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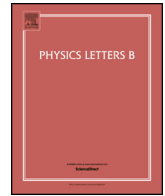
Bekenstein bound and uncertainty relations

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ABSTRACT

The non–zero value of Planck constant h underlies the emergence of several inequalities that must be satisfied in the quantum realm, the most prominent one being Heisenberg Uncertainty Principle. Among these inequalities, Bekenstein bound provides a universal limit on the entropy that can be contained in a localized quantum system of given size and total energy. In this Letter, we explore how Bekenstein bound is affected when Heisenberg uncertainty relation is deformed so as to accommodate gravitational effects close to Planck scale (Generalized Uncertainty Principle). By resorting to general thermodynamic arguments, and in regimes where the equipartition theorem still holds, we derive in this way a "generalized Bekenstein bound". Physical implications of this result are discussed for both cases of positive and negative values of the deformation parameter.

1. Introduction

In 1981 Jacob Bekenstein proposed a universal upper bound on the entropy S of a localized quantum system [1]

$$S \leq \frac{2\pi k_B R E}{\hbar c} , \qquad (1)$$

where E is the total energy of the system and $R = \sqrt{A/4\pi}$ its circumferential radius, with A being the area of the enclosing surface. Clearly, for $\hbar \to 0$, one obtains $S \leq \infty$, which tells us that, classically speaking, the entropy of a system is unbounded from above. The result (1) was the last offspring of a revolutionary decade of investigation, which started with the puzzling proposal of Bekenstein himself about the entropy of a black hole [2], then the formulation of black hole thermodynamics [3], and culminated with the renowned discovery of Hawking thermal radiation [4].

A key assumption in Bekenstein's derivation of the bound is that the gravitational self–interaction of the system can be neglected. Indeed, Eq. (1) does not contain Newton constant G_N,

even though it was obtained in regimes of strong gravity with gedanken experiments involving black holes. Remarkably, the inequality is exactly saturated by Schwarzschild black holes, whose entropy is given by $S = k_B A_H/(2\ell_p)^2$, where A_H denotes the horizon area and $\ell_p = \sqrt{\hbar G_N/c^3}$ the Planck length. Although many arguments [5] have been suggested to support the validity of Eq. (1), also several counterexamples have been brought forward, thereby enriching a lively debate which is still ongoing [6]. Further years of intuitions and studies have then led to the formulation of the well-known Holographic Principle [7–9], the Covariant [10] and Causal [11] Entropy Bounds, and finally to the rigorous quantum field theoretical proof of Bekenstein bound in flat spacetime [12]. For a general influence of the ideas of Bekenstein on quantum information theory, we remand the reader to Refs. [13–15]. Connections of various entropy bounds with cosmology [11,16–19], perturbative unitarity [20] and the Pauli principle [21] have also been addressed with non-trivial results.

In the last four decades, predictions from string theory, loop quantum gravity, deformed special relativity, non-commutative geometry and black hole physics [22–34] have converged on a feasible generalization of Heisenberg Uncertainty Principle (HUP), which is expected to simultaneously account for quantum and gravitational effects at Planck scale. In this framework, the stan-

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dard uncertainty relation for a quantum system should be modified as follows

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \beta \left(\frac{\Delta p}{m_p c} \right)^2 \right], \tag{2}$$

where Δx and Δp are the position and momentum uncertainties of the system, respectively, $m_p = \hbar/\ell_p c$ is the Planck mass and β the so called deformation parameter, which is considered to be of order unity in most of quantum gravity models [35]. The inequality (2) is commonly known as Generalized Uncertainty Principle (GUP). One of its most important implications is the significant modification of the behavior of Δx as a function of Δp in the regime $\Delta p/(m_p c) \simeq 1$. This results in the prediction of a minimum observable length $\Delta x \sim \sqrt{\beta} \ell_p$ occurring for $\beta > 0$ [24,25]. However, scenarios with $\beta < 0$ have been extensively discussed [36–38], along with various remarkable consequences. To further substantiate the soundness of the GUP framework not only at the theoretical level, we also mention that several experiments have been carried out or proposed to test the effects predicted by Eq. (2) (see e.g. Refs. [39]). Of course the GUP should not be intended as a complete theory of Quantum Gravity, fully valid at the Planck scale, neither it claims to be so. A prudent attitude, underlying the most wise literature on this topic, interprets the GUP as an instrument able to describe physics at energies closer to Planck scale, better than what the standard HUP can do.

