_p^k$ with $\|\rho\|_p < 1$ (because $p < s$), we obtain that

$$\sum_{k=1}^{\infty} k^2 \big\|\Delta(k,\omega) - \Delta\big(k,\omega'\big)\big\|_p < \infty. \tag{A.38}$$

To verify the last condition note that, since $\Delta(j,\theta^{-k}\omega) = \Delta(j-k,\omega)$, by expanding (A.34) and using the stationarity of $\Delta(k,\omega)$, we get

$$\sigma_{\mu,2}^2 = E_0^{\mu}\big[\Delta(0,\omega)^2\big] + 2\sum_{k\in\mathbb{N}} E_0^{\mu}\big[\Delta(0,\omega)\Delta(k,\omega)\big]. \tag{A.39}$$

Since

$$\Sigma\big(\theta^k\omega\big) = \frac{1}{\omega_k} + \frac{1}{\omega_{k-1}}\rho_k + \cdots + \frac{1}{\omega_1}\rho_k \times \cdots \times \rho_2 + \rho_k \times \cdots \times \rho_1 \Sigma(\omega), \tag{A.40}$$

by (A.5) it follows that

$$E_0^{\mu}\big[\Sigma(\omega)\Sigma\big(\theta^k\omega\big)\big] = v_\mu^{-2}\big(1 - \langle\rho\rangle^k\big) + \langle\rho\rangle^k E_0^{\mu}\big[\Sigma(\omega)^2\big]. \tag{A.41}$$

Since, for non-degenerate α, $E_0^{\mu}[\Sigma(\omega)^2] > E_0^{\mu}[\Sigma(\omega)]^2 = v_\mu^{-2}$, we obtain that

$$\sigma_{\mu,2}^2 \sum_{k\in\mathbb{N}} E_0^{\mu}\big[\Delta(0,\omega)\Delta(k,\omega)\big] > 0, \tag{A.42}$$

which implies that $\sigma_{\mu,2}^2 > 0$. Conditions (A.36), (A.37), (A.38) and (A.42) allow us to apply the result in [7] and obtain (A.33). By substituting (A.33) into the right-hand side of (A.21) we obtain (A.9) and thereby conclude the proof of Theorem 4.

Appendix B: Oscillations of mean displacement

B.1. Asymmetry in the Sinai regime

We prove Theorem 5(I).

Proof. To show that

$$\{\alpha\colon \langle \log\rho\rangle = 0,\, E_0^{\mu}[Z_n] \neq 0 \text{ i.o.}\} \neq \varnothing, \tag{B.1}$$

define, for $x \in (0,1)$, $\alpha_x := x\delta_x + (1-x)\delta_{\eta(x)}$, where $\eta(x) \in (0,1)$ is defined by the relation $\langle \log\rho\rangle = 0$, which makes α_x recurrent. Let $\mu_x = \alpha_x^{\mathbb{Z}}$ (recall (1.1)), and consider the sets

$$A_n := \{x \in (0,1)\colon E_0^{\mu_x}[Z_n] = 0\}, \quad n \in \mathbb{N}. \tag{B.2}$$

By the implicit function theorem, $x \mapsto \eta(x)$ is analytic. Therefore A_n is finite (otherwise $x \mapsto \mathbb{E}_0^{\mu_x}[Z_n]$ would be constant equal to 0, which is not the case because $\lim_{x\uparrow 1} E_0^{\mu_x}[Z_n] = n$ and $\lim_{x\downarrow 1} E_0^{\mu_x}[Z_n] = -n$). Consequently, $A := \bigcup_{n\in\mathbb{N}} A_n$ is countable and hence $A^c := (0,1) \setminus A \neq \varnothing$. Now (B.1) follows because, for any $x \in A^c$, $E_0^{\mu_x}[Z_n] \neq 0 \forall n \in \mathbb{N}$. □

B.2. Asymmetry in the Gaussian regime

In this section, we prove Theorem 5(II).

Proof. Fix $s \in (2,\infty)$. To show that

$$\{\alpha\colon \langle \log\rho\rangle < 0, \langle \rho^s\rangle = 1,\, E_0^{\mu}[Z_n] \neq v_\mu n \text{ i.o.}\} \neq \varnothing, \tag{B.3}$$

we proceed as above. Define α_x as in B.1, but define $\eta(x) \in (0,1)$ to satisfy

$$x\left(\frac{1-x}{x}\right)^s + (1-x)\left(\frac{1-\eta(x)}{\eta(x)}\right)^s = 1, \tag{B.4}$$

which implies that η_x satisfies $\langle\rho\rangle < 0$. Let $\mu_x = \alpha_x^{\mathbb{Z}}$, and consider the sets

$$B_n := \{x \in (0,1)\colon E_0^{\mu_x}[Z_n] = v_\mu n\}, \quad n \in \mathbb{N}. \tag{B.5}$$

By the implicit function theorem, $x \mapsto \eta(x)$ is analytic. Consequently, $B := \bigcup_{n\in\mathbb{N}} B_n$ is countable and hence $B^c := (0,1) \setminus B \neq \varnothing$. Now (B.3) follows because, for any $x \in B^c$, $E_0^{\mu_x}[Z_n] \neq v_\mu n \forall n \in \mathbb{N}$. □

Appendix C: Bound on recurrent fluctuations

We prove Theorem 5(III). The line of proof was suggested by Zhan Shi.

Proof. Throughout this section, C is a constant that does not depend on n and may vary from line to line.

