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## Branched $SL(2, \mathbb{Z})$ duality

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**ABSTRACT:** We investigate how  $SL(2, \mathbb{Z})$  duality is realized in nonrelativistic type IIB superstring theory, which is a self-contained corner of relativistic string theory. Within this corner, we realize manifestly  $SL(2, \mathbb{Z})$ -invariant  $(p, q)$ -string actions. The construction of these actions imposes a branching between strings of opposite charges associated with the two-form fields. The branch point is determined by these charges and the axion background field. Both branches must be incorporated in order to realize the full  $SL(2, \mathbb{Z})$  group. Besides these string actions, we also construct D-instanton and D3-brane actions that manifestly realize the branched  $SL(2, \mathbb{Z})$  symmetry.

**KEYWORDS:** D-Branes, String Duality

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**1 Introduction**

Dualities have played a key role in probing the different non-perturbative corners of string/M-theory. In particular, type IIB superstring theory exhibits an  $SL(2, \mathbb{Z})$  symmetry connecting its perturbative and non-perturbative sectors. This  $SL(2, \mathbb{Z})$  symmetry survives in the low-energy supergravity limit, which greatly facilitates the study of the spectrum of supergravity solutions.<sup>1</sup> A realization of this  $SL(2, \mathbb{Z})$  symmetry was given in the context of the string sigma model by [8, 9] and in the context of the effective worldvolume actions of

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<sup>1</sup>Historically, the discovery went in the other direction:  $SU(1,1) \cong SL(2, \mathbb{R})$  was observed as a global symmetry of type IIB supergravity first in [1]. A decade later, this symmetry group was finally proposed as a symmetry in type IIB superstring theory in [2] (also see [3]). See e.g. page 695 of [4] for a review of the history. See also [5–7] for earlier references on the  $SL(2, \mathbb{Z})$  duality in lattice models. We thank Paul Townsend for pointing out this early occurrence of the  $SL(2, \mathbb{Z})$  duality.

D-branes by [10, 11]. D-branes are useful probes for understanding how the  $SL(2, \mathbb{Z})$  group acts on the background fields of type IIB superstring theory. In this paper, we will follow this approach as our guiding principle and use D-branes to study the  $SL(2, \mathbb{Z})$  duality in the less well-understood case of nonrelativistic string theory [12–14].

Recently, a systematic construction of the non-perturbative duality web in nonrelativistic string/M-theory has been initiated in [15]. Nonrelativistic string theory arises as a corner of relativistic string theory, under a zero Regge slope limit where the Kalb-Ramond field is fine-tuned to its critical value that matches the string tension [12–14]. This leads to a string theory with a spectrum satisfying a Galilean-invariant dispersion relation. This corner defines a self-consistent and ultra-violet complete string theory on its own [13]. Furthermore, it has been established that this theory provides a first-principles definition for string theory in the discrete light cone quantization (DLCQ) [16]. DLCQ plays an important role in nonperturbative approaches to string/M-theory such as Matrix theory [17–20].

Lately, effective field theories that arise from nonrelativistic string theory have been intensively studied, leading to novel nonrelativistic gravity and gauge theories [21–33] (for further references, see the review [34]). In [16, 21], it is shown that the spacetime geometry underlying nonrelativistic closed string theory is non-Riemannian and has a codimension-two foliation structure.<sup>2</sup> The spacetime equations of motion that govern the dynamics of the target-space geometry have been studied in [25, 35–38] by imposing quantum Weyl invariance. Extending the discussion to open strings, the effective worldvolume actions for D-branes coupled to the bosonic closed string background fields including the Ramond-Ramond fields have been studied in [15, 29]. This provides a framework that unifies nonrelativistic Yang-Mills theory [29], noncommutative open string theory and noncommutative Yang-Mills theory [39]. Analyses of various T- and S-duality transformations of nonrelativistic D-branes have appeared in [15, 27, 40]. In particular, a nonrelativistic  $(p, q)$ -string action has been derived in [15].<sup>3</sup>

So far, the full  $SL(2, \mathbb{Z})$  symmetry in nonrelativistic string theory has not yet been thoroughly studied.<sup>4</sup> It is the purpose of this paper to fill this gap and to investigate how this  $SL(2, \mathbb{Z})$  duality manifests itself in the context of nonrelativistic string theory. Surprisingly, the  $SL(2, \mathbb{Z})$  symmetry is realized in a nontrivial way where a branching arises, which splits the  $(p, q)$  parameter space into two halves. The  $SL(2, \mathbb{Z})$  transformations connect these two branches. Crucially, the branching is characterized not only by the values of  $p$  and  $q$ , but also by the position  $x$  in spacetime via the background axion Ramond-Ramond field  $C^{(0)}(x)$ , through the sign of the quantity

$$p - q C^{(0)}. \tag{1.1}$$

Note that this condition itself transforms under  $SL(2, \mathbb{Z})$ , which therefore splits this global symmetry group into two halves, in an  $x$ -dependent way, to those that preserve and those

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<sup>2</sup>Here, we have a slight abuse of the terminology “foliation,” which is technically only a “distribution” in mathematics. This distinction is not important in this paper.

<sup>3</sup>Here,  $p$  and  $q$  are two coprime integers that transform as a doublet under the  $SL(2, \mathbb{Z})$  duality symmetry. We use the convention where  $(p, q) = (1, 0)$  corresponds to the fundamental string.

<sup>4</sup>See e.g. [41–44] for studies of  $SL(2, \mathbb{Z})$  duality in the context of noncommutative open string theory.

that flip the sign of  $p - qC^{(0)}$ . The inter-branch case where  $p - qC^{(0)} = 0$  requires careful attention and will be discussed separately in this paper.

The paper is organized as follows. In section 2, we review some essential ingredients in nonrelativistic string theory. In section 3, we derive the nonrelativistic  $(p, q)$ -string action in eqs. (3.4) and (3.8) and show that its  $(p, q)$ -space splits into two branches. We then derive the associated  $SL(2, \mathbb{Z})$  transformations in eq. (3.30) (and eq. (4.25)), which are also branched. At the end of this section, we study a tensionless limit of nonrelativistic  $(p, q)$ -strings with the inter-branch condition  $p - qC^{(0)} = 0$ . In section 4, we extend our discussion to other nonrelativistic  $Dp$ -branes in type IIB superstring theory, including the D-instanton and D3-brane described by the actions (4.7) and (4.20), respectively. We also study the inter-branch case for the nonrelativistic D3-brane action with  $p - qC^{(0)} = 0$ . We conclude the paper in section 5. This paper has two appendices. In appendix A, we present a systematic derivation of the  $SL(2, \mathbb{Z})$ -invariants in nonrelativistic string theory. In appendix B, we give an alternative derivation of the main results of this paper via a nonrelativistic limit of relativistic string theory.

## 2 Nonrelativistic string theory and D-Branes

We start with a review of some basic ingredients of nonrelativistic string theory and the relevant  $Dp$ -brane actions, which will be essential for our later constructions of manifestly  $SL(2, \mathbb{Z})$ -invariant  $Dp$ -brane actions in nonrelativistic string theory.

### 2.1 Nonrelativistic closed string theory

Nonrelativistic closed string theory is defined by a two-dimensional (relativistic) sigma model that maps the worldsheet  $\Sigma$  to a  $d$ -dimensional spacetime manifold  $\mathcal{M}$ . Define the worldsheet coordinates to be  $\sigma^\alpha$ ,  $\alpha = 0, 1$  and the worldsheet metric  $h_{\alpha\beta}$ . In addition to the worldsheet fields  $X^\mu(\sigma)$ ,  $\mu = 0, \dots, d - 1$  representing the spacetime coordinates, the sigma model describing nonrelativistic strings also contains a pair of one-form fields  $\lambda$  and  $\bar{\lambda}$ . Moreover, it is useful to introduce the zweibein field  $e_\alpha^a$ ,  $a = 0, 1$  associated with  $h_{\alpha\beta}$ , such that  $h_{\alpha\beta} = e_\alpha^a e_\beta^b \eta_{ab}$ . In flat spacetime, the sigma model is given by [13, 16]

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^{A'} \partial_\beta X^{B'} \delta_{A'B'} + \epsilon^{\alpha\beta} (\lambda e_\alpha \partial_\beta X + \bar{\lambda} \bar{e}_\alpha \partial_\beta \bar{X}) \right], \quad (2.1)$$

where  $\alpha'$  is the Regge slope,  $e_\alpha = e_\alpha^0 + e_\alpha^1$  and  $\bar{e}_\alpha = e_\alpha^0 - e_\alpha^1$ . Moreover,  $X = X^0 + X^1$  and  $\bar{X} = X^0 - X^1$  are the lightlike coordinates in the longitudinal sector of the spacetime manifold  $\mathcal{M}$ . The index  $A' = 2, \dots, d - 1$  represents the transverse sector. Since the transverse sector is of Euclidean signature, we will not distinguish upper and lower transverse indices. Locally, in the operatorial formalism,  $\lambda$  and  $\bar{\lambda}$  are associated with derivatives of the T-dual coordinates of  $X$  and  $\bar{X}$ , respectively [38]. While  $X$  and  $\bar{X}$  are conjugate to the momenta, the T-dual coordinates are conjugate to the windings. The closed string sector has a nontrivial spectrum if the longitudinal spatial direction  $X^1$  is compactified over a circle of radius  $R$ . The dispersion relation and the level matching condition are [13]

$$E = \frac{\alpha'}{2wR} \left[ K^{A'} K^{A'} + \frac{2}{\alpha'} (N + \tilde{N} - 2) \right], \quad \tilde{N} - N = nw. \quad (2.2)$$

Here,  $E$  is the energy,  $K^{A'}$  is the transverse momentum,  $w$  is the winding number in  $X^1$ ,  $n$  is the Kaluza-Klein momentum number in the  $X^1$  circle, and  $N$  and  $\tilde{N}$  are the string excitation numbers. T-dualizing nonrelativistic string theory along  $X^1$  defines the discrete light cone quantization (DLCQ) of relativistic string theory, where the longitudinal spacelike circle is mapped to a lightlike circle [16].

In curved spacetime, the action (2.1) is generalized to be [16, 45]

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu E_{\mu\nu} + \epsilon^{\alpha\beta} \left( \lambda e_\alpha \partial_\beta X^\mu \tau_\mu + \bar{\lambda} \bar{e}_\alpha \partial_\beta X^\mu \bar{\tau}_\mu \right) \right] - \frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} R \Phi. \quad (2.3)$$

Here, we require that the symmetric two-tensor  $E_{\mu\nu}$  take a restricted form with

$$E_{\mu\nu} = E_\mu^{A'} E_\nu^{A'}, \quad (2.4)$$

where  $E_\mu^{A'}$  plays the role of the transverse vielbein in spacetime.<sup>5</sup> We also introduced the longitudinal vielbein  $\tau_\mu^A$ ,  $A = 0, 1$  and defined

$$\tau_\mu = \tau_\mu^0 + \tau_\mu^1, \quad \bar{\tau}_\mu = \tau_\mu^0 - \tau_\mu^1. \quad (2.5)$$

Finally, the theory is also coupled to the spacetime dilaton field  $\Phi$  via the worldsheet Einstein-Hilbert term. Such geometrical data encoded by  $\tau_\mu^A$  and  $E_\mu^{A'}$ , together with other background fields including the Kalb-Ramond field  $B_{\mu\nu}$  and dilaton field  $\Phi$ , defines the so-called string Newton-Cartan geometry [21], which is endowed with a codimension-two foliation structure. This geometry naturally generalizes Newton-Cartan geometry with a codimension-one foliation structure for particles. There is no graviton in the string spectrum, and the only gravitational force is the instantaneous Newton-like interaction between winding strings [13, 46].

The one-form fields  $\lambda$  and  $\bar{\lambda}$  in eq. (2.3) play the role of Lagrange multipliers that impose the constraint equations,

$$\epsilon^{\alpha\beta} e_\alpha \tau_\beta = \epsilon^{\alpha\beta} \bar{e}_\alpha \bar{\tau}_\beta = 0, \quad (2.6)$$

where we have introduced the pullback  $\tau_\alpha^A = \partial_\alpha X^\mu \tau_\mu^A$ .<sup>6</sup> Up to an undetermined scale factor, these constraints are solved by

$$e_\alpha \propto \tau_\alpha, \quad \bar{e}_\alpha \propto \bar{\tau}_\alpha. \quad (2.7)$$

In the absence of the Einstein-Hilbert term, we use these equations to eliminate the auxiliary worldsheet metric  $h_{\alpha\beta}$  in eq. (2.3) to find the following Nambu-Goto action [21]:

$$S_{\text{NG}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left( \sqrt{-\tau} \tau^{\alpha\beta} E_{\alpha\beta} + \epsilon^{\alpha\beta} B_{\alpha\beta} \right). \quad (2.8)$$

<sup>5</sup>An arbitrary symmetric two-tensor  $E_{\mu\nu}$  can always be brought into the form of  $E_{\mu\nu} = E_\mu^{A'} E_\nu^{A'}$  by redefining the one-form fields  $\lambda$  and  $\bar{\lambda}$  together with the Kalb-Ramond field  $B_{\mu\nu}$  [34].

<sup>6</sup>When we use form notation, we do not distinguish between the worldvolume and target-space. It should be clear from the context which one we mean.

Here, we have defined the longitudinal metric  $\tau_{\mu\nu} \equiv \tau_\mu^A \tau_\nu^B \eta_{AB}$  and its pullback to the worldsheet  $\tau_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \tau_{\mu\nu}$ . The metric determinant and inverse metric are denoted as  $\tau = \det \tau_{\alpha\beta}$  and  $\tau^{\alpha\beta}$ , respectively. We also introduced the pullbacks  $E_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu E_{\mu\nu}$  and  $B_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}$ .

## 2.2 Open strings and nonrelativistic D-Branes

In the case where the worldsheet  $\Sigma$  has a boundary  $\partial\Sigma$ , it is also possible to introduce open string vertex operators whose coherent states give rise to the boundary action. To be concrete, we map the worldsheet to the upper half plane, with the boundary  $\partial\Sigma$  identified with the real axis at  $\sigma = 0$ . Open strings have to end on D-branes. For simplicity, we consider the case of a single Dp-brane, which is a submanifold embedded in the spacetime manifold  $\mathcal{M}$ , parametrized by

$$X^\mu|_{\partial\Sigma} = f^\mu(Y^\alpha), \quad \alpha = 0, \dots, p, \quad (2.9)$$

where  $Y^\alpha$  are coordinates on the D-brane and  $f^\mu$  is the embedding function. It is illuminating to consider a D( $d-2$ )-brane, which is transverse to the longitudinal spatial direction and extends in the remaining directions. The covariant boundary action is [29]

$$S_b = \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau \left[ \frac{1}{2} (\lambda - \bar{\lambda}) \nu + \partial_\tau Y^\alpha A_\alpha^B \right], \quad \alpha = 0, 2, \dots, d-1, \quad (2.10)$$

where  $\nu$  corresponds to a scalar mode that perturbs the shape of the D-brane and  $A_\alpha^B$  is a U(1) gauge field on the D-brane. This theory describes nonrelativistic open strings [46]. String amplitudes in nonrelativistic open string theory have been recently studied in [47], where a new KLT relation between nonrelativistic closed and open string amplitudes is realized.

In flat spacetime, and with the longitudinal spatial direction being compactified over a circle of radius  $R$ , the open string states enjoy a Galilean-invariant dispersion relation [46],

$$E = \frac{\alpha'}{2wR} \left[ K^{A'} K_{A'} + \frac{1}{\alpha'} (N-1) \right]. \quad (2.11)$$

Requiring that the theory is consistent at the quantum level sets the beta-functionals of the open string background fields  $\nu$  and  $A_\alpha^B$  to zero, giving rise to the spacetime equations of motion that determine the low-energy dynamics of the D-brane [29]. It is shown in [29] that these equations of motion arise from a Dirac-Born-Infeld-like (DBI) action

$$S_{D(d-2)} = -T_{d-2} \int d^{d-1}Y e^{-\Phi} \sqrt{-\det \begin{pmatrix} 0 & \partial_\beta f^\nu \tau_\nu \\ \partial_\alpha f^\mu \bar{\tau}_\mu & \partial_\alpha f^\mu \partial_\beta f^\nu E_{\mu\nu} + \mathcal{F}_{\alpha\beta} \end{pmatrix}}, \quad (2.12)$$

where  $\mathcal{F}_{\alpha\beta} = \partial_\alpha f^\mu \partial_\beta f^\nu B_{\mu\nu} + F_{\alpha\beta}$ , with  $F_{\alpha\beta} = \partial_\alpha A_\beta^B - \partial_\beta A_\alpha^B$ . Formally, (2.12) looks rather similar to the relativistic case; however, a crucial difference makes this action fundamentally distinct from the standard DBI action: the matrix is  $d \times d$  instead of  $(d-1) \times (d-1)$  as

is the case for relativistic  $D(d-2)$ -branes! This D-brane action can be readily generalized to any  $Dp$ -branes, also in the presence of Ramond-Ramond potentials, with [15]

$$S_{Dp} = -T_p \int d^{p+1} \sigma e^{-\Phi} \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta \\ \bar{\tau}_\alpha & E_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \end{pmatrix}} - T_p \int \sum_q C^{(q)} \wedge e^{\mathcal{F}} \Big|_{p+1}, \quad (2.13)$$

where

$$T_p = \frac{1}{(2\pi)^p (\alpha')^{(p+1)/2}}. \quad (2.14)$$

It is understood that only  $(p+1)$ -forms are kept in the Wess-Zumino term. Here,  $p$  is even for type IIA and odd for type IIB superstring theory. We use the basis in which the action (2.13) is invariant under the Ramond-Ramond gauge transformations,

$$\delta_{RR} C^{(q)} = d\zeta^{(q-1)} + dB \wedge \zeta^{(q-3)}, \quad (2.15)$$

where  $\zeta^{(q)} = 0$  if  $q \leq 0$ . The action is also invariant under the Kalb-Ramond two-form transformation,

$$\delta_{KR} B = d\xi, \quad \delta_{KR} A^B = -\xi. \quad (2.16)$$

In flat spacetime,<sup>7</sup> the effective nonrelativistic  $Dp$ -brane tension is given by

$$T_{Dp} = \frac{T_p}{g_s}, \quad g_s = e^{\langle \Phi \rangle}. \quad (2.17)$$

Note that the action (2.13) also captures the D-brane configuration that extends in the longitudinal spatial direction, which leads to noncommutative open string theory with a relativistic string spectrum and noncommutativity between space and time [46].

### 3 Nonrelativistic $(p, q)$ -strings and $SL(2, \mathbb{Z})$ duality

In this section, we construct the  $SL(2, \mathbb{Z})$ -invariant  $(p, q)$ -string action in nonrelativistic string theory and show that it contains two different branches representing strings carrying opposite two-form charges. However, the strings in either branch do not form any closed group representation individually. Instead, the two branches transform into each other under the full  $SL(2, \mathbb{Z})$  group. We will show how this works at the level of the string action and the string states (see after eq. (3.25)). While the background fields transform under the  $SL(2, \mathbb{Z})$ , the boundary between the two branches also rotates accordingly. As we will see below, this implies that the  $SL(2, \mathbb{Z})$  transformation is realized in a highly non-conventional way in nonrelativistic string theory. In appendix B, we will show how this branched phenomenon in nonrelativistic string theory arises from a well-defined limit of relativistic string theory.

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<sup>7</sup>In the case where the longitudinal spatial direction is compactified over a circle, we fix its radius to be unity, such that  $\tau_\mu^A = \delta_\mu^A$ .

### 3.1 Nonrelativistic $(p, q)$ -strings

It is known that the relativistic  $(p, q)$ -string action can be derived from dualizing the  $U(1)$  gauge potential on the D1-brane action in relativistic string theory [8, 48, 49]. The same mechanism has also been realized in nonrelativistic string theory [15], which we review in the following. This analysis gives rise to the nonrelativistic  $(p, q)$ -string action, which enables our analysis of the  $SL(2, \mathbb{Z})$  duality in nonrelativistic string theory.

From the nonrelativistic  $Dp$ -brane action (2.13), we read off the action for  $q$  decoupled nonrelativistic D1-branes

$$S_{D1} = -|q| T_1 \int d^2\sigma e^{-\Phi} \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta \\ \bar{\tau}_\alpha & E_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \end{pmatrix}} - q T_1 \int (C^{(2)} + C^{(0)} \mathcal{F}), \quad (3.1)$$

where, for later use, we allow  $q$  to be negative. Here,  $T_1$  denotes the fundamental string tension. It then follows that the tension for a single D1-string is

$$T_{D1} = \frac{T_1}{g_s}. \quad (3.2)$$

Note that the  $q$  in front of the DBI-like term has to appear as an absolute value for the theory to be positive definite. Next, we perform a worldsheet duality transformation of the one-form gauge field  $A_\alpha$  by introducing a parent action that includes a generating functional as follows:

$$S_{\text{parent}} = S_{D1} + \frac{T_1}{2} \int d^2\sigma \tilde{H}^{\alpha\beta} (F_{\alpha\beta} - 2 \partial_{[\alpha} A_{\beta]}^B). \quad (3.3)$$

Here,  $F$  is treated as an independent field. Integrating out the auxiliary anti-symmetric two-tensor  $\tilde{H}^{\alpha\beta}$  imposes  $F = dA^B$  and leads back to the original action (3.1). Integrating out  $A_\alpha^B$  imposes the equation  $\partial_\alpha \tilde{H}^{\alpha\beta} = 0$ . In two dimensions, this equation can be solved locally by setting  $\tilde{H}^{\alpha\beta} = p \epsilon^{\alpha\beta}$ , where  $p$  will turn out to be an integer that counts the number of fundamental strings in the dual theory. Performing the electromagnetic duality transformation by further integrating out  $F$ , we find the dual  $(p, q)$ -string action [15],<sup>8</sup>

$$S_{\text{string}}^+ = -\frac{T_1}{2} \int d^2\sigma (p - q C^{(0)}) \sqrt{-\tau} \tau^{\alpha\beta} E_{\alpha\beta} - T_1 \int \left( p B + q C^{(2)} + \frac{1}{2} \frac{q^2 e^{-2\Phi} \ell}{p - q C^{(0)}} \right) \quad (3.4)$$

Here,  $\ell$  is a two-form that is defined in components to be

$$\ell_{\alpha\beta} = \tau_\alpha^A \tau_\beta^B \epsilon_{AB}. \quad (3.5)$$

Without loss of generality, we assumed that

$$\star \ell = \frac{1}{2} \epsilon^{\alpha\beta} \tau_\alpha^A \tau_\beta^B \epsilon_{AB} > 0. \quad (3.6)$$

This restriction is consistent with the connected component of the local  $SO(1, 1)$  in the two-dimensional longitudinal sector of the target-space geometry. The  $(p, q)$ -string action

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<sup>8</sup>The procedure of dualizing the worldsheet  $U(1)$  potential only corresponds to an  $SL(2, \mathbb{Z})$  transformation when  $q = \pm 1$ .

has to be supplemented with the condition  $p - q C^{(0)} > 0$ . This is because the equation of motion from varying  $F$  in eq. (3.3) gives

$$p - q C^{(0)} = |q| e^{-\Phi} \sqrt{-\tau} \left[ -\det \begin{pmatrix} 0 & \tau_\beta \\ \bar{\tau}_\alpha & E_{\alpha\beta} + \mathcal{F}_{\alpha\beta} \end{pmatrix} \right]^{-\frac{1}{2}} > 0. \quad (3.7)$$

Moreover, the condition  $p - q C^{(0)} > 0$  is necessary for the kinetic part of the action (3.4) to be positive definite. For this reason, we labeled the  $(p, q)$ -string action (3.4) with “+” as  $S_{\text{string}}^+$ . Quantum mechanically,  $p$  and  $q$  are required to be integers and label the number of fundamental strings and D1-strings in the composite  $(p, q)$ -string state. Note that, for the above procedure to make sense, we have assumed that  $q \neq 0$  and  $p - q C^{(0)} \neq 0$ . However, when  $p \neq 0$  and  $q = 0$ , the action (3.4) still holds and describes  $p$  fundamental nonrelativistic strings, in agreement with the Nambu-Goto action (2.8). This implies that the condition  $q \neq 0$  arises only from a shortcoming of the D1-string action (3.1) instead of any physical limitation, which can be avoided by using the Hamiltonian formalism (see page 47 in [48] for an extensive argument).

Performing the same duality transformation for the action (3.1) but with  $q$  replaced by  $-q$  and substituting the solution  $\tilde{H}^{\alpha\beta} = -p \epsilon^{\alpha\beta}$ , we are led to the following action:

$$S_{\text{string}}^- = \frac{T_1}{2} \int d^2\sigma (p - q C^{(0)}) \sqrt{-\tau} \tau^{\alpha\beta} E_{\alpha\beta} + T_1 \int \left( p B + q C^{(2)} + \frac{1}{2} \frac{q^2 e^{-2\Phi} \ell}{p - q C^{(0)}} \right) \quad (3.8)$$

This action is supplemented with the condition

$$p - q C^{(0)} < 0, \quad (3.9)$$

opposite to the condition (3.7). The string states captured by  $S_{\text{string}}^+$  and  $S_{\text{string}}^-$  carry the opposite  $(p, q)$  charges, and are related to each other via the transformation  $(p, q) \rightarrow (-p, -q)$ , leaving the background fields unchanged. We will see that both  $S_{\text{string}}^+$  and  $S_{\text{string}}^-$  are required to realize the  $\text{SL}(2, \mathbb{Z})$  transformation in nonrelativistic string theory. We will comment on the boundary case with  $p - q C^{(0)} = 0$  in section 3.4 for nonrelativistic  $(p, q)$ -strings and section 4.3 for nonrelativistic D3-branes.

Generically, it can be a nontrivial matter to extract the effective tension of the  $(p, q)$ -string from the action (3.4) (or eq. (3.8)). The tension is the coefficient of the kinetic term of the Nambu-Goldstone bosons describing the fluctuations of the string. In the following, we compute the effective tension in the special case when the  $(p, q)$ -string extends in the longitudinal spatial direction. In flat spacetime with  $\tau_\mu^A = \delta_\mu^A$  and  $E_\mu^{A'} = \delta_\mu^{A'}$ , the string configuration is described by

$$X^0 = \sigma^0, \quad X^1 = \sigma^1, \quad X^{A'} = X_0^{A'} + \pi^{A'}(\sigma^\alpha), \quad (3.10)$$

where the  $(p, q)$ -string is localized at the transverse directions  $X_0^{A'}$ . The fields  $\pi^{A'}$  denote the associated Nambu-Goldstone bosons that arise from spontaneously breaking the translational symmetries, and they describe the fluctuations of the shape of the  $(p, q)$ -string. Expanding the action (3.4) (or eq. (3.8)) to quadratic order in  $\pi^{A'}$ , we find,

$$S_{\text{string}}^\pm \sim \mp \frac{T_1}{2} \int d^2\sigma (p - q C^{(0)}) \partial^\alpha \pi^{A'} \partial_\alpha \pi^{A'} + \dots \quad (3.11)$$

In this special case, the quantity

$$T_{p,q} = T_1 |p - q \langle C^{(0)} \rangle| \quad (3.12)$$

plays the role of the effective tension of the  $(p, q)$ -string. Such  $(p, q)$ -string tensions satisfy the following triangle inequality:

$$T_{p_1+p_2, q_1+q_2} \leq T_{p_1, q_1} + T_{p_2, q_2}. \quad (3.13)$$

This inequality is saturated if all the  $(p, q)$ -strings involved in eq. (3.13) fall in the same branch of  $(p, q)$ -space. The  $(p, q)$ -strings saturating the inequality are marginally stable bound states [50].

### 3.2 A branched $SL(2, \mathbb{Z})$ duality

Now, we are ready to study the  $SL(2, \mathbb{Z})$  transformations of the background fields and analyze how these transformations act on the nonrelativistic  $(p, q)$ -string actions (3.4) and (3.8) in different branches. This analysis will uncover a branched structure of the  $SL(2, \mathbb{Z})$  duality in nonrelativistic string theory.

**The positive branch.** To observe how the nonrelativistic  $(p, q)$ -string transforms under the  $SL(2, \mathbb{Z})$  group, it is useful to introduce variables in the Einstein frame, with

$$(\tau_{\alpha\beta}^E, E_{\alpha\beta}^E, \ell^E) = e^{-\Phi/2} (\tau_{\alpha\beta}, E_{\alpha\beta}, \ell), \quad (3.14)$$

We then rewrite the action (3.4) as

$$S_{\text{string}}^+ = -\frac{T_1}{2} \int d^2\sigma (\Theta^\top \mathcal{C}) \sqrt{-\tau^E} \tau_E^{\alpha\beta} E_{\alpha\beta}^E - T_1 \int \Theta^\top \Sigma, \quad (3.15)$$

where

$$\Theta = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \mathcal{C} = e^{\Phi/2} \begin{pmatrix} 1 \\ -C^{(0)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} + \frac{q}{p - q C^{(0)}} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix}. \quad (3.16)$$

We require that the vielbeine be  $SL(2, \mathbb{Z})$  invariant in the Einstein frame. Here, we assume that  $(p, q)$  forms a doublet under the  $SL(2, \mathbb{Z})$  group and transforms as

$$\Theta' = \Lambda \Theta, \quad (3.17)$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (3.18)$$

Here,  $a, b, c, d \in \mathbb{Z}$  are  $SL(2, \mathbb{Z})$  group parameters satisfying  $ad - bc = 1$ . For the first term in (3.15) to be  $SL(2, \mathbb{Z})$  invariant,  $\mathcal{C}$  has to be a doublet as well and thus transforms as

$$\mathcal{C}' = (\Lambda^{-1})^\top \mathcal{C}, \quad (3.19)$$

which further implies that

$$C'^{(0)} = \frac{a C^{(0)} + b}{c C^{(0)} + d}, \quad e^{\Phi'/2} = (c C^{(0)} + d) e^{\Phi/2}. \quad (3.20)$$

This transformation only makes sense when  $cC^{(0)} + d > 0$ . This is also the requirement that preserves the condition  $p - qC^{(0)} > 0$ : under  $\text{SL}(2, \mathbb{Z})$ , we have

$$p' - q' C'^{(0)} = \frac{p - q C^{(0)}}{c C^{(0)} + d} > 0. \quad (3.21)$$

It looks like only the  $\text{SL}(2, \mathbb{Z})$  transformations that preserve the condition  $p - qC^{(0)} > 0$  survive. Note that  $C^{(0)}$  also transforms nontrivially as in eq. (3.20).

An obvious question is: where does the other half of the  $\text{SL}(2, \mathbb{Z})$  transformations satisfying the condition  $cC^{(0)} + d < 0$  go?<sup>9</sup> Before answering this question, let us first stick to the branch satisfying  $cC^{(0)} + d > 0$  and understand how the two-form fields  $B$  and  $C^{(2)}$  transform under the action of  $\text{SL}(2, \mathbb{Z})$ , such that the second term in eq. (3.15) is invariant. Naïvely, one would expect that the quantity  $\Sigma$  in eq. (3.16) has to transform as a doublet, with  $\Sigma' = (\Lambda^{-1})^\top \Sigma$ , for the term  $\Theta^\top \Sigma$  to be  $\text{SL}(2, \mathbb{Z})$  invariant. However, the definition of  $\Sigma$  in eq. (3.16) contains a  $(p, q)$ -dependent term, and forcing  $\Sigma$  to be a doublet would require that  $\text{SL}(2, \mathbb{Z})$  act on both  $B$  and  $C^{(2)}$  in a  $(p, q)$ -dependent way. Such a  $(p, q)$ -dependent transformation of the supergravity fields  $B$  and  $C^{(2)}$  would not form a symmetry of type IIB nonrelativistic supergravity: the background fields would have to transform differently depending on how many fundamental and D1-strings are present, even in the case where no back reaction to the background fields is included!

Therefore, since the quantity  $\Sigma$  is not a supergravity field, it makes perfect sense that we find that  $\Sigma$  is a “quasi”-doublet, meaning that it transforms as a doublet up to a  $(p, q)$ -dependent term, i.e.,

$$\Sigma' = (\Lambda^{-1})^\top (\Sigma + \alpha \Theta^\perp), \quad \Theta^\perp = \begin{pmatrix} q \\ -p \end{pmatrix}. \quad (3.22)$$

This already guarantees that the combination  $\Theta^\top \Sigma$  is invariant. Requiring that both the transformed background fields  $B'$  and  $C'^{(2)}$  be independent of  $p$  and  $q$ , we find a unique solution to  $\alpha$ , with

$$\alpha = \frac{c\ell}{2e^{2\Phi}(cC^{(0)} + d)(p - qC^{(0)})}. \quad (3.23)$$

Together with eq. (3.20), we find a well-defined set of  $\text{SL}(2, \mathbb{Z})$  transformations of the background fields  $B$ ,  $C^{(0)}$ ,  $C^{(2)}$  and  $\Phi$ , given by

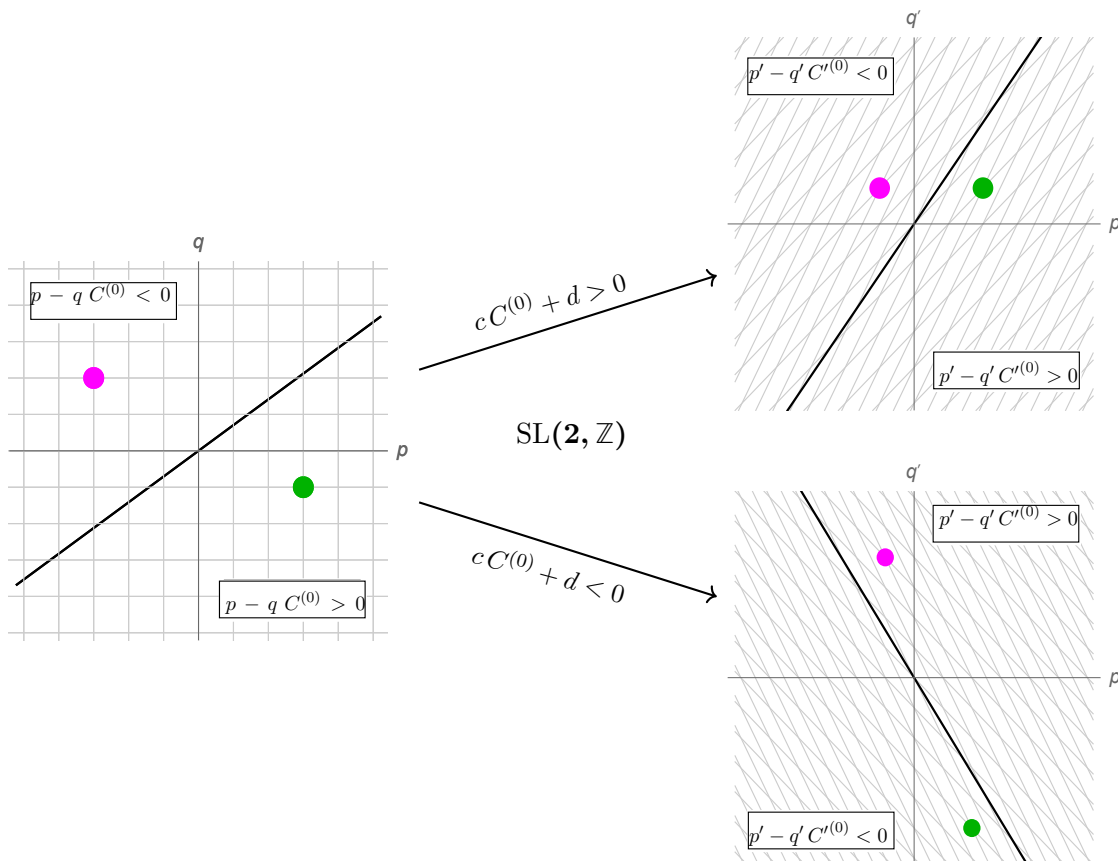
$$C'^{(0)} = \frac{aC^{(0)} + b}{cC^{(0)} + d}, \quad \Phi' = \Phi + 2 \ln(cC^{(0)} + d), \quad (3.24a)$$

$$\begin{pmatrix} B' \\ C'^{(2)} \end{pmatrix} = (\Lambda^{-1})^\top \left[ \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} - \frac{c}{cC^{(0)} + d} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix} \right] - \frac{c}{(cC^{(0)} + d)^2} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix}, \quad (3.24b)$$

which are independent of  $p$  and  $q$ . These transformations have to satisfy  $cC^{(0)} + d > 0$  for the  $(p, q)$ -string action (3.4) to be invariant. In appendix B, we show that the same transformations in eq. (3.24) can be obtained equivalently by taking a well-defined limit of the  $\text{SL}(2, \mathbb{Z})$  transformations in relativistic string theory.

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<sup>9</sup>When  $C^{(0)}$  is rational, it is also possible that  $cC^{(0)} + d = 0$ . We exclude this singular case.



**Figure 1.** The  $SL(2, \mathbb{Z})$  action in  $(p, q)$ -space. We marked one point in each branch and plotted the location to which they are mapped under an  $SL(2, \mathbb{Z})$  transformation satisfying  $cC^{(0)} + d > 0$  and another one satisfying  $cC^{(0)} + d < 0$ . In the first case, the colored points remain in the same branch, whereas in the second, they switch branches. The grids are aids to visualize how the branched  $SL(2, \mathbb{Z})$  transformation acts on the plane.

**The negative branch.** As we have discussed earlier, the transformations (3.24) with the condition  $cC^{(0)} + d > 0$  does not yet constitute the full group of  $SL(2, \mathbb{Z})$ . What if we apply the same transformations (3.24) to the  $(p, q)$ -string action (3.4), but now with the condition  $cC^{(0)} + d < 0$ ? This procedure maps the action  $S_{\text{string}}^+$  in eq. (3.15) to  $S_{\text{string}}^-$  in eq. (3.8), i.e.,

$$S_{\text{string}}^- = \frac{T_1}{2} \int d^2\sigma (\Theta^\top \mathcal{C}) \sqrt{-\tau^E} \tau_E^{\alpha\beta} E_{\alpha\beta}^E + T_1 \int \Theta^\top \Sigma, \quad (3.25)$$

where we performed a discrete transformation  $B' \rightarrow -B'$  and  $C'^{(2)} \rightarrow -C'^{(2)}$  and dropped the primes that label the  $SL(2, \mathbb{Z})$ -dual fields in the result. In this dual frame, according to the transformation in eq. (3.21), we are now in the branch of  $p - qC^{(0)} < 0$ , which is precisely the condition for the kinetic part of  $S_{\text{string}}^-$  to be positive definite. The  $SL(2, \mathbb{Z})$  transformations that keep  $S_{\text{string}}^-$  invariant are the same as the ones given in eq. (3.24). We therefore conclude the following:

- the  $\text{SL}(2, \mathbb{Z})$  transformations satisfying  $cC^{(0)} + d > 0$  preserve the string action  $S_{\text{string}}^+$  ( $S_{\text{string}}^-$ ) in the positive (negative)  $p - qC^{(0)}$  branch in the space of  $(p, q)$ -string states;
- the  $\text{SL}(2, \mathbb{Z})$  transformations satisfying  $cC^{(0)} + d < 0$  map between  $S_{\text{string}}^+$  and  $S_{\text{string}}^-$ .

Therefore, the  $\text{SL}(2, \mathbb{Z})$  duality in type IIB nonrelativistic superstring theory relates two different branches representing nonrelativistic  $(p, q)$ -string states satisfying  $p - qC^{(0)} > 0$  and the states satisfying  $p - qC^{(0)} < 0$ , respectively. These two branches are separated by the line  $p - qC^{(0)} = 0$  in the  $(p, q)$ -plane and they are described by the  $(p, q)$ -string actions  $S_{\text{string}}^+$  and  $S_{\text{string}}^-$ , respectively. The boundary  $p - qC^{(0)} = 0$  itself changes with respect to the  $\text{SL}(2, \mathbb{Z})$  transformations as  $C^{(0)}$  transforms nontrivially. The  $\text{SL}(2, \mathbb{Z})$  transformations are also branched: the transformations satisfying  $cC^{(0)} + d > 0$  map between string states in the same branch, while the transformations satisfying  $cC^{(0)} + d < 0$  map between different branches. See figure 1 for a summary of the  $\text{SL}(2, \mathbb{Z})$  action in  $(p, q)$ -space.

### 3.3 Manifestly $\text{SL}(2, \mathbb{Z})$ invariant string action

Collectively, the above branched realization of  $\text{SL}(2, \mathbb{Z})$  invariance can be summarized as follows: the invariant action that unifies  $S_{\text{string}}^\pm$  is

$$S_{\text{string}} = -\frac{T_1}{2} \int d^2\sigma |\Theta^\top \mathcal{C}| \sqrt{-\tau^{\text{E}}} \tau_{\text{E}}^{\alpha\beta} E_{\alpha\beta}^{\text{E}} - T_1 \int (\Theta^\top \Sigma) \text{sgn}(p - qC^{(0)}). \quad (3.26)$$

The  $\text{SL}(2, \mathbb{Z})$  transformations are

$$\Theta' = \Lambda \Theta, \quad \mathcal{C}' = \text{sgn}(cC^{(0)} + d) \Lambda \mathcal{C}, \quad (3.27)$$

and

$$\Sigma' = \text{sgn}(cC^{(0)} + d) (\Lambda^{-1})^\top \left[ \Sigma + \frac{c\ell}{2e^{2\Phi}(cC^{(0)} + d)(p - qC^{(0)})} \Theta^\perp \right]. \quad (3.28)$$

The variables  $(\tau_{\alpha\beta}^{\text{E}}, E_{\alpha\beta}^{\text{E}}, \ell^{\text{E}})$  in Einstein's frame are invariant. Recall that the quantities  $\Theta$ ,  $\mathcal{C}$  and  $\Sigma$  are defined in eq. (3.16). Moreover, as defined in eq. (3.22),  $\Theta^\perp = (q, -p)^\top$ . The  $\text{SL}(2, \mathbb{Z})$  invariance of eq. (3.26) is made manifest by noticing that

$$\text{sgn}(p' - q' C'^{(0)}) = \text{sgn}\left(\frac{p - qC^{(0)}}{cC^{(0)} + d}\right). \quad (3.29)$$

Note that the quasi-doublet  $\Sigma$  does not only transform as a doublet up to a term orthogonal to  $\Theta$ , but also up to an overall sign, depending on the value of the quantity  $cC^{(0)} + d$ . In components, we find the complete  $\text{SL}(2, \mathbb{Z})$  transformations that generalize eq. (3.24),

$$C'^{(0)} = \frac{aC^{(0)} + b}{cC^{(0)} + d}, \quad \Phi' = \Phi + 2 \ln |cC^{(0)} + d|, \quad (3.30a)$$

$$\begin{pmatrix} B' \\ C'^{(2)} \end{pmatrix} = \text{sgn}(cC^{(0)} + d) \left\{ (\Lambda^{-1})^\top \left[ \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} - \frac{c}{cC^{(0)} + d} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix} \right] - \frac{c}{(cC^{(0)} + d)^2} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix} \right\}. \quad (3.30b)$$

These transformations will later be extended to include the one acting on the four-form Ramond-Ramond potential  $C^{(4)}$  in eq. (4.25).

### 3.4 Winding strings in a tensionless limit

So far, we only considered nonrelativistic  $(p, q)$ -strings satisfying the condition  $p - q C^{(0)} \neq 0$ . In this last subsection, we consider a class of inter-branch strings at  $p - q C^{(0)} = 0$ . One such  $p - q C^{(0)} \rightarrow 0$  limit arises if the tension  $T_1$  is simultaneously scaled to zero, while keeping finite the  $\text{SL}(2, \mathbb{Z})$ -invariant quantity

$$T_{\text{eff}} \equiv \lim_{\substack{T_1 \rightarrow 0 \\ C^{(0)} \rightarrow \frac{p}{q}}} T_1 e^{-\Phi} \left| \frac{q}{p - q C^{(0)}} \right|. \quad (3.31)$$

The  $\text{SL}(2, \mathbb{Z})$ -invariance of such an effective tension  $T_{\text{eff}}$  at  $p - q C^{(0)} = 0$  follows directly from the  $\text{SL}(2, \mathbb{Z})$  transformations (3.17) and (3.24) in section 3.3. Since  $T_1 = (2\pi\alpha')^{-1}$  (see eq. (2.14)), we are essentially taking an infinite Regge slope limit. Under this double scaling limit, the resulting action is

$$S_{\text{tensionless}} = \frac{T_{\text{eff}}}{2} \int |q| e^{-\Phi} \ell. \quad (3.32)$$

This theory is  $\text{SL}(2, \mathbb{Z})$ -invariant and purely topological. Just as in the previous analysis, this action splits into two branches depending on the sign of  $q$ . As before, each branch is mapped into itself under an  $\text{SL}(2, \mathbb{Z})$  transformation satisfying the condition  $c C^{(0)} + d > 0$ , and one branch is mapped into the other branch and vice versa if  $c C^{(0)} + d < 0$ .

In order to decode the physics described by the new action (3.32), we consider a simple string configuration in a constant dilaton background with  $g_s = e^\Phi$  and in the flat geometrical background fields with  $\tau_\mu^A = \delta_\mu^A$  and  $E_\mu^{A'} = \delta_\mu^{A'}$ . We focus on the winding  $(p, q)$ -string configuration defined by

$$X^0 = \sigma^0, \quad X^1 = w R \sigma^1, \quad X^{A'} = X_0^{A'} + \pi^{A'}(\sigma^\alpha), \quad (3.33)$$

where  $X^1$  is compactified over a circle of radius  $R$  and the  $(p, q)$ -string winds  $w$  times around  $X^1$ , with  $w \in \mathbb{Z}$ . In this case, the Lagrangian density of the action (3.32) evaluates to

$$\mathcal{L}_{\text{tensionless}} = w R \frac{|q| T_{\text{eff}}}{g_s}, \quad (3.34)$$

which is proportional to how many times the  $(p, q)$ -string winds around the  $X^1$  direction.

## 4 $\text{SL}(2, \mathbb{Z})$ symmetric actions for D-instanton and D3-brane

We have discussed the  $\text{SL}(2, \mathbb{Z})$  transformations of various background fields in type IIB nonrelativistic supergravity, using the  $(p, q)$ -string as a probe. However, the consideration of the  $(p, q)$ -strings does not suffice for establishing the complete  $\text{SL}(2, \mathbb{Z})$  duality in type IIB nonrelativistic supergravity, which also contains a Ramond-Ramond potential  $C^{(4)}$ . This four-form gauge field is not visible to the  $(p, q)$ -string, but it is coupled to D3-branes. In this section, we construct a manifestly  $\text{SL}(2, \mathbb{Z})$  invariant D3-brane action in nonrelativistic string theory. In appendix B, we will show how the action arises from its relativistic counterpart. This formalism will fortify the branched  $\text{SL}(2, \mathbb{Z})$  realization

in nonrelativistic string theory, which we have already learned about from nonrelativistic  $(p, q)$ -strings. Moreover, it will grant us access to studying the inter-branch states satisfying  $p - q C^{(0)} = 0$ .

Just as in the relativistic case, in order to formulate a manifestly  $SL(2, \mathbb{Z})$ -symmetric D3-brane action, it is convenient to introduce another doublet [11]

$$\tilde{\Theta} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} \tag{4.1}$$

that is conjugate to  $\Theta = (p \ q)^\top$ , satisfying the condition

$$p \tilde{q} - q \tilde{p} = 1. \tag{4.2}$$

Note that  $\tilde{\Theta}$  is defined up to an arbitrary shift in  $\Theta$ , forming the equivalence class

$$[\tilde{\Theta}] : \quad \tilde{\Theta} \sim \tilde{\Theta} + \beta \Theta, \quad \beta \in \mathbb{R}. \tag{4.3}$$

We have chosen the normalization of  $\tilde{\Theta}$  such that the expression in eq. (4.2) is unity.<sup>10</sup> Just as  $\Theta$ , the new quantity  $\tilde{\Theta}$  transforms under the  $SL(2, \mathbb{Z})$  symmetry as

$$\tilde{\Theta}' = \Lambda \tilde{\Theta}. \tag{4.4}$$

The condition (4.2) is manifestly  $SL(2, \mathbb{Z})$  invariant.

In the special case where  $(p, q) = (1, 0)$ , i.e., the  $(p, q)$ -string reduces to a single fundamental string, a conjugate state is the single D-string state with  $(\tilde{p}, \tilde{q}) = (0, 1)$ . In this case,  $\Theta$  and  $\tilde{\Theta}$  are projections to the fundamental and D-string state, respectively.

The new ingredient  $\tilde{\Theta}$  is already required for the construction of a  $(p, q)$ -symmetric action for the nonrelativistic D(-1)-brane, which represents an instanton state. We will study this D-instanton first, from which useful ingredients for building a  $(p, q)$ -symmetric nonrelativistic D3-brane action will be obtained.

#### 4.1 Nonrelativistic D-instanton

The nonrelativistic D-instanton action can be read off from the general  $Dp$ -brane action (2.13) by setting  $p = -1$ , which gives

$$S_{D(-1)} = -T_{-1} C^{(0)}. \tag{4.5}$$

This action is not manifestly invariant under the  $SL(2, \mathbb{Z})$  transformation (3.24), which acts nontrivially on  $C^{(0)}$ . In order to achieve the  $SL(2, \mathbb{Z})$  symmetry, eq. (4.5) has to be covariantized as follows:

$$S_{D(-1)} = T_{-1} \frac{\tilde{\Theta}^\top \mathcal{C}}{\Theta^\top \mathcal{C}}. \tag{4.6}$$

Since  $\mathcal{C}$  defined in Eq. (3.16) is an  $SL(2, \mathbb{Z})$  doublet up to  $\text{sgn}(c C^{(0)} + d)$ , the above action is manifestly  $SL(2, \mathbb{Z})$  invariant. More explicitly, the instanton action (4.6) gives<sup>11</sup>

$$S_{D(-1)} = T_{-1} \frac{\tilde{p} - \tilde{q} C^{(0)}}{p - q C^{(0)}}. \tag{4.7}$$

<sup>10</sup>When  $p \tilde{q} - q \tilde{p} = 0$ , we have  $\tilde{\Theta} \propto \Theta$  and such a  $\tilde{\Theta}$  does not constitute a linearly independent doublet.

<sup>11</sup>Note that  $T_{-1} = 2\pi$  (see eq. (2.14)) and thus there is *no* analog of the tensionless  $T_1$  limit as in section 3.4.

In the case where  $\Theta = (1 \ 0)^\top$  and  $\tilde{\Theta} = (0 \ 1)^\top$ , the D-instanton action (4.5) is recovered. Note that the action becomes independent of  $\tilde{\Theta}$  after the condition (4.2) is imposed, except for an overall constant shift of the action.

## 4.2 Nonrelativistic D3-brane

We are finally ready to construct a manifestly  $SL(2, \mathbb{Z})$ -invariant nonrelativistic D3-brane action. Setting  $p = 3$  in the general nonrelativistic  $Dp$ -brane action (2.13), and rewriting the geometric data in terms of the Einstein-frame variables, we find

$$S_{D3} = -T_3 \int d^4\sigma \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta^E \\ \bar{\tau}_\alpha^E & E_{\alpha\beta}^E + e^{-\Phi/2} \mathcal{F}_{\alpha\beta} \end{pmatrix}} - T_3 \int \left( C^{(4)} + C^{(2)} \wedge \mathcal{F} + \frac{1}{2} C^{(0)} \mathcal{F} \wedge \mathcal{F} \right), \quad (4.8)$$

where  $\mathcal{F} = B + dA^B$ . This action is invariant under the Ramond-Ramond and Kalb-Ramond gauge symmetries given in eqs. (2.15) and (2.16).

We would like to write down a manifestly  $SL(2, \mathbb{Z})$ -invariant D3-brane action by replacing the quantities in the action (4.8) with their  $SL(2, \mathbb{Z})$ -invariant counterparts.<sup>12</sup> The invariant form of  $C^{(0)}$  has already been given in section 4.1, where we essentially performed the following replacement:

$$C^{(0)} \rightarrow -\frac{\tilde{\Theta}^\top \mathcal{C}}{\Theta^\top \mathcal{C}}, \quad (4.9)$$

which reduces to  $C^{(0)}$  when  $\Theta = (1 \ 0)^\top$  and  $\tilde{\Theta} = (0 \ 1)^\top$ . Similarly, the  $SL(2, \mathbb{Z})$ -invariant counterpart of the dilaton term  $e^\Phi$  is given by

$$e^\Phi \rightarrow (\Theta^\top \mathcal{C})^2. \quad (4.10)$$

We also learned in section 3 that the  $SL(2, \mathbb{Z})$ -invariant version of the  $B$ -field is given by eq. (3.26), with

$$B \rightarrow \text{sgn}(p - q C^{(0)}) (\Theta^\top \Sigma), \quad (4.11)$$

where  $\Sigma$  is defined in eq. (3.16). This implies that the quantity  $\mathcal{F}$  should be replaced as

$$\mathcal{F} \rightarrow \text{sgn}(p - q C^{(0)}) (\Theta^\top \mathcal{F}), \quad \mathcal{F} = \Sigma + d\mathcal{A}, \quad (4.12)$$

where  $\mathcal{A} = (A^B \ A^C)^\top$ , with  $A^B$  and  $A^C$  the Born-Infeld potentials associated with  $B$  and  $C^{(2)}$ , respectively. The Born-Infeld vector  $\mathcal{A}$  transforms under the combined Ramond-Ramond and Kalb-Ramond gauge symmetries as

$$\delta\mathcal{A} = -\Xi, \quad \Xi \equiv \begin{pmatrix} \xi \\ \zeta^{(1)} \end{pmatrix}, \quad (4.13)$$

such that  $\mathcal{F}$  is gauge invariant. We require that  $\mathcal{A}$  be an  $SL(2, \mathbb{Z})$  doublet up to a sign, satisfying the following transformation:

$$\mathcal{A}' = \text{sgn}(c C^{(0)} + d) (\Lambda^{-1})^\top \mathcal{A}. \quad (4.14)$$

<sup>12</sup>See appendix A for further justification of the following replacements.

The introduction of the extra field  $A^C$  does not change the degrees of freedom as long as  $\mathcal{A}$  is only contracted with  $\Theta$  (but not  $\tilde{\Theta}$ ) [11].

We still need to construct an  $\text{SL}(2, \mathbb{Z})$ -invariant counterpart of  $C^{(2)}$ , which has to involve the doublet  $\tilde{\Theta}$ . This requires building a new quasi-doublet  $\tilde{\Sigma}$  that transforms as a doublet up to a  $(\tilde{p}, \tilde{q})$ -dependent term. In analogy with the quantity  $\Sigma$  in eq. (3.28), we have

$$\tilde{\Sigma}' = \text{sgn}(cC^{(0)} + d) (\Lambda^{-1})^\top (\tilde{\Sigma} + \tilde{\alpha} \tilde{\Theta}^\perp), \quad \tilde{\Theta}^\perp = \begin{pmatrix} \tilde{q} \\ -\tilde{p} \end{pmatrix}. \quad (4.15)$$

We have introduced  $\tilde{\Theta}^\perp$  that is orthogonal to  $\tilde{\Theta}$ , satisfying  $\tilde{\Theta}^\top \tilde{\Theta}^\perp = 0$ . Such a quasi-doublet  $\tilde{\Sigma}$  defined with respect to  $\tilde{\Theta}$  is derived in appendix A, with

$$\tilde{\Sigma} = \Sigma + \frac{q\tilde{q}^{-1}}{(p-qC^{(0)})^2} \begin{pmatrix} 0 \\ \frac{1}{2}\ell e^{-2\Phi} \end{pmatrix}, \quad (4.16)$$

from which we find that the parameter  $\tilde{\alpha}$  in eq. (4.15) is given by

$$\tilde{\alpha} = \frac{c e^{-2\Phi} \ell}{2(cC^{(0)} + d)(p-qC^{(0)})(c\tilde{p} + d\tilde{q})} \left( cp + dq + \frac{q}{\tilde{q}} \frac{cC^{(0)} + d}{p-qC^{(0)}} \right). \quad (4.17)$$

The appropriate replacement of  $C^{(2)}$  is therefore

$$C^{(2)} \rightarrow \text{sgn}(p-qC^{(0)}) (\tilde{\Theta}^\top \tilde{\Sigma}), \quad (4.18)$$

which is manifestly  $\text{SL}(2, \mathbb{Z})$ -invariant and gives back  $C^{(2)}$  when  $\Theta = (1 \ 0)^\top$  and  $\tilde{\Theta} = (0 \ 1)^\top$ .

Applying the replacements (4.9), (4.10), (4.12), and (4.18) to the D3-brane action (4.8), we find the following manifestly  $\text{SL}(2, \mathbb{Z})$ -invariant action:

$$S_{\text{D3}} = -T_3 \int d^4\sigma \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta^E \\ \bar{\tau}_\alpha^E & E_{\alpha\beta}^E + \frac{\Theta^\top \mathcal{F}_{\alpha\beta}}{|\Theta^\top \mathcal{C}|} \end{pmatrix}} \\ - T_3 \int \left[ \mathcal{C}^{(4)} + (\tilde{\Theta}^\top \tilde{\Sigma}) \wedge (\Theta^\top \mathcal{F}) - \frac{1}{2} \frac{\tilde{\Theta}^\top \mathcal{C}}{\Theta^\top \mathcal{C}} (\Theta^\top \mathcal{F}) \wedge (\Theta^\top \mathcal{F}) \right]. \quad (4.19)$$

We have introduced an  $\text{SL}(2, \mathbb{Z})$  singlet  $\mathcal{C}^{(4)}$  that replaces  $C^{(4)}$ . We will later derive the explicit form of  $\mathcal{C}^{(4)}$  in eq. (4.22) by analyzing the gauge symmetries. This action already combines the positive and negative  $p-qC^{(0)}$  branches described by the following actions:

$$S_{\text{D3}}^\pm = -T_3 \int d^4\sigma \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta^E \\ \bar{\tau}_\alpha^E & E_{\alpha\beta}^E \pm \frac{\Theta^\top \mathcal{F}_{\alpha\beta}}{\Theta^\top \mathcal{C}} \end{pmatrix}} \\ - T_3 \int \left[ \mathcal{C}^{(4)} + (\tilde{\Theta}^\top \tilde{\Sigma}) \wedge (\Theta^\top \mathcal{F}) - \frac{1}{2} \frac{\tilde{\Theta}^\top \mathcal{C}}{\Theta^\top \mathcal{C}} (\Theta^\top \mathcal{F}) \wedge (\Theta^\top \mathcal{F}) \right] \quad (4.20)$$

Here,  $S_{\text{D3}}^+$  describes D3-branes coupled to the  $(p, q)$ -strings with  $p-qC^{(0)} > 0$ , while  $S_{\text{D3}}^-$  describes D3-branes coupled to the  $(p, q)$ -strings with  $p-qC^{(0)} < 0$ . Just like the

$(p, q)$ -string case discussed in section 3.2, the  $SL(2, \mathbb{Z})$  duality is branched: the  $SL(2, \mathbb{Z})$  transformations satisfying  $cC^{(0)} + d > 0$  map  $S_{D3}^+$  ( $S_{D3}^-$ ) to itself, and the  $SL(2, \mathbb{Z})$  transformations satisfying  $cC^{(0)} + d < 0$  map  $S_{D3}^+$  and  $S_{D3}^-$  to each other. .

In order to compensate for the variation of the action (4.20) with respect to the gauge transformation rule  $\delta\tilde{\Sigma} = d\Xi$ , the  $SL(2, \mathbb{Z})$ -invariant  $\mathcal{C}^{(4)}$  has to transform under the Ramond-Ramond and Kalb-Ramond gauge symmetries as

$$\delta\mathcal{C}^{(4)} = dZ^{(3)} + (\Theta^\top d\Sigma) \wedge (\tilde{\Theta}^\top \Xi), \quad (4.21)$$

where  $Z^{(3)}$  is a parameter characterizing the three-form gauge transformation. The relation between  $\mathcal{C}^{(4)}$  and the objects that we have defined, namely,  $C^{(4)}$  and  $\Sigma$ , can be fixed by matching the gauge transformations, which gives

$$\mathcal{C}^{(4)} = C^{(4)} + \frac{1}{4} \Sigma^\top \wedge (\rho \Sigma) - \frac{1}{2} (\Theta^\top \Sigma) \wedge (\tilde{\Theta}^\top \Sigma). \quad (4.22)$$

Here, we defined the constant matrix

$$\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.23)$$

The gauge transformation of  $\mathcal{C}^{(4)}$  can be found in eq. (2.15), where the gauge parameter  $\zeta^{(3)}$  is related to  $Z^{(3)}$  in eq. (4.21) via

$$Z^{(3)} = \zeta^{(3)} - \frac{1}{2} \Sigma^\top \wedge (\Theta \tilde{\Theta}^\top + \tilde{\Theta} \Theta^\top) \Xi + \frac{1}{2} \Sigma^\top \wedge (\rho \Xi). \quad (4.24)$$

From the  $SL(2, \mathbb{Z})$ -invariant expression (4.22), we can solve for the  $SL(2, \mathbb{Z})$  transformation of  $\mathcal{C}^{(4)}$ , which gives,

$$\mathcal{C}'^{(4)} = \mathcal{C}^{(4)} + \frac{1}{2} \mathcal{K}^\top \begin{pmatrix} bd & -bc \\ -bc & ac \end{pmatrix} \wedge \mathcal{K}, \quad (4.25)$$

where

$$\mathcal{K} = \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} - \frac{c}{cC^{(0)} + d} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix}. \quad (4.26)$$

Note that the action (4.19) becomes independent of  $\tilde{\Theta}$  after the condition (4.2) is imposed, up to a boundary term that we ignore. Moreover, under the condition  $\Theta = (1 \ 0)^\top$ , as expected, the action reduces to the original nonrelativistic D3-brane action (4.8). Alternatively, when  $\Theta = (0 \ 1)^\top$ , the action (4.19) becomes

$$\begin{aligned} \tilde{S}_{D3} = & -T_3 \int d^4\sigma e^{-\Phi} \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta \\ \tilde{\tau}_\alpha & E_{\alpha\beta} + \tilde{\mathcal{F}}_{\alpha\beta} \end{pmatrix}} \\ & - T_3 \int \left( \tilde{C}^{(4)} + \tilde{C}^{(2)} \wedge \tilde{\mathcal{F}} + \frac{1}{2} \tilde{C}^{(0)} \tilde{\mathcal{F}} \wedge \tilde{\mathcal{F}} \right), \end{aligned} \quad (4.27)$$

where

$$\tilde{C}^{(0)} = -C^{(0)}, \quad \tilde{\mathcal{F}} = -\frac{C^{(2)} + dA^C}{C^{(0)}} + \frac{\ell}{2(e^\Phi C^{(0)})^2}, \quad (4.28a)$$

$$\tilde{C}^{(2)} = C^{(0)} \left[ B - \frac{\ell}{2(e^\Phi C^{(0)})^2} \right], \quad \tilde{C}^{(4)} = C^{(4)} + \left( C^{(2)} - \frac{e^{-2\Phi} \ell}{2C^{(0)}} \right) \wedge B. \quad (4.28b)$$

This is the S-dual action of the original nonrelativistic D3-brane action (4.8), which can be equivalently obtained by dualizing the U(1) gauge field on the D3-brane [15].

### 4.3 Inter-branch D3-branes

In section 3.1, we found the actions  $S^\pm$  in eqs. (3.4) and (3.8), which describe nonrelativistic  $(p, q)$ -strings satisfying  $p - q C^{(0)} \neq 0$ . Although the  $(p, q)$ -string action is well defined in both the branches, it becomes singular at the boundary defined by  $p - q C^{(0)} = 0$ . In contrast, intriguingly, when the D3-brane action (4.19) is concerned, the limit  $p - q C^{(0)} \rightarrow 0$  appears to be non-singular (e.g., see [15] when  $\Theta = (0 \ 1)^\top$ ). In the following, we study explicitly such inter-branch D3-branes and uncover the resulting finite action in the  $p - q C^{(0)} \rightarrow 0$  limit.

We start with defining  $\delta \equiv p - q C^{(0)}$ . It matters from which side that we approach the boundary. We first consider the limit with  $\delta \rightarrow 0^+$ . Expanding eq. (4.19) with respect to  $\delta$ , we find that the DBI-like part of the Lagrangian gives

$$\begin{aligned} \mathcal{L}_{\text{DBI}} &\equiv - \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta^E \\ \bar{\tau}_\alpha^E & E_{\alpha\beta}^E + \frac{\Theta^\top \mathcal{F}_{\alpha\beta}}{|\Theta^\top \mathcal{C}|} \end{pmatrix}} \\ &= \frac{1}{2\delta^2} \frac{|q \text{tr}(\star \ell \mathcal{X})|}{e^{2\Phi}} + \frac{1}{4\delta q} \text{tr}(\star \mathcal{X} \mathcal{X}) \text{sgn}[q \text{tr}(\star \ell \mathcal{X})] \\ &\quad + \frac{q^2 e^{-2\Phi} G + \text{tr}(\star \mathcal{X} \tau \star \mathcal{X} E) - \frac{1}{16} q^{-2} e^{2\Phi} [\text{tr}(\star \mathcal{X} \mathcal{X})]^2}{|q \text{tr}(\star \ell \mathcal{X})|} + O(\delta). \end{aligned} \tag{4.29}$$

Since the worldvolume is flat, the indices are raised (lowered) by the Minkowski metric  $\eta^{\alpha\beta}$  ( $\eta_{\alpha\beta}$ ). Here,  $(\tau)_{\alpha\beta} = \tau_{\alpha\beta}$ ,  $(E)_{\alpha\beta} = E_{\alpha\beta}$ , and

$$\mathcal{X} = p(B + dA^B) + q(C^{(2)} + dA^C), \tag{4.30a}$$

$$G = -\frac{1}{4} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon^{\beta_1 \beta_2 \beta_3 \beta_4} \tau_{\alpha_1 \beta_1} \tau_{\alpha_2 \beta_2} E_{\alpha_3 \beta_3} E_{\alpha_4 \beta_4}. \tag{4.30b}$$

The Hodge duals of  $\ell$  and  $\mathcal{X}$  are defined to be

$$(\star \ell)_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} \ell_{\gamma\delta}, \quad (\star \mathcal{X})_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} \mathcal{X}_{\gamma\delta}. \tag{4.31}$$

Similarly, the Wess-Zumino part of the Lagrangian in eq. (4.19) gives

$$\begin{aligned} \mathcal{L}_{\text{WZ}} &\equiv - \left[ \mathcal{C}^{(4)} + (\tilde{\Theta}^\top \tilde{\Sigma}) \wedge (\Theta^\top \mathcal{F}) - \frac{1}{2} \frac{\tilde{\Theta}^\top \mathcal{C}}{\Theta^\top \mathcal{C}} (\Theta^\top \mathcal{F}) \wedge (\Theta^\top \mathcal{F}) \right] \\ &= -\frac{1}{\delta^2} \frac{q}{e^{2\Phi}} \ell \wedge \mathcal{X} - \frac{1}{2\delta q} \mathcal{X} \wedge \mathcal{X} \\ &\quad - \mathcal{X}^{(4)} - (\tilde{p} B + \tilde{q} C^{(2)}) \wedge \mathcal{X} + \frac{\tilde{q}}{2q} \mathcal{X} \wedge \mathcal{X} + O(\delta). \end{aligned} \tag{4.32}$$

Here,

$$\mathcal{X}^{(4)} = C^{(4)} + \frac{1}{2} B \wedge C^{(2)} - \frac{1}{2} (p B + q C^{(2)}) \wedge (p B + q C^{(2)}), \tag{4.33}$$

in analogy with the definition (4.22). For the combination  $\mathcal{L}_{\text{DBI}} + \mathcal{L}_{\text{WZ}}$  to be finite, we require

$$q \operatorname{tr}(\star \ell \mathcal{X}) > 0, \quad \text{for } p - q C^{(0)} \rightarrow 0^+. \quad (4.34)$$

In particular, we require that  $q \neq 0$ . As a result, in the limit  $\delta \rightarrow 0^+$ , we find a nonsingular action for the inter-branch D3-brane as

$$S = T_3 \int d^4 \sigma \frac{q^2 e^{-2\Phi} G + \operatorname{tr}(\star \mathcal{X} \tau \star \mathcal{X} E) - \frac{1}{16} q^{-2} e^{2\Phi} [\operatorname{tr}(\star \mathcal{X} \mathcal{X})]^2}{|q \operatorname{tr}(\star \ell \mathcal{X})|} - T_3 \int \left[ \mathcal{X}^{(4)} + (\tilde{p} B + \tilde{q} C^{(2)}) \wedge \mathcal{X} - \frac{\tilde{q}}{2q} \mathcal{X} \wedge \mathcal{X} \right]. \quad (4.35)$$

Note that we have introduced an absolute value in eq. (4.35) such that it also incorporates the other limit by sending  $\delta \rightarrow 0^-$ , instead of  $\delta \rightarrow 0^+$ . For the limit  $\delta \rightarrow 0^-$  to be well-defined, the condition (4.34) has to be replaced with

$$q \operatorname{tr}(\star \ell \mathcal{X}) < 0, \quad \text{for } p - q C^{(0)} \rightarrow 0^-. \quad (4.36)$$

The quantity  $|q \operatorname{tr}(\star \ell \mathcal{X})|$  plays the role of the effective coupling of the inter-branch D3-brane.

In the case where  $\Theta = (0, 1)$  and  $\tilde{\Theta} = (-1, 0)$ , the action (4.35) becomes

$$S = T_3 \int d^4 \sigma \frac{e^{-2\Phi} G + \operatorname{tr}(\star \mathcal{X} \tau \star \mathcal{X} E) - \frac{1}{16} e^{2\Phi} [\operatorname{tr}(\star \mathcal{X} \mathcal{X})]^2}{|\operatorname{tr}(\star \ell \mathcal{X})|} - T_3 \int \left[ C^{(4)} + B \wedge (C^{(2)} + dA^C) \right]. \quad (4.37)$$

This is the action that has been studied in [15], which is S-dual to the nonrelativistic D3-brane action (4.8) at  $C^{(0)} \rightarrow 0$ . In other words, eq. (4.37) coincides with the S-dual action (4.27) in the  $C^{(0)} \rightarrow 0$  limit. This theory has a sector that describes noncommutative Yang-Mills (NCYM) theory [15, 39]. In order to make this NCYM sector manifest, we focus on flat spacetime and choose the following background field configuration:

$$\tau_\mu^A = \delta_\mu^A, \quad E_\mu^{A'} = \delta_\mu^{A'}, \quad C^{(q)} = 0, \quad B_{\alpha\beta} = - \begin{pmatrix} 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & b \epsilon_{ij} \end{pmatrix}, \quad (4.38)$$

where  $i, j = 2, 3$ . Here,  $b$  is a constant and  $\epsilon_{ij}$  is a Levi-Civita symbol defined by  $\epsilon_{23} = -\epsilon_{32} = 1$ . Moreover, we choose the D-brane configuration such that  $X^\alpha = \sigma^\alpha$ ,  $\alpha = 0, \dots, 3$ , such that the D3-brane extends in the target-space longitudinal directions. Using the Seiberg-Witten map [51], we find that the effective open string coupling is given by

$$G_o = \sqrt{2\pi g_s b}, \quad (4.39)$$

where  $g_s = e^{\langle \Phi \rangle}$  is the closed string coupling. Moreover, the worldvolume acquires non-commutativity in the  $\sigma^2$  and  $\sigma^3$  directions, with

$$[\sigma^2, \sigma^3] \propto \frac{1}{b}, \quad (4.40)$$

controlled by the inverse of the “magnetic”  $B$ -field. This implies that the resulting theory in the chosen configuration describes NCYM on the worldvolume, which is S-dual to non-commutative open string theory [39].<sup>13</sup> The NCYM coupling is given by the open string coupling  $G_o$  in eq. (4.39), which is proportional to the square-root of the magnetic  $B$ -field.

## 5 Conclusions

In this paper, we constructed manifestly  $SL(2, \mathbb{Z})$ -invariant actions in nonrelativistic string theory for D-instantons in eq. (4.7),  $(p, q)$ -strings in eqs. (3.4) and (3.8) and D3-branes in eq. (4.20), where the branes are coupled to a non-Lorentzian background geometry equipped with a codimension-two foliation structure. Intriguingly, such a realization requires a branching in the  $(p, q)$ -space, determined by the sign of  $p - qC^{(0)}$ , with  $C^{(0)}$  the zero-form axion background field. The states in the different branches are described by distinct actions, related to each other by sending  $(p, q)$  to  $(-p, -q)$  without changing the background fields. The  $SL(2, \mathbb{Z})$  group itself is also branched. Consider the  $SL(2, \mathbb{Z})$  group element  $\Lambda$  in eq. (3.18), which satisfies  $cC^{(0)} + d > 0$ . Such a transformation maps covariantly each of the branches in  $(p, q)$ -space to itself. In contrast, an  $SL(2, \mathbb{Z})$  transformation satisfying  $cC^{(0)} + d < 0$  maps between one branch in  $(p, q)$ -space and the other.

We also discussed the inter-branch D3-branes that arise from the limit  $p - qC^{(0)} \rightarrow 0$ , resulting in the non-singular,  $SL(2, \mathbb{Z})$ -invariant brane action (4.35). In flat spacetime with zero Ramond-Ramond potentials and  $(p, q) = (0, 1)$ , the inter-branch D3-brane action leads to noncommutative Yang-Mills theory [15, 39].

It should be noted that the non-relativistic  $SL(2, \mathbb{Z})$  transformations we constructed in eqs. (3.30) and (4.25) are a deformation of the standard relativistic duality rules containing extra terms. It is well-known that these relativistic  $SL(2, \mathbb{Z})$  duality transformations have a geometric interpretation, from a nine-dimensional point of view, as the geometric transformations of an M-theory two-torus [3]. It would be interesting to see what the geometric interpretation of the extra terms in the non-relativistic  $SL(2, \mathbb{Z})$  duality rules are, given the fact that the general coordinate transformations do not have non-relativistic corrections.

The fact that the  $SL(2, \mathbb{Z})$  duality in nonrelativistic string theory involves a branching depending on  $C^{(0)}$  may suggest that a background D7-brane is needed for a thorough understanding of this branched duality. The magnetic dual of  $C^{(0)}$  is an eight-form field, which naturally couples to the D7-brane. It would therefore be natural to examine the role that such D7-branes might play in describing the dynamics of  $(p, q)$ -strings in non-relativistic string/F-theory. Moreover, it would be interesting to study the implications of our studies of various bound states for the Hagedorn transition along the lines of [52].

It is important to extend the analysis of this project to include supersymmetry and study the corresponding type IIB supergravity background fields. This will be an extension of the construction of nonrelativistic minimal supergravity in [30, 32] to the case of maximal supersymmetry. It would be intriguing to understand the realization of the branched  $SL(2, \mathbb{Z})$  symmetry in that theory. Once the  $SL(2, \mathbb{Z})$  duality is understood in supergravity,

<sup>13</sup>The fact that the S-dual relation in [39] between NCYM and NCOS only arises when the inter-branch condition  $p - qC^{(0)} = 0$  is satisfied has been observed in e.g. [44].

it is natural to apply it together with the nonrelativistic T-duality transformations [15, 16] to construct new half-BPS background solutions out of known solutions, such as a nonrelativistic version of the well-known D-brane solutions, along the lines of [33]. We expect that these solutions will provide an important input for a top-down construction of nonrelativistic holography. We are currently working on a partner article addressing these questions.

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## A Derivation of $SL(2, \mathbb{Z})$ invariants

In our construction of the manifestly  $SL(2, \mathbb{Z})$ -invariant nonrelativistic D3-brane action in section 4.2, the following  $SL(2, \mathbb{Z})$ -invariant is introduced in eq. (4.18):

$$\text{sgn}(p - q C^{(0)}) (\tilde{\Theta}^\top \tilde{\Sigma}), \tag{A.1}$$

where  $\tilde{\Theta} = (\tilde{p} \ \tilde{q})^\top$  and

$$\tilde{\Sigma} = \Sigma + \frac{q \tilde{q}^{-1}}{(p - q C^{(0)})^2} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-\Phi} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} + \frac{q}{p - q C^{(0)}} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix}. \tag{A.2}$$

It is straightforward to check the invariance of eq. (A.1) under the  $SL(2, \mathbb{Z})$  transformations (3.30). In this appendix, we present a systematic procedure of constructing the  $SL(2, \mathbb{Z})$ -invariants that appeared in the bulk of the paper by starting with the transformation rules of the background fields, using which the invariant (A.1) can also be naturally derived.

### A.1 Branched $SL(2, \mathbb{Z})$ transformations

We start with summarizing the  $SL(2, \mathbb{Z})$  transformations that have been given in eqs. (3.30) and (4.25), which act on the Ramond-Ramond potentials and the  $B$ -field as follows:<sup>14</sup>

$$C'^{(0)} = \frac{aC^{(0)} + b}{cC^{(0)} + d}, \tag{A.4a}$$

$$\begin{pmatrix} B' \\ C'^{(2)} \end{pmatrix} = \text{sgn}(cC^{(0)} + d) \left[ (\Lambda^{-1})^\top \mathcal{K} - \frac{c}{(cC^{(0)} + d)^2} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix} \right], \tag{A.4b}$$

$$C'^{(4)} = C^{(4)} + \frac{1}{2} \mathcal{K}^\top \begin{pmatrix} bd & -bc \\ -bc & ac \end{pmatrix} \wedge \mathcal{K}, \tag{A.4c}$$

where

$$\mathcal{K} = \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} - \frac{c}{cC^{(0)} + d} \begin{pmatrix} 0 \\ \frac{1}{2} \ell e^{-2\Phi} \end{pmatrix}. \tag{A.5}$$

It is also useful to recall that

$$\ell' = |cC^{(0)} + d| \ell, \quad \Phi' = 2 \ln |cC^{(0)} + d|, \tag{A.6}$$

which are needed for explicitly checking the  $SL(2, \mathbb{Z})$ -invariance of the quantities that we will construct in the following.

### A.2 Zero-form invariants

In order to demonstrate the procedure of deriving  $SL(2, \mathbb{Z})$ -invariants, we start with using the  $SL(2, \mathbb{Z})$  transformation of the zero-form  $C^{(0)}$  in eq. (A.4a) to derive a family of  $SL(2, \mathbb{Z})$  invariants. Recall that  $\Theta$  and  $\tilde{\Theta}$  are both  $SL(2, \mathbb{Z})$ -doublets, which transform under the action of the  $SL(2, \mathbb{Z})$  matrix  $\Lambda$  as

$$\Theta' = \Lambda \Theta, \quad \tilde{\Theta}' = \Lambda \tilde{\Theta}. \tag{A.7}$$

First, we fix all the primed background fields and the primed doublets  $\Theta'$  and  $\tilde{\Theta}'$ , which take the following reference values:

$$\Theta' = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \tilde{\Theta}' = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \tag{A.8}$$

where the constant integers  $x, y, \tilde{x}, \tilde{y}$  satisfy  $x\tilde{y} - y\tilde{x} = 1$ . Then, the matrix  $\Lambda$  can be expressed in terms of  $\Theta$  and  $\tilde{\Theta}$ , with

$$\Lambda = \begin{pmatrix} x\tilde{q} - \tilde{x}q & \tilde{x}p - x\tilde{p} \\ y\tilde{q} - \tilde{y}q & \tilde{y}p - y\tilde{p} \end{pmatrix}. \tag{A.9}$$

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<sup>14</sup>It is interesting to note the curious fact that the transformation (A.4c) of  $C^{(4)}$  is reminiscent of its counterpart in relativistic string theory [53]. However,  $\mathcal{K}$  is *not* an  $SL(2, \mathbb{Z})$ -doublet:  $\mathcal{K}$  is only defined with respect to a particular  $SL(2, \mathbb{Z})$  transformation, with explicit dependence on the Lie group parameters  $c$  and  $d$ . Moreover, the transformation (A.4b) of the two-form fields can be brought into the form analogous to the relativistic case as

$$\mathcal{K}' = \text{sgn}(cC^{(0)} + d) (\Lambda^{-1})^\top \mathcal{K}. \tag{A.3}$$

Note that  $\mathcal{K}'$  contains  $c' = -c$  and  $d' = a$ . Here,  $\mathcal{K}'$  depends on the  $SL(2, \mathbb{Z})$  transformation  $\Lambda'$  that maps the primed fields to the unprimed fields, i.e.,  $\Lambda' = \Lambda^{-1}$ .

Plugging eq. (A.9) into the transformation rule (A.4a) for  $C^{(0)}$ , we find

$$C'^{(0)} = \frac{\tilde{x}(p - qC^{(0)}) - x(\tilde{p} - \tilde{q}C^{(0)})}{y(\tilde{p} - \tilde{q}C^{(0)}) - \tilde{y}(p - qC^{(0)})}. \quad (\text{A.10})$$

Since we have fixed the primed background fields including the zero-form  $C'^{(0)}$ , the r.h.s. of eq. (A.10) takes the same value for any  $\Theta$  and  $\tilde{\Theta}$ , which are related to the fixed  $\Theta'$  and  $\tilde{\Theta}'$  in eq. (A.8) via an  $\text{SL}(2, \mathbb{Z})$  transformation. Therefore, due to the associativity of group actions, the r.h.s. of eq. (A.10) defines an  $\text{SL}(2, \mathbb{Z})$  invariant.

Note that the invariant implied by eq. (A.10) is defined with respect to the arbitrary but fixed reference parameters in eq. (A.8), which ensures that it is  $\text{SL}(2, \mathbb{Z})$ -invariant, but does not necessarily mean that it must arise in the D-instanton Lagrangian in nonrelativistic string theory. The actual terms that arise in the nonrelativistic D(-1)-brane action correspond to fixing the reference values of  $\Theta'$  and  $\tilde{\Theta}'$  in eq. (A.8) to be

$$\Theta' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\Theta}' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{A.11})$$

Then, the  $\text{SL}(2, \mathbb{Z})$  matrix eq. (A.9) becomes

$$\Lambda = \begin{pmatrix} \tilde{q} & -\tilde{p} \\ -q & p \end{pmatrix}. \quad (\text{A.12})$$

Henceforth, we will fix  $\Theta'$  and  $\tilde{\Theta}'$  to the values in eq. (A.11). In this case, the  $\text{SL}(2, \mathbb{Z})$ -invariant implied by eq. (A.10) becomes

$$\frac{\tilde{p} - \tilde{q}C^{(0)}}{p - qC^{(0)}}, \quad (\text{A.13})$$

which reproduces the  $\text{SL}(2, \mathbb{Z})$ -invariant that we already constructed in eq. (4.7) for nonrelativistic D-instantons.

### A.3 Higher-form invariants

We are now ready to apply the same method developed in section A.2 to the  $\text{SL}(2, \mathbb{Z})$  transformations of the higher-form fields and derive the associated  $\text{SL}(2, \mathbb{Z})$  invariants. Again, the actual terms that arise in the Dp-brane action correspond to fixing  $\Theta'$  and  $\tilde{\Theta}'$  as in eq. (A.11).

We first consider the  $\text{SL}(2, \mathbb{Z})$  transformations of the two-form fields in eq. (A.4b). Using the reference values of  $\Theta'$  and  $\tilde{\Theta}'$  in eq. (A.11), and plugging eq. (A.12) into eq. (A.4b), we find

$$\begin{pmatrix} B' \\ C'^{(2)} \end{pmatrix} = \text{sgn}(p - qC^{(0)}) \begin{pmatrix} \Theta^\top \Sigma \\ \tilde{\Theta}^\top \tilde{\Sigma} \end{pmatrix}, \quad (\text{A.14})$$

where  $\Sigma$  and  $\tilde{\Sigma}$  are given in eq. (A.2). Since the primed background fields  $B'$  and  $C'^{(2)}$  are both fixed, we find two independent  $\text{SL}(2, \mathbb{Z})$ -invariants,

$$\text{sgn}(p - qC^{(0)}) (\Theta^\top \Sigma), \quad \text{sgn}(p - qC^{(0)}) (\tilde{\Theta}^\top \tilde{\Sigma}). \quad (\text{A.15})$$

The first expression in eq. (A.15) reproduces the Wess-Zumino term in the nonrelativistic  $(p, q)$ -string action (3.26). Moreover, the second expression in eq. (A.15) gives the  $\text{SL}(2, \mathbb{Z})$ -invariant that we quoted in eq. (4.18), which is important to the construction of the nonrelativistic D3-brane action (4.19).

Finally, plugging eq. (A.12) into the  $\text{SL}(2, \mathbb{Z})$  transformation (A.4c) of the four-form field  $C^{(4)}$ , we find

$$C'^{(4)} = C^{(4)} - \frac{1}{2} \Sigma^\tau \begin{pmatrix} p\tilde{p} & q\tilde{p} \\ q\tilde{p} & q\tilde{q} \end{pmatrix} \wedge \Sigma = \mathcal{C}^{(4)}, \quad (\text{A.16})$$

where  $\mathcal{C}^{(4)}$  is defined in eq. (4.22), which is the  $\text{SL}(2, \mathbb{Z})$ -singlet desired for the construction of the nonrelativistic D3-brane action (4.19).

## B Nonrelativistic string limit

In this appendix, we show how the D-instanton action (4.7), the  $(p, q)$ -string actions (3.4) and (3.8), and the D3-brane action (4.20) in nonrelativistic string theory follow from the nonrelativistic string limit of relativistic string theory. We start by reviewing the corresponding actions for relativistic instantons,  $(p, q)$ -strings, and D3-branes, which are given by [8, 11]

$$\hat{S}_{\text{D}(-1)} = T_{-1} \frac{\tilde{\Theta}^\tau \hat{\mathcal{M}}^{-1} \Theta}{\Theta^\tau \hat{\mathcal{M}}^{-1} \Theta}, \quad (\text{B.1a})$$

$$\hat{S}_{\text{string}} = -T_1 \int d^2\sigma \sqrt{\Theta^\tau \hat{\mathcal{M}}^{-1} \Theta} \sqrt{-\det \hat{G}_{\alpha\beta}^{\text{E}}} - T_1 \int \Theta^\tau \hat{\Sigma}, \quad (\text{B.1b})$$

$$\begin{aligned} \hat{S}_{\text{D3}} = & -T_3 \int d^4\sigma \sqrt{-\det \left( \hat{G}^{\text{E}} + \frac{\Theta^\tau \hat{\mathcal{F}}}{\sqrt{\Theta^\tau \hat{\mathcal{M}}^{-1} \Theta}} \right)} \\ & - T_3 \int \left[ \hat{\mathcal{C}}^{(4)} + (\tilde{\Theta}^\tau \hat{\Sigma}) \wedge (\Theta^\tau \hat{\mathcal{F}}) - \frac{1}{2} \frac{\tilde{\Theta}^\tau \hat{\mathcal{M}}^{-1} \Theta}{\Theta^\tau \hat{\mathcal{M}}^{-1} \Theta} (\Theta^\tau \hat{\mathcal{F}}) \wedge (\Theta^\tau \hat{\mathcal{F}}) \right], \end{aligned} \quad (\text{B.1c})$$

The  $\text{SL}(2, \mathbb{Z})$  doublets  $\Theta$  and  $\tilde{\Theta}$  have already been defined in the bulk of the paper, see equations (3.16), (4.1), and (4.2). The Einstein frame metric and its pullback are defined as

$$\hat{G}_{\alpha\beta}^{\text{E}} = \partial_\alpha X^\mu \partial_\beta X^\nu \hat{G}_{\mu\nu}^{\text{E}}, \quad \hat{G}_{\mu\nu}^{\text{E}} = e^{-\hat{\Phi}/2} \hat{G}_{\mu\nu}, \quad (\text{B.2})$$

where  $\alpha, \beta$  are the respective worldvolume indices. We furthermore use the following definitions of the matrix  $\hat{\mathcal{M}}$  and the doublet of two-forms  $\hat{\Sigma}$ :

$$\hat{\mathcal{M}} = e^{\hat{\Phi}} \begin{pmatrix} (\hat{C}^{(0)})^2 + e^{-2\hat{\Phi}} \hat{C}^{(0)} & \\ \hat{C}^{(0)} & 1 \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} \hat{B} \\ \hat{C}^{(2)} \end{pmatrix}. \quad (\text{B.3})$$

Together with a doublet of Born-Infeld vectors  $\mathcal{A}$ , we also introduced the following world-volume two-form  $\hat{\mathcal{F}}$  and  $\text{SL}(2, \mathbb{Z})$  singlet four-form  $\mathcal{C}^{(4)}$ :

$$\hat{\mathcal{F}} = \hat{\Sigma} + d\mathcal{A}, \quad \hat{\mathcal{C}}^{(4)} = \hat{C}^{(4)} + \frac{1}{2} \hat{B} \wedge \hat{C}^{(2)} - \frac{1}{2} (\Theta^\tau \hat{\Sigma}) \wedge (\tilde{\Theta}^\tau \hat{\Sigma}). \quad (\text{B.4})$$

The instanton,  $(p, q)$  string, and D3 brane actions (B.1) are invariant<sup>15</sup> under the action of  $SL(2, \mathbb{Z})$ , where  $\Theta$  and  $\tilde{\Theta}$  transform as doublets, while the Einstein frame metric  $\hat{G}^E$  and the four-form potential  $\hat{C}^{(4)}$  transform as singlets. Moreover, the quantities  $\hat{\mathcal{M}}$ ,  $\hat{\Sigma}$ , and  $\hat{\mathcal{F}}$  transform as

$$\hat{\mathcal{M}}' = \Lambda \hat{\mathcal{M}} \Lambda^\top, \quad \hat{\Sigma}' = (\Lambda^{-1})^\top \hat{\Sigma}, \quad \hat{\mathcal{F}}' = (\Lambda^{-1})^\top \hat{\mathcal{F}}. \quad (\text{B.5})$$

In terms of the string coupling  $\hat{g}_s = \langle e^{\hat{\Phi}} \rangle$ , the tension of the  $(p, q)$ -string action (B.1b) is given by

$$\hat{T}_{p,q} = T_1 \sqrt{\left( p - q \langle \hat{C}^{(0)} \rangle \right)^2 + \frac{q^2}{\hat{g}_s^2}}, \quad (\text{B.6})$$

which matches the result from type IIB supergravity [3].

Nonrelativistic string theory arises from the nonrelativistic limit of relativistic string theory [13]. To facilitate such a limit, we introduce a real dimensionless parameter  $\omega > 0$  by reparametrizing the relativistic metric field  $\hat{G}_{\mu\nu}$ , Kalb-Ramond field  $\hat{B}_{\mu\nu}$ , dilaton field  $\hat{\Phi}$ , and Ramond-Ramond forms  $\hat{C}^{(q)}$  as follows [15, 24, 25]:<sup>16</sup>

$$\hat{G}_{\mu\nu} = \omega^2 \tau_{\mu\nu} + E_{\mu\nu}, \quad \hat{\Phi} = \Phi + \ln \omega, \quad (\text{B.7a})$$

$$\hat{B} = -\omega^2 \ell + B, \quad (\text{B.7b})$$

$$\hat{C}^{(q)} = \omega^2 C^{(q-2)} \wedge \ell + C^{(q)}. \quad (\text{B.7c})$$

Here, we define the two-form  $\ell$  to be in the target space as opposed to eq. (3.5), which is on the worldsheet:

$$\ell_{\mu\nu} = \tau_\mu^A \tau_\nu^B \epsilon_{AB}. \quad (\text{B.8})$$

Moreover, for the purpose of this paper, we only care about  $q = 0, 2, 4$ . After taking the limit  $\omega \rightarrow \infty$ , the fields  $\tau_{\mu\nu}$ ,  $E_{\mu\nu}$ ,  $B_{\mu\nu}$ ,  $\Phi$ , and  $C^{(q)}$  are interpreted as the background fields of nonrelativistic string theory that we have reviewed in section 2.1. Note that all hatted symbols represent quantities in relativistic string theory. The nonrelativistic string sigma model (2.3) and nonrelativistic D-brane action (2.13) then arise from taking the  $\omega \rightarrow \infty$  limit of their associated relativistic theories [15, 21, 25, 29, 31]. We emphasize that these resulting extended objects in nonrelativistic string theory are coupled to string Newton-Cartan geometry with a codimension-two foliation. This is distinct from the general  $p$ -brane limits considered in the literature, where a codimension- $(p + 1)$  foliation is induced in the target space [13, 54].

We now apply the nonrelativistic string limit to the manifestly  $SL(2, \mathbb{Z})$ -invariant actions for relativistic instantons, strings, and D3-branes in eq. (B.1). The  $\omega \rightarrow \infty$  limit of the relativistic D-instanton action (B.1a) proceeds in a rather straightforward way, which leads to the nonrelativistic instanton action (4.7). In the following, we discuss how the nonrelativistic  $(p, q)$ -string and D3-brane actions arise from the nonrelativistic string limit.

<sup>15</sup>In the literature, the action (B.1b) is sometimes called covariant [8], when the action of  $SL(2, \mathbb{Z})$  on  $\Theta$  is considered to be passive.

<sup>16</sup>It is understood that  $C^{(q)} = 0$  for  $q < 0$ . In particular, for the purpose of this paper,  $C^{(-2)} = 0$ .

Expanding the relativistic  $(p, q)$ -string action (B.1b) with respect to a large  $\omega$ , we find

$$\hat{S}_{\text{string}} = -\omega^2 T_1 \int \ell \left[ |p - q C^{(0)}| - (p - q C^{(0)}) \right] + O(\omega^0), \quad (\text{B.9})$$

which implies that the  $\omega \rightarrow \infty$  limit is well-defined only if the condition  $p - q C^{(0)} > 0$  is satisfied. Note that the leading  $O(\omega^2)$  term in eq. (B.9) receives contributions from both the kinetic and the Wess-Zumino part of the relativistic action (B.1b). As a result,

$$\lim_{\omega \rightarrow \infty} \hat{S}_{\text{string}} = S_{\text{string}}^+, \quad \text{if } p - q C^{(0)} > 0, \quad (\text{B.10})$$

where  $S_{\text{string}}^+$  is given in eq. (3.4). This shows that the  $\omega \rightarrow \infty$  limit effectively decouples half of the relativistic states — namely, those with  $p - q C^{(0)} < 0$ . Moreover, taking the nonrelativistic string limit of the relativistic  $(p, q)$ -string tension (B.6), where  $\hat{g}_s = \omega g_s$  due to the rescaling of the dilaton, leads to the nonrelativistic  $(p, q)$ -string tension (3.12) for the effective action (3.11).

Applying the  $\omega \rightarrow \infty$  limit to the relativistic D3-brane action (B.1c), we find that the Wess-Zumino term leads to a finite contribution. In contrast, for the  $\omega \rightarrow \infty$  limit of the kinetic term to be finite, the same condition  $p - q C^{(0)} > 0$  has to hold. It then follows that

$$\lim_{\omega \rightarrow \infty} \hat{S}_{\text{D3}} = S_{\text{D3}}^+, \quad \text{if } p - q C^{(0)} > 0, \quad (\text{B.11})$$

where the nonrelativistic D3-brane action  $S_{\text{D3}}^+$  is given in eq. (4.20).

In the bulk of this paper, we have seen that the nonrelativistic string and brane actions  $S_{\text{string}}^+$  and  $S_{\text{D3}}^+$  are not invariant under the full  $\text{SL}(2, \mathbb{Z})$ -duality group by themselves. Instead, under  $\text{SL}(2, \mathbb{Z})$  transformations satisfying the condition  $c C^{(0)} + d < 0$ , the positive-branch actions  $S_{\text{string}}^+$  and  $S_{\text{D3}}^+$  with  $p - q C^{(0)} > 0$  are respectively mapped to the negative-branch actions  $S_{\text{string}}^-$  and  $S_{\text{D3}}^-$  with  $p - q C^{(0)} < 0$ . It is natural to wonder whether these negative-branch actions also arise from a similar nonrelativistic string limit. This can be achieved by slightly changing the reparametrization by inverting the signs of the two-forms. Effectively, one can define the negative-branch limit by replacing the rules for the two forms in eq. (B.7) by

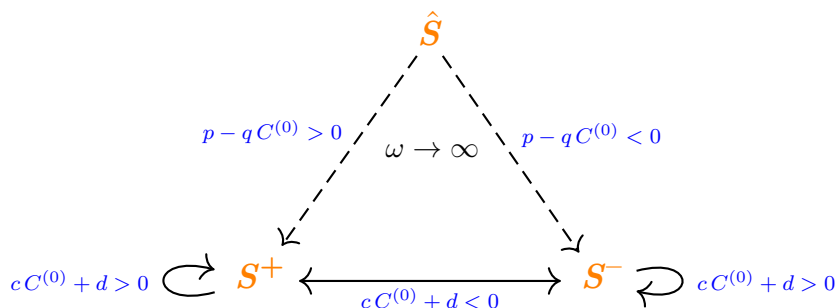
$$\hat{B} = \omega^2 \ell - B, \quad \hat{C}^{(2)} = -\omega^2 C^{(0)} \ell - C^{(2)}. \quad (\text{B.12a})$$

Using this parametrization, we find in the  $\omega \rightarrow \infty$  limit that

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \hat{S}_{\text{string}} &= S_{\text{string}}^-, \\ \lim_{\omega \rightarrow \infty} \hat{S}_{\text{D3}} &= S_{\text{D3}}^-, \end{aligned} \quad \text{if } p - q C^{(0)} < 0, \quad (\text{B.13})$$

where  $S_{\text{string}}^-$  is given in eq. (3.8) and  $S_{\text{D3}}^-$  is given in eq. (4.20).

We observe that the branching of the  $(p, q)$ -space naturally follows from the occurrence of a divergence in the nonrelativistic limit. Physically, this can be understood as follows: for a given  $\hat{C}^{(0)}$ , certain states in the  $(p, q)$ -space become infinitely heavy as  $\omega \rightarrow \infty$ . The  $\omega \rightarrow \infty$  limit associated with the background field reparametrization (B.7) decouples all the states with  $p - q C^{(0)} < 0$ . The  $\omega \rightarrow \infty$  limit associated with the reparametrization (B.12),



**Figure 2.** This figure illustrates the limiting procedure for nonrelativistic  $Dp$ -branes. Here,  $\hat{S}$  denotes a  $Dp$ -brane action in relativistic string theory. The positive branch arises from the  $\omega \rightarrow \infty$  limit with the reparametrization (B.7) of the background fields in relativistic string theory, which leads to the nonrelativistic  $Dp$ -brane action  $S^+$ , satisfying the condition  $p - qC^{(0)} > 0$ . The negative branch arises from the  $\omega \rightarrow \infty$  limit with the reparametrization (B.12), which leads to the action  $S^-$ , satisfying the condition  $p - qC^{(0)} < 0$ . While the  $SL(2, \mathbb{Z})$  transformations satisfying  $cC^{(0)} + d > 0$  map  $S^+$  ( $S^-$ ) to itself, the transformations satisfying  $cC^{(0)} + d < 0$  map between  $S^+$  and  $S^-$ .

on the other hand, decouples all states with  $p - qC^{(0)} > 0$ . The surviving states are described by  $S_{\text{string}}^+$  and  $S_{\text{string}}^-$ , respectively. Similarly, the limit of the D3-branes branches and the surviving states are described by  $S_{\text{D3}}^+$  and  $S_{\text{D3}}^-$ , respectively. This limiting procedure is exemplified in figure 2.

Similar to the non-relativistic limit of the relativistic action we have discussed above, the non-relativistic  $SL(2, \mathbb{Z})$  transformations can be obtained by a large  $\omega$  expansion of the relativistic fields in eq. (B.3). The relativistic  $SL(2, \mathbb{Z})$  transformations in eq. (B.5) and the transformation of  $\hat{C}^{(4)}$  can be expressed as

$$\begin{aligned}
 \hat{G}'_{\alpha\beta} &= \left[ (c\hat{C}^{(0)} + d)^2 + c^2 e^{-2\hat{\Phi}} \right]^{\frac{1}{2}} \hat{G}_{\alpha\beta}, & \hat{B}' &= d\hat{B} - c\hat{C}^{(2)}, \\
 e^{\hat{\Phi}'} &= \left[ (c\hat{C}^{(0)} + d)^2 + c^2 e^{-2\hat{\Phi}} \right] e^{\hat{\Phi}}, & \hat{C}^{(2)'} &= a\hat{C}^{(2)} - b\hat{B}, \\
 \hat{C}^{(0)'} &= \frac{(a\hat{C}^{(0)} + d)(c\hat{C}^{(0)} + d) + ac e^{-2\hat{\Phi}}}{(c\hat{C}^{(0)} + d)^2 + c^2 e^{-2\hat{\Phi}}}, & \hat{C}'^{(4)} &= \hat{C}^{(4)} + \frac{1}{2} \hat{\Sigma}^T \begin{pmatrix} bd & -bc \\ -bc & ac \end{pmatrix} \wedge \hat{\Sigma}.
 \end{aligned}
 \tag{B.14}$$

Substituting the expansion (B.7) into both sides of the transformations in eq. (B.14), we get the non-relativistic  $SL(2, \mathbb{Z})$  transformation in eqs. (3.30) and (4.25) (or, equivalently, (A.4) and (A.6)) for  $cC^{(0)} + d > 0$ , which is a required condition for the resulting branched  $SL(2, \mathbb{Z})$ -transformations to be finite.

We note that there is a complementary but equivalent procedure where instead of taking two inequivalent limits of one action one takes one limit of two inequivalent actions. These actions differ from each other by the sign of the Wess-Zumino term.

Finally, it is also possible to reproduce the topological action (3.32) by starting with the relativistic  $(p, q)$ -string action (B.1b) satisfying the condition  $p - qC^{(0)} = 0$ . Using the parametrizations in eq. (B.7), together with the additional ansatz,

$$T_1 = \frac{T_{\text{eff}}}{\omega}, \tag{B.15}$$

we find that the action (3.32) arises as the  $\omega \rightarrow \infty$  limit of the relativistic action.

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