Let us remark that Bekenstein bound has been rigorously proved by assuming standard principles of quantum mechanics and quantum field theory in flat spacetime [12]. In this Letter, we are interested in understanding how such an entropy bound is affected when HUP is replaced by the GUP in Eq. (2). To this aim, first we show how Bekenstein inequality (1) can be directly connected to HUP on the basis of general thermodynamic arguments. The present derivation, being elementary and based on first principles, should make it clear why Bekenstein bound has such a wide range of validity. The obtained result is then generalized to the context of GUP, leading to a *generalized Bekenstein bound* which by construction takes into account also quantum gravitational effects. Physical implications are finally investigated for both positive and negative values of the deformation parameter β, highlighting the different predictions of the two settings.

2. Bekenstein bound and HUP

Let us consider an isolated quantum system localized inside a finite region of circumferential radius R. From the basics of thermodynamics, it follows that if the relation between the energy E, the entropy S and the volume V of the system is known, then its temperature T can be easily calculated as follows

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_V. \tag{3}$$

By establishing Eq. (3), we explicitly exclude systems that may possess a negative temperature, otherwise it would result problematic to even introduce the elementary assumptions listed in what follows. Of course, Eq. (3) entails also the differentiability of the function $S(E, V)$.

Henceforth, the main working hypotheses underlying our analysis are:

1. We consider a regime where, on average, the energy μ of each component of the system is approximately given by

$$\mu \simeq k_B T, \tag{4}$$

   according to the equipartition theorem.

2. The momentum p of each component of our system satisfies the de Broglie relation

$$p = \frac{\hbar}{\lambda}, \tag{5}$$

   where λ denotes the corresponding wavelength.

Note that the second condition only makes sense for intrinsically quantum particles. From Eq. (5), it is a simple text-book exercise to derive Heisenberg relation between the momentum and position uncertainties. This is a crucial point in the present analysis, since Eq. (5) provides the springboard for the extension of the Bekenstein result to the GUP framework.

Concerning the first condition, it is well-known that the equipartition theorem is a classical statement. However, it also holds true that, for a large majority of physical systems in regimes close to the classical one, the energy μ of each component can be approximately described by $\mu \simeq k_B T$. In other words, Maxwell-Boltzmann statistics is a good approximation of quantum statistics in most of the systems in semiclassical regimes. For example, a gas of bosons at low frequencies or high temperatures is well described by the standard Maxwell-Boltzmann statistics.

Now, for the above two prescriptions to be valid and from an inspection of the quantum statistics distribution formula, we can infer that the energy $k_B T$ should satisfy the condition

$$k_B T \gtrsim \frac{\hbar c}{\lambda} = pc. \tag{6}$$

Given that our system is completely localized inside a volume of radius R, the inequality $\lambda \lesssim 2R$ holds true, so that from Eqs. (3) and (6) we obtain

$$\frac{\partial S}{\partial E} = \frac{1}{T} \lesssim \frac{k_B \lambda}{\hbar c} \lesssim \frac{2k_B R}{\hbar c}, \tag{7}$$

where it is understood that the derivative is taken at constant volume.

In general, R and E can be regarded as independent variables, therefore we can easily integrate the above relation with the condition[1] $S(E = 0) = 0$, obtaining

$$S \lesssim \frac{2\alpha k_B R E}{\hbar c}, \tag{8}$$

where we have inserted a "calibration factor" α in order to account for all the approximations performed so far. Note that this factor cannot be exactly fixed by our thermodynamic argument. However, the magnitude of the calibration factor will be obtained in the next Section by means of consistency arguments. Indeed, in analogy with the derivation of the modified Hawking temperature in Ref. [27], α can be chosen *a posteriori* by requiring that the generalized entropy bound obtained in the GUP framework recovers Bekenstein inequality (1) for a vanishing deformation parameter β (see below). This occurs for $\alpha = \pi$.