Scaled potential process. Define $U^{\omega,n}(t) := \frac{1}{\sigma_0 \log n} U^{\omega}(\lfloor t \log^2 n\rfloor)$, where

$$U^{\omega}(k) = \begin{cases} \sum_{i=1}^{k} \log\rho_i, & k \in \mathbb{N}, \\ 0, & k = 0, \\ -\sum_{i=k+1}^{0} \log\rho_i, & k \in -\mathbb{N}. \end{cases} \tag{C.1}$$

From (1.1) and (1.2) it follows that $t \mapsto U^{\omega,n}(t)$ converges weakly to a Brownian motion. Let $\overline{b}^n$ be the position of the bottom of the smallest valley $(\overline{a}^n, \overline{b}^n, \overline{c}^n)$ of the process $(U^{\omega,n}(t))_{t\in\mathbb{R}}$, which contains the origin and has depth larger than

1 (for a formall definition of the smallest valley see [19, Section 2.5]). Similarly, for any $\delta > 0$, let $(\overline{a}^n_\delta, \overline{b}^n_\delta, \overline{c}^n_\delta)$ be the smallest valley containing the origin with depth larger than $1+\delta$. We start with the decomposition

$$\frac{Z_n}{\log^2 n} = \left(\frac{Z_n}{\log^2 n} - \overline{b}^n\right) + \overline{b}^n =: \bar{B}_n + \overline{b}^n. \tag{C.2}$$

To control the left-hand side above, it suffices to show that for any $\varepsilon > 0$ there is a $C \in (0, \infty)$ such that

$$E_\mu\big[\overline{b}^n\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}, \tag{C.3}$$

$$E_0^\mu[\bar{B}_n] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}. \tag{C.4}$$

• *decay of* $E_\mu[\overline{b}^n]$. The proof of (C.3) is done via a Skorohod embedding. It is organised in three parts. In the first part we define the Skorohod embedding. In the second part, using the Skorohod embedding we compare the bottom of the valley $\overline{b}^n$ of the scaled potential process with the bottom of the valley $\hat{b}^n$ embedded potential process. In this part we use Kolmogorov's inequality combined with estimates on the random times that define the embedding. The third part consists of comparing the bottom of the embedded valley with the bottom of the underlying Brownian motion that we used for the embedding. This part relies on the control of the oscilations between the random times in the embedding together with the relation between conditioned Brownian motion and the Bessel bridge.

Skorohod embedding. Let $(B_t)_{t\in\mathbb{R}}$ be a two sided Brownian motion with $B_0 := 0$ defined on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t\in\mathbb{R}}, \hat{P})$, endowed with the double sided filtration generated by $(B_t)_{t\in\mathbb{R}}$ starting from 0, i.e., $\hat{\mathcal{F}}_t = \sigma(B_{\frac{st}{|t|}}, 0 \le s \le |t|)$. By the Skorokhod embedding [5, Thm 7.6.3, p. 404] for each n, there is a sequence of stopping times, $(\hat{T}_{n,k})_{k\in\mathbb{Z}}$ with $\hat{T}_{n,0} = 0$ and satisfying

$$U^{\omega,n}\left(\frac{k}{\log^2 n}\right) \overset{(d)}{=} B_{\hat{T}_{n,k}}. \tag{C.5}$$

Let $t_{n,k} := k \log^{-2} n$ denote the jump times of the scaled potential process. From now on

$$\big(\hat{U}^{\omega,n}(t)\big)_{t\in\mathbb{R}} \tag{C.6}$$

refers to the *embedded potential process* determined by $(B_t)_{t\in\mathbb{R}}$ with jump times $\hat{T}_{n,k}$. We denote by $(\hat{a}^n, \hat{b}^n, \hat{c}^n)$ the smallest valley of the process $(\hat{U}^{\omega,n}(t))_{t\in\mathbb{R}}$ that contains the origin and has depth larger than 1. We write $\hat{E}$ to denote expectation w.r.t. the embedded random variables $(\log \rho_i)_{i\in Z}$ that regulate the jumps of the scaled and embedded potential processes.

Let $(\hat{a}, \hat{b}, \hat{c})$ be the smallest valley of depth 1 containing the origin of the Brownian motion $(B_t)_{t\ge 0}$. Note that the distribution $\hat{b}$ is given by (1.7) and by symmetry:

$$\hat{E}[\hat{b}] = 0. \tag{C.7}$$

Note first that $\bar{b}^n \le \overline{c}^n - \overline{a}^n$. As shown in [3, Appendix C], the random variable $\bar{J}^n := \overline{c}^n - \overline{a}^n$ satisfies $\sup_p E[|\bar{J}^n|^p] < \infty$. Note next that

$$\max\{|\bar{b}^n|, |\hat{b}^n|, |\hat{b}|\} \le \bar{J}^n. \tag{C.8}$$

The general idea to prove (C.3) is to find sets A_n for which

$$\hat{E}\big[\bar{b}^n \mathbb{1}_{A_n}\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}, \qquad \hat{P}\big[A_n^c\big] \le \frac{C}{\log^{\frac{2}{3}-\frac{\varepsilon}{2}} n}. \tag{C.9}$$

