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# The law of the iterated logarithm for a piecewise deterministic Markov process assured by the properties of the Markov chain given by its post-jump locations

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**ABSTRACT**

In the paper, we consider some piecewise deterministic Markov process, whose continuous component evolves according to semiflows, which are switched at the jump times of a Poisson process. The associated Markov chain describes the states of this process directly after the jumps. Certain ergodic properties of these two dynamical systems have been already investigated in our recent papers. We now aim to establish the law of the iterated logarithm for the aforementioned continuous-time process. Moreover, we intend to do this using the already proven properties of the discrete-time system. The abstract model under consideration has interesting interpretations in real-life sciences, such as biology. Among others, it can be used to describe the stochastic dynamics of gene expression.

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## Introduction

The law of the iterated logarithm (LIL) characterizes essentially the maximal fluctuations around the mean of a stochastic process in discrete or continuous time. It is intimately related to the strong law of large numbers (SLLN) and the central limit theorem (CLT). The history of results on the LIL dates back to the work by Khinchin [1], in the specific context of dyadic representations of numbers, and to the one by Kolmogorov [2], for general sequences of independent, non-necessarily identically distributed, random variables that satisfy a particular "asymptotic boundedness" condition. Kolomogorov's results for identically distributed random variables with finite second moment were further generalized into the version of the LIL known as the Hartman–Wintner Theorem [3]. See also [4] and e.g. [5] for the review of results (at the time of writing) on the LIL in the case of independent random variables.

The main goal of this paper is to prove the validity of the LIL for a class of piecewise deterministic Markov processes (PDMPs). In this setting, the associated random variables are neither independent, nor identically distributed. Our method of proof is

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intentionally such that the result for the PDMP is derived from the validity of the LIL for the Markov chain given by its post-jump locations. The latter has been established in [6] (see also the references mentioned there).

PDMPs have been introduced by Davis [7] as a general class of stochastic processes. They are encountered as suitable mathematical models for processes in the physical world around us, e.g. in biology, as stochastic models for gene expression [8], gene regulation [9], excitable membranes [10] or population dynamics [11], as well as in resource allocation and service provisioning (queuing, cf. [7]). Questions of ergodicity of PDMPs defined on locally compact state spaces have been studied in detail in [12–15]. The case of non-locally compact state space has been studied much less so far (see e.g. [9, 11, 16, 17]). A similar statement applies to the study of limit theorems (see [10, 18]). For more information on the validity of limit theorems (SLLN, CLT or LIL) for non-stationary processes one may consult [6, 16, 19–21].

A PDMP consists of deterministic motion in a state space (a Polish metric space in our case) that is alternated at random times of intervention with a random jump in state. In general, the distribution of the next intervention time and the jump can be both state dependent (cf. e.g. [9]). Here, and e.g. also in [11], only the jump is distributed conditionally given the current state of the system. The process examined in this paper (described in detail in Section 2) involves jumps that occur at random time points according to a Poisson process. Any post-jump location is attained by transforming a pre-jump state using a randomly selected function, and, further, by adding a random shift to the resulting state. Between any two consecutive jumps, the system is driven deterministically by one of a finite number of flows, which are switched at the jump times. If the state space is augmented with an index set of the applied movements, then the chain obtained by pairing the state just after the jump with the index of the movement that is applied thereafter yields a Markov chain, which intuitively should contain "all information" about the PDMP. Therefore, it is enlightening to show how properties of the PDMP can be proven from relevant properties of the Markov chain constituted by the post-jump states.

Essentially, our method of proof splits the problem into subproblems that can be analyzed separately. One subproblem can be addressed using a version of the LIL for certain square integrable martingales, whose proof draws heavily on [22, Theorem 1] and uses the coupling methods applied for establishing [20, Lemma 2.2] (cf. also [23]). Another builds on the validity of the LIL for Markov chains associated to PDMPs in the abstract model class, which has been obtained recently (cf. [6, Theorem 4.1]).

We believe that the class of dynamical systems under study is broad enough to cover models of suitable real-life systems, e.g. biological systems, such as those describing chemotactic movement of bacteria or ameba (related to the study of so-called velocity-jump models, employing particular Fokker–Planck equations, see e.g. [24–26]). Discussion and the detailed study of such application are beyond the scope of this paper, but they shall be the subject of our further research collaboration.

## 1. Preliminaries

Let us first introduce a piece of notation, as well as gather the most important definitions and facts, used in this paper.

### 1.1. Some notation and basic definitions

For any point $x$ and any set $A$, the symbols $\delta_x$ and $\mathbb{1}_A$ will denote the Dirac measure at $x$ and the indicator function of $A$, respectively.

Suppose that $(E, \varrho_E)$ is a Polish metric space, and let $\mathcal{B}_E$ denote the $\sigma$-field of all its Borel subsets. Let $B_b(E)$ stand for the space of all bounded, Borel measurable functions $f : E \to \mathbb{R}$ equipped with the supremum norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$. We shall also refer to certain subspaces of $B_b(E)$, namely $C_b(E)$, consisting of all continuous functions an $Lip_b(E)$, consisting of all Lipschitz continuous functions, as well as the set given by

$$Lip_{FM}(E) = \{f \in Lip_b(E) : \ \|f\|_{BL} \leq 1\},$$

where $\|\cdot\|_{BL}$ is the norm given by$\|f\|_{BL} = \max\{|f|_{Lip}, \ \|f\|_\infty\}$, and $|f|_{Lip}$ stands for the minimal Lipschitz constant of $f$ for every $f \in Lip_b(E)$. Finally, we will also consider the space $\bar{B}_b(E)$ of functions $f : E \to \mathbb{R}$ which are Borel measurable and bounded below.

The spaces of finite and probability Borel measures on $E$ will be denoted by $\mathcal{M}_{fin}(E)$ and $\mathcal{M}_1(E)$, respectively. Further, we also define

$$\mathcal{M}_{1,r}^V(E) = \left\{\mu \in \mathcal{M}_1(E) : \ \int_E V^r(x) \ \mu(dx) < \infty\right\}$$

for any $r > 0$ and any given Lyapunov function $V : E \to [0, \infty)$, that is, a function which is continuous, bounded on bounded sets, and, in the case of unbounded $E$, satisfies $\lim_{\varrho_E(x,\bar{x})\to\infty} V(x) = \infty$ for some fixed point $\bar{x} \in E$. For brevity, for any $f \in \bar{B}_b(E)$ and any signed Borel measure $\mu$ on $E$, we will write $\langle f, \mu\rangle$ for $\int_E f(x)\, \mu(dx)$. As usual, supp $\mu$ will stand for the support of $\mu \in \mathcal{M}_{fin}(E)$.

To evaluate the distance between probability measures, we will use the so-called Fortet–Mourier distance (see e.g. [27]), defined as follows:

$$d_{FM}(\mu_1, \mu_2) = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : \ f \in Lip_{FM}(E)\} \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1(E).$$

Let us indicate that, under the assumption that $(E, \varrho_E)$ is a Polish space, the convergence in $d_{FM}$ is equivalent to the weak convergence of probability measures, and also the space $(\mathcal{M}_1(E), d_{FM})$ is complete (for the proofs of both these facts see e.g. [28]).

### 1.2. Markov operators and the semigroups of Markov operators

A function $P : E \times \mathcal{B}_E \to [0, 1]$ is called a (sub)stochastic kernel, if, for any fixed $A \in \mathcal{B}_E$, $P(\cdot, A) : E \to [0, 1]$ is a Borel measurable map, and, for any fixed $x \in E$, $P(x, \cdot) : \mathcal{B}_E \to [0, 1]$ is a (sub)probability Borel measure. For any two kernels $P : E \times \mathcal{B}_E \to [0, 1]$ and $R : E \times \mathcal{B}_E \to [0, 1]$ we can define their composition $PR : E \times \mathcal{B}_E \to [0, 1]$ given by

$$PR(x, A) = \int_E P(y, A)R(x, dy) \ \text{ for } \ x \in E \ \text{ and } \ A \in \mathcal{B}_E. \tag{1.1}$$

Following this rule, for any (sub)stochastic kernel $P : E \times \mathcal{B}_E \to [0, 1]$, we can define its $n$-th step kernels $P^n : E \times \mathcal{B}_E \to [0, 1]$, inductively on $n \in \mathbb{N}$, by setting $P^n = PP^{n-1}$, where $P^0$ is given by $P^0(x, A) = \delta_x(A)$ for every $x \in E$ and any $A \in \mathcal{B}_E$.

Moreover, for any stochastic kernel $P$, we can define a regular Markov operator $(\cdot)P : \mathcal{M}_{fin}(E) \to \mathcal{M}_{fin}(E)$ and its dual operator $P(\cdot) : B_b(E) \to B_b(E)$ in the following way:

$$\mu P(A) = \int_E P(x, A)\, \mu(dx) \;\text{ for }\; \mu \in \mathcal{M}_{fin}(E),\;\; A \in \mathcal{B}_E, \tag{1.2}$$

$$Pf(x) = \int_E f(y)\, P(x, dy) \;\text{ for }\; f \in B_b(E),\;\; x \in E. \tag{1.3}$$

Obviously, $\langle f, \mu P\rangle = \langle Pf, \mu\rangle$ for any $f \in B_b(E)$ and any $\mu \in \mathcal{M}_{fin}(E)$. Moreover, note that any operator $P(\cdot)$ of the form (1.3) can be extended, in the usual way, to a linear operator on $\bar{B}_b(E)$, preserving the above-mentioned duality property, and hence it is reasonable to apply $P(\cdot)$ to any Lyapunov function. For notational simplicity, we shall use the same symbol for the extension as for the original operator on $B_b(E)$. An operator $(\cdot)P$, of the form (1.2), is said to be Markov–Feller if $Pf \in C_b(E)$ for every $f \in C_b(E)$.

We call $\mu_* \in \mathcal{M}_{fin}(E)$ an invariant measure of $(\cdot)P$ if $\mu_* P = \mu_*$. If $(\cdot)P$ has a unique invariant measure $\mu_* \in \mathcal{M}_1(E)$ and there exists $q \in (0, 1)$ such that

$$d_{FM}(\mu P^n, \mu_*) \le c(\mu) q^n \;\text{ for any }\; \mu \in \mathcal{M}^V_{1,1}(E),\;\; n \in \mathbb{N},$$

where $c(\mu)$ is a constant depending only on $\mu$, then $(\cdot)P$ is said to be exponentially ergodic in $d_{FM}$.

Let us consider $E^{\mathbb{N}_0}$ with the product topology. For every $n \in \mathbb{N}_0$ define $\phi_n : E^{\mathbb{N}_0} \to E$ by the formula $\phi_n(\omega) = e_n$, where $\omega = (e_0, e_1, \ldots) \in E^{\mathbb{N}_0}$. According to [29, Theorem 2.8], for any $\mu \in \mathcal{M}_1(E)$ and any stochastic kernel $P : E \times \mathcal{B}_E \to [0, 1]$, there exists $\mathbb{P} \in \mathcal{M}_1(E^{\mathbb{N}_0})$ such that $(\phi_n)_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain on the probability space $(E^{\mathbb{N}_0}, \mathcal{B}_{E^{\mathbb{N}_0}}, \mathbb{P})$ with transition function $P$ and initial measure $\mu$, that is

$$P^n(x, A) = \mathbb{P}(\phi_{k+n} \in A | \phi_k = x) \;\text{ for every }\; x \in E,\;\; A \in \mathcal{B}_E,\;\; n, k \in \mathbb{N}_0, \tag{1.4}$$

and

$$\mu(A) = \mathbb{P}(\phi_0 \in A) \;\text{ for any }\; A \in \mathcal{B}_E.$$

The chain defined as above shall be further called the canonical Markov chain. Clearly, $\mathbb{P}(B)$ may be read as the probability of the event $\{(\phi_n)_{n\in\mathbb{N}_0} \in B\}$ for any $B \in \mathcal{B}_{E^{\mathbb{N}_0}}$.

Conversely, it is clear that the one-step transition law of any time-homogeneous Markov chain determines a stochastic kernel and the corresponding $n$-step kernels, which satisfy (1.4). As far as the dual operator $P(\cdot)$ is concerned, we have

$$P^n f(x) = \mathbb{E}\big(f(\phi_n) | \phi_0 = x\big) \;\text{ for }\; x \in E,\;\; f \in B_b(E),\;\; n \in \mathbb{N}.$$

A regular Markov semigroup $(P_t)_{t\in\mathbb{R}_+}$ is a family of regular Markov operators $(\cdot)P_t : \mathcal{M}_{fin}(E) \to \mathcal{M}_{fin}(E)$, $t \in \mathbb{R}_+$, which form a semigroup (under composition) with the identity transformation $(\cdot)P_0$ as the unity element. Provided that $(\cdot)P_t$ is a Markov-Feller operator for every $t \in \mathbb{R}_+$, the semigroup $(P_t)_{t\in\mathbb{R}_+}$ is said to be Markov-Feller, too. If, for some $\mu_* \in \mathcal{M}_{fin}(E)$, $\mu_* P_t = \mu_*$ for every $t \in \mathbb{R}_+$, then we call $\mu_*$ an invariant measure of $(P_t)_{t\in\mathbb{R}_+}$.

Let $(\phi(t))_{t\in\mathbb{R}_+}$ be an $E$-valued time-homogeneous Markov process, defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with continuous time parameter $t \in \mathbb{R}_+$. Suppose that, for any $t \in \mathbb{R}_+$, $P_t : E \times \mathcal{B}_E \to [0, 1]$ is defined by

$$P_t(x,A) = \mathbb{P}(\phi(t) \in A | \phi(0) = x) \text{ for } x \in E, \ A \in \mathcal{B}_E, \ t \in \mathbb{R}_+. \tag{1.5}$$

It is well-known that these transition probability functions form a semigroup of stochastic kernels under the composition operation defined by (1.1). Thus, the family of the corresponding Markov operators $(P_t)_{t\in\mathbb{R}_+}$ is a regular Markov semigroup. The dual operator of $P_t$, $t \in \mathbb{R}_+$, can be expressed in the form

$$P_t f(x) = \mathbb{E}\big(f(\phi(t))|\phi(0) = x\big) \text{ for } x \in E, \ f \in B_b(E)$$

Now, let $(\phi_n)_{n\in\mathbb{N}_0}$ be a Markov chain with transition law $P$, and let $(\phi_n^{(1)})_{n\in\mathbb{N}_0}$, $(\phi_n^{(2)})_{n\in\mathbb{N}_0}$ be its copies with initial distributions $\mu_1 \in \mathcal{M}_1(E)$, $\mu_2 \in \mathcal{M}_1(E)$, respectively. A time-homogeneous Markov chain $(\phi_n^{(1)}, \phi_n^{(2)})_{n\in\mathbb{N}_0}$ evolving on $E^2$ (endowed with the product topology) is said to be a Markovian coupling of $(\phi_n^{(1)})_{n\in\mathbb{N}_0}$ and $(\phi_n^{(2)})_{n\in\mathbb{N}_0}$ whenever its transition law $C : E^2 \times \mathcal{B}_{E^2} \to [0,1]$ satisfies

$$C((x,y), A\times E) = P(x,A), \ C((x,y), E\times A) = P(y,A) \text{ for any } x,y \in E, \ A \in \mathcal{B}_E,$$

and its initial distribution $\alpha \in \mathcal{M}_1(E^2)$ is such that

$$\alpha(A\times E) = \mu_1(A), \ \alpha(E\times A) = \mu_2(A) \text{ for any } A \in \mathcal{B}_E.$$

In what follows we always assume that the coupling is defined canonically on the coordinate space $((E^2)^{\mathbb{N}_0}, \mathcal{B}_{(E^2)^{\mathbb{N}_0}})$ endowed with an appropriately constructed measure $\mathbb{C} \in \mathcal{M}_1((E^2)^{\mathbb{N}_0})$.

### 1.3. The law of the iterated logarithm for Markov processes

Consider an $E$-valued time-homogeneous Markov chain $(\phi_n)_{n\in\mathbb{N}_0}$ with initial distribution $\mu \in \mathcal{M}_1(E)$ and an $E$-valued time-homogeneous Markov process $(\phi(t))_{t\in\mathbb{R}_+}$ with initial distribution $\nu \in \mathcal{M}_1(E)$. For any function $g \in Lip_b(E)$, let us introduce $(s_n(g))_{n\in\mathbb{N}_0}$ and $(s(g)(t))_{t\in\mathbb{R}_+}$, given by

$$s_n(g) = \frac{\sum_{i=0}^{n-1} g(\phi_i)}{\sqrt{2n\ln(\ln(n))}} \text{ for } n > e \qquad \text{and} \qquad s_n(g) = 0 \text{ for } n \le e; \tag{1.6}$$

$$s(g)(t) = \frac{\int_0^t g(\phi(s))ds}{\sqrt{2t\ln(\ln(t))}} \text{ for } t > e \qquad \text{and} \qquad s(g)(t) = 0 \text{ for } t \le e. \tag{1.7}$$

Suppose that $\mu_* \in \mathcal{M}_1(E)$ and $\nu_* \in \mathcal{M}_1(E)$ are the unique invariant measures for the Markov operator and the Markov semigroup induced by the transition laws of $(\phi_n)_{n\in\mathbb{N}_0}$ and $(\phi(t))_{t\in\mathbb{R}_+}$, respectively. We say that the Markov chain $(g(\phi_n))_{n\in\mathbb{N}_0}$ satisfies the LIL if, for $\hat{g} := g - \langle g, \mu_*\rangle$ and some $\sigma(\hat{g}) \in (0,\infty)$,

$$\limsup_{n\to\infty} s_n(\hat{g}) = \sigma(\hat{g}) \text{ and } \liminf_{n\to\infty} s_n(\hat{g}) = -\sigma(\hat{g}) \ \mathbb{P}\text{-a.s.}$$

Accordingly, we say that the Markov process $(g(\phi(t)))_{t\in\mathbb{R}_+}$ satisfies the LIL if, for $\bar{g} := g - \langle g, \nu_*\rangle$ and some $\bar{\sigma}(\bar{g}) \in (0,\infty)$,

$$\limsup_{t\to\infty} s(\bar{g})(t) = \bar{\sigma}(\bar{g}) \text{ and } \liminf_{t\to\infty} s(\bar{g})(t) = -\bar{\sigma}(\bar{g}) \ \mathbb{P}\text{-a.s.}$$

## 2. An abstract model

In the beginning, we shall discuss the structure and assumptions of the model under consideration. Let us indicate that this model was initially introduced in [16], where we have also elaborated on its possible applications. Further, let us summarize the already known results that are used further in this paper.

### 2.1. The structure of the model and the undertaken assumptions

Consider a separable Banach space $(H, \|\cdot\|)$ and a closed subset $Y$ of $H$. For any $x \in H$ and any $r > 0$, let $B(x, r)$ denote an open ball in $H$ centered at $x$ and of radius $r$. Let us also fix a topological measure space $(\Theta, \mathscr{B}_\Theta, \vartheta)$ with a finite Borel measure $\vartheta$. With a slight abuse of notation, we will further write $d\theta$ only, instead of $\vartheta(d\theta)$. Finally, fix $m \in \mathbb{N}$ and introduce the set of indexes $I := \{1, \dots, m\}$ equipped with the metric $d$ given by

$$d(i,j) = \begin{cases} 1, & i \neq j \\ 0, & i = j. \end{cases}$$

We shall investigate a random dynamical system $(Y(t))_{t\in\mathbb{R}_+}$ evolving through random jumps, occurring at random moments $\tau_n$, $n \in \mathbb{N}$, which coincide with the jump times of a Poisson process with a given intensity $\lambda$. In every time interval $[\tau_{n-1}, \tau_n)$, where $\tau_0 = 0$, the system is driven by one of the given continuous semiflows $S_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$. The current semiflow, say $S_i$, is switched at a jump time to another (or the same) one $S_j$ with probability $\pi_{ij}(y)$, depending on the post-jump state $y$. We assume that these place-dependent probabilities constitute a matrix of continuous functions $\pi_{ij} : Y \to [0, 1]$, $i, j \in I$, such that

$$\sum_{j\in I} \pi_{ij}(y) = 1 \text{ for any } y \in Y, \ i \in I.$$

The above description can be shortly formalized by the following formula:

$$Y(t) = S_{\xi_n}(t - \tau_n, Y_n) \text{ for } t \in [\tau_n, \tau_{n+1}), \tag{2.1}$$

where $\xi_n$ is an $I$-valued random variable indicating which semiflow has been chosen after the $n$-th jump, and $Y_n := Y(\tau_n)$ is a result of some transformation of the state $Y(\tau_n-)$ just before the jump. The transformation is attained by a function $w_\theta : Y \to Y$, selected randomly among all possible ones $\{w_\theta : \theta \in \Theta\}$, and further disturbed by adding some random shift $H_n$. Therefore, we can formally write

$$Y_n = w_{\theta_n}(Y(\tau_n-)) + H_n.$$

It is assumed that, given $Y(\tau_n-) = y$, the probability of choosing $w_\theta$ (at the jump time $\tau_n$) is determined by the density function $\theta \mapsto p(y, \theta)$ such that $p : Y \times \Theta \to [0, \infty)$ is a continuous map. Moreover, it is required that the map $(y, \theta) \mapsto w_\theta(y)$ is continuous. Further, we also assume that, for some $\varepsilon > 0$, all the variables $H_n$, $n \in \mathbb{N}$, have a common distribution $\nu^\varepsilon \in \mathscr{M}_1(H)$ supported on $B(0, \varepsilon) \subset H$, and that

$$w_\theta(y) + h \in Y \text{ for any } h \in \operatorname{supp}(\nu^\varepsilon), \ \theta \in \Theta, \ y \in Y.$$

We therefore formally consider a stochastic process $(Y(t))_{t\in\mathbb{R}_+}$ of the form (2.1), defined as an interpolation of the discrete-time process $(Y_n)_{n\in\mathbb{N}_0}$ determined by the recursive formula

$$Y_n = Y(\tau_n) = w_{\theta_n}(S_{\xi_{n-1}}(\Delta\tau_n, Y_{n-1})) + H_n \text{ with } \Delta\tau_n := \tau_n - \tau_{n-1} \text{ for } n \in \mathbb{N}, \tag{2.2}$$

where $(\tau_n)_{n\in\mathbb{N}_0}$, $(\theta_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}_0}$ and $(H_n)_{n\in\mathbb{N}}$ are certain sequences of random variables (specified below) with values in $\mathbb{R}_+$, $\Theta$, $I$ and $H$, respectively.

The distribution of $(Y_0, \xi_0)$ is fixed arbitrarily. The sequence $(\tau_n)_{n\in\mathbb{N}_0}$, wherein $\tau_0 = 0$ a.s., is such that $\tau_n \uparrow \infty$ a.s., as $n \to \infty$. The increments $\Delta\tau_n$, $n \in \mathbb{N}$, are, in turn, assumed to be mutually independent and identically distributed according to the exponential distribution with rate $\lambda > 0$. Moreover, the disturbances $(H_n)_{n\in\mathbb{N}}$ are identically distributed with $\nu^\varepsilon$, introduced above. Finally, the sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(\theta_n)_{n\in\mathbb{N}}$ are defined, inductively on $n \in \mathbb{N}_0$, as follows:

$$\mathbb{P}(\xi_{n+1} = j \mid Y_{n+1} = y,\ \xi_n = i;\ W_n) = \pi_{ij}(y) \text{ for } y \in Y,\ i,j \in I,$$

$$\mathbb{P}(\theta_{n+1} \in D \mid S_{\xi_n}(\Delta\tau_{n+1}, Y_n) = y;\ W_n) = \int_D p(y,\theta)\,d\theta \text{ for } D \in \mathscr{B}_\Theta,\ y \in Y,$$

where

$$W_0 = (Y_0,\ \xi_0),\ W_n = (W_0,\ H_1,\dots,H_n,\ \tau_1,\dots,\tau_n,\ \theta_1,\dots,\theta_n,\ \xi_1,\dots,\xi_n) \text{ for } n \in \mathbb{N}.$$

We also demand that, for any $n \in \mathbb{N}_0$, the variables $\Delta\tau_{n+1}$, $H_{n+1}$, $\theta_{n+1}$ and $\xi_{n+1}$ are (mutually) conditionally independent given $W_n$, and that $\Delta\tau_{n+1}$ and $H_{n+1}$ are independent of $W_n$.

Let us now consider the space $X := Y \times I$ with the metric $\varrho_c$, given by

$$\varrho_c\big((y_1,i_1),(y_2,i_2)\big) = \|y_1 - y_2\| + c\,d(i_1,i_2) \quad \text{for} \quad (y_1,i_1),(y_2,i_2) \in X, \tag{2.3}$$

with a sufficiently large constant $c \geq 1$ (defined explicitly in [16]). Now, define

$$X_n := (Y_n, \xi_n) \text{ for } n \in \mathbb{N}_0.$$

Assuming that $\mu \in \mathscr{M}_1(X)$ is the distribution of $X_0 = (Y_0, \xi_0)$ and putting $\Delta\tau_0 := 0$, it is not hard to check that $(X_n, \Delta\tau_n)_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain with initial distribution $\mu \otimes \delta_0$ and transition law $\Pi : (X \times \mathbb{R}_+) \times \mathscr{B}_{X\times\mathbb{R}_+} \to [0,1]$ given by

$$\begin{aligned}\Pi((x,s),D) = \int_0^\infty \lambda e^{-\lambda t} \int_\Theta p(S_i(t,y),\theta) \int_{\mathrm{supp}(\nu^\varepsilon)} \Bigg( &\sum_{j\in I} \mathbb{1}_D(w_\theta(S_i(t,y)) + h, j, t) \\ &\times \pi_{ij}(w_\theta(S_i(t,y)) + h) \Bigg) \nu^\varepsilon(dh)\,d\theta\,dt\end{aligned} \tag{2.4}$$

for any $x = (y,i) \in X, s \in \mathbb{R}_+$ and $D \in \mathscr{B}_{X\times\mathbb{R}_+}$. In our further analysis, $(X_n)_{n\in\mathbb{N}_0}$ will be viewed as a suitable canonical Markov chain defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, wherein $\Omega := (X \times \mathbb{R}_+)^{\mathbb{N}_0}, \mathscr{F} := \mathscr{B}_\Omega$ and $\mathbb{P}$ is an appropriate probability measure on $\mathscr{F}$ such that $\mathbb{P}((X_0, \tau_0) \in D) = (\mu \otimes \delta_0)(D)$ for any $D \in \mathscr{B}_{X\times\mathbb{R}_+}$.

Note that $(X_n)_{n\in\mathbb{N}_0}$ itself is also a time-homogeneous Markov chain with transition law $P : X \times \mathscr{B}_X \to [0,1]$ satisfying

$$P(x,A) = \Pi((x,s), A \times \mathbb{R}_+) \text{ for any } x \in X,\ s \in \mathbb{R}_+ \text{ and } A \in \mathscr{B}_X. \tag{2.5}$$

Moreover, we have

$$\Pi((x,s), X \times B) = \int_B \lambda e^{-\lambda t} dt \text{ for any } (x,s) \in X \times \mathbb{R}_+ \text{ and } B \in \mathscr{B}_{\mathbb{R}_+}.$$

Now, define the continuous-time process $(X(t))_{t\in\mathbb{R}_+}$ on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$, by setting

$$X(t) = (Y(t), \xi(t)) = \left(S_{\xi_n}(t - \tau_n, Y_n), \xi_n\right) \text{ for } t \in [\tau_n, \tau_{n+1}). \tag{2.6}$$

One may check that $(X(t))_{t\in\mathbb{R}_+}$ is an $X$-valued time-homogeneous Markov process such that

$$X(\tau_n) = X_n \text{ for any } n \in \mathbb{N}_0.$$

The Markov transition semigroup associated with the process $(X(t))_{t\in\mathbb{R}_+}$ shall be denoted by $(P_t)_{t\in\mathbb{R}_+}$.

Summarizing this part of the paper, let us indicate that, if $X_0$ is distributed according to some measure $\mu \in \mathscr{M}_1(X)$, then we get

$$\begin{aligned} &\mathbb{P}((X_n, \Delta\tau_n) \in D) = (\mu \otimes \delta_0)P^n(D) \text{ for any } D \in \mathscr{B}_{X\times\mathbb{R}_+}, \ n \in \mathbb{N}, \\ &\mathbb{P}(\Delta\tau_n \in B) = \int_B \lambda e^{-\lambda t} dt \text{ for any } B \in \mathscr{B}_{\mathbb{R}_+}, \ n \in \mathbb{N}, \end{aligned} \tag{2.7}$$

$$\mathbb{P}(X(t) \in A) = \mu P_t(A) \text{ for any } A \in \mathscr{B}_X, \ t \in \mathbb{R}_+. \tag{2.8}$$

Let us further assume that there exist a point $\bar{y} \in Y$ and constants $L, \bar{L}, L_w, L_\pi, L_p, C_\pi, C_p \in (0, \infty), r \in (0, 2)$ such that

$$L^{2+r} L_w + (2 + r)\frac{\alpha}{\lambda} < 1, \tag{2.9}$$

and, for all $i, i_1, i_2 \in I, \ y_1, y_2 \in Y, \ t \in \mathbb{R}_+$, the following conditions hold:

$$\sup_{y\in Y} \int_0^\infty e^{-\lambda t} \int_\Theta \|w_\theta(S_i(t, \bar{y})) - \bar{y}\|^{2+r} p(S_i(t, y), \theta)\, d\theta\, dt < \infty, \tag{A1}$$

$$\|S_{i_1}(t, y_1) - S_{i_2}(t, y_2)\| \le L e^{\alpha t}\|y_1 - y_2\| + t\bar{L}\, d(i_1, i_2), \tag{A2}$$

$$\int_\Theta p(y_1, \theta)\|w_\theta(y_1) - w_\theta(y_2)\|^{2+r}\, d\theta \le L_w\|y_1 - y_2\|^{2+r}, \tag{A3}$$

$$\sum_{j\in I} |\pi_{ij}(y_1) - \pi_{ij}(y_2)| \le L_\pi\|y_1 - y_2\|, \quad \int_\Theta |p(y_1, \theta) - p(y_2, \theta)|\, d\theta \le L_p\|y_1 - y_2\|, \tag{A4}$$

$$\sum_{j\in I} \min\{\pi_{i_1,j}(y_1), \pi_{i_2,j}(y_2)\} \ge C_\pi, \quad \int_{\Theta(y_1,y_2)} \min\{p(y_1, \theta), p(y_2, \theta)\}\, d\theta \ge C_p, \tag{A5}$$

where $\Theta(y_1, y_2) := \{\theta \in \Theta : \ \|w_\theta(y_1) - w_\theta(y_2)\| \le L_w\|y_1 - y_2\|\}$. Hypotheses (A1)–(A5) and their reasonableness are discussed in detail e.g. in [6, 16, 30].

### 2.2. Certain properties of the model under consideration

Suppose that hypotheses (A1)–(A5) hold with constants satisfying (2.9). Then [16, Theorem 4.1] implies that the Markov operator $P$, determined by (2.5), is exponentially ergodic in $d_{FM}$ induced by the metric $\varrho_c$, given by (2.3). In fact, the exponential ergodicity itself can be obtained even under slightly weaker assumptions than (A1)–(A5) (cf. [16]). To be more precise (A1), (A3) and (2.9) may be considered in their weaker versions, wherein $r = -1$. However, to establish the LIL, we need them as given in [6] and also in this paper.

Let $\mu_*$ stand for the unique invariant probability measure of $P$, and suppose that the chain $(X_n)_{n\in\mathbb{N}_0}$, governed by $P$, is initially distributed according to $\mu \in \mathcal{M}_{1,2+r}^V(X)$, where $r \in (0,2)$ is the constant appearing in (2.9), and $V : X \to [0,\infty)$ is the Lyapunov function given by

$$V(y,i) = \|y - \bar{y}\| \quad \text{for every} \quad (y,i) \in X, \tag{2.10}$$

where $\bar{y}$ is determined by (A1). Referring to [6, Theorem 4.1], we know that, for any $g \in Lip_b(X)$, the chain $(g(X_n))_{n\in\mathbb{N}_0}$ satisfies the invariance principle for the LIL, and hence also the LIL itself (cf. [6, Section 3.2]), whenever $g$ is not constant $\mu_*$-almost everywhere (a.e.), or, equivalently, it is non-constant on a set of positive measure $\mu_*$.

In [16, Corollary 4.5] we have proven that there is a one-to-one correspondence between invariant measures of the operator $P$ and those of the semigroup $(P_t)_{t\in\mathbb{R}_+}$. This obviously implies that $(P_t)_{t\in\mathbb{R}_+}$ has a unique invariant distribution if and only if $P$ admits the one, which holds, in particular, whenever conditions (A1)–(A5) and (2.9) are satisfied. The above-mentioned correspondence can be described explicitly, using the Markov operators associated with the stochastic kernels $G, W : X \times \mathcal{B}_X \to [0,1]$ defined as follows:

$$G((y,i),A) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{1}_A(S_i(t,y),i)\,dt, \tag{2.11}$$

$$W((y,i),A) = \int_\Theta p(y,\theta) \int_{\text{supp}(\nu^\varepsilon)} \sum_{j\in I} \mathbb{1}_A(w_\theta(y)+h,j)\ \pi_{ij}\big(w_\theta(y)+h\big)\ \nu^\varepsilon(dh)\,d\theta \tag{2.12}$$

for any $(y,i) \in X,\ A \in \mathcal{B}_X$. More precisely [16, Theorem 4.4] says that if $\mu_* \in \mathcal{M}_1(X)$ is an invariant measure of the Markov operator $P$, then $\nu_* := \mu_* G$ is an invariant measure of the Markov semigroup $(P_t)_{t\in\mathbb{R}_+}$. Conversely, if $\nu_* \in \mathcal{M}_1(X)$ is an invariant measure of $(P_t)_{t\in\mathbb{R}_+}$, then $\mu_* := \nu_* W$ is an invariant measure of $P$.

Finally, let us denote the renewal counting process with arrival times $\tau_n$, $n \in \mathbb{N}_0$, by $(N_t)_{t\in\mathbb{R}_+}$, i.e.

$$N_t := \max\{n \in \mathbb{N}_0 : \tau_n \le t\} \ \text{for} \ t \in \mathbb{R}_+. \tag{2.13}$$

## 3. The main result

Consider the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with transition law $P$, given by (2.5), as well as the piecewise deterministic Markov process $(X(t))_{t\in\mathbb{R}_+}$, defined by (2.6), whose transition semigroup has been denoted by $(P_t)_{t\in\mathbb{R}_+}$. Further, recall that under hypotheses (A1)–(A5) and (2.9) both the semigroup $(P_t)_{t\in\mathbb{R}_+}$ and the operator $P$ possess unique invariant distributions, denoted by $\nu_* \in \mathcal{M}_1(X)$ and $\mu_* \in \mathcal{M}_1(X)$, respectively. Moreover, we know that $\nu_* = \mu_* G$, where $G$ is defined in (2.11).

Let $g \in Lip_b(X)$ be an arbitrary function, and define $\bar{g} = g - \langle g, \nu_* \rangle$. Following (1.6) and (1.7), we can introduce

$$s_n(G\bar{g}) = \frac{\sum_{i=0}^{n-1} G\bar{g}(X_i)}{\sqrt{2n \ln(\ln(n))}} \ \text{for} \ n > e, \qquad s_n(G\bar{g}) = 0 \ \text{for} \ n \le e, \tag{3.1}$$

$$s(\bar{g})(t) = \frac{\int_0^t \bar{g}(X(s))ds}{\sqrt{2t\ln(\ln(t))}} \text{ for } t > e, \qquad \text{and} \qquad s(\bar{g})(t) = 0 \text{ for } t \le e. \tag{3.2}$$

We are now ready to state our main result, whose proof is presented in the remainder of the paper.

**Theorem 3.1.** *Suppose that conditions (A1)–(A5) hold with constants satisfying (2.9), and let $\mu^*$ be the unique invariant probability measure of $P$. Then, for any function $g \in Lip_b(X)$ that is not constant $\mu^*$-a.e. and any initial distribution $\mu \in \mathcal{M}_{1,2+r}^V(X)$ with $V$ given by (2.10), the process $(g(X(t)))_{t\in\mathbb{R}_+}$ satisfies the LIL.*

### *3.1. The proof of the main result*

Let $\mu^*$ stand for the unique invariant probability measure for $P$, and let $g \in Lip_b(X)$ be an arbitrary function that is not constant $\mu^*$-a.e. According to the definition introduced in Section 1.3, we need to prove that

$$\limsup_{t\to\infty} s(\bar{g})(t) = \bar{\sigma}(\bar{g}) \text{ and } \liminf_{t\to\infty} s(\bar{g})(t) = -\bar{\sigma}(\bar{g}) \ \mathbb{P}\text{-a.s.}$$

for some $\bar{\sigma}(\bar{g}) \in (0, \infty)$.

Recall that, for any $t \in \mathbb{R}_+$, $N_t$ is given by (2.13). Further, note that whenever $t \ge \tau_3$, which in other words means that $N_t > e$, we have

$$s(\bar{g})(t) = \frac{\sqrt{2N_t\ln(\ln(N_t))}}{\sqrt{2t\ln(\ln(t))}}\left(\frac{1}{\sqrt{2N_t\ln(\ln(N_t))}}\sum_{i=0}^{N_t-1}\int_{\tau_i}^{\tau_{i+1}}\bar{g}(X(s))ds + R_t(\bar{g})\right),$$

where

$$R_t(\bar{g}) := \frac{1}{\sqrt{2N_t\ln(\ln(N_t))}}\int_{\tau_{N_t}}^{t}\bar{g}(X(s))ds.$$

We can further write

$$s(\bar{g})(t) = \frac{\sqrt{N_t\ln(\ln(N_t))}}{\sqrt{t\ln(\ln(t))}}\left(\frac{1}{\sqrt{2N_t\ln(\ln(N_t))}}\sum_{i=0}^{N_t-1}\left(\int_{\tau_i}^{\tau_{i+1}}\bar{g}(X(s))ds - \frac{1}{\lambda}G\bar{g}(X_i)\right)\right.$$
$$\left. + R_t(\bar{g}) + \frac{1}{\lambda}s_{N_t}(G\bar{g})\right), \tag{3.3}$$

where $s_{N_t}(G\bar{g})$ is defined as in (3.1). Referring to the elementary renewal theorem, which says that

$$\lim_{t\to\infty}\frac{N_t}{t} = \lambda \ \mathbb{P}\text{-a.s.}, \tag{3.4}$$

one can easily conclude that

$$\lim_{t\to\infty}\frac{\sqrt{N_t\ln(\ln(N_t))}}{\sqrt{t\ln(\ln(t))}} = \sqrt{\lambda} \ \mathbb{P}\text{-a.s.} \tag{3.5}$$

For any $t \in \mathbb{R}_+$, let $I_j(t)$, $j\in\{1,2,3\}$, be random variables defined by

$$I_1(t) := \frac{1}{\sqrt{2N_t \ln(\ln(N_t))}} \sum_{i=0}^{N_t-1} \left( \int_{\tau_i}^{\tau_{i+1}} \bar{g}(X(s))ds - \frac{1}{\lambda} G\bar{g}(X_i) \right),$$

$$I_2(t) := R_t(\bar{g}), \qquad I_3(t) := \frac{1}{\lambda} s_{N_t}(G\bar{g}),$$

whenever $\tau_3 \le t$. The asymptotic behavior of each of these components shall be analyzed separately.

First of all, we have

$$|R_t(\bar{g})| \le \|\bar{g}\|_\infty \frac{\Delta\tau_{N_t+1}}{\sqrt{2N_t \ln(\ln(N_t))}} \quad \text{for } t \ge \tau_3. \tag{3.6}$$

Observe that the right-hand side of the above inequality tends to zero, as $t \to \infty$. Indeed, note that

$$\sum_{n=3}^{\infty} \mathbb{P}\left( \frac{\Delta\tau_{n+1}}{\sqrt{2n \ln(\ln(n))}} \ge \varepsilon \right) = \sum_{n=3}^{\infty} e^{-\lambda\varepsilon\sqrt{2n\ln(\ln(n))}} < \infty \text{ for any } \quad \varepsilon > 0.$$

Hence, due to the Borel–Cantelli lemma,

$$\lim_{n\to\infty} \frac{\Delta\tau_{n+1}}{\sqrt{2n \ln(\ln(n))}} = 0 \ \mathbb{P}\text{-a.s.},$$

whence also

$$\lim_{t\to\infty} \frac{\Delta\tau_{N_t+1}}{\sqrt{2N_t \ln(\ln(N_t))}} = 0 \ \mathbb{P}\text{-a.s.},$$

which follows from (3.4). Finally, referring to (3.6), we see that

$$\lim_{t\to\infty} I_2(t) = 0 \ \mathbb{P}\text{-a.s.} \tag{3.7}$$

While investigating $I_3$, we shall refer to [6, Theorem 4.2]. Note that the Markov chain $(X_n)_{n\in\mathbb{N}_0}$, for which the sequence $(s_n(G\bar{g}))_{n\in\mathbb{N}_0}$ is defined, satisfies all the assumptions required in [6, Theorem 4.2]. Therefore, the only conditions that need to be proven are $G\bar{g} \in Lip_b(X)$ and $\langle G\bar{g}, \mu_*\rangle = 0$, where the latter follows immediately from the definition of $\bar{g}$ and the fact that $\langle G\bar{g}, \mu_*\rangle = \langle \bar{g}, \nu_*\rangle$ (cf. [16, Theorem 4.4]). Since the boundedness of $G\bar{g}$ is also obvious, it now remains to show its Lipschitz continuity. Note that, according to (A2), for any $(y_1, i_1), (y_2, i_2) \in X$, we have

$$\begin{aligned}
|G\bar{g}(y_1, i_1) - G\bar{g}(y_2, i_2)| &\le \int_0^\infty \lambda e^{-\lambda t} \left| g\left(S_{i_1}(t, y_1), i_1\right) - g\left(S_{i_2}(t, y_2), i_2\right) \right| dt \\
&\le |g|_{Lip} \int_0^\infty \lambda e^{-\lambda t} \left( \|S_{i_1}(t, y_1) - S_{i_2}(t, y_2)\| + cd(i_1, i_2) \right) dt \\
&\le |g|_{Lip} \left( \lambda L \|y_1 - y_2\| \int_0^\infty e^{-(\lambda-\alpha)t} dt + d(i_1, i_2) \bar{L} \int_0^\infty \lambda e^{-\lambda t} t \, dt + cd(i_1, i_2) \right) \\
&= |g|_{Lip} \left( \frac{\lambda L}{\lambda - \alpha} \|y_1 - y_2\| + d(i_1, i_2) \left( \frac{\bar{L}}{\lambda} + c \right) \right) \\
&\le |g|_{Lip} \left( \frac{\lambda L}{\lambda - \alpha} + \frac{\bar{L}}{\lambda} + c \right) \varrho_c((y_1, i_1), (y_2, i_2)),
\end{aligned}$$

which guarantees that $G\bar{g} \in Lip_b(X)$. Therefore, it follows from [6, Theorem 4.2] that

$$\limsup_{n\to\infty} s_n(G\bar{g}) = \sigma(G\bar{g}) \text{ and } \liminf_{n\to\infty} s_n(G\bar{g}) = -\sigma(G\bar{g}) \ \mathbb{P}\text{-a.s.}, \tag{3.8}$$

where, for any function $h \in Lip_b(X)$,

$$\sigma^2(h) = \mathbb{E}_{\mu_*}\left(\left(\sum_{i=0}^{\infty} P^i h(X_1) - \sum_{i=0}^{\infty} P^i h(X_0) + h(X_0)\right)^2\right),$$

and $\mathbb{E}_{\mu_*}$ is the expected value corresponding to the probability measure $\mathbb{P}_{\mu_*}$ defined by $\mathbb{P}_{\mu_*}(F) := \int_X \mathbb{P}(F|X_0 = x)\mu_*(dx)$ for $F \in \mathscr{F}$. Hence, due to (3.8) and (3.4), we obtain

$$\limsup_{t\to\infty} I_3(t) = \frac{1}{\lambda}, \quad \limsup_{t\to\infty} s_{N_t}(G\bar{g}) = \frac{1}{\lambda}\sigma(G\bar{g}) \tag{3.9}$$

and

$$\liminf_{t\to\infty} I_3(t) = -\frac{1}{\lambda}\sigma(G\bar{g})$$

Note that $\sigma(G\bar{g}) < \infty$, which is explained in details in [6].

Finally, to analyze the asymptotic behavior of $I_1$, we need to appeal to [22, Theorem 1], whose assertion guarantees the LIL for certain square integrable martingales. Let us first introduce the sequence $(M_n(\bar{g}))_{n\in\mathbb{N}_0}$ given by

$$M_0(\bar{g}) = 0 \text{ and } M_n(\bar{g}) = \sum_{k=0}^{n-1}\left(\int_{\tau_k}^{\tau_{k+1}} \bar{g}(X(s))\,ds - \frac{1}{\lambda}G\bar{g}(X_k)\right) \text{ for } n \in \mathbb{N}. \tag{3.10}$$

Note that $(M_n(\bar{g}))_{n\in\mathbb{N}_0}$ is a martingale with respect to the natural filtration $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$ of $(X_n, \Delta\tau_n)_{n\in\mathbb{N}_0}$. Indeed, we have

$$\begin{aligned} Z_{n+1}(\bar{g}) := M_{n+1}(\bar{g}) - M_n(\bar{g}) &= \int_{\tau_n}^{\tau_{n+1}} \bar{g}(X(s))ds - \frac{1}{\lambda}G\bar{g}(X_n) \\ &= \int_{\tau_n}^{\tau_{n+1}} \bar{g}\big(S_{\xi_n}(s-\tau_n, Y_n), \xi_n\big)ds - \frac{1}{\lambda}G\bar{g}(Y_n, \xi_n) \\ &= \int_0^{\Delta\tau_{n+1}} \bar{g}\big(S_{\xi_n}(s, Y_n), \xi_n\big)ds - \frac{1}{\lambda}G\bar{g}(Y_n, \xi_n), \end{aligned} \tag{3.11}$$

whence, appealing to (2.7), for any $(y, i) \in X$ and $u \in \mathbb{R}_+$, we get

$$\begin{aligned} \mathbb{E}\big(Z_{n+1}(\bar{g})|Y_n = y, \xi_n = i, \Delta\tau_n = u\big) &= \int_{\mathbb{R}}\int_0^t \bar{g}\big(S_i(s,y), i\big)ds \ \mathbb{P}(\Delta\tau_{n+1} \in dt) \\ &\quad - \frac{1}{\lambda}G\bar{g}(y,i) \\ &= \int_0^\infty \lambda e^{-\lambda t}\int_0^t \bar{g}\big(S_i(s,y), i\big)ds\,dt - \frac{1}{\lambda}G\bar{g}(y,i) \\ &= \int_0^\infty\int_s^\infty \lambda e^{-\lambda t}dt \ \bar{g}\big(S_i(s,y), i\big)ds - \frac{1}{\lambda}G\bar{g}(y,i) \\ &= \int_0^\infty e^{-\lambda s}\bar{g}\big(S_i(s,y), i\big)ds - \frac{1}{\lambda}G\bar{g}(y,i) = 0, \end{aligned}$$

which, by the Markov property of the chain $(X_n, \Delta\tau_n)_{n\in\mathbb{N}_0}$, implies that $(M_n(\bar{g}))_{n\in\mathbb{N}_0}$ is a martingale. Further, applying the Jensen inequality, we obtain

$$\mathbb{E}\Big(Z_{n+1}^2(\bar{g})\Big) \le 2\mathbb{E}\left(\left(\int_0^{\Delta\tau_{n+1}} \bar{g}\big(S_{\xi_n}(s, Y_n), \xi_n\big)ds\right)^2\right) + 2\mathbb{E}\left(\left(\frac{1}{\lambda}G\bar{g}(Y_n, \xi_n)\right)^2\right)$$
$$\le 2\|\bar{g}\|_\infty^2 \mathbb{E}\big((\Delta\tau_{n+1})^2\big) + \frac{3}{\lambda^2}\|\bar{g}\|_\infty^2 = \frac{8}{\lambda^2}\|\bar{g}\|_\infty^2,$$

which means that the martingale increments $Z_n(\bar{g}) = M_n(\bar{g}) - M_{n-1}(\bar{g})$, $n \in \mathbb{N}$, are uniformly bounded in the $\mathscr{L}^2(\mathbb{P})$-norm, and thus the martingale itself is square-integrable, as required in [22, Theorem 1].

Now, define

$$h_n^2(\bar{g}) := \mathbb{E}\big(M_n^2(\bar{g})\big) \text{ for } n \in \mathbb{N}_0.$$

We need to show that $h_n^2(\bar{g}) \to \infty$, as $n\to\infty$, and establish the following conditions:

$$\lim_{n\to\infty} \frac{1}{h_n^2(\bar{g})} \sum_{l=1}^{n} Z_l^2(\bar{g}) = 1 \ \mathbb{P}\text{-a.s.}, \tag{3.12}$$

$$\sum_{n=\bar{n}}^{\infty} h_n^{-4}(\bar{g})\mathbb{E}\Big(Z_n^4(\bar{g})\mathbb{1}_{\{|Z_n(\bar{g})|<\upsilon h_n(\bar{g})\}}\Big) < \infty \text{ for every } \upsilon > 0, \tag{3.13}$$

$$\sum_{n=\bar{n}}^{\infty} h_n^{-1}(\bar{g})\mathbb{E}\Big(|Z_n(\bar{g})|\mathbb{1}_{\{|Z_n(\bar{g})|\ge\vartheta h_n(\bar{g})\}}\Big) < \infty \text{ for every } \vartheta > 0, \tag{3.14}$$

where $\bar{n}$ is so large that $h_n(\bar{g}) > 0$ for any $n \ge \bar{n}$. Then [22, Theorem 1] will imply the LIL for the martingale $(M_n(\bar{g}))_{n\in\mathbb{N}_0}$. To be more precise, according to [22, Theorem 1], the sequence $(M_n(\bar{g}))_{n\in\mathbb{N}_0}$ satisfies the Strassen invariance principle for the LIL with the normalizing factors

$$\frac{1}{\sqrt{2h_n^2(\bar{g})\ln\big(\ln\big(h_n^2(\bar{g})\big)\big)}}, \ n \ge \bar{n}.$$

In particular, it also satisfies the LIL itself, which, in this case, means that

$$\limsup_{n\to\infty} \frac{M_n(\bar{g})}{\sqrt{2h_n^2(\bar{g})\ln\big(\ln\big(h_n^2(\bar{g})\big)\big)}} = 1, \quad \liminf_{n\to\infty} \frac{M_n(\bar{g})}{\sqrt{2h_n^2(\bar{g})\ln\big(\ln\big(h_n^2(\bar{g})\big)\big)}} = -1 \ \mathbb{P}\text{-a.s.},$$

and so, according to (3.4), we further obtain

$$\limsup_{t\to\infty} \frac{M_{N_t}(\bar{g})}{\sqrt{2h_{N_t}^2(\bar{g})\ln\big(\ln\big(h_{N_t}^2(\bar{g})\big)\big)}} = 1 \ \mathbb{P}\text{-a.s.},$$
$$\liminf_{t\to\infty} \frac{M_{N_t}(\bar{g})}{\sqrt{2h_{N_t}^2(\bar{g})\ln\big(\ln\big(h_{N_t}^2(\bar{g})\big)\big)}} = -1 \ \mathbb{P}\text{-a.s.}$$

Let the part of the proof in which we verify the limit $h_n^2(\bar{g}) \to \infty$ and conditions (3.12)–(3.14) be postponed into the subsequent section, namely Section 3.2 In this section, we will also prove that

$$\lim_{t\to\infty}\frac{\sqrt{h_{N_t}^2(\bar{g})\ln\left(\ln\left(h_{N_t}^2(\bar{g})\right)\right)}}{\sqrt{N_t\ln\left(\ln\left(N_t\right)\right)}}=\widetilde{\sigma}(\bar{g})\ \mathbb{P}\text{-a.s.,} \tag{3.15}$$

where

$$\widetilde{\sigma}^2(\bar{g}):=\mathbb{E}_{\mu_*}\left(Z_1^2(\bar{g})\right)=\mathbb{E}_{\mu_*}\left(M_1^2(\bar{g})\right)\in(0,\infty). \tag{3.16}$$

Then, provided that (3.12)–(3.14) and (3.15) are established, we obtain

$$\limsup_{t\to\infty} I_1(t)=\widetilde{\sigma}(\bar{g})\ \text{ and }\ \liminf_{t\to\infty} I_1(t)=-\widetilde{\sigma}(\bar{g})\ \mathbb{P}\text{-a.s.} \tag{3.17}$$

Finally, combining (3.3) with (3.5), (3.7), (3.9) and (3.17), we can conclude

$$\limsup_{t\to\infty} s(\bar{g})(t)=\bar{\sigma}(\bar{g})\ \text{ and }\ \liminf_{t\to\infty} s(\bar{g})(t)=-\bar{\sigma}(\bar{g})\ \mathbb{P}\text{-a.s.,} \tag{3.18}$$

where

$$\bar{\sigma}(\bar{g}):=\sqrt{\lambda}\left(\frac{1}{\lambda}\sigma(G\bar{g})+\widetilde{\sigma}(\bar{g})\right)\in(0,\infty).$$

The proof of Theorem 3.1 is therefore completed (provided that the limit $h_n(\bar{g})\to\infty$ and conditions (3.12)–(3.15) are established, which shall be done in the upcoming section).

### 3.2. The proof of the LIL for an appropriate martingale

Let us consider

$$\mathscr{X}:=\{((x_1,t),(x_2,s))\in(X\times\mathbb{R}_+)^2:\ \ t=s\},$$

and, for any $D\in\mathscr{B}_{X^2}$, define

$$(D)_{\mathscr{X}}:=\left\{((x_1,t),(x_2,t))\in\mathscr{X}:\ \ (x_1,x_2)\in D\right\}.$$

Further, introduce $\widetilde{Q}:\mathscr{X}\times\mathscr{B}_{\mathscr{X}}\to[0,1]$ given by

$$\begin{aligned}\widetilde{Q}\left(((x_1,s),(x_2,s)),Z\right)&=\int_0^\infty\lambda e^{-\lambda t}\int_\Theta\mathbf{p}(x_1,x_2,t,\theta)\\&\times\int_{\operatorname{supp}(\nu^\varepsilon)}\left(\sum_{j\in I}\mathbb{1}_Z\left(\mathbf{w}_j(x_1,x_2,t,\theta,h)\right)\boldsymbol{\pi}_j(x_1,x_2,t,\theta,h)\right)\ \nu^\varepsilon(dh)\,d\theta\,dt\end{aligned} \tag{3.19}$$

for $((x_1,s),(x_2,s))\in\mathscr{X}$ and $Z\in\mathscr{B}_{\mathscr{X}}$, where $x_1=(y_1,i_1)\in X, x_2=(y_2,i_2)\in X$ and

$$\begin{aligned}\mathbf{w}_j(x_1,x_2,t,\theta,h)&:=\left(\left(w_\theta\left(S_{i_1}(t,y_1)+h\right),j,t\right),\left(w_\theta\left(S_{i_2}(t,y_2)+h\right),j,t\right)\right),\\ \boldsymbol{\pi}_j(x_1,x_2,t,\theta,h)&:=\pi_{i_1,j}\left(w_\theta\left(S_{i_1}(t,y_1)\right)+h\right)\wedge\pi_{i_2,j}\left(w_\theta\left(S_{i_2}(t,y_2)\right)+h\right),\\ \mathbf{p}(x_1,x_2,t,\theta)&:=p(S_{i_1}(t,y_1),\theta)\wedge p(S_{i_2}(t,y_2),\theta).\end{aligned}$$

Note that $\widetilde{Q}$ is a substochastic kernel, and, for any $x_1,x_2\in X,\ \ t\in\mathbb{R}_+,\ \ B\in\mathscr{B}_X,$ satisfies the following properties:

$$\begin{aligned}\widetilde{Q}\left(((x_1,t),(x_2,t)),(B\times X)_{\mathscr{X}}\right)&\le\Pi((x_1,t),B\times\mathbb{R}_+),\\ \widetilde{Q}\left(((x_1,t),(x_2,t)),(X\times B)_{\mathscr{X}}\right)&\le\Pi((x_2,t),B\times\mathbb{R}_+).\end{aligned}$$

For any given distribution $\mathbf{m} \in \mathcal{M}_1(X^2)$, on the coordinate space $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ associated with $\mathscr{X}$, we can now construct a probability measure $\widetilde{\mathbb{C}}$ so that

$$\widetilde{\mathbb{C}}\left(\left(\widetilde{X}_0^{(1)}, \widetilde{X}_0^{(2)}\right) \in D,\ \widetilde{\Delta\tau}_0 = 0\right) = \mathbf{m}(D) \text{ for any } D \in \mathscr{B}_{X^2},$$

and the canonical coupling $((\widetilde{X}_n^{(1)}, \widetilde{\Delta\tau}_n), (\widetilde{X}_n^{(2)}, \widetilde{\Delta\tau}_n))_{n\in\mathbb{N}_0}$ of two copies of Markov chain with transition law $\Pi$, defined on the space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \mathbb{C})$, is governed by the transition probability kernel of the form

$$\widetilde{C} = \widetilde{Q} + \widetilde{R},$$

where $\widetilde{Q}$ is defined by (3.19), and $\widetilde{R}$ stands for a complementary substochastic kernel on $\mathscr{X} \times B_{\mathscr{X}}$. The latter can be specified by defining the corresponding family of measures on rectangles $\{(A \times B)_{\mathscr{X}} : A, B \in \mathscr{B}_X\}$ as follows:

$$\begin{aligned}\widetilde{R}\big(((x_1,t),(x_2,t)),(A\times B)_{\mathscr{X}}\big) &= \frac{1}{1-\widetilde{Q}(((x_1,t),(x_2,t)),\mathscr{X})}\\ &\times\Big(\Pi((x_1,t),A\times\mathbb{R}_+) - \widetilde{Q}\big(((x_1,t),(x_2,t)),(A\times X)_{\mathscr{X}}\big)\Big)\\ &\times\Big(\Pi((x_2,t),B\times\mathbb{R}_+) - \widetilde{Q}\big(((x_1,t),(x_2,t)),(X\times B)_{\mathscr{X}}\big)\Big),\end{aligned}$$

when $\widetilde{Q}(((x_1,t),(x_2,t)),\mathscr{X}) < 1$, and $\widetilde{R}(((x_1,t),(x_2,t)),(A\times B)_{\mathscr{X}}) = 0$ otherwise.

Now, define $Q : X^2 \times \mathscr{B}_{X^2} \to [0,1]$ and $C : X^2 \times \mathscr{B}_{X^2} \to [0,1]$ as the kernels which, for any $(x_1, x_2) \in X^2$, $t \in \mathbb{R}_+$ and $D \in \mathscr{B}_{X^2}$, satisfy

$$\begin{aligned}Q\big((x_1,x_2),D\big) &= \widetilde{Q}\big(((x_1,0),(x_2,0)),(D)_{\mathscr{X}}\big) = \widetilde{Q}\big(((x_1,t),(x_2,t)),(D)_{\mathscr{X}}\big),\\ C\big((x_1,x_2),D\big) &= \widetilde{C}\big(((x_1,0),(x_2,0)),(D)_{\mathscr{X}}\big) = \widetilde{C}\big(((x_1,t),(x_2,t)),(D)_{\mathscr{X}}\big).\end{aligned} \tag{3.20}$$

Later on in this paper, we will write $\widetilde{\mathbb{E}}_{x_1,x_2}$ for the expected value corresponding to the measure

$$\widetilde{\mathbb{C}}_{x_1,x_2} := \widetilde{\mathbb{C}}\left(\cdot\middle|\widetilde{X}_0^{(1)} = x_1, \widetilde{X}_0^{(2)} = x_2\right),\quad x_1,x_2 \in X.$$

Let us indicate that the model under consideration enjoys all the hypotheses assumed in [31, Theorem 2.1] (see the proof of [16, Theorem 4.1], where these conditions are verified), which, in particular, means that

(B0) The Markov operator $P$ is Feller.
(B1) There exist constants $a \in (0,1)$ and $b \in (0,\infty)$ such that

$$PV(x) \le aV(x) + b \text{ for every } x \in X,$$

where $V$ is given by (2.10).

Moreover, letting

$$F = \left\{((y_1,i_1),(y_2,i_2)) \in X^2 :\ i_1 = i_2 \text{ or } V((y_1,i_1)) + V((y_2,i_2)) < \frac{4b}{1-a}\right\},$$

the following statements hold:

(B2) We have $\operatorname{supp} Q(x_1, x_2, \cdot) \subset F$ and, for some $\beta \in (0,1)$,

$$\int_{X^2} \varrho_c(u,v)\ Q(x_1,x_2,du \times dv) \le \beta \varrho_c(x_1,x_2) \text{ for any } (x_1,x_2) \in F.$$

(B3) Letting $U(r) := \{(u,v) \in F : \varrho_c(u,v) \le r\}$ for any $r > 0$, we have

$$\inf_{(x_1,x_2)\in F} Q(x_1,x_2,U(\beta\varrho_c(x_1,x_2))) > 0.$$

(B4) There exists $l > 0$ such that

$$Q(x_1,x_2,X^2) \ge 1 - l\varrho_c(x_1,x_2) \quad \text{for every} \quad (x_1,x_2) \in F.$$

(B5) There exist $\gamma \in (0,1)$ and $\widehat{c} > 0$ such that

$$\widetilde{\mathbb{E}}_{x_1,x_2}(\gamma^{-\rho}) \le \widehat{c}, \quad \text{whenever} \quad V(x_1) + V(x_2) < 4b(1-a)^{-1},$$

where $V$ is given by (2.10) and

$$\rho = \inf\left\{n \in \mathbb{N}_0 : \left(\widetilde{X}_n^{(1)}, \widetilde{X}_n^{(2)}\right) \in F \text{ and } V\left(\widetilde{X}_n^{(1)}\right) + V\left(\widetilde{X}_n^{(2)}\right) < \frac{4b}{1-a}\right\}. \tag{3.21}$$

For the martingale $(Z_n(\bar{g}))_{n\in\mathbb{N}}$, given by (3.11), let us consider the sequences of their copies $(\widetilde{Z}_n^{(i)}(\bar{g}))_{n\in\mathbb{N}}$, $i \in \{1,2\}$, defined on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{C}})$ as follows:

$$\widetilde{Z}_n^{(i)}(\bar{g}) = Z_n(\bar{g})\left(\left(\widetilde{X}_0^{(i)}, \widetilde{\Delta\tau}_0\right), \left(\widetilde{X}_1^{(i)}, \widetilde{\Delta\tau}_1\right), \ldots\right) \text{ for } n \in \mathbb{N}_0 \text{ and } i \in \{1,2\}. \tag{3.22}$$

According to [6, Lemmas 3.4 and 3.5], we can now state the following result.

**Lemma 3.2.** Let $g \in Lip_b(X)$ be a function that is not constant $\mu_*$-a.e, and suppose that (3.16) holds. Further, assume that

$$\sum_{n=1}^{\infty} \widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g}) - \widetilde{Z}_n^{(2)}(\bar{g})\right| < \infty \textit{ for all } x_1, x_2 \in X, \tag{3.23}$$

*and that there exists* $r \in (0,2)$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}|Z_n(\bar{g})|^{2+r} < \infty. \tag{3.24}$$

*Then*

$$\lim_{n\to\infty} \frac{h_n^2(\bar{g})}{n} = \widetilde{\sigma}(\bar{g}) > 0, \tag{3.25}$$

*which in turn yields that*

$$\lim_{n\to\infty} h_n^2(\bar{g}) = \infty, \qquad \lim_{n\to\infty} \frac{\sqrt{h_n^2(\bar{g}) \ln\left(\ln\left(h_n^2(\bar{g})\right)\right)}}{\sqrt{n \ln\left(\ln(n)\right)}} = \widetilde{\sigma}(\bar{g}),$$

*and consequently* (3.15) *holds. Moreover, conditions* (3.24) and (3.25) *imply that hypotheses* (3.12)–(3.14) *hold. Hence, due to* [22, Theorem 1], *the martingale* $(M_n(\bar{g}))_{n\in\mathbb{N}_0}$, *given by* (3.10), *satisfies the LIL.*

In view of the above lemma, to finalize the proof of Theorem 3.1, it remains to establish hypotheses (3.16), (3.23) and (3.24).

Let us introduce the function $F(\bar{g}) : X \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$F(\bar{g})(x,t) = \int_0^t \bar{g}\big(S_i(s,y),i\big)ds \text{ for any } x=(y,i)\in X,\ \ t\in\mathbb{R}_+. \tag{3.26}$$

We will first show that condition (3.16) holds. Having in mind the definition of $G$, for every $x=(y,i)\in X$, we can write

$$\begin{aligned}\frac{1}{\lambda}G\bar{g}(x) &= \int_0^\infty e^{-\lambda s}\bar{g}(S_i(t,y),i)\ ds = \int_0^\infty\int_s^\infty \lambda e^{-\lambda t}\bar{g}(S_i(s,y),i)\ dt\ ds\\ &= \int_0^\infty \lambda e^{-\lambda t}\int_0^t \bar{g}(S_i(s,y),i)\ ds\ dt = \int_0^\infty \lambda e^{-\lambda t}F(\bar{g})(x,t)\ dt.\end{aligned}$$

It now follows that, for any $x\in X$,

$$\begin{aligned}\mathbb{E}_x\big[Z_1^2(\bar{g})\big] &:= \mathbb{E}[Z_1^2(\bar{g})\ |\ X_0=x] = \mathbb{E}\Big[\big(F(\bar{g})(x,\Delta\tau_1)-\tfrac{1}{\lambda}G\bar{g}(x)\big)^2\Big]\\ &= \int_0^\infty \lambda e^{-\lambda t}F^2(\bar{g})(x,t)\ dt - \frac{2}{\lambda}G\bar{g}(x)\int_0^\infty \lambda e^{-\lambda t}F(\bar{g})(x,t)\ dt + \frac{1}{\lambda^2}(G\bar{g})^2(x)\\ &= \int_0^\infty \lambda e^{-\lambda t}F^2(\bar{g})(x,t)\ dt - 2\Big(\int_0^\infty \lambda e^{-\lambda t}F(\bar{g})(x,t)\ dt\Big)^2 + \Big(\int_0^\infty \lambda e^{-\lambda t}F(\bar{g})(x,t)\ dt\Big)^2\\ &= \int_0^\infty \lambda e^{-\lambda t}F^2(\bar{g})(x,t)\ dt - \Big(\int_0^\infty \lambda e^{-\lambda t}F(\bar{g})(x,t)\ dt\Big)^2.\end{aligned}$$

Obviously, the right-hand side of the latter equality is 0 if and only if $\mathbb{R}_+\ni t\mapsto F(\bar{g})(x,t)$ is constant almost everywhere. But note that, if $x=(y,i)\in X$ and $\bar{g}(x)=\bar{g}(y,i)\neq 0$, then, due to continuity of $t\mapsto S_i(t,y)$ and the identity $S_i(0,y)=y$, we have $\bar{g}(S_i(s,y),i)\neq 0$ for any $s\in[0,\bar{s})$ and some $\bar{s}>0$. This gives

$$\frac{d}{dt}F(\bar{g})(x,t) = \bar{g}(S_i(t,y),i)\neq 0 \quad \text{for any} \quad t\in(0,\bar{s}),$$

and therefore yields that $t\mapsto F(\bar{g})(x,t)$ is injective on $(0,\bar{s})$. Hence, we have shown that

$$\mathbb{E}_x[Z_1^2(\bar{g})]>0 \quad \text{whenever} \quad \bar{g}(x)\neq 0.$$

From the assumption that $g$ is non-constant $\mu_*$-a.e. it follows that $\bar{g}(x)=g(x)-\langle g,\nu_*\rangle\ /\!= 0$ on some subset of $X$ with positive measure $\mu_*$. This finally shows that

$$\tilde{\sigma}^2(\bar{g}) = \mathbb{E}_{\mu_*}[Z_1^2(\bar{g})] = \int_X \mathbb{E}_x[Z_1^2(\bar{g})]\mu_*(dx)>0.$$

Let us now proceed with the proof of condition (3.23). According to (3.11) and (3.22), we have

$$\begin{aligned}\widetilde{\mathbb{E}}_{x_1,x_2}\Big|\widetilde{Z}_{n+1}^{(1)}(\bar{g})-\widetilde{Z}_{n+1}^{(2)}(\bar{g})\Big| &\le \widetilde{\mathbb{E}}_{x_1,x_2}\Big|F(\bar{g})\Big(\widetilde{X}_n^{(1)},\widetilde{\Delta\tau}_{n+1}\Big)-F(\bar{g})\Big(\widetilde{X}_n^{(2)},\widetilde{\Delta\tau}_{n+1}\Big)\Big|\\ &\quad+\frac{1}{\lambda}\widetilde{\mathbb{E}}_{x_1,x_2}\Big|G\bar{g}\Big(\widetilde{X}_n^{(1)}\Big)-G\bar{g}\Big(\widetilde{X}_n^{(2)}\Big)\Big|,\ \ x_1,x_2\in X.\end{aligned}\tag{3.27}$$

Let us estimate each component on the right-hand side of (3.27) separately. First of all, according to (3.20) and (2.7), for $x_1,x_2\in X,$ we have

$$\widetilde{\mathbb{E}}_{x_1,x_2}\left|F(\bar{g})\left(\widetilde{X}_n^{(1)},\widetilde{\Delta\tau}_{n+1}\right)-F(\bar{g})\left(\widetilde{X}_n^{(2)},\widetilde{\Delta\tau}_{n+1}\right)\right|$$
$$=\int_{X^2}\left(\int_0^\infty \lambda e^{-\lambda t}|F(\bar{g})(u,i,t)-F(\bar{g})(v,j,t)|\,dt\right)C^n\left((x_1,x_2),(du\times di)\times(dv\times dj)\right).$$

Further, according to (3.26), we get

$$\int_0^\infty \lambda e^{-\lambda t}|F(\bar{g})(u,i,t)-F(\bar{g})(v,j,t)|\,dt$$
$$\le\int_0^\infty \lambda e^{-\lambda t}\int_0^t\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|ds\,dt$$
$$=\int_0^\infty\int_s^\infty \lambda e^{-\lambda t}\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|dt\,ds$$
$$=\int_0^\infty\left(\int_s^\infty \lambda e^{-\lambda t}dt\right)\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|ds$$
$$=\int_0^\infty e^{-\lambda s}\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|ds,$$

and therefore, for any $x_1,x_2\in X,$ we obtain

$$\widetilde{\mathbb{E}}_{x_1,x_2}\left|F(\bar{g})\left(\widetilde{X}_n^{(1)},\widetilde{\Delta\tau}_{n+1}\right)-F(\bar{g})\left(\widetilde{X}_n^{(2)},\widetilde{\Delta\tau}_{n+1}\right)\right|$$
$$\le\int_{X^2}\int_0^\infty e^{-\lambda s}\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|ds \tag{3.28}$$
$$\times C^n\left((x_1,x_2),(du\times di)\times(dv\times dj)\right).$$

The second component on the right-hand side of (3.27) can be estimated similarly, i.e.

$$\frac{1}{\lambda}\widetilde{\mathbb{E}}_{x_1,x_2}\left|G\bar{g}\left(\widetilde{X}_n^{(1)}\right)-G\bar{g}\left(\widetilde{X}_n^{(2)}\right)\right|$$
$$\le\int_{X^2}\int_0^\infty e^{-\lambda s}\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|ds \tag{3.29}$$
$$\times C^n\left((x_1,x_2),(du\times di)\times(dv\times dj)\right)\ \text{ for all } x_1,x_2\in X.$$

Combining (3.28) and (3.29), gives

$$\widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g})-\widetilde{Z}_n^{(2)}(\bar{g})\right|$$
$$\le 2\int_{X^2}\int_0^\infty e^{-\lambda s}\left|\bar{g}(S_i(s,u),i)-\bar{g}\left(S_j(s,v),j\right)\right|ds \tag{3.30}$$
$$\times C^n\left((x_1,x_2),(du\times di)\times(dv\times dj)\right)\ \text{ for all } x_1,x_2\in X.$$

Consider $\widehat{\mathscr{X}}:=\widehat{\mathscr{X}}_Q\cup\widehat{\mathscr{X}}_R,$ where $\widehat{\mathscr{X}}_Q:=\mathscr{X}\times\{1\}$ and $\widehat{\mathscr{X}}_R:=\mathscr{X}\times\{0\}$. Then, there exists some probability space $(\widehat{\Omega},\widehat{\mathscr{F}},\widehat{\mathbb{C}}),$ on which we can construct a time-homogeneous canonical Markov chain $((\widehat{X}_n^{(1)},\widehat{\Delta\tau}_n),(\widehat{X}_n^{(2)},\widehat{\Delta\tau}_n),\zeta_n)_{n\in\mathbb{N}_0}$ with $\widehat{\Delta\tau}_0=0$ and $\zeta_0=0,$

evolving on $\widehat{\mathscr{X}}$, and such that its transition law $\widehat{C}$ is given by

$$\widehat{C}(((x_1,t),(x_2,t),\zeta),Z) = \left(\widetilde{Q}(((x_1,t),(x_2,t)),\cdot)\otimes\delta_1\right)(Z) \\ +\left(\widetilde{R}(((x_1,t),(x_2,t)),\cdot)\otimes\delta_0\right)(Z)$$

for $((x_1,t),(x_2,t),\zeta)\in\widehat{\mathscr{X}}$ and $Z\in\mathscr{B}_{\widehat{\mathscr{X}}}$ (cf. e.g. [16, 20, 31]). By convention, we will further write $\widehat{\mathbb{C}}_{x_1,x_2}(\cdot)$ for $\widehat{\mathbb{C}}(\cdot|\widehat{X}_0^{(1)}=x_1,\widehat{X}_0^{(2)}=x_2)$, and we will denote the corresponding expected value by $\widehat{\mathbb{E}}_{x_1,x_2}$, $x_1,x_2\in X$.

Let $\rho$ be given by (3.21), and, for $N\in\mathbb{N}$, define

$$\rho_N := \inf\left\{n\geq N:\ \left(\widehat{X}_n^{(1)},\widehat{X}_n^{(2)}\right)\in F \text{ and } V\left(\widehat{X}_n^{(1)}\right)+V\left(\widehat{X}_n^{(2)}\right)<\frac{4b}{1-a}\right\}.$$

Moreover, introduce

$$\tau := \inf\left\{n\in\mathbb{N}:\ \left(\left(\widehat{X}_k^{(1)},\widehat{\Delta\tau}_k\right),\left(\widehat{X}_k^{(2)},\widehat{\Delta\tau}_k\right),\zeta_k\right)\in\widehat{\mathscr{X}}_Q \text{ for all } k\geq n\right\},$$

and

$$H_{N,n}=\bigcap_{j=N}^{n}\{\zeta_j=1\} \text{ for } n,N\in\mathbb{N} \text{ such that } n>N.$$

Note that, for any $x_1,x_2\in X$,

$$\widehat{\mathbb{C}}_{x_1,x_2}\left(\widehat{\Omega}\backslash H_{N,n}\right)=\widehat{\mathbb{C}}_{x_1,x_2}\left(\bigcup_{j=N}^{n}\{\zeta_j=0\}\right)\leq\widehat{\mathbb{C}}_{x_1,x_2}(\tau>N),\ \ n>N,\ \ n,N\in\mathbb{N}. \tag{3.31}$$

Now, fix $n$, $N$, $M$ such that $n>M>N$ and introduce

$$\widehat{\mathbb{C}}^{n,M,N}_{x_1,x_2}(\cdot) := \widehat{\mathbb{C}}_{x_1,x_2}(\cdot\cap\{\rho_N\leq M\}\cap H_{N,n}) \text{ for any } x_1,x_2\in X.$$

Following the reasoning presented e.g. in [20], and applying the estimate (3.31), we obtain

$$\widehat{\mathbb{C}}_{x_1,x_2}(\cdot)\leq\widehat{\mathbb{C}}^{n,M,N}_{x_1,x_2}(\cdot)+\widehat{\mathbb{C}}_{x_1,x_2}(\cdot\cap\{\rho_N>M\})+\widehat{\mathbb{C}}_{x_1,x_2}\left(\cdot\cap\widehat{\Omega}\backslash H_{N,n}\right),\ \ x_1,x_2\in X,$$

and therefore, using (3.30) and referring to the fact that $\bar{g}\in Lip_b(X)$, we get

$$\begin{aligned}\widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g})-\widetilde{Z}_n^{(2)}(\bar{g})\right| \leq 2|\bar{g}|_{Lip}\int_{X^2}&\left(\int_0^\infty e^{-\lambda s}\varrho_c\big((S_i(s,u),i),(S_j(s,v),j)\big)ds\right)\\ &\times\widehat{\mathbb{C}}^{n,M,N}_{x_1,x_2}\left(\widehat{X}_n^{(1)}\in du\times di,\widehat{X}_n^{(2)}\in dv\times dj\right)\\ &+\frac{4\|\bar{g}\|_\infty}{\lambda}\left(\widehat{\mathbb{C}}_{x_1,x_2}(\rho_N>M)+\widehat{\mathbb{C}}_{x_1,x_2}(\tau>N)\right),\end{aligned} \tag{3.32}$$

where $\varrho_c$ is given by (2.3). Further, condition (A2) implies the following:

$$\begin{aligned}\int_0^\infty & e^{-\lambda s}\varrho_c\big((S_i(s,u),i),(S_j(s,v),j)\big)ds\\&=\int_0^\infty e^{-\lambda s}\big(\|S_i(s,u)-S_j(s,v)\|+cd(i,j)\big)ds\\&\le\int_0^\infty e^{-\lambda s}\big(Le^{\alpha s}\|u-v\|+s\bar{L}d(i,j)+cd(i,j)\big)ds\\&=L\|u-v\|\int_0^\infty e^{-(\lambda-\alpha)s}ds+d(i,j)\int_0^\infty(\bar{L}se^{-\lambda s}+ce^{-\lambda s})ds\\&=\frac{L}{\lambda-\alpha}\|u-v\|+\left(\frac{\bar{L}}{\lambda^2}+\frac{c}{\lambda}\right)d(i,j)\\&\le\left(\frac{L}{\lambda-\alpha}+\frac{\bar{L}}{\lambda^2}+\frac{1}{\lambda}\right)\varrho_c\big((u,i),(v,j)\big).\end{aligned}\tag{3.33}$$

Note that the last inequality holds, since $c\geq 1$. According to (3.32) and (3.33), we obtain, for any $x_1, x_2 \in X$,

$$\begin{aligned}\widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g})-\widetilde{Z}_n^{(2)}(\bar{g})\right| &\le 2|\bar{g}|_{Lip}\left(\frac{L}{\lambda-\alpha}+\frac{\bar{L}}{\lambda^2}+\frac{1}{\lambda}\right)\int_{X^2}\varrho_c\big((u,i),(v,j)\big)\\&\quad\times\widehat{\mathbb{C}}^{n,M,N}_{x_1,x_2}\left(\widehat{X}_n^{(1)}\in du\times di,\widehat{X}_n^{(2)}\in dv\times dj\right)\\&\quad+\frac{4\|\bar{g}\|_\infty}{\lambda}\left(\widehat{\mathbb{C}}_{x_1,x_2}(\rho_N>M)+\widehat{\mathbb{C}}_{x_1,x_2}(\tau>N)\right).\end{aligned}\tag{3.34}$$

Due to [20, Lemma 2.2], there exist constants $c_1, c_2, c_3 > 0$, $q_1, q_2, q_3 \in (0, 1)$ and $p \geq 1$ such that, for any $x_1, x_2 \in X$ and $n, N, M \in \mathbb{N}$ satisfying $n > N > M$, the following inequalities hold:

$$\begin{aligned}\int_{X^2}\varrho_c((u,i),(v,j))\ \widehat{\mathbb{C}}^{n,M,N}_{x_1,x_2}\left(\widehat{X}_n^{(1)}\in du\times di,\widehat{X}_n^{(2)}\in dv\times dj\right)&\le c_1q_1^{n-M},\\ \widehat{\mathbb{C}}_{x_1,x_2}(\rho_N>M)&\le c_2q_2^{M-pN}(1+V(x_1)+V(x_2)),\\ \widehat{\mathbb{C}}_{x_1,x_2}(\tau>N)&\le c_3q_3^N(1+V(x_1)+V(x_2)),\end{aligned}$$

which, together with (3.34), imply that

$$\begin{aligned}\widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g})-\widetilde{Z}_n^{(2)}(\bar{g})\right| &\le 2|\bar{g}|_{Lip}\left(\frac{L}{\lambda-\alpha}+\frac{\bar{L}}{\lambda^2}+\frac{1}{\lambda}\right)c_1q_1^{n-M}\\&\quad+\frac{4\|\bar{g}\|_\infty}{\lambda}\left(c_2q_2^{M-pN}+c_3q_3^N\right)(1+V(x_1)+V(x_2))\\&\le C\|\bar{g}\|_\infty\left(q_1^{n-M}+q_2^{M-pN}+q_3^N\right)(1+V(x_1)+V(x_2))\end{aligned}$$

with

$$C:=2c_1\left(\frac{L}{\lambda-\alpha}+\frac{\bar{L}}{\lambda^2}+\frac{1}{\lambda}\right)+\frac{4}{\lambda}(c_2+c_3).$$

Now, define $n_0=\lceil 4p\rceil$ and fix an arbitrary $n>n_0$. Letting $N=\lfloor n/(4p)\rfloor$ and $M=\lceil n/2\rceil$, we obtain

$$\widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g}) - \widetilde{Z}_n^{(2)}(\bar{g})\right| \le \bar{C}\|\bar{g}\|_{BL} q^n(1+V(x_1)+V(x_2)) \text{ for every } x_1,x_2 \in X,$$

where $\bar{C} := C\max\{q_1^{-1}, q_2^{-1}\}$ and $q := \max\{q_1^{1/2}, q_2^{1/4}, q_3^{1/(4p)}\} \in (0,1)$. Since $\bar{g}$ is bounded, the above estimation also holds (with some $\widehat{C}$ in the place of $\bar{C}$) for $n \le n_0$. We finally get

$$\sum_{n=1}^{\infty}\widetilde{\mathbb{E}}_{x_1,x_2}\left|\widetilde{Z}_n^{(1)}(\bar{g}) - \widetilde{Z}_n^{(2)}(\bar{g})\right| < \infty \text{ for every } x_1,x_2 \in X,$$

which proves (3.23).

It now remains to establish (3.24). Referring to (3.11) and (2.11), for every $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\mathbb{E}|Z_n(\bar{g})|^{2+r} &= \mathbb{E}\left|\int_0^{\Delta\tau_{n+1}} \bar{g}\left(S_{\xi_n}(s,Y_n),\xi_n\right)ds - \frac{1}{\lambda}G\bar{g}(X_n)\right|^{2+r}\\ &= \mathbb{E}\left|\int_0^{\Delta\tau_{n+1}} \bar{g}\left(S_{\xi_n}(s,Y_n),\xi_n\right)ds - \int_0^{\infty} e^{-\lambda s}\bar{g}\left(S_{\xi_n}(s,Y_n),\xi_n\right)ds\right|^{2+r}.\end{aligned}$$

Since $\bar{g}$ is bounded, we further get

$$\mathbb{E}|Z_n(\bar{g})|^{2+r} \le \|\bar{g}\|_{\infty}^{2+r}\mathbb{E}\left(\Delta\tau_{n+1} + \frac{1}{\lambda}\right)^{2+r} \text{ for every } n \in \mathbb{N}.$$

One can easily prove that, for $r>0$, there exists some $\kappa \in (2,\infty)$ such that

$$(\psi_1+\psi_2)^{2+r} \le \kappa\left(\psi_1^{2+r}+\psi_2^{2+r}\right) \text{ for any } \psi_1,\psi_2 \ge 0,$$

whence

$$\mathbb{E}|Z_n(\bar{g})|^{2+r} \le \kappa\|\bar{g}\|_{\infty}^{2+r}\left(\mathbb{E}\left((\Delta\tau_1)^{2+r}\right) + \frac{1}{\lambda^{2+r}}\right) \text{ for every } n \in \mathbb{N},$$

which is finite, due to the fact that $\widetilde{\Delta\tau_1}$ has the exponential distribution. Finally, we get

$$\sup_{n\in\mathbb{N}}\mathbb{E}|Z_n(\bar{g})|^{2+r} < \infty,$$

and the proof is completed.

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