Remarkably, the above considerations and the ensuing bound (8) also encompass the case in which R and E are related via an equation of state. In fact, for a general and physically plausible radius-energy relation of the form $R = R(E)$, with $R(E)$ being a monotonically non-decreasing function of E, one can prove that the inequality (8) is still satisfied (see the Appendix for the proof).

Let us also mention that the Bekenstein bound can be saturated for a system composed by soft quanta, *i.e.*, of wavelength $\lambda \sim 2R$. According to the corpuscular models [40–43], this can

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[1] For the sake of completeness, we emphasize that the ansatz $S(E = 0) = 0$ naturally contains the hidden assumption of a unique ground state.

represent the case of a black hole whose constituents are soft gravitons of energy $\mu \sim \hbar c/\lambda \sim \hbar c/R_s$, with $R_s = 2G_N M/c^2$ being Schwarzschild radius.

Before turning to the calculation of GUP corrections, we stress that our result (8) has been derived by relying on quite general hypotheses. Furthermore, we have made no explicit reference to the particular behavior of the entropy as a function of the energy and/or the number of the elementary constituents. Less complete attempts to trace the Bekenstein bound back to HUP can be found in Refs. [44,45].

It is worth mentioning that the inverse implication, *i.e.*, a derivation of HUP from the Bekenstein bound, can also be achieved, as outlined in Ref. [46]. In a nutshell, let us consider a particle of rest mass m described by a wave-packet of spatial size R, and suppose it is marginally relativistic, namely $p \simeq E/c$. For that particle, the inequality (1) can be recast as

$$R\,p \geq \frac{\hbar}{2}\frac{S}{\pi k_B} \gtrsim \frac{\hbar}{2}, \qquad (9)$$

which applies to any system for which $S \gtrsim \mathcal{O}(k_B)$.[2] Of course, the above inequality holds up to a calibration factor which again results equal to π, but that cannot be determined with this heuristic approach.

Now, since the direction of motion of our particle is unknown *a priori*, we can safely suppose $\Delta p_x \simeq p$, and of course $\Delta x \simeq R$, as for the uncertainty on its position. Therefore Bekenstein inequality (9) can be read as

$$\Delta x \Delta p_x \gtrsim \frac{\hbar}{2}, \qquad (10)$$

which is the standard HUP for the particle in question. Therefore, together with the implication previously shown, the latter argument highlights a full consistency between Bekenstein bound and Heisenberg Uncertainty Principle.

3. Generalized Bekenstein bound

Let us now extend the previous considerations to the case in which the underlying theory is built upon the GUP (2). In particular, we wonder how the inequality (8) would appear when taking into account gravity effects at Planck scale via the GUP. Clearly, in order to consistently generalize calculations, we need to revise the de Broglie relation in Eq. (5).

In the same fashion as HUP is in one–to–one correspondence with the de Broglie relation, it is reasonable to expect that the GUP is consistent with a gravitationally modified de Broglie equation. This issue has been considered in Ref. [23] and in particular in [47], where the author obtained a generalized wave-particle duality relation of the form

$$\lambda \simeq \frac{\hbar}{p}\left[1 + \beta\left(\frac{p}{m_p c}\right)^2\right]. \qquad (11)$$

Note that a similar expression is encountered when using the GUP in the astrophysical regime, where it gives rise to the so-called "GUP stars" [48].

Equation (11) provides the starting point of our next analysis. By solving it with respect to the momentum p, we readily obtain

$$p \simeq \frac{\hbar \lambda}{2\beta \ell_p^2}\left[1 \pm \sqrt{1 - 4\beta\left(\frac{\ell_p}{\lambda}\right)^2}\right]. \qquad (12)$$

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[2] For instance, an electron can be in two possible states (spin up and spin down) and therefore its entropy is given by $S = k_B \log 2 \sim \mathcal{O}(k_B)$.

This reduces to the standard de Broglie relation (5) in the limit $\beta \ell_p/\lambda \to 0$ if the negative sign is chosen, whereas the positive sign has no evident physical meaning. Thus, in what follows we only work with the solution corresponding to the minus sign.

We now have all the necessary ingredients to derive a generalized Bekenstein bound. Hence, by following the same reasoning as done above, we assume that the energy of each quantum constituent is given by $\mu \simeq k_B T \gtrsim pc$ and that the system is well-localized inside a radius R, *i.e.*, $\lambda \lesssim 2R$. A comment is here in order. We are still assuming the validity of the equipartition theorem, and considering a regime where $k_B T > pc$. Since we are dealing with the GUP, we are surely closer to Planck energy than what we could reach by describing things just only with the simple HUP. However, we *should not* assume that $pc \sim E_{Planck}$, otherwise this would imply $T > T_{Planck}$, a nonsense. As specified before, the GUP formalism can be trusted for energies enough smaller than E_{Planck}, where therefore a regime with $k_B T > pc$ is still imaginable, without running into the oddities of $T \sim T_{Planck}$. Thus, the analogue of Eq. (7) is given by

$$\frac{\partial S}{\partial E} = \frac{1}{T} \lesssim \frac{k_B \beta \ell_p^2}{\hbar R c}\left[1 - \sqrt{1 - \beta\frac{\ell_p^2}{R^2}}\right]^{-1}, \qquad (13)$$

where we have exploited the fact that the r.h.s. of Eq. (12) is a monotonically decreasing function of λ. Note that Eq. (13) consistently reduces to Eq. (7) in the limit $\beta \ell_p^2/R^2 \to 0$.

In what follows, we discuss separately the two cases of $\beta > 0$ and $\beta < 0$.

3.1. Case $\beta > 0$

For positive values of the deformation parameter, the momentum p in Eq. (12) takes real values only when $\lambda \geq 2\ell_p \sqrt{\beta}$, the minimal size allowed by the GUP. We can now integrate Eq. (13) under the general assumption that R is independent of E, and the usual condition $S(E = 0) = 0$, thus we obtain

$$S \lesssim \frac{\alpha k_B \beta \ell_p^2 E}{\hbar R c}\left[1 - \sqrt{1 - \beta\frac{\ell_p^2}{R^2}}\right]^{-1}, \qquad (14)$$

that represents the generalized Bekenstein inequality in the case of $\beta > 0$. Once again, we see that the obtained bound is determined up to a factor α which can be set by requiring that Eq. (1) is recovered in the limit of vanishing β, and a direct comparison yields $\alpha = \pi$. Furthermore, as in HUP framework, Eq. (14) still holds true for any monotonic non-decreasing radius-energy relation $R = R(E)$.

If we now expand the square root to the next-to-leading order in $\beta \ell_p/R \ll 1$, Eq. (14) yields

$$S \leq \frac{2\pi k_B R E}{\hbar c}\left[1 - \frac{\beta}{4}\left(\frac{\ell_p}{R}\right)^2\right], \qquad (15)$$

where we have inserted the exact numerical factor $\alpha = \pi$. The above relation provides us with the effective expression of the generalized Bekenstein bound in the presently accessible regime, which is far above the Planck scale (we are assuming $\beta \sim \mathcal{O}(1)$).

The behavior of the generalized Bekenstein bound (14) as a function of the radius R is shown in Fig. 1 (orange dashed line). Note that the plot stops at $R \sim \ell_p$ (we choose $\beta = 1$ for simplicity), consistently with the emergence of a minimal length at this scale. We point out that the GUP correction for $\beta > 0$ lowers the standard Bekenstein limit, thus giving rise to a more stringent condition on the entropy which can be stored in a system
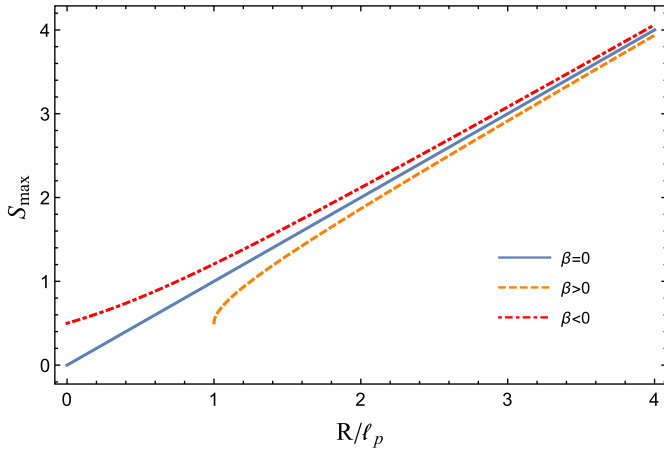
Fig. 1. Behavior of the upper bound $S_{\max}$ as a function of the scaled radius R/ℓ_p for $\beta = 0$ (blue solid line), $\beta = 1$ (orange dashed line) and $\beta = -1$ (red dot-dashed line). We have set $\hbar c/(2\pi E k_B \ell_p) = 1$. As expected, the discrepancy between the three plots narrows as R/ℓ_p increases.

of given size and total energy. Consequently, one may then suspect that Schwarzschild black holes would violate the generalized bound. However, this is not true, due to the fact that deformations of HUP (2) affect not only Bekenstein bound, but also the black hole entropy. Indeed, if one considers the GUP-modified expression of the black hole entropy [27], it is straightforward to check that it is still consistent with our bound.

Therefore, from Eqs. (14)-(15), it follows that if a system satisfies the generalized Bekenstein bound, it automatically complies with the standard Bekenstein bound too. In a broader sense, such a result is in line with physical intuition. Indeed, it is expected that the existence of a minimal length can reduce the number of microstates within a definite volume, thus decreasing the total amount of information associated with a system of given size. In other words, if there is no minimum length, then one can divide the volume more finely, thus allowing for higher entropy. Clearly, for radii R far above the Planck scale, GUP effects become increasingly negligible, and in fact the generalized and standard Bekenstein bounds tend to coincide.

3.2. Case $\beta < 0$

Let us now consider the case of negative deformation parameter, $\beta < 0$ (which means $\beta = -|\beta|$). In this framework there is no minimal size allowed by the GUP, as it can be seen from Eq. (2). Besides this caveat, whose implications are discussed below, calculations and general concepts are the same as in the previous analysis.

By integrating Eq. (13) with the generic assumption of R independent from E, we obtain the following upper bound on the entropy

$$ S \lesssim \frac{\alpha\, k_B\, |\beta|\, \ell_p^2\, E}{\hbar\, R\, c} \left[\sqrt{1 + |\beta| \frac{\ell_p^2}{R^2}} - 1 \right]^{-1}. \tag{16} $$

As before, we set $\alpha = \pi$ by requiring consistency with Eq. (1) for $\beta \to 0$. Again, inequality (16) is still true for any relation $R = R(E)$ obeying the very plausible property of being monotonic non-decreasing in E. The plot of the new GUP-corrected Bekenstein bound is shown in Fig. 1 (red dot-dashed line). For radii R such that $|\beta|\ell_p/R \ll 1$, the above expression can be expanded to the next-to-leading order in β, obtaining

$$ S \leq \frac{2\pi k_B R E}{\hbar c} \left[1 + \frac{|\beta|}{4} \left(\frac{\ell_p}{R} \right)^2 \right], \tag{17} $$

which is consistent with Eq. (15) with the sign of β reversed. On the other hand, the usual Bekenstein bound is recovered for $R \gg \ell_p$, as it should be.

Now, from a comparison with the $\beta > 0$ model, we can draw very interesting considerations. Indeed, by looking at Eq. (17), we immediately notice a striking physical implication: because of the positive sign in front of the GUP correction, the generalized Bekenstein bound with $\beta < 0$ allows the entropy S of a system to exceed the upper limit predicted by Bekenstein. Of course, this violation is suppressed as $(\ell_p/R)^2$, so that any experimental test appears to be problematic, at least at present. However, we emphasize that such an exotic behavior is not surprising, if we think that HUP itself can be violated for negative values of the deformation parameter. Indeed, from Eq. (2), it is clear that, for $\Delta p \sim m_p c$ and $\beta < 0$, we can have $\Delta x \Delta p \geq 0$, which is typical of a classical regime. As a matter of fact, the possibility of a quantum-to-classical throwback at Planck scale has been explored in literature, e.g., by considering $\hbar$ as a dynamical field that vanishes in the Planckian limit [49,50]. Moreover, the scenario in which the universe at Planck energies appears to be deterministic rather than being dominated by quantum fluctuations is the vision at the core of 't Hooft's "deterministic" quantum mechanics [51–55]. In terms of momentum and wavelength, this means that the quantum wave-packet of an object with momentum $p \simeq m_p c$ can be maximally localized, *i.e.*, $\lambda \simeq 0$, consistently with the fact that a GUP with $\beta < 0$ does not predict any minimal length [36].

Finally, in connection with the possibility of accessing arbitrarily short distances in the case of $\beta < 0$, let us observe that the upper bound in Eq. (16) converges to $\pi k_B \sqrt{|\beta|} E\ell_p/(\hbar c)$ for $R \to 0$. This would imply that a small - but finite - amount of entropy/information may be packed in a region of whatever small size, contrary to intuitive expectations. However, such a result is most likely just a signal that we are trying to extrapolate our considerations outside their domain of validity. It is actually evident that, for $R = 0$, the energy of the system cannot but be zero. This means that $S(R = 0) = S(E = 0) = 0$, according to the normalization we have adopted. Moreover, as shown in Ref. [36], the GUP with $\beta < 0$ seems to be implied by a reticular structure of the spacetime, which would make in any case the limit $R \to 0$ essentially meaningless. Surely the above aspects deserve deeper attention and will be better investigated elsewhere.

4. Concluding remarks

In this Letter we have presented arguments in favor of a full consistency between Heisenberg Uncertainty Principle and Bekenstein bound on the entropy of a localized system with a given size and total energy. Such a result has paved the way for the generalization of the Bekenstein inequality close to the Planck scale, where both quantum and gravitational effects are expected to come into play. In particular, we have argued that, if the underlying theory has a Generalized Uncertainty Principle built in, and in regimes where the equipartition theorem still holds, then Bekenstein bound turns out to be non-trivially modified; corrections have been computed in both cases of positive and negative values of the deformation parameter, see Eqs. (15) and (17), paying great attention to the issue of the minimal length emerging when $\beta > 0$. Apart from the well-known Holographic Bound (which is meant to apply to the most general spacetimes of any curvature), to the best of our knowledge this is a first concrete attempt towards a derivation of an upper bound on the entropy that takes into account both quantum and gravitational effects close to the Planck scale,

thus going beyond the flat-space proof based on standard quantum field theory with canonical commutation relations [12].

Apart from its intrinsic relevance, we point out that the obtained result finds applications in several other contexts. For instance, it may have significant implications on the holographic bound [7–10]. In fact, by assuming the absence of gravitational instability, or in other words that the size of the system R is larger than the corresponding gravitational radius, Eq. (15) leads to a generalization of the holographic bound, with potential connections to the world of quantum information theory (see Refs. [14,15]). Finally, we expect that the inequality (15), once properly extended to black hole physics, would allow us to establish a link with the theory of black hole remnants [56,57]. Remnants have been thought to be good candidates to model dark matter [58] and could also play an important rôle in the resolution of the information loss paradox (see, for instance, Ref. [57] and therein). This and further aspects are presently under active investigation and will be discussed elsewhere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Appendix A

In this Appendix we show that our derivation of the inequalities (8), (14), (16) holds for any monotonically non-decreasing function $R = R(E)$. Let $R(\varepsilon)$ and $g(R)$ be two positive, monotonically non-decreasing functions of ε (with $0 \le \varepsilon \le E$) and R, respectively. By introducing the partial derivative $S'(\varepsilon) := \partial S / \partial \varepsilon$, the inequalities (7) and (13) can be written in the following compact form

$$S'(\varepsilon) \lesssim g(R(\varepsilon)) \,. \tag{18}$$

We can now integrate the above inequality with the usual condition $S(\varepsilon = 0) = 0$ and obtain

$$S(E) = \int_0^E \mathrm{d}\varepsilon \, S'(\varepsilon) \lesssim \int_0^E \mathrm{d}\varepsilon \, g(R(\varepsilon)) \le E \, g(R(E)) \,, \tag{19}$$

where we used the fact that also $g(R(\varepsilon))$ is a monotonically non-decreasing function of ε as it is a composition of two monotonically non-decreasing functions. Therefore, we proved that $S(E) \lesssim E \, g(R(E))$, which resumes the inequalities (8), (14), (16).

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