To obtain (C.3), we use Hölder's inequality to bound $\hat{E}[\bar{b}^n \mathbb{1}_{A_n^c}]$ by

$$\hat{E}\big[\bar{J}^n \mathbb{1}_{A_n^c}\big] \le \hat{E}\big[|\bar{J}^n|^p\big]^{\frac{1}{p}} \left(\frac{1}{\log^{\frac{2}{3}-\frac{\varepsilon}{2}} n}\right)^{\frac{p-1}{p}} \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}, \tag{C.10}$$

where the last inequality follows by taking p sufficiently large. More specifically, to prove (C.3) will show that there are sets A_n and E_n for which

$$\hat{E}\big[\big|\bar{b}^n - \hat{b}^n\big|\mathbb{1}_{A_n}\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}, \qquad \hat{P}\big(A_n^c\big) \le \frac{C}{\log^{\frac{2}{3}-\frac{\varepsilon}{2}} n}, \tag{C.11}$$

$$\hat{E}\big[\big|\hat{b}^n - \hat{b}\big|\mathbb{1}_{(A_n\cap E_n^c)}\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}, \qquad \hat{P}\big(A_n^c \cup E_n\big) \le \frac{C}{\log^{\frac{2}{3}-\frac{\varepsilon}{2}} n}. \tag{C.12}$$

Reasoning as in (C.9)–(C.10), using (C.7) (C.8), (C.11) and (C.12), we obtain

$$\hat{E}\big[\overline{b}^n\big] \le \hat{E}\big[\big|\overline{b}^n - \hat{b}^n\big|\big] + \hat{E}\big[\big|\hat{b}^n - \hat{b}\big|\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}. \tag{C.13}$$

In the next two paragraphs we will show (C.11) by comparing the deterministic times $t_{n,k}$ with the random times $\hat{T}_{n,k}$ with the help of moment estimates. After that we will show (C.12) by comparing the location of the embedded minimum $\hat{b}^n$ with the location of the true minimum $\hat{b}$ with the help of estimates on Bessel bridges.

Comparing $t_{n,k}$ with $\hat{T}_{n,k}$. Let $a^n := \bar{a}^n \log^2 n$, $b^n := \bar{b}^n \log^2 n$ and $c^n := \bar{c}^n \log^2 n$ and let $J(n) := c^n - a^n$. The times $(\hat{\tau}_{n,k} := \hat{T}_{n,k} - \hat{T}_{n,k-1})_{k\in\mathbb{Z}}$ defined by the Skorokhod embedding theorem stated in [5, Thm 7.6.3] are i.i.d. and satisfy

$$\begin{aligned} &\hat{E}[\hat{\tau}_{n,k}] = \hat{E}\bigg[\bigg(\frac{\log\rho_0}{\sigma_0\log n}\bigg)^2\bigg] = \frac{1}{\log^2 n}, \\ &\hat{E}\big[\hat{\tau}_{n,k}^2\big] \le C\hat{E}\bigg[\bigg(\frac{\log\rho_0}{\sigma_0\log n}\bigg)^4\bigg] < \frac{C}{\log^4 n}. \end{aligned} \tag{C.14}$$

Furthermore, since $B^{2k} - p_k(t)$ is a martingale for some polynomial $p_k(t)$ of degree k, the optional stopping theorem and (1.2) give

$$\hat{E}\big[(\hat{\tau}_{n,k})^k\big] \le C\hat{E}\big[B^{2k}_{\hat{\tau}_{n,k}}\big] = C\hat{E}\bigg[\bigg(\frac{\log\rho_0}{\sigma_0\log n}\bigg)^{2k}\bigg] \le \frac{C}{\log^{2k} n}. \tag{C.15}$$

Therefore, by Markov's inequality, for any $k \in \mathbb{N}$,

$$\hat{P}\bigg(\hat{\tau}_{n,k} > 2\frac{\sigma_\mu}{\log^{2-\varepsilon} n}\bigg) \le \frac{(\log n)^{k(2-\varepsilon)}}{\sigma_\mu^2}\frac{C}{\log^{2k} n} \le \frac{C}{\log^{k\varepsilon} n}. \tag{C.16}$$

For $k\varepsilon - 2 > 2 + 2\varepsilon$, and any fixed $J_0 > 0$, a union bound gives that

$$\hat{P}\bigg(\exists k \le J(n)\colon\ \hat{\tau}_{n,k} > 2\frac{\sigma_\mu}{\log^{2-\varepsilon} n}, \frac{J(n)}{\log^2 n} \le J_0\bigg) \le \frac{C}{\log^{2+2\varepsilon} n}. \tag{C.17}$$

Abbreviate $\bar{J}^n := J(n)\log^{-2} n$ and define the set

$$A_n := \bigg\{\omega\colon \sup_{k\le J(n)} \hat{\tau}_{n,k} < 2\frac{\sigma_\mu}{\log^{2-\varepsilon} n}, \bar{J}(n) \le \big(\log\log^4 n\big)\bigg\}. \tag{C.18}$$

We have

$$\hat{P}\big(\bar{J}_n > \log\log^4 n\big) \le c_1\hat{P}\Big(\sup_{t\in[0,\log\log^4 n]} |B_t| < 1\Big) \le \frac{C}{\log^4 n}, \tag{C.19}$$

where c_1 stands for a constant that takes into account the double-sided necessary estimates to the right and to the left of the origin. Furthermore, the constant c_1 also absorbs the uniform approximation error of the discrete walk, with respect to the Brownian motion. From (C.17) and (C.19) it follows that

$$\hat{P}\big(A_n^c\big) \le \frac{C}{\log^{2+2\varepsilon} n}. \tag{C.20}$$

Therefore, on A_n, using that $(\hat{\tau}_{n,k} - \log^{-2} n)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. mean zero random variables, by Kolmogorov's inequality and (C.14) it follows that, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\hat{P}\Big(\sup_{a^n\le j\le c^n} t_{n,j} - \hat{T}_{n,j} > \frac{1}{\log n}, A_n\Big)\\
&\quad\le \hat{P}\Bigg[\sup_{j\le \log\log^4 n}\sum_{k=0}^{j}\hat{\tau}_{n,k} - \log^{-2} n > \frac{1}{\log n}\Bigg]\\
&\quad\le \big(\log^2 n\big)\hat{E}\Bigg[\Bigg(\sum_{k=0}^{\log(\log^4 n)}\hat{\tau}_{n,k} - \log^{-2} n\Bigg)^2\Bigg]\\
&\quad\le \big(\log\log^4 n\big)\log^2 n\frac{C}{\log^4 n} \le \frac{C}{\log^{2-\frac{\varepsilon}{2}} n}.
\end{aligned}
\tag{C.21}
$$

Let

$$
A_{n,\le} := \Big\{\omega\colon \sup_{a^n\le j\le c^n} t_{n,j} - \hat{T}_{n,j} \le \frac{1}{\log n}\Big\}. \tag{C.22}
$$

Since $\hat{b}_n = \hat{T}_{n,b^n}$, by (C.20) and (C.21), and arguing as in (C.9)–(C.10), we get that

$$
\begin{aligned}
\hat{E}\big[\big|\overline{b}^n - \hat{b}^n\big|\big] &\le \hat{E}\big[\big|\overline{b}^n - \hat{b}^n\big|\mathbb{1}_{A_n}\big] + \hat{E}\big[\big|\overline{b}^n - \hat{b}^n\big|\mathbb{1}_{A_n^c}\big]\\
&\le \frac{1}{\log n} + \hat{E}\big[\big|\bar{J}^n\big|\mathbb{1}_{A_{n,\le}^c}\mathbb{1}_{A_n}\big] + \hat{E}\big[\big|\bar{J}^n\big|\mathbb{1}_{A_n^c}\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}.
\end{aligned}
\tag{C.23}
$$

Comparing $\hat{b}^n$ with $\hat{b}$. To prove (C.3) it suffices to show that

$$
\hat{E}\big[|\hat{b}_n - \hat{b}|\mathbb{1}_{A_n}\big] \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}. \tag{C.24}
$$

To prove (C.24) we first note that, conditioned on $\hat{b}$ being the bottom of the valley $(\hat{a}, \hat{b}, \hat{c})$ of depth 1, the trajectory of the Brownian motion B_t behaves as a two-sided Bessel bridge of dimension 3 (see [12]). Taking the point $(\hat{b}, B_{\hat{b}})$ to be the origin, we see that the two bridges we are considering can be described by

$$
dX_t = \Big(\frac{1}{X_t} + \frac{1}{1-X_t} + \frac{X_t - 1}{t - (\hat{c} - \hat{b})}\Big)dt + dB_t \quad \text{for } t \le \hat{c} - \hat{b}, \tag{C.25}
$$

$$
dX_t = \Big(\frac{1}{X_t} + \frac{1}{1-X_t} + \frac{X_t - 1}{t - (\hat{b} - \hat{a})}\Big)dt + dB_t \quad \text{for } t \le \hat{b} - \hat{a}. \tag{C.26}
$$

By the symmetry of Brownian motion, it suffices to analyse (C.25), which, in integral form, for $t \le (\hat{c} - \hat{b})$ reads as

$$
Y_t = \int_0^t \Big(\frac{1}{X_s} + \frac{1}{X_s - 1} + \frac{1 - X_s}{s - (\hat{c} - \hat{b})}\Big)ds + B_t. \tag{C.27}
$$

By the invariance of Brownian motion, $B_1 \overset{(d)}{=} -B_1 \overset{(d)}{=} \frac{B_t}{\sqrt{t}}$. Then, by (C.27), for any $\delta > 0$ and $\eta \in (0, \frac{1}{4})$ we get

$$
\begin{aligned}
\hat{P}\Big(\sup_{t\in[0,\delta]} Y_t < \eta\Big) &\le \hat{P}\Big(\frac{1}{2}\eta^{-1}\delta + B_\delta < \eta\Big)\\
&= \hat{P}\Big(\frac{1}{2}\eta^{-1}\delta^{\frac{1}{2}} - \eta\delta^{-\frac{1}{2}} < B_1\Big).
\end{aligned}
\tag{C.28}
$$

Next, by taking $\delta_n = \log^{-\alpha} n$, $\eta_n = \log^{-\beta} n < \frac{1}{4}$ with $\beta = \frac{1}{3} - \frac{1}{4}\varepsilon$ and $\alpha = \frac{2}{3} - \varepsilon$, with $\varepsilon > 0$ sufficiently small, we get

$$\begin{aligned}\hat{P}(\exists t \le \delta_n \text{ with } Y_t > \eta_n) &= 1 - \hat{P}\Big(\sup_{t\in[0,\delta_n]} Y_t < \eta_n\Big) \\ &\ge 1 - \hat{P}\bigg(\frac{1}{2}\log^{\frac{\varepsilon}{2}} n - \log^{-\frac{\varepsilon}{2}} n < B_1\bigg) \\ &\ge 1 - \frac{1}{\exp(c\log^{\varepsilon} n)},\end{aligned} \tag{C.29}$$

for some $c > 0$. Now let $B(\delta_n, \eta_n) := \{\omega\colon \exists t \le \delta_n \text{ with } Y_t > \eta_n\}$ and note that

$$\hat{P}\big(B(\delta_n, \eta_n)\big) \le \frac{1}{\exp(c\log^{\varepsilon} n)}. \tag{C.30}$$

By the construction of the Skorohod embedding and by (1.2),

$$\sup_{k\in\mathbb{N}} \sup_{t\in[T_{n,k-1},T_{n,k}]} |B_t - B_{\hat{T}_{n,k-1}}| \le \log\frac{1-\mathfrak{c}}{\mathfrak{c}}\frac{1}{\log n}. \tag{C.31}$$

So, on the event $B(\delta_n, \eta_n)$, $|\hat{b}^n - \hat{b}| > \frac{1}{\log^{\frac{2}{3}-\varepsilon} n}$ implies

$$\inf_{t\le\hat{c}-\hat{b}} Y_t < \frac{1}{\log^{1-\varepsilon} n} \quad \text{when } Y_0 = \frac{1}{\log^{\frac{1}{3}-\frac{\varepsilon}{4}} n}. \tag{C.32}$$

Let

$$E_n := \bigg\{\omega\colon \inf_{t\le\hat{c}-\hat{b}} Y_t < \frac{1}{\log^{1-\varepsilon} n} \text{ given } Y_0 = \frac{1}{\log^{\frac{1}{3}-\frac{\varepsilon}{4}} n}\bigg\}. \tag{C.33}$$

From the hitting times for Bessel processes [8, Problem 3.3.23, p. 162] it follows that

$$\hat{P}(E_n) \le \frac{C}{\log^{\frac{2}{3}-\frac{3\varepsilon}{4}} n}. \tag{C.34}$$

Noting that $\bar{J}_n \le \log\log^4 n$ on A_n and using (C.8) and (C.32), we get

$$\begin{aligned}&\hat{E}\big[|\hat{b}_n - \hat{b}|\mathbb{1}_{A_n}\big] \\ &\quad\le \frac{1}{\log^{\frac{2}{3}-\varepsilon} n} + \hat{E}\big[|\hat{b}_n - \hat{b}|\mathbb{1}_{A_n}\mathbb{1}_{\{|\hat{b}^n-\hat{b}|>\log^{-(\frac{2}{3}-\varepsilon)} n\}}\big] \\ &\quad\le \frac{1}{\log^{\frac{2}{3}-\varepsilon} n} + \log\log^4 n\big(\hat{E}[1_{E_n}] + \hat{E}[\mathbb{1}_{(B(\delta_n,\eta_n))^c}]\big) \\ &\quad\le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n},\end{aligned} \tag{C.35}$$

where the last inequality uses (C.30) and (C.34). Using (C.35), (C.23), and (C.13), we conclude the proof of (C.3).

• *decay of* $E_0^{\mu}[\bar{B}_n]$. It remains to show (C.4). We follow [19, pp. 249–251], with appropriate modifications. We define the set of "good environments", with δ and J both depending on n, as

$$A_n^{J,\delta} := \left\{\omega\colon \begin{array}{l} \overline{b}^n = \overline{b}^n_\delta, \\ \text{any refinement}(a,b,c) \text{ of } (\overline{a}^n_\delta, \overline{b}^n, \overline{c}^n_\delta) \\ \text{with } b \ne \overline{b}^n \text{ has depth } < 1-\delta, \\ |\overline{a}^n_\delta| + |\overline{c}^n_\delta| \le J, \\ \inf_{t-\bar{b}^n>\delta} B_t - B_{\hat{b}} > \delta^{\frac{3}{2}}, \end{array}\right\} \tag{C.36}$$

with δ and J chosen as

$$\delta = \delta(n) := \frac{1}{\log^r n}, \qquad J = J(n) := \log\log^4 n, \tag{C.37}$$

with $r \in (0,1)$ a parameter to be fixed later. From now on, we simply write P and E for the annealed measure and corresponding expectation, as well as for the measure of the underlying Brownian motion that was used for the embedding in the previous paragraph and its corresponding expectation.

Recall that $\bar{B}_n = \frac{Z_n - b^n}{\log^2 n}$. Let

$$G_n := \big\{\omega\colon (X_i), 0 \le i \le n \text{ hits the boundary of } \big[a_\delta^n, c_\delta^n\big]\big\}. \tag{C.38}$$

With these definitions, we split $E[|\bar{B}_n|]$ as

$$\begin{aligned} E\big[|\bar{B}_n|\big] &= E\big[|\bar{B}_n|\mathbb{1}_{(A_n^{J,\delta})^c}\big] + E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\big] \\ &= \mathrm{I}_n + E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n<0\}}\big] + E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n>0\}}\big] \\ &= \mathrm{I}_n + \mathrm{II}_n + E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n>0\}}\mathbb{1}_{G_n^c}\big] \\ &\quad + E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n>0\}}\mathbb{1}_{G_n}\big] \\ &= \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n + \mathrm{IV}_n, \end{aligned} \tag{C.39}$$

where

$$\begin{aligned} &\mathrm{I}_n := E\big[|\bar{B}_n|\mathbb{1}_{(A_n^{J,\delta})^c}\big], \qquad \mathrm{III}_n := E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n>0\}}\mathbb{1}_{G_n^c}\big], \\ &\mathrm{II}_n := E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n<0\}}\big], \qquad \mathrm{IV}_n := E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\overline{b}^n>0\}}\mathbb{1}_{G_n}\big]. \end{aligned} \tag{C.40}$$

To prove (C.4), it suffices to show that there is a constant $C > 0$ for which

$$\max\{\mathrm{I}_n, \mathrm{II}_n, \mathrm{III}_n, \mathrm{IV}_n\} \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}. \tag{C.41}$$

In what follows we will show that this bound holds for each of the above terms.

Estimate of I_n. The estimate of IV_n follows directly from the definition of $A_n^{J,\delta}$. By Hölder's inequality, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\mathrm{IV}_n \le E\big[|\bar{B}_n|^p\big]^{\frac{1}{p}} P\big(\big(A_n^{J,\delta}\big)^c\big)^{\frac{1}{q}}. \tag{C.42}$$

Since $\sup_{n\in\mathbb{N}} E[|\bar{B}_n|^p] < \infty$ (see [3]), it suffices to estimate $P((A_n^{J,\delta})^c)$. The definition of $A_n^{J,\delta}$ consists of four conditions. Therefore we estimate

$$\begin{aligned} P\big(\big(A_n^{J,\delta}\big)^c\big) \le P\big(\bar{b}^n \ne \bar{b}_\delta^n\big) &\\ &+ P\big(\exists \text{ refinement } (a,b,c) \text{ of } \big(\overline{a}_\delta^n, \overline{b}^n, \overline{c}_\delta^n\big) \text{ with } b \ne \overline{b}^n \text{ and depth } > 1-\delta\big) \\ &+ P\big(\big|\overline{a}_\delta^n\big| + \big|\overline{c}_\delta^n\big| > J\big) \\ &+ P\Big(\inf_{t-\bar{b}^n>\delta} B_t - B_{\hat{b}} < \delta^{\frac{3}{2}}\Big). \end{aligned} \tag{C.43}$$

Note that

$$\big\{\bar{b}^n \ne \bar{b}_\delta^n\big\} \subset \big\{\exists \text{ refinement } (a,b,c) \text{ of } \big(\overline{a}_\delta^n, \overline{b}^n, \overline{c}_\delta^n\big) \text{ with } b \ne \overline{b}^n \text{ and depth } > 1-\delta\big\}. \tag{C.44}$$

Furthermore, the probability of having a valley of depth larger than $1-\delta$ is bounded from above by the probability for a Bessel bridge starting from 1 to reach a value smaller than δ, which in turn is bounded from above by the probability for the infimum of a Bessel process of dimension 3 starting from 1 to be smaller than δ. By the estimate for hitting times of Bessel process, it follows that

$$\begin{aligned} &P\big(\exists \text{ refinement } (a,b,c) \text{ of } \big(\overline{a}^n_\delta, \overline{b}^n, \overline{c}^n_\delta\big) \text{ with } b \neq \overline{b}^n \text{ and depth } > 1-\delta\big) \\ &\quad \leq 2P(\text{ Bessel process of dimension 3 started from 1 reaches a value smaller than } \delta) \\ &\quad \leq 2\delta \leq \frac{2}{\log^r n}, \end{aligned} \tag{C.45}$$

where the factor 2 takes into account the double-sided necessary estimates (to the right and to the left of $\bar{b}^n$). Combining (C.44) with (C.45), we get

$$P\big(\big(A_n^{J,\delta}\big)^c\big) \leq \frac{C}{\log^r n} + P\big(\big|\overline{a}^n_\delta\big| + \big|\overline{c}^n_\delta\big| > J\big) + P\Big(\inf_{t-\bar{b}^n > \delta} B_t - B_{\hat{b}} < \delta^{\frac{3}{2}}\Big). \tag{C.46}$$

To estimate the remaining terms in (C.43), we first note that, by (C.19),

$$P\big(\big|\bar{a}^n_\delta\big| + \big|\bar{c}^n_\delta\big| > J\big) \leq \frac{C}{\log^4 n}. \tag{C.47}$$

The last term in (C.43) can be bounded via the same reasoning used in (C.30) and (C.34), and so we get that

$$P\Big(\inf_{t-\bar{b}^n > \delta} B_t - B_{\hat{b}} > \delta^{\frac{3}{2}}\Big) \leq \frac{C}{\log^r n}. \tag{C.48}$$

Therefore, with $r = \frac{2}{3}$ in (C.37) it follows from (C.46), (C.47) and (C.48) that there is a choice of p, q in (C.42) such that $\mathrm{I}_n \leq \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}$.

Estimate of II_n. The estimate of II_n is analogous to $\mathrm{III}_n + \mathrm{IV}_n$.

Estimate of III_n. Before proving the this estimate, we recall the expression for the hitting times as stated in [19, p. 196 (2.1.4)]): for $a < x < b$,

$$\begin{aligned} P_x^\omega(H_a < H_b) &= \frac{\sum_{i=x}^{b-1} \exp U^\omega(i)}{\sum_{i=a}^{b-1} \exp U^\omega(i)}, \\ P_x^\omega(H_b < H_a) &= \frac{\sum_{i=a}^{x-1} \exp U^\omega(i)}{\sum_{i=a}^{b-1} \exp U^\omega(i)}, \end{aligned} \tag{C.49}$$

where, for any $y \in \mathbb{Z}$, $H_y := \inf\{i \in \mathbb{N}_0\colon Z_i = y\}$. On the event $E_n^c \cap A_n^{J,\delta} \cap \{\bar{b}^n > 0\}$ the random walk $(Z_t)_{t\in\mathbb{N}_0}$ is equivalent to the reflecting random walk at a^n denoted by $(\tilde{Z}_t)_{t\in\mathbb{N}_0}$. More formally $\tilde{Z}_t$ is the random walk in the environment $\overline{\omega}_z := \omega_z$ for $z > a^n_\delta$, $\overline{\omega}^+_{a^n_\delta} = 1$ and $\overline{\omega}^+_{a^n_\delta - 1} = 0$. Therefore, for $\omega \in A_n^{J,\delta}$,

$$\begin{aligned} &E_0^\omega\left(\left|\frac{Z_t}{\log^2 n} - \overline{b}^n\right|, G_n^c\right) \\ &\quad \leq \log\log^4 n P_0^\omega(H_{b^n} > n) + E_0^\omega\left(\left|\frac{\tilde{Z}_t}{\log^2 n} - \overline{b}^n\right| \mathbb{1}_{\{T_{b^n} < n\}}\right) \\ &\quad \leq \log\log^4 n P_0^\omega(T_{b^n} > n) + \max_{t\in[0,n]\cap\mathbb{Z}} E_{b^n}^\omega\left(\left|\frac{\tilde{Z}_t}{\log^2 n} - \overline{b}^n\right|\right). \end{aligned} \tag{C.50}$$

The arguments that lead to [19, Eqs. (2.4.4)–2.5.5, pp. 249-250], imply that

$$P_0^\omega(H_{b^n} > n) \leq \frac{C}{\exp(2^{-1}\delta_n \log n)} \leq \frac{C}{\exp(2^{-1}\log^{1-r} n)}. \tag{C.51}$$

Therefore, for any $r < 1$ there is a $C > 0$ for which

$$\big(\log\log^4 n\big) P_0^{\omega}(T_{b^n} > n) \le \frac{C}{\log^{\frac{2}{3}-\varepsilon} n}. \tag{C.52}$$

To estimate the second term in the right-hand side of (C.50), we follow [19, pp.250–251]. Define

$$\begin{aligned} f(z) &:= \frac{\prod_{a_\delta^n+1\le i<z}\omega_i}{\prod_{a_\delta^n+1\le i<z}(1-\omega_{i+1})} = \frac{(1-\omega_{a_\delta^n+1})}{\omega_z} n^{-[U^{\omega,n}(z)-U^{\omega,n}(a_\delta^n)]}, \\ \overline{f}(z) &:= \frac{f(z)}{f(b^n)}. \end{aligned} \tag{C.53}$$

For $g\colon \mathbb{Z}\to\mathbb{R}$, let $\nu_g = \sum_{z\in\mathbb{Z}} \delta_z g(z)$, where δ_z is the Dirac measure concentrated at z. The one-step transition operator of this process $\mathcal{A}$ acts on a measures on $\mathbb{Z}$ as follows:

$$(\nu\mathcal{A})(z) := \overline{\omega}_{z-1}\nu(z-1) + (1-\overline{\omega}_{z+1})\nu(z+1). \tag{C.54}$$

Note that, by (C.53) and (C.54), $\nu_{\overline{f}}\mathcal{A} = \nu_{\overline{f}}$. In words, $\nu_{\overline{f}}$ is an invariant measure for the reflecting random walk $(\overline{Z}_t)_{t\ge 0}$. Since $\overline{f}(z) \ge \mathbb{1}_{b^n}(z)$ for all z, and $g\mathcal{A}\ge 0$ for all $g \ge 0$, we obtain that

$$\begin{aligned} P_{b^n}^{\omega}(\overline{Z}_t = z) &= \nu_{\mathbb{1}_{b^n}}\mathcal{A}^t(z) \le \nu_{\overline{f}}\mathcal{A}^t(z) = f(z) \\ &= \frac{\omega_{b^n}^+}{\omega_z^+} n^{-[U^{\omega,n}(z)-V(b^n)]} \le \frac{1}{\mathfrak{c}} n^{-[U^{\omega,n}(z)-U^{\omega,n}(b^n)]}, \end{aligned} \tag{C.55}$$

the last inequality being a consequence of the uniform ellipticity assumption ($\omega_0^+ \ge \mathfrak{c}$). Note now that, for $\omega \in A_n^{J,\delta}$,

$$\big|z-b^n\big| > \delta_n \quad \Longrightarrow \quad U^{\omega,n}(z) - U^{\omega,n}(\bar{b}_n) \ge \delta_n^{\frac{3}{2}} = \frac{1}{\log^{\frac{3r}{2}} n}. \tag{C.56}$$

Hence, uniformly in all $t \ge \mathbb{Z}_+$, for any $r < \frac{2}{3}$ there is a $C > 0$ for which

$$\begin{aligned} E_{b^n}^{\omega}\left(\left|\frac{\overline{Z}_t}{\log^2 n} - \overline{b}^n\right|\right) &= \sum_{z\in[a_\delta^n,c_\delta^n]\cap\mathbb{Z}} P_{b^n}^{\omega}(\overline{Z}_t = z)\left|\frac{z}{\log^2 n} - \overline{b}^n\right| \\ &\le \frac{1}{\mathfrak{c}\log^2 n} \sum_{z\in[a_\delta^n,c_\delta^n]\cap\mathbb{Z}} \big|z-b^n\big| n^{-[U^{\omega,n}(z)-U^{\omega,n}(b^n)]} \\ &\le 2\delta_n + \frac{(J\log^2 n)^2}{\mathfrak{c}\log^2 n} \sup_{z-b^n>\delta\log^2 n} e^{-\log n[V(z)-v(b^n)]} \\ &\le \frac{2}{\log^r n} + C\log^2 n\big(\log\log^4 n\big)e^{-\log^{(1-\frac{3r}{2})} n} \le \frac{C}{\log^r n}, \end{aligned} \tag{C.57}$$

which yields the bound for III_n.

Estimate of IV_n. By Hölder's inequality, for $p, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$,

$$\begin{aligned} \mathrm{I}_n &\le E\big[|\bar{B}_n|\mathbb{1}_{A_n^{J,\delta}}\mathbb{1}_{\{\bar{b}^n>0\}}\mathbb{1}_{G_n}\big] \\ &\le E\big[|\bar{B}_n|^p\big]^{\frac{1}{p}}\mathbb{P}\big(\big\{\bar{b}^n > 0\big\}\cap A_n^{J,\delta}\cap G_n\big)^{\frac{1}{q}}. \end{aligned} \tag{C.58}$$

As $\sup_n E[|\bar{B}_n|^p] < \infty$, we estimate $\mathbb{P}(\{\bar{b}^n > 0\}\cap A_n^{J,\delta}\cap G_n)$. Define

$$\begin{aligned} H_{b,n} &:= \inf\big\{i \ge 0\colon Z_i = b^n\big\}, \\ H_{a,b,n} &:= \inf\big\{i \ge 0\colon Z_i = b^n \text{ or } Z_i = a_\delta^n\big\}. \end{aligned} \tag{C.59}$$

Then, by (C.49), we have (this is the same inequality as in [19, Eq. (2.5.4)])

$$P_0^\omega\big(Z_{H_{a,b,n}} = a_\delta^n\big) \le \frac{J\log^2 n}{n^\delta}. \tag{C.60}$$

Again, denote by $(\overline{Z}_t)_{t\ge 0}$ the random walk in random environment with a reflecting barrier at a_δ^n, and let $\overline{H}_{a,b,n}$ be the analogue of $H_{a,b,n}$ for $(\overline{Z}_t)_{t\ge 0}$, and $\overline{H}_{a,n}$ be the hitting time of b^n by the reflecting walk. Then, by (1.2),

$$\begin{aligned} E_0^\omega[H_{a,b,n}] \le E_0^\omega[\overline{H}_{b,n}] &= \sum_{i=1}^{b^n}\sum_{j=0}^{i-1-a_\delta^n}\frac{1}{\omega_{i-j-1}}\prod_{k=1}^{j}\rho(i-k)\\ &\le \frac{1}{\mathfrak{c}}\sum_{i=1}^{b^n}\sum_{j=0}^{i-1-a_\delta^n}\exp^{(U^{\omega,n}(i)-U^{\omega,n}(i-j))\log n}\\ &\le \frac{(2J\log^2 n)^2}{\mathfrak{c}}\exp^{(1-\delta)\log n}, \end{aligned} \tag{C.61}$$

see [19, p. 250]. Consequently, for $\omega \in A_n^{J,\delta}$ satisfying $\overline{b}^n > 0$, by (C.60) and (C.61) and Markov' s inequality, we obtain

$$\begin{aligned} &P^\omega(H_{b,n}\ge n) \le P^\omega\big(H_{a,b,n}<n, Z_{H_{a,b,n}} = a_\delta^n\big) + P^\omega(H_{a,b,n}\ge n),\\ &\frac{J\log^2 n}{n^\delta} + \frac{1}{n}\frac{2(J\log n)^2}{\mathfrak{c}}e^{(1-\delta)(\log n)} = \frac{J\log^2 n + \frac{2(J\log n)^2}{\mathfrak{c}}}{n^\delta}. \end{aligned} \tag{C.62}$$

This is the analogue of [19, Eq. (2.5.5)] and says that, with overwhelming probability, the random walk hits b^n before time n. Let us now argue that, with overwhelming probability, after hitting b^n, the random walk will come back to b^n before hitting either a_δ^n or c_δ^n. By (C.49), for all $\omega \in A_n^{J,\delta}$,

$$\begin{aligned} &P_{b^n-1}^\omega\big((Z_i)_{i\in\mathbb{N}_0} \text{ hits } a_\delta^n \text{ before } b^n\big) \le n^{-(1+\frac{\delta}{2})},\\ &P_{b^n+1}^\omega\big((Z_i)_{i\in\mathbb{N}_0} \text{ hits } c_\delta^n \text{ before } b^n\big) \le n^{-(1+\frac{\delta}{2})}. \end{aligned} \tag{C.63}$$

Compare with [19, Eq. (2.5.6)].] As such, for $\omega \in A_n^{J,\delta}$ the P^ω-probability of the event that "after hitting b^n, the random walk exits $[a_\delta^n, c_\delta^n]$ within the next n steps" is bounded by

$$1-\big(1-n^{-(1+\frac{\delta}{2})}\big)^n \le \frac{C}{n^{\frac{\delta}{2}}}, \tag{C.64}$$

Combined with (C.62), this gives

$$P^\omega\big(A_n^{J,\delta}\cap G_n\big) \le \frac{J\log^2 n + \frac{2(J\log n)^2}{\mathfrak{c}}}{n^\delta} + \frac{2}{n^{\frac{\delta}{2}}} \le \frac{C}{n^{\frac{\delta}{2}}}. \tag{C.65}$$

Applying Hölder's inequality and the L^p bound $\sup_{p,n\in\mathbb{N}} E[|\bar{B}_n|^p] < \infty$, we find that $\mathrm{IV}_n \le \frac{C}{n^{\frac{\delta}{3}}} \le \frac{C}{\log^4 n}$, which proves the desired estimate in (C.4). □

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