Intermittency and number expansions for random interval maps

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PART II

EXTENSIONS OF LOCHS' THEOREM TO RANDOM SYSTEMS

CHAPTER 5

A Lochs Theorem for random interval maps

This chapter is based on: [KVZ22].

Abstract

In 1964 Lochs proved a theorem on the number of continued fraction digits of a real number x that can be determined from just knowing its first n decimal digits. In 2001 this result was generalised to a dynamical systems setting by Dajani and Fieldsteel, where it compares sizes of cylinder sets for different transformations. In this chapter we prove a version of Lochs' Theorem for a broad class of random dynamical systems and under additional assumptions we prove a corresponding Central Limit Theorem as well. The main ingredient for the proof is an estimate on the asymptotic size of the cylinder sets of the random system in terms of the fiber entropy. To compute this entropy we provide a random version of Rokhlin's formula for entropy.

§5.1 Introduction

§5.1.1 Extension of Lochs' Theorem to number theoretic fibered maps

Real numbers can be represented in many different ways, e.g. by binary, decimal or continued fraction expansions, and one can wonder about the amount of information that each one of these expansions carries. In 1964 Lochs considered a specific question of this form: Given the first n decimal digits of a further unknown irrational number $x \in (0,1)$, what is the largest number $m = m(n,x)$ of regular continued fraction digits of x is that can be determined from this information. Lochs answered this question in [L64] for the limit $n \to \infty$ by showing that for Lebesgue almost every $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{m(n,x)}{n} = \frac{6 \log 2 \log 10}{\pi^2}. \tag{5.1}$$

Over the years Lochs' result has been refined and generalised in many directions. Let λ denote the Lebesgue measure on $[0,1)$. In [F98] Faivre established a Central Limit Theorem associated to Lochs' Theorem:

$$\lim_{n\to\infty} \lambda\Bigg(\Big\{x \in (0,1) : \frac{m(n,x) - n\frac{6\log 2 \log 10}{\pi^2}}{\sigma\sqrt{n}} \le u\Big\}\Bigg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} dt \tag{5.2}$$

holds for some constant $\sigma > 0$. See [F97, F01, W06, W08] for other results related to the limit in (5.1) and [LW08, BI08, FWL16, FWL19] for results where the decimal expansions in (5.1) are replaced by β-expansions.

In [BDK99] Bosma, Dajani and Kraaikamp highlighted that Lochs' Theorem can be seen as a dynamical statement. This viewpoint was further developed in [DF01], where Dajani and Fieldsteel gave the dynamical equivalent of the local limit statement from (5.1) for what they called *number theoretic fibered maps (NTFM)*. An NTFM is a triple (T, μ, α) where $T : [0,1) \to [0,1)$ is a surjective map, μ is a Borel measure on $[0,1)$ and $\alpha = \{A_j : j \in D\}$ is an at most countable interval partition of $[0,1)$ indexed by some set D, such that

(n1) $T|_{A_j}$ is continuous and strictly monotone for each $j \in D$;

(n2) μ is an ergodic invariant probability measure for T that is equivalent to λ with a density that is bounded and bounded away from zero;

(n3) the partition α generates the Borel σ-algebra $\mathcal{B}$ on $[0,1)$ in the sense that if for each n we use

$$\alpha_n = \bigvee_{k=0}^{n-1} T^{-k}\alpha = \{A_{j_1} \cap T^{-1}A_{j_2} \cap \cdots \cap T^{-(n-1)}A_{j_n} \,:\, A_{j_k} \in \alpha,\, 1 \le k \le n\} \tag{5.3}$$

to denote the *level n cylinders* of T, then the smallest σ-algebra containing all these sets for all $n \ge 1$, denoted by $\sigma\big(\bigcup_{n\in\mathbb{N}} \alpha_n\big)$, equals $\mathcal{B}$ up to sets of Lebesgue measure zero;

(n4) the entropy $-\sum_{A\in\alpha}\mu(A)\log\mu(A)$ of α with respect to μ is finite.

The name NTFM refers to the fact that an NTFM generates for each $x \in [0,1)$ a digit sequence $(d_n^T(x))_{n\geq 1}$ with digits in D by setting

$$d_n^T(x) = j \quad \text{if } T^{n-1}x \in A_j,\ j \in D, \tag{5.4}$$

and for certain maps T these sequences correspond to well-known number expansions. The procedure from (5.4) can in words be described as follows: After assigning digit j to subinterval A_j for each $j \in D$, for each $x \in [0,1)$ the digit sequence from (5.4) is obtained by writing down in order the digits corresponding to the elements of α that the orbit of x under T visits.

Example 5.1.1 (N-adic transformations). Let $T_N : [0,1) \to [0,1)$ with integer $N \geq 2$ be the N-adic transformation from Example 1.3.1, λ the Lebesgue measure on $[0,1)$ and $\alpha = \{A_j : j \in \{0,1,\dots,N-1\}\}$ the partition given by

$$A_j = \Big[\frac{j}{N}, \frac{j+1}{N}\Big), \quad j \in \{0,1,\dots,N-1\}. \tag{5.5}$$

Then (T_N, λ, α) is an NTFM and digit sequences $(d_n^{T_N}(x))_{n\geq 1}$ in $\{0,1,\dots,N-1\}^{\mathbb{N}}$ can then be obtained by following the procedure from (5.4). This is illustrated in Figure 5.1 for the case that $N = 2$. For each $x \in [0,1)$ this sequence $(d_n^{T_N}(x))_{n\geq 1}$ yields the *expansion in integer base* N of x given by

$$x = \sum_{n=1}^{\infty} \frac{d_n^{T_N}(x)}{N^n}. \tag{5.6}$$

Indeed, note that

$$T_N^n(x) = N \cdot T_N^{n-1}(x) - d_n^{T_N}(x)$$

holds for each $n \in \mathbb{N}$, so that recursively, this gives

$$x = \frac{d_1^{T_N}(x)}{N} + \frac{d_2^{T_N}(x)}{N^2} + \cdots + \frac{d_n^{T_N}(x)}{N^n} + \frac{T_N^n(x)}{N^n}.$$

Since $0 \leq T_N^n x \leq 1$, this yields (5.6) by taking $n \to \infty$.

Example 5.1.2 (Gauss map). Let $G : [0,1) \to [0,1)$ and μ_G be the Gauss map and Gauss probability measure from Example 1.3.2, respectively, and let $\alpha = \{A_j : j \in \mathbb{N}\}$ be the partition of $(0,1]$ given by

$$A_j = \Big(\frac{1}{j+1}, \frac{1}{j}\Big], \quad j \in \mathbb{N}.$$

The triple (G, μ_G, α) is an NTFM and for each irrational $x \in [0,1)$ the digit sequence $(d_n^G(x))_{n\geq 1}$ in $\mathbb{N}^{\mathbb{N}}$ generated as in (5.4) gives the *regular continued fraction expansion*

$$x = \cfrac{1}{d_1^G(x) + \cfrac{1}{d_2^G(x) + \cfrac{1}{d_3^G(x) + \ddots}}}.$$

See e.g. [DK21, Section 8.1] for a justification.

Chapter 5

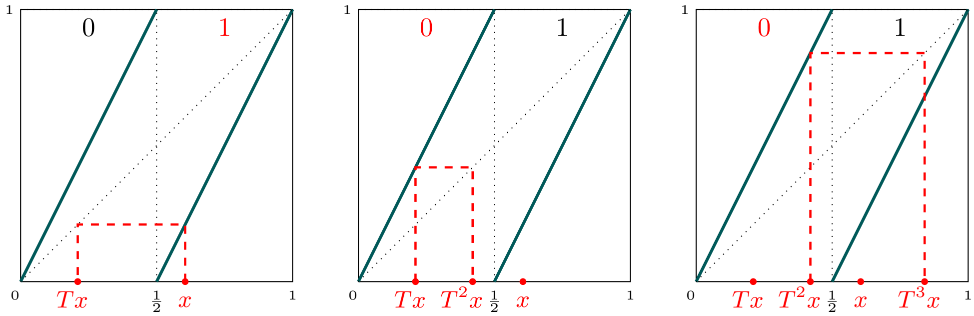


Figure 5.1: Illustration of the generation of digit sequences $(d_n^T(x))_{n\geq 1}$ given by (5.4) for the doubling map. The point x depicted satisfies $d_1^T(x) = 1$, $d_2^T(x) = 0$, $d_3^T(x) = 0$, etc.

Other examples of number expansions that can be obtained by an NTFM include β-expansions, various Lüroth-type expansions and various types of continued fraction expansions.

If for two NTFM's (T, μ, α) and $(S, \tilde{\mu}, \gamma)$ we define the number

$$m_{T,S}(n, x) = \sup\{m \in \mathbb{N} : \alpha_n(x) \subseteq \gamma_m(x)\},$$

where $\alpha_n(x)$ and $\gamma_m(x)$ denote the elements of the partitions α_n and γ_m as in (5.3) that contain x, respectively, then one can interpret $m_{T,S}(n, x)$ as the largest m so that the level m cylinder for S containing x can be determined from knowing only the level n cylinder for T that contains x. Equivalently, if we use $(d_k^T(x))_{k\geq 1}$ and $(d_k^S(x))_{k\geq 1}$ to denote the digit sequences produced by T and S, $m_{T,S}(n, x)$ is the largest m such that the digits $d_1^S(x), \dots, d_m^S(x)$ can be determined from knowing $d_1^T(x), \dots, d_n^T(x)$ of a further unknown $x \in [0, 1)$. The authors of [DF01] proved that for any two NTFM's (T, μ, α) and $(S, \tilde{\mu}, \gamma)$ with measure theoretic entropies $h_\mu(T), h_{\tilde{\mu}}(S) > 0$ it holds that

$$\lim_{n\to\infty} \frac{m_{T,S}(n, x)}{n} = \frac{h_\mu(T)}{h_{\tilde{\mu}}(S)} \qquad \lambda\text{-a.e.} \tag{5.7}$$

Lochs' original result given in (5.1) can be recovered by taking for T the map T_N from Example 5.1.1 with $N = 10$ and for S the Gauss map S from Example 5.1.2. Similar to the result in (5.2) by Faivre, Herczegh [H09] proved a Central Limit Theorem for the statement in (5.7) for a specific class of pairs of NTFM's. For two NTFM's (T, μ, α) and $(S, \tilde{\mu}, \gamma)$ that satisfy $h_\mu(T), h_{\tilde{\mu}}(S) > 0$ and several additional conditions, he proved in [H09, Corollary 2.1] that for each $u \in \mathbb{R}$,

$$\lim_{n\to\infty} \tilde{\mu}\left(\left\{x \in (0,1) : \frac{m_{T,S}(n,x) - n\frac{h_\mu(T)}{h_{\tilde{\mu}}(S)}}{\sigma\sqrt{n}} \leq u\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt \tag{5.8}$$

for some appropriate constant $\sigma > 0$.

§5.1.2 Number expansions generated by random interval maps

Random interval maps that generate number expansions have received increasing attention in the past few decades. This is partly due to the fact that often such systems generate for a typical real number not just one, but uncountably many different number expansions of a given type. We describe the general procedure to obtain such expansions.

Let $\mathcal{T} = \{T_i : [0,1) \to [0,1)\}_{i\in I}$ be a family of interval maps on $[0,1)$ and suppose for each $i \in I$ there is a partition α_i of $[0,1)$ into finitely or countably many subintervals. For each $i \in I$ we assign to each subinterval of α_i a digit and write D_i for the collection of these digits. We then write $A_{i,j}$ for the subinterval corresponding to the digit $j \in D_i$, so that for each $i \in I$ we can write $\alpha_i = \{A_{i,j} : j \in D_i\}$. We now define digit sequences by following random orbits and at each time step recording, before a map T_i is applied, the digit that corresponds to the partition element of α_i in which the random orbit at that moment is located. More precisely, for each $(\omega, x) \in I^{\mathbb{N}} \times [0,1)$ we define the digit sequence $(d_n^{\mathcal{T}}(\omega,x))_{n\geq 1}$ where $d_n^{\mathcal{T}}(\omega,x) \in D_{\omega_n}$ is the label of the partition element of α_{ω_n} in which $T_\omega^{n-1}(x)$ lies, i.e.

$$d_n^{\mathcal{T}}(\omega,x) = j \qquad \text{if } T_\omega^{n-1}(x) \in A_{\omega_n,j}, \quad j \in D_{\omega_n}. \tag{5.9}$$

We clarify this method with an example.

Example 5.1.3 (Random integer base transformations). Let $I = \{2,3\}$ and $\mathcal{T} = \{T_i\}_{i\in I}$ where $T_2, T_3 : [0,1] \to [0,1]$ are as in Example 5.1.1, i.e. $T_2(x) = 2x \bmod 1$ and $T_3(x) = 3x \bmod 1$. For the method described above, we associate to T_2 and T_3 the partitions $\alpha_2 = \{A_{2,0}, A_{2,1}\}$ and $\alpha_3 = \{A_{3,0}, A_{3,1}, A_{3,2}\}$ given by (5.5), meaning that before applying T_2 we assign digits 0 and 1 to $[0,\frac{1}{2})$ and $[\frac{1}{2},1)$, respectively, and before applying T_3 we assign digits 0,1 and 2 to $[0,\frac{1}{3})$, $[\frac{1}{3},\frac{2}{3})$ and $[\frac{2}{3},1)$, respectively. For each $(\omega,x) \in I^{\mathbb{N}} \times [0,1]$ we then obtain a digit sequence $(d_n^{\mathcal{T}}(\omega,x))_{n\geq 1}$ as given by (5.9). This is illustrated in Figure 5.2. For each $(\omega,x) \in I^{\mathbb{N}} \times [0,1]$ and $n \in \mathbb{N}$ we have

$$T_\omega^n(x) = \omega_n \cdot T_\omega^{n-1}(x) - d_n^{\mathcal{T}}(\omega,x),$$

so that similar as in Example 5.1.1 this iteratively yields

$$x = \frac{d_1^{\mathcal{T}}(\omega,x)}{\omega_1} + \frac{d_2^{\mathcal{T}}(\omega,x)}{\omega_1\omega_2} + \cdots + \frac{d_n^{\mathcal{T}}(\omega,x)}{\omega_1\cdots\omega_n} + \frac{T_\omega^n(x)}{\omega_1\cdots\omega_n}.$$

From $\lim_{n\to\infty} \left|\frac{T_\omega^n(x)}{\omega_1\cdots\omega_n}\right| \leq \lim_{n\to\infty} \frac{1}{2^n} = 0$, it follows that

$$x = \sum_{n=1}^{\infty} \frac{d_n^{\mathcal{T}}(\omega,x)}{2^{c_n(\omega)}3^{n-c_n(\omega)}},$$

where $c_n(\omega) = \#\{1 \leq k \leq n : \omega_k = 2\}$. Thus, the procedure from (5.9) with $\mathcal{T} = \{T_2, T_3\}$ generates number expansions in mixed integer base 2 and 3.

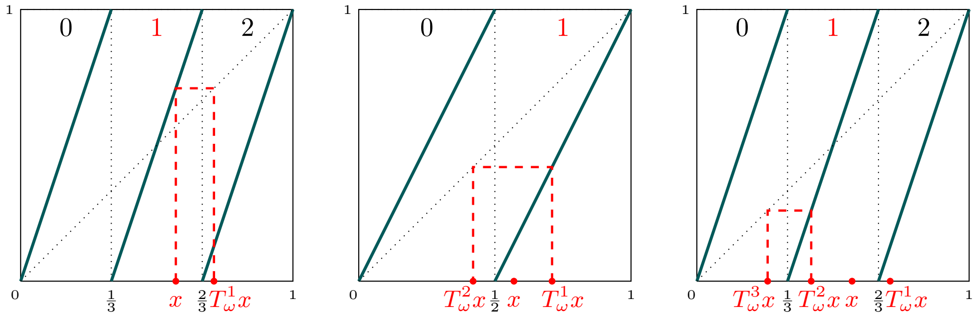


Figure 5.2: Illustration of the generation of digit sequences $(d_n^{\mathcal{T}}(\omega, x))_{n\geq 1}$ given by (5.9) for the random map from Example 5.1.3 given by random compositions of T_2 and T_3. The point x depicted satisfies, for $\omega = (3, 2, 3, \dots)$, $d_1^{\mathcal{T}}(\omega, x) = 1$, $d_2^{\mathcal{T}}(\omega, x) = 1$, $d_3^{\mathcal{T}}(\omega, x) = 1$, etc.

There are various other random dynamical systems like the above example related to number expansions. The random β-transformation was first introduced in [DK03] and then further investigated in [DdV05, DdV07, DK07, DK13, K14, BD17, DJ17, S19, DKM21]. Interesting features of this system are its relation to Bernoulli convolutions, see e.g. [JSS11, DK13, K14], and to β-encoders, see Chapter 6 and e.g. [DDGV02, DGWY10, G12, KHTA12, JMKA13, MIS$^+$15, SJO15, JM16] and the references therein. A random system producing binary expansions was studied in [DK20], random dynamical systems related to continued fraction expansions appear in [KKV17, DO18, BRS20, AFGTV21, DKM21, KMTV22] and random Lüroth maps are considered in [KM22a, KM22b].

In this chapter we extend the version of Lochs' Theorem in (5.7) to the setting of random dynamical systems and give a corresponding Central Limit Theorem in the spirit of (5.8). The class of random dynamical systems we consider, which we call *random number systems*, contains the class of deterministic NTFM's and all of the random dynamical systems related to number expansions mentioned above. A random number system consists of a family of maps $\mathcal{T} = \{T_i : [0, 1) \to [0, 1)\}_{i\in I}$, where the index set I is a possibly uncountable Polish space, each map $T_i : [0, 1) \to [0, 1)$, $i \in I$, admits an appropriate partition $\alpha_i = \{A_{i,0}, A_{i,1}, \dots\}$ of $[0, 1)$ and there exists an appropriate probability measure μ on $I^{\mathbb{N}} \times [0, 1)$. (The precise definition will be given in Section 5.2.) Thus a random number system is a quintuple $(I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$, where $\mathbb{P}$ is the probability law on $I^{\mathbb{N}}$ determining the random choices. For ease of notation we also write $\mathcal{T}$ for this quintuple.

Generating a digit sequence for x by means of (5.9) gives information about the location of x in the following way:

$$(d_1^{\mathcal{T}}(\omega, x), \dots, d_n^{\mathcal{T}}(\omega, x)) = (j_1, \dots, j_n) \quad \Rightarrow \quad x \in \bigcap_{k=1}^{n} T^{-1}_{\omega_1\cdots\omega_{k-1}} A_{\omega_k, j_k}. \tag{5.10}$$

As the right side shows, this information about the location of x provided by the digit sequence is under the assumption that $\omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}$ is known. This setting is interesting from a number theoretic point of view and will be investigated in this chapter, but for more practical applications where $\omega \in I^{\mathbb{N}}$ models e.g. some noise in the system, it is not very feasible to assume that ω is known or can be observed with full accuracy. In such cases the following model might be more realistic:

$$(j_1, \ldots, j_n) \text{ observed for unknown } x \quad \Rightarrow \quad x \in \bigcup_{(\omega_1,\ldots,\omega_n)\in I^n} \bigcap_{k=1}^{n} T^{-1}_{\omega_1\cdots\omega_{k-1}} A_{\omega_k, j_k}, \tag{5.11}$$

where $A_{i,j} = \emptyset$ if $j \notin D_i$. We consider the setting of (5.11) in Chapter 6 for a specific random interval map that models the A/D conversion in a β-encoder and where fluctuations in the system are due to noise and thus unknown, see footnote 1 on page 164. Finally, note that the models (5.10) and (5.11) coincide if the corresponding digit sets $\{D_i\}_{i\in I}$ are pairwise disjoint.

§5.1.3 Main results

For a random number system $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ we define analogous to (5.3) the *random level n cylinders* as follows: For each $\omega \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$ we define the partition

$$\begin{aligned}\alpha_{\omega,n} &= \bigvee_{k=0}^{n-1} (T^k_\omega)^{-1} \alpha_{\omega_{k+1}} \\ &= \{A_{\omega_1,j_1} \cap T^{-1}_{\omega_1} A_{\omega_2,j_2} \cap \cdots \cap T^{-1}_{\omega_1\cdots\omega_{n-1}} A_{\omega_n,j_n} \;:\; A_{\omega_k,j_k} \in \alpha_{\omega_k},\ 1 \le k \le n\}.\end{aligned} \tag{5.12}$$

Furthermore, we write $\alpha_{\omega,n}(x)$ for the random (n,ω)-cylinder that contains x. Given two random number systems $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ and $\mathcal{S} = (J, \mathbb{Q}, \{S_j\}_{j\in J}, \rho, \{\gamma_j\}_{j\in J})$, for each $n \in \mathbb{N}$, $\omega \in I^{\mathbb{N}}$, $\tilde{\omega} \in J^{\mathbb{N}}$ and $x \in [0,1)$, let

$$m_{\mathcal{T},\mathcal{S}}(n, \omega, \tilde{\omega}, x) = \sup\{m \in \mathbb{N} : \alpha_{\omega,n}(x) \subseteq \gamma_{\tilde{\omega},m}(x)\}. \tag{5.13}$$

This quantity can be interpreted as follows: For given $\omega \in I^{\mathbb{N}}$ and $\tilde{\omega} \in J^{\mathbb{N}}$, $m_{\mathcal{T},\mathcal{S}}(n, \omega, \tilde{\omega}, x)$ is the largest level m for which we can determine the random $(m, \tilde{\omega})$-cylinder for $\mathcal{S}$ containing x from knowing only the random (n,ω)-cylinder for $\mathcal{T}$ that contains x. Alternatively, it is the largest m such that $d^{\mathcal{S}}_1(\tilde{\omega}, x), \ldots, d^{\mathcal{S}}_m(\tilde{\omega}, x)$ can be determined from knowing the digits $d^{\mathcal{T}}_1(\omega, x), \ldots, d^{\mathcal{T}}_n(\omega, x)$ of a further unknown $x \in [0,1)$. In this chapter we obtain the following Random Lochs' Theorem, where the measure theoretic entropy from (5.7) is replaced by *fiber entropy*, which for a random number system $\mathcal{T}$ is a quantity $h^{\text{fib}}(\mathcal{T}) \in [0,\infty)$ and will be defined in Section 5.5.

Theorem 5.1.4. *Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$, $\mathcal{S} = (J, \mathbb{Q}, \{S_j\}_{j\in J}, \rho, \{\gamma_j\}_{j\in J})$ be two random number systems. If $h^{\text{fib}}(\mathcal{T}), h^{\text{fib}}(\mathcal{S}) > 0$, then*

$$\lim_{n\to\infty} \frac{m_{\mathcal{T},\mathcal{S}}(n, \omega, \tilde{\omega}, x)}{n} = \frac{h^{\text{fib}}(\mathcal{T})}{h^{\text{fib}}(\mathcal{S})} \qquad \lambda\text{-a.e.}$$

for $\mathbb{P} \times \mathbb{Q}$-a.a. $(\omega, \tilde{\omega}) \in I^{\mathbb{N}} \times J^{\mathbb{N}}$.

We like to make two remarks about this result. Firstly, the quotient of measure theoretic entropies that appears as the value of the limit in the deterministic setting has been replaced by a quotient of fiber entropies in the random setting. Secondly, the setup allows for the index set I of the family $\{T_i\}_{i\in I}$ to be uncountable, so that the results apply to e.g. random β-transformations where the value of β can range over a whole interval, see Example 5.7.5 below. This makes the proofs more involved.

In [DF01] an essential ingredient to prove (5.7) is the following general result on interval partitions. If $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a sequence of interval partitions and $c \geq 0$, we say that $\mathcal{P}$ *has entropy* c λ*-a.e.* if

$$\lim_{n\to\infty} -\frac{\log \lambda\big(P_n(x)\big)}{n} = c \qquad \lambda\text{-a.e.},$$

where $P_n(x)$ denotes the element of the partition P_n containing x.

Theorem 5.1.5 (Theorem 4 of [DF01]). *Let $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ and $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ be two sequences of interval partitions. For each $n \in \mathbb{N}$ and $x \in [0,1)$, put*

$$m_{\mathcal{P},\mathcal{Q}}(n,x) = \sup\{m \in \mathbb{N} : P_n(x) \subseteq Q_m(x)\}.$$

Suppose that $\mathcal{P}$ has entropy $c \in (0,\infty)$ λ-a.e. and $\mathcal{Q}$ has entropy $d \in (0,\infty)$ λ-a.e. Then

$$\lim_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} = \frac{c}{d} \qquad \lambda\text{-a.e.}$$

The proof of (5.7) goes roughly along the following lines. An application of the Kolmogorov-Sinai Theorem and of the Shannon-McMillan-Breiman Theorem to the NTFM's T and S provides the appropriate asymptotics for the size of the cylinder sets from (5.3) for both maps T and S to establish the positive entropy conditions and then Theorem 5.1.5 completes the proof. To achieve Theorem 5.1.4 we also employ Theorem 5.1.5 and therefore the main achievement here is obtaining the right asymptotics for the size of the random cylinder sets from (5.12). More precisely Theorem 5.1.4 will appear as a corollary of the following theorem.

Theorem 5.1.6. *Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system. The following hold,*

(i) For $\mathbb{P}$-a.a. $\omega \in I^{\mathbb{N}}$ we have

$$\lim_{n\to\infty} -\frac{\log \lambda\big(\alpha_{\omega,n}(x)\big)}{n} = h^{\mathrm{fib}}(\mathcal{T}), \qquad \lambda\text{-a.e.}$$

(ii) Let ν denote the marginal of μ on $I^{\mathbb{N}}$. Furthermore, let F be the skew product on $I^{\mathbb{N}} \times [0,1)$ given by $F(\omega,x) = (\tau\omega, T_{\omega_1}(x))$, where τ denotes the left shift on $I^{\mathbb{N}}$. If $h_\nu(\tau) < \infty$, then

$$h^{\mathrm{fib}}(\mathcal{T}) = h_\mu(F) - h_\nu(\tau).$$

(iii) If for each $i \in I$ and $A \in \alpha_i$ the restriction $T_i|_A$ is differentiable, then

$$h^{\mathrm{fib}}(\mathcal{T}) = \int_{I^{\mathbb{N}} \times [0,1)} \log |DT_{\omega_1}(x)| \, d\mu(\omega, x).$$

The first part of this theorem gives the required estimates for the asymptotic sizes of the cylinder sets from (5.12) and, when combined with Theorem 5.1.5, leads to Theorem 5.1.4. The limit from Theorem 5.1.4 is expressed in terms of the fiber entropies of the two random number systems. Parts (ii) and (iii) of Theorem 5.1.6 give different ways to determine this limit. The second part works in case the entropy of the marginal of μ on $I^{\mathbb{N}}$ is finite. The third part gives a random version of Rokhlin's Formula for entropy.

We also prove a Central Limit Theorem for Theorem 5.1.4 in case we compare the digits obtained from a random number system $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i \in I}, \mu, \{\alpha_i\}_{i \in I})$ to those from an NTFM $(S, \tilde{\mu}, \gamma)$ under additional assumptions on both systems. To be more specific, for such systems we obtain that for all $u \in \mathbb{R}$,

$$\lim_{n \to \infty} \mu\Big(\Big\{(\omega, x) \in I^{\mathbb{N}} \times [0,1) : \frac{m_{\mathcal{T},S}(n, \omega, x) - n\frac{h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)}}{\kappa\sqrt{n}} \leq u\Big\}\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} dt, \tag{5.14}$$

for an appropriate constant $\kappa > 0$.

The remainder of this chapter is organised as follows. In Section 5.2 we give a precise definition of random number systems. We then consider in Section 5.3 a special but wide class of random number systems for which the proof of Theorem 5.1.6 is relatively easy. In Sections 5.4 and 5.5 we provide some preliminaries for the proof of the general case and give a precise definition of fiber entropy. We prove Theorem 5.1.6 in Section 5.6 and obtain Theorem 5.1.4 as a corollary and we prove the Central Limit Theorem from (5.14). In Section 5.7 we provide some examples.

§5.2 Random number systems

In this section we define the dynamical systems that we are interested in. Let $(I^{\mathbb{N}}, \mathcal{B}_I^{\mathbb{N}}, \mathbb{P})$ be a base space where I is a Polish space with associated Borel σ-algebra $\mathcal{B}_I$ and where $\mathbb{P}$ is a Borel probability measure on the product σ-algebra $\mathcal{B}_I^{\mathbb{N}}$ such that the left shift τ on $I^{\mathbb{N}}$ is non-singular with respect to $\mathbb{P}$, i.e. $\mathbb{P}(\tau^{-1}A) = 0$ if and only if $\mathbb{P}(A) = 0$ for all $A \in \mathcal{B}_I^{\mathbb{N}}$. For each $i \in I$, let $T_i : [0,1) \to [0,1)$ be a Borel measurable transformation. Let $\mathcal{B}$ denote the Borel σ-algebra on $[0,1)$ and λ the Lebesgue measure on $[0,1)$. Associated to the family $\{T_i : [0,1) \to [0,1)\}_{i \in I}$ let F be the skew product transformation

$$F : I^{\mathbb{N}} \times [0,1) \to I^{\mathbb{N}} \times [0,1), \; (\omega, x) \mapsto \big(\tau\omega, T_{\omega_1}(x)\big).$$

Let μ be an invariant probability measure for F on $I^{\mathbb{N}} \times [0,1)$. For each $i \in I$ let $\alpha_i = \{A_{i,0}, A_{i,1}, \ldots\}$ be a partition of $[0,1)$ by countably many subintervals of

$[0,1)$, possibly containing empty sets.[1] A *random number system* is a collection $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ on $[0,1)$ that satisfies the following conditions.

(r1) The map $I \times [0,1) \ni (i,x) \mapsto T_i(x) \in [0,1)$ is measurable.

(r2) For each $i \in I$ and $A \in \alpha_i$, $T_i|_A$ is strictly monotone and continuous.

(r3) The partition $\Delta = \{\Delta(j) : j \geq 0\}$ of $I^{\mathbb{N}} \times [0,1)$ given by

$$\Delta(j) = \{(\omega, x) : x \in A_{\omega_1,j}\} = \bigcup_{i\in I} [i] \times A_{i,j} \quad \text{for each } j \geq 0 \tag{5.15}$$

is measurable, i.e. $\Delta(j) \in \mathcal{B}_I^{\mathbb{N}} \times \mathcal{B}$ for all $j \geq 0$.

(r4) For $\mathbb{P}$-a.a. $\omega \in I^{\mathbb{N}}$ we have that, for all $B \in \mathcal{B}$, $\lambda(T_{\omega_1}^{-1}B) = 0$ if $\lambda(B) = 0$.

(r5) For $\mathbb{P}$-a.a. $\omega \in I^{\mathbb{N}}$ we have $\sigma\big(\bigcup_{n\in\mathbb{N}} \alpha_{\omega,n}\big) = \mathcal{B}$ up to sets of λ-measure zero.

(r6) The F-invariant measure μ is ergodic and equivalent to $\mathbb{P} \times \lambda$.

(r7) The entropy of Δ w.r.t. μ, i.e. $H_\mu(\Delta) = -\sum_{j\geq 0} \mu(\Delta(j)) \log \mu(\Delta(j))$, is finite.

Most of the conditions (r1)–(r7) are easily verified in specific applications and not very restrictive. We give some comments on them.

- Conditions (r1) and (r3) are typical measurability conditions and are immediate in case I is at most countable (and equipped with the discrete topology). It easily follows from (r1) that the skew product F is measurable.

- Condition (r2) is needed to get digit sequences $(d_n^{\mathcal{T}}(\omega,x))_{n\geq 1}$ as in (5.9). It follows from (r5) that, for $\mathbb{P}$-a.a. $\omega \in I^{\mathbb{N}}$, knowing $(d_n^{\mathcal{T}}(\omega,x))_{n\geq 1}$ determines $x \in [0,1)$ uniquely λ-a.e.

- Condition (r4) is a form of fiberwise non-singularity and from (r6) it follows that μ is the only probability measure that is both F-invariant and absolutely continuous w.r.t. $\mathbb{P} \times \lambda$ as can be seen from Theorem 1.2.6. In case I is countable, then it is easy to verify that (r4) already follows from only assuming (r6).

- If we let $\pi_I : I^{\mathbb{N}} \times [0,1) \to I^{\mathbb{N}}$ be the canonical projection onto the first coordinate and write $\nu = \mu \circ \pi_I^{-1}$ for the marginal of the invariant measure μ on $I^{\mathbb{N}}$, then from (r6) it follows that ν is τ-invariant, ergodic and equivalent to $\mathbb{P}$. In particular, again by Theorem 1.2.6, if $\mathbb{P}$ is τ-invariant, then $\nu = \mathbb{P}$.

- Condition (r7) guarantees that the fiber entropy defined later on is well defined. Note that if Δ is a finite set (that is, if Δ contains a finite number of non-empty elements), then (r7) is automatically satisfied.

We now present some classes of systems for which the assumptions from the definition of random number system are satisfied.

[1] For notational convenience we take $\{0,1,2,\ldots\}$ as digit set for each T_i. If for a map T_i the digit set would naturally be a finite set, then we take for α_i a collection that contains countably many empty sets.

- A class of maps that satisfy (r2) and (r4) and that are well studied is the class of *Lasota-Yorke type maps.* A Lasota-Yorke type map is a map $T : [0,1) \to [0,1)$ that is piecewise monotone C^2 and non-singular with $|DT(x)| > 0$ for all x where the derivative is defined. In that case an obvious candidate for the partition from (r2) is the partition of $[0,1)$ given by the maximal intervals on which T is monotone.

- Given a family $\{T_i\}_{i\in I}$ of Lasota-Yorke type maps for some appropriate index set I, a sufficient condition for Δ to be a generator in the sense of (r5) is that $\inf_{(i,x)} |DT_i(x)| > 1$. In case I is finite, this is equivalent to the condition that each T_i is expanding. We can allow for neutral fixed points as well and still get (r5) if we assume that the branches of the maps are full, i.e. map onto the whole interval $(0,1)$, and expanding outside each neighborhood of the neutral fixed point. Examples include the random Gauss-Rényi map from Example 1.4.2 that we again will encounter in Example 5.7.4 below and random Manneville-Pomeau maps.

- There exist various sets of conditions under which the existence of an invariant measure μ for the skew product F that satisfies (r6) is guaranteed. See Section 1.4 and the references mentioned there for some results in this direction for the case that $\mathbb{P}$ is a Bernoulli measure.

- The results from [KM22a] give an algorithm for determining explicit formulae for invariant probability measures of the form $\mathbb{P} \times \rho$ with $\rho \ll \lambda$ in case all maps T_i are piecewise linear Lasota-Yorke type maps satisfying some further conditions. Having an explicit formula facilitates the computation of the entropy of Δ and the verification of (r7).

Remark 5.2.1. If I consists of only one element, then the random number system reduces to an interval map. In this case, conditions (r2), (r5) and (r7) are equivalent with assuming that (n1), (n3) and (n4) hold for this interval map, respectively. Moreover, it follows from (r6) that the interval map is onto $[0,1)$ up to some λ-measure zero set and it follows from (r6) that this interval map satisfies (n2) except that the density does not necessarily have to be bounded and bounded away from zero. Thus in particular, each NTFM is an example of a random number system where the index set consists of only one element. Furthermore, note that in case $I = \{1\}$ contains only one element, then $h_\nu(\tau) = 0$ and Theorem 5.1.6(ii) gives that $h^{\text{fib}}(\mathcal{T}) = h_\rho(T_1)$, where ρ is the ergodic invariant probability measure for T_1 equivalent to λ. Hence, Theorem 5.1.4 is an extension of the result in (5.7) from [DF01] and shows that (5.7) remains true for two NTFM's for which the condition in (n2) on the bounds on the density is dropped.

§5.3 A special case

Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system. In this section we put the following additional four conditions on $\mathcal{T}$:

(s1) The index set I is countable.

(s2) Writing $\iota = \{[i] : i \in I\}$ for the countable partition of $I^{\mathbb{N}}$ given by the 1-cylinders, we assume that the entropy of ι w.r.t. the marginal ν of μ on $I^{\mathbb{N}}$ is finite, i.e. $H_\nu(\iota) < \infty$.

(s3) For all $\omega \in I^{\mathbb{N}}$ we have $\sigma\big(\bigcup_{n\in\mathbb{N}} \alpha_{\omega,n}\big) = \mathcal{B}$.

(s4) The density $\frac{d\mu}{d\nu\times\lambda}$ is bounded and bounded away from zero.

Note that (s3) is a slight strengthening of (r5). A class of random interval maps can be derived from [P84, I12, M85a, M85b, P84] that satisfy the conditions of a random number system as well as the additional conditions (s1)-(s4). These random interval maps are expanding on average and are composed of Lasota-Yorke type maps such that the corresponding invariant density is bounded, and admit a suitable covering property such that the density is bounded away from zero, thus satisfying (s4). Moreover, the base space then consists of a finite or countable index set I equipped with a Bernoulli measure or Markov measure.

Let $\mathcal{T}$ be as above satisfying (s1)-(s4). We will prove Theorem 5.1.6 for $\mathcal{T}$. It is clear that ι is a generator for the left shift τ on $I^{\mathbb{N}}$ w.r.t. ν. Furthermore, we define $\bar{\iota} = \{[i]\times[0,1) : i \in I\}$ being the partition ι embedded into $I^{\mathbb{N}}\times[0,1)$, and we define the countable partition ξ of $I^{\mathbb{N}}\times[0,1)$ given by

$$\xi = \big\{[i]\times A_{i,j} : i \in I, j \in \mathbb{N}_0\big\},$$

where as before $A_{i,j}$ are the partition elements of α_i. Note that ξ is the common refinement of $\bar{\iota}$ and the partition Δ given by (5.15), i.e. $\xi = \bar{\iota}\vee\Delta$.

Lemma 5.3.1. *The partition ξ is a generator for F.*

Proof. We write $\xi_n = \bigvee_{k=0}^{n-1} F^{-k}\xi$ and $\iota_n = \bigvee_{k=0}^{n-1}\tau^{-k}\iota$ for each $n\in\mathbb{N}$. Then for each $(\omega,x)\in I^{\mathbb{N}}\times[0,1)$ and $n\in\mathbb{N}$ we have

$$\xi_n(\omega,x) = [\omega_1\cdots\omega_n]\times\alpha_{\omega,n}(x) = \iota_n(\omega)\times\alpha_{\omega,n}(x), \tag{5.16}$$

where $\xi_n(\omega,x)$ denotes the partition element of ξ_n containing x, and a similar meaning for $\iota_n(\omega)$ and $\alpha_{\omega,n}(x)$. Let $(\omega,x),(\tilde{\omega},y)\in I^{\mathbb{N}}\times[0,1)$. If $\omega\neq\tilde{\omega}$, then there exists $n\in\mathbb{N}$ such that $\iota_n(\omega)\neq\iota_n(\tilde{\omega})$ and thus $\xi_n(\omega,x)\neq\xi_n(\tilde{\omega},y)$. If $\omega=\tilde{\omega}$ and $x\neq y$, then according to (s3) there exists $n\in\mathbb{N}$ such that $\alpha_{\omega,n}(x)\neq\alpha_{\omega,n}(y)$ and thus $\xi_n(\omega,x)\neq\xi_n(\tilde{\omega},y)$. Hence $\{\xi_n\}$ separates points, so ξ is a generator for F. □

It follows from (s2) that $h_\nu(\tau)<\infty$. Furthermore, we obtain from (s2) and (r7) that

$$H_\mu(\xi) \le H_\mu(\bar{\iota}) + H_\mu(\Delta) = H_\nu(\iota) + H_\mu(\Delta) < \infty,$$

so that by Lemma 5.3.1 we have $h_\mu(F)<\infty$. We have the following two results:

Proposition 5.3.2. *For $\mathbb{P}$-a.a. $\omega\in I^{\mathbb{N}}$ we have*

$$\lim_{n\to\infty} -\frac{\log\lambda\big(\alpha_{\omega,n}(x)\big)}{n} = h_\mu(F) - h_\nu(\tau), \qquad \lambda\text{-a.e.}$$

Proof. Combining Lemma 5.3.1 and $H_\mu(\xi) < \infty$ with the Kolmogorov-Sinai Theorem (Theorem 1.2.17) and the Shannon-McMillan-Breiman Theorem (Theorem 1.2.22) yields that

$$\lim_{n\to\infty} -\frac{\log \mu\big(\xi_n(\omega,x)\big)}{n} = h_\mu(F), \qquad \mu\text{-a.e.} \tag{5.17}$$

By (r6) we can replace in (5.17) μ-a.e. with $\mathbb{P}\times\lambda$-a.e. and by (s4) we can replace $\mu\big(\xi_n(\omega,x)\big)$ with $\nu\times\lambda\big(\xi_n(\omega,x)\big)$, i.e.

$$\lim_{n\to\infty} -\frac{\log \nu\times\lambda\big(\xi_n(\omega,x)\big)}{n} = h_\mu(F), \qquad \mathbb{P}\times\lambda\text{-a.e.} \tag{5.18}$$

Also, we obtain from the Kolmogorov-Sinai Theorem and the Shannon-McMillan-Breiman Theorem together with the equivalence between ν and $\mathbb{P}$ that

$$\lim_{n\to\infty} -\frac{\log \nu\big(\iota_n(\omega)\big)}{n} = h_\nu(\tau), \qquad \mathbb{P}\text{-a.e.} \tag{5.19}$$

Combining (5.16), (5.18) and (5.19) now yields the result. □

Proposition 5.3.3. *If for each $i \in I$ and $A \in \alpha_i$ the restriction $T_i|_A$ is differentiable, then*

$$h_\mu(F) = \int_{I^{\mathbb{N}}\times[0,1)} \log|DT_{\omega_1}(x)|d\mu(\omega,x) + h_\nu(\tau).$$

Proof. Note that the partition ι of $I^{\mathbb{N}}$ consists of invertibility domains of τ. It follows from the Rokhlin Formula (Theorem 1.2.21) that

$$h_\nu(\tau) = \int_{I^{\mathbb{N}}} \log J_\nu\tau d\nu,$$

where $J_\nu\tau$ is the Jacobian of τ w.r.t. ν. Furthermore, note that ξ is a partition of $I^{\mathbb{N}}\times[0,1)$ by invertibility domains of F. For each $i \in I$, $j \in \mathbb{N}_0$, $C \in \mathcal{B}_I^{\mathbb{N}} \cap [i]$ and $D \in \mathcal{B} \cap A^T_{i,j}$ we have

$$\begin{aligned}\nu\times\lambda(F(C\times D)) &= \nu\times\lambda(\tau(C)\times T_i(D))\\ &= \int_{C\times D} J_\nu\tau(\omega)|DT_{\omega_1}(x)|d\nu\times\lambda(\omega,x).\end{aligned}$$

Using standard arguments we can show from this that for each $A \in \xi$ and each measurable $B \subseteq A$ we have

$$\nu\times\lambda(F(B)) = \int_B J_\nu\tau(\omega)|DT_{\omega_1}(x)|d\nu\times\lambda(\omega,x),$$

so the Jacobian $J_{\nu\times\lambda}F$ of F w.r.t. $\nu\times\lambda$ exists and is given by

$$J_{\nu\times\lambda}F = J_\nu\tau(\omega)|DT_{\omega_1}(x)|, \qquad \text{for } \nu\times\lambda\text{-a.e. } (\omega,x)\in I^{\mathbb{N}}\times[0,1).$$

Furthermore, by the change of variables formula from Lemma 1.2.20(a) this gives for each $A \in \xi$ and each measurable $B \subseteq A$ that

$$\begin{aligned}\mu(F(B)) &= \int_{F(B)} \frac{d\mu}{d\nu\times\lambda} d\nu\times\lambda \\ &= \int_B \Big(\frac{d\mu}{d\nu\times\lambda}\circ F\Big) J_{\nu\times\lambda}F \frac{d\nu\times\lambda}{d\mu} d\mu,\end{aligned}$$

so the Jacobian $J_\mu F$ of F w.r.t. μ exists and is given by

$$J_\mu F = \big(\frac{d\mu}{d\nu\times\lambda}\circ F\big) J_{\nu\times\lambda}F \frac{d\nu\times\lambda}{d\mu}, \qquad \mu\text{-a.e.}$$

Using Lemma 5.3.1 and $H_\mu(\xi) < \infty$, it follows from the Rokhlin Formula that

$$h_\mu(F) = \int_{I^{\mathbb{N}}\times[0,1)} \log J_\mu F(\omega,x) d\mu(\omega,x).$$

We conclude that

$$\begin{aligned}h_\mu(F) &= \int \log\Big(\frac{d\mu}{d\nu\times\lambda}\circ F\Big)d\mu + \int \log\big(J_{\nu\times\lambda}F\big)d\mu + \int \log\Big(\frac{d\nu\times\lambda}{d\mu}\Big)d\mu \\ &= \int \log\Big(\frac{d\mu}{d\nu\times\lambda}\frac{d\nu\times\lambda}{d\mu}\Big)d\mu + \int \log|DT_{\omega_1}(x)|d\mu(\omega,x) \\ &\quad + \int \log\big(J_\nu\tau(\omega)\big)d\mu(\omega,x) \\ &= \int \log|DT_{\omega_1}(x)|d\mu(\omega,x) + h_\nu(\tau).\end{aligned}$$

So the above two propositions prove Theorem 5.1.6 in the special case that $\mathcal{T}$ satisfies (s1)-(s4).

Remark 5.3.4. We used condition (s4) for replacing $\mu\big(\xi_n(\omega,x)\big)$ in (5.17) with $\nu\times\lambda\big(\xi_n(\omega,x)\big)$. For this purpose, instead of assuming (s4), it is also sufficient to assume that $\nu\times\lambda$-a.a. $(\omega,x)\in I^{\mathbb{N}}\times[0,1)$ is a *density point* for $\frac{d\mu}{d\nu\times\lambda}$ w.r.t. $\nu\times\lambda$, i.e.

$$\lim_{r\downarrow 0} \frac{1}{\nu\times\lambda(B((\omega,x),r))}\int_{B((\omega,x),r)} \frac{d\mu}{d\nu\times\lambda} d\nu\times\lambda = \frac{d\mu}{d\nu\times\lambda}(\omega,x), \tag{5.20}$$

where $B((\omega,x),r)$ denotes the open ball centered at (ω,x) with radius $r>0$ with respect to a compatible metric, e.g. the one of the form as in (3.3). Then combined with Lemma 5.3.1 it follows that for $\mathbb{P}\times\lambda$-a.a. (ω,x) and every $\varepsilon>0$ there exists an $N\in\mathbb{N}$ such that for each integer $n\geq N$ we have

$$\frac{d\mu}{d\nu\times\lambda}(\omega,x)-\varepsilon \leq \frac{\mu\big(\xi_n(\omega,x)\big)}{\nu\times\lambda\big(\xi_n(\omega,x)\big)} \leq \frac{d\mu}{d\nu\times\lambda}(\omega,x)+\varepsilon.$$

Using this and the fact that $\frac{d\mu}{d\nu\times\lambda}(\omega,x)\in(0,\infty)$ holds for $\mathbb{P}\times\lambda$-a.e. $(\omega,x)\in I^{\mathbb{N}}\times[0,1)$ by (r6), we can indeed then replace $\mu\big(\xi_n(\omega,x)\big)$ in (5.17) with $\nu\times\lambda\big(\xi_n(\omega,x)\big)$. An example when (5.20) is satisfied $\nu \times \lambda$-a.e. is when

$$\limsup_{r\downarrow 0} \frac{\nu \times \lambda\big(B((\omega,x),2r)\big)}{\nu \times \lambda\big(B(x,r)\big)} < \infty, \qquad \nu \times \lambda\text{-a.e. } (\omega,x) \in I^{\mathbb{N}} \times [0,1),$$

see e.g. [HKST15, Section 3.4]. This is for instance the case if $\mathbb{P}$ is a Bernoulli measure or Markov measure (recall that $\nu = \mathbb{P}$ if $\mathbb{P}$ is τ-invariant).

Note that the reasoning in this section does not work for proving Theorem 5.1.6 in full generality, because the classical Kolmogorov-Sinai Theorem, Shannon-McMillan-Breiman Theorem and Rokhlin Formula as applied in this section require a partition that is finite or countable with finite entropy. To overcome this problem, we will apply instead a fiberwise version of those theorems. However, the fiberwise Shannon-McMillan-Breiman Theorem and Rokhlin Formula we will use require that the underlying base map of the skew product is invertible. So we first need to extend the dynamics on the base space so that it becomes invertible, which we will do in the next section.

§5.4 Invertible base maps and invariant measures

Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system. One of the consequences of (r1)–(r7) is that $\mathcal{T}$ admits a *system of conditional measures*, i.e. a family of measures $\{\mu_\omega\}_{\omega\in I^{\mathbb{N}}}$ such that

- μ_ω is a probability measure on $([0,1),\mathcal{B})$ for ν-a.a. $\omega \in I^{\mathbb{N}}$,
- for any $f \in L^1(I^{\mathbb{N}} \times [0,1), \mu)$ the map $\omega \mapsto \int_{[0,1)} f(\omega,x)\, d\mu_\omega(x)$ on $I^{\mathbb{N}}$ is measurable and

$$\int_{I^{\mathbb{N}}\times[0,1)} f\, d\mu = \int_{I^{\mathbb{N}}} \Big(\int_{[0,1)} f(\omega,x)\, d\mu_\omega(x) \Big) d\nu(\omega). \tag{5.21}$$

Moreover, if $\{\tilde{\mu}_\omega\}_{\omega\in I^{\mathbb{N}}}$ is another system of conditional measures for μ, then $\mu_\omega = \tilde{\mu}_\omega$ for ν-a.a. $\omega \in I^{\mathbb{N}}$. (See [A97, Theorem 1.0.8] together with [VO16, Proposition 5.1.7] for a justification.)

The dynamics on the base space I of a random number system is given by the left shift τ on the set $I^{\mathbb{N}}$, which is not invertible. This setup corresponds to the setup for random systems associated to number expansions that is adopted in most of the references mentioned in the introduction. To prove Theorem 5.1.4, however, we employ known theory on random systems and fiber entropy that is available for skew products with invertible dynamics on the base space. One can easily extend the one-sided shift in the first coordinate of F to a two-sided (thus invertible) shift and as we shall see next, this has no profound effect on the invariant measures.

Let $\hat{\tau}$ denote the left shift on $I^{\mathbb{Z}}$ and extend the skew product F to a map $\hat{F}$ that is invertible in the first coordinate by setting

$$\hat{F} : I^{\mathbb{Z}} \times [0,1) \to I^{\mathbb{Z}} \times [0,1),\ (\hat{\omega}, x) \mapsto (\hat{\tau}(\hat{\omega}), T_{\hat{\omega}_1}(x)).$$

Let $\mathcal{B}_I^{\mathbb{Z}}$ denote the Borel σ-algebra on $I^{\mathbb{Z}}$. Use $\pi : I^{\mathbb{Z}} \to I^{\mathbb{N}}$ to denote the canonical projection. To keep notation simple for two-sided sequences $\hat{\omega} \in I^{\mathbb{Z}}$ and $n \geq 0$ we use the same notation for $T_{\hat{\omega}}^n$, $\alpha_{\hat{\omega},n}$ and $\mu_{\hat{\omega}}$ as for one-sided sequences, i.e.

$$T_{\hat{\omega}}^n = T_{\pi(\hat{\omega})}^n, \quad \alpha_{\hat{\omega},n} = \alpha_{\pi(\hat{\omega}),n}, \quad \mu_{\hat{\omega}} = \mu_{\pi(\hat{\omega})}.$$

The skew product $\hat{F}$ is measurable due to (r1). The next proposition gives a relation between the (ergodic) invariant measures of F and those of $\hat{F}$. It can be found in a slightly more restrictive setting in Appendix A of [GH17] (see also the references therein), but the proof carries over unchanged to our setting. We reproduce the statement here for our setting for convenience. Use $\hat{\pi}_I : I^{\mathbb{Z}} \times [0,1) \to I^{\mathbb{Z}}$ and $\Pi : I^{\mathbb{Z}} \times [0,1) \to I^{\mathbb{N}} \times [0,1)$ to denote the respective canonical projections.

Proposition 5.4.1 ([GH17, Proposition A.1 and Remark A.2]). *Let μ be an F-invariant probability measure with marginal $\nu = \mu \circ \pi_I^{-1}$ and system of conditional measures $\{\mu_\omega\}_{\omega \in I^{\mathbb{N}}}$. Then the following statements hold.*

(i) There exists an $\hat{F}$-invariant probability measure $\hat{\mu}$ with marginal $\hat{\nu} = \hat{\mu} \circ \hat{\pi}_I^{-1}$ and a system of conditional measures $\{\hat{\mu}_{\hat{\omega}}\}_{\hat{\omega} \in I^{\mathbb{Z}}}$ such that, for $\hat{\nu}$-a.a. $\hat{\omega} \in I^{\mathbb{Z}}$,

$$\hat{\mu}_{\hat{\omega}}(B) = \lim_{n \to \infty} \mu_{\hat{\tau}^{-n}\hat{\omega}}\Big((T_{\hat{\tau}^{-n}\hat{\omega}}^n)^{-1}(B)\Big), \qquad B \in \mathcal{B}.$$

(ii) Conversely, let $\hat{\mu}$ be an $\hat{F}$-invariant probability measure with marginal $\hat{\nu} = \hat{\mu} \circ \hat{\pi}_I^{-1}$. Then the probability measure

$$\tilde{\mu} = \hat{\mu} \circ \Pi^{-1}$$

is F-invariant and has marginal $\tilde{\nu} = \hat{\nu} \circ \pi^{-1}$.

(iii) The correspondence $\mu \leftrightarrow \hat{\mu}$ given by (i) and (ii) is one-to-one and has the property that μ is ergodic for F if and only if $\hat{\mu}$ is ergodic for $\hat{F}$.

From Proposition 5.4.1 and (r6) we obtain an $\hat{F}$-invariant and ergodic probability measure $\hat{\mu}$ with a system of conditional measures $\{\hat{\mu}_{\hat{\omega}}\}_{\hat{\omega} \in I^{\mathbb{Z}}}$ for the marginal $\hat{\nu} = \hat{\mu} \circ \hat{\pi}_I^{-1}$. The following lemma will be used later.

Lemma 5.4.2. *For $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$ it holds that $\hat{\mu}_{\hat{\omega}} \ll \lambda$. Moreover, $\hat{\mu} \ll \hat{\nu} \times \lambda$ and for $\hat{\nu} \times \lambda$-a.e. $(\hat{\omega}, x) \in I^{\mathbb{Z}} \times [0,1)$ we have*

$$\frac{d\hat{\mu}}{d\hat{\nu} \times \lambda}(\hat{\omega}, x) = \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x).$$

Proof. Combining that τ is non-singular with respect to $\mathbb{P}$ with (r4) gives that for each $n \geq 1$,

$$\mathbb{P}(\{\omega \in I^{\mathbb{N}} : \exists B \in \mathcal{B} \quad \lambda(B) = 0 \text{ and } \lambda((T_\omega^n)^{-1}B) > 0\}) = 0.$$

Recall that ν and $\mathbb{P}$ are equivalent. Furthermore, the measure $\hat{\nu}$ is invariant with respect to $\hat{\tau}$ and $\hat{\nu} \circ \pi^{-1} = \nu$. This gives for each $m \in \mathbb{Z}$ that

$$\hat{\nu}(\hat{\tau}^{-m}\pi^{-1}\{\omega \in I^{\mathbb{N}} : \exists B \in \mathcal{B} \quad \lambda(B) = 0 \text{ and } \lambda((T_\omega^n)^{-1}B) > 0\}) = 0.$$

Taking $m = -n$ it follows for $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$ and $n \geq 1$ that

$$\lambda(B) = 0 \quad \Rightarrow \quad \lambda((T^n_{\hat{\tau}^{-n}(\hat{\omega})})^{-1}B) = 0, \qquad \forall B \in \mathcal{B}. \tag{5.22}$$

The measures μ and $\nu \times \lambda$ are equivalent by condition (r6). Let $A \in \mathcal{B}_I^{\mathbb{N}}$ and $B \in \mathcal{B}$. Then by (5.21)

$$\int_A \mu_\omega(B)\, d\nu(\omega) = \mu(A \times B) = \int_A \int_B \frac{d\mu}{d\nu \times \lambda}(\omega, x)\, d\lambda(x)\, d\nu(\omega),$$

which means that for any $B \in \mathcal{B}$ we can find a ν-null set N_B, such that for all $\omega \in I^{\mathbb{N}} \setminus N_B$ we have

$$\mu_\omega(B) = \int_B \frac{d\mu}{d\nu \times \lambda}(\omega, x)\, d\lambda(x).$$

Since $\mathcal{B}$ is countably generated, we can find a ν-null set N, such that for each $\omega \in I^{\mathbb{N}} \setminus N$ and all $B \in \mathcal{B}$,

$$\mu_\omega(B) = \int_B \frac{d\mu}{d\nu \times \lambda}(\omega, x)\, d\lambda(x).$$

Hence, $\mu_\omega \ll \lambda$ for ν-a.e. $\omega \in I^{\mathbb{N}}$ and for those ω,

$$\frac{d\mu_\omega}{d\lambda}(x) = \frac{d\mu}{d\nu \times \lambda}(\omega, x) \qquad \lambda\text{-a.e.}$$

It immediately follows that $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$ satisfies $\mu_{\hat{\omega}} \ll \lambda$ and since $\hat{\nu}$ is $\hat{\tau}$-invariant, we get that for $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$, $\mu_{\hat{\tau}^{-n}\hat{\omega}} \ll \lambda$ for each n. Combining this with (5.22) and Proposition 5.4.1(i) gives that $\hat{\mu}_{\hat{\omega}} \ll \lambda$ for $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$. Since $\{\hat{\mu}_{\hat{\omega}}\}$ is a system of conditional invariant measures, we get that for each $A \in \mathcal{B}_I^{\mathbb{Z}} \times \mathcal{B}$ (cf. (5.21)),

$$\hat{\mu}(A) = \int_{I^{\mathbb{Z}}} \hat{\mu}_{\hat{\omega}}(A_{\hat{\omega}})\, d\hat{\nu}(\hat{\omega}) = \int_A \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x)\, d\hat{\nu} \times \lambda(\hat{\omega}, x),$$

where $A_{\hat{\omega}} = \{x \in [0,1) : (\hat{\omega}, x) \in A\}$. This means that $\hat{\mu} \ll \hat{\nu} \times \lambda$ and that for $\hat{\nu} \times \lambda$-a.e. $(\hat{\omega}, x)$ it holds that $\frac{d\hat{\mu}}{d\hat{\nu} \times \lambda}(\hat{\omega}, x) = \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x)$. □

§5.5 Fiber entropy

The concept of fiber entropy was introduced in [AR66]. Here, as well as in the later works [B82] and [M86], the entropy of a skew product is studied for the case that the associated transformations are all measure preserving with respect to the same measure. In [K86] and [LY88], the notion fiber entropy is considered for skew products of transformations with a Bernoulli measure on the base space. These two settings are extended in the works [B92] and [BC92], where the invariant measure of the skew product admits a system of conditional measures.

Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system. Here we introduce fiber entropy for $\mathcal{T}$ following the approach of Bogenschütz [B93]. Two standing assumptions in [B93] (and one of the reasons why we extended F to $\hat{F}$) are that the dynamics on the base space are invertible and that the σ-algebra considered on the first coordinate is countably generated. By our definition of $\hat{F}$ and the assumption that I is a Polish space, we satisfy both these assumptions.

Consider the sub-σ-algebra $\mathcal{A} := \mathcal{B}_I^{\mathbb{Z}} \times [0,1)$ of $\mathcal{B}_I^{\mathbb{Z}} \times \mathcal{B}$. From $\hat{\tau} \circ \hat{\pi}_I = \hat{\pi}_I \circ \hat{F}$ we see that $\hat{F}^{-1}\mathcal{A} \subseteq \mathcal{A}$ and in this situation we can define the *conditional entropy* of a partition $\mathcal{P}$ of $I^{\mathbb{Z}} \times [0,1)$ given $\mathcal{A}$ as in [B93] using the conditional expectation by

$$H_{\hat{\mu}}(\mathcal{P}|\mathcal{A}) = -\int_{I^{\mathbb{Z}}\times[0,1)} \sum_{P\in\mathcal{P}} \mathbb{E}_{\hat{\mu}}(1_P|\mathcal{A}) \log \mathbb{E}_{\hat{\mu}}(1_P|\mathcal{A})\, d\hat{\mu}.$$

Then $H_{\hat{\mu}}(\mathcal{P}|\mathcal{A}) \leq H_{\hat{\mu}}(\mathcal{P})$ (see e.g. [K86, Lemma II.1.2(vi)]). The *fiber entropy* of $\mathcal{T}$ is defined in [B93, Definition 2.2.1] as

$$h^{\text{fib}}(\mathcal{T}) := \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{n} H_{\hat{\mu}}\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\mathcal{P}|\mathcal{A}\Big),$$

where the supremum is taken over all partitions $\mathcal{P}$ satisfying $H_{\hat{\mu}}(\mathcal{P}|\mathcal{A}) < \infty$.

As usual for entropy it is often not very practical to compute the fiber entropy of a system directly from the definition. One way to determine $h^{\text{fib}}(\mathcal{T})$ follows from the main theorem of [BC92], which gives the *Abramov-Rokhlin Formula*

$$h_{\hat{\mu}}(\hat{F}) = h_{\hat{\nu}}(\hat{\tau}) + h^{\text{fib}}(\mathcal{T}). \tag{5.23}$$

This leads to an expression for $h^{\text{fib}}(\mathcal{T})$ in case $h_{\hat{\nu}}(\hat{\tau}) < \infty$. Furthermore, the literature provides two versions of the Kolmorogov-Sinai Theorem for random systems, namely [K86, Lemma II.1.5] and [B93, Theorem 2.3.3]. The first of these results requires a generating partition $\mathcal{P}$ of the product space $I^{\mathbb{Z}} \times [0,1)$ in the sense that

$$\sigma\Big(\bigvee_{k\geq 0} \hat{F}^{-k}\mathcal{P}\Big) \vee \mathcal{A} = \mathcal{B}_I^{\mathbb{Z}} \times \mathcal{B} \;\text{ (up to sets of } \hat{\mu}\text{-measure zero).} \tag{5.24}$$

The latter requires a partition γ of $[0,1)$ that satisfies for $\hat{\nu}$-a.a. $\hat{\omega} \in I^{\mathbb{Z}}$

$$\sigma\Big(\bigvee_{k\geq 0} (T_{\hat{\omega}}^k)^{-1}\gamma\Big) = \mathcal{B} \;\text{ (up to sets of } \hat{\mu}_{\hat{\omega}}\text{-measure zero).}$$

For random number systems a natural candidate for a generating partition is provided by the family of partitions $\{\alpha_i\}_{i\in I}$ and the corresponding partition Δ on $I^{\mathbb{N}} \times [0,1)$ from (5.15). Δ is easily extended to a partition $\hat{\Delta}$ of $I^{\mathbb{Z}} \times [0,1)$ by setting

$$\hat{\Delta} = \{\Pi^{-1}\Delta(j) : j \geq 0\}.$$

The property (r5) is not enough to guarantee that the conditions of [K86, Lemma II.1.5] are satisfied by $\hat{\Delta}$ and the conditions of [B93, Theorem 2.3.3] are not satisfied either, since we consider a family of partitions $\{\alpha_i\}_{i\in I}$ on $[0,1)$ rather than a single partition γ. Nevertheless, since from (r7) we have

$$H_{\hat{\mu}}(\hat{\Delta}|\mathcal{A}) \leq H_{\hat{\mu}}(\hat{\Delta}) = H_{\mu}(\Delta) < \infty, \tag{5.25}$$

and from Lemma 5.4.2 we know that $\hat{\mu}_{\hat{\omega}} \ll \lambda$ holds for $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$, by a reasoning completely analogous to the proof of [B93, Theorem 2.3.3] we still obtain that

$$h^{\text{fib}}(\mathcal{T}) = \lim_{n\to\infty} \frac{1}{n} H_{\hat{\mu}}\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\hat{\Delta}|\mathcal{A}\Big). \tag{5.26}$$

Hence, $\hat{\Delta}$ serves enough as a generating partition to have a Kolmogorov-Sinai type result and therefore an expression of $h^{\text{fib}}(\mathcal{T})$ in terms of $\hat{\Delta}$. The reason we have put (r5) as a property of random number systems and not a condition like (5.24) is that compared to condition (5.24) from [K86, Lemma II.1.5], condition (r5) is easier to verify.

The sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n = H_{\hat{\mu}}\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\hat{\Delta}|\mathcal{A}\Big)$ is subadditive (see e.g. [K86, Theorem II.1.1]), thus it follows from Fekete's Subadditive Lemma together with (5.26) that

$$h^{\text{fib}}(\mathcal{T}) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_{\hat{\mu}}\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\hat{\Delta}|\mathcal{A}\Big).$$

In particular, (r7) implies via (5.25) that $h^{\text{fib}}(\mathcal{T}) < \infty$.

Using standard arguments (see e.g. [VO16, Lemma 9.1.12]) one can deduce that

$$\lim_{n\to\infty} \frac{1}{n} H_{\hat{\mu}}\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\hat{\Delta}|\mathcal{A}\Big) = \lim_{n\to\infty} H_{\hat{\mu}}\Big(\hat{\Delta}|\sigma\Big(\bigvee_{k=1}^{n-1} \hat{F}^{-k}\hat{\Delta}\Big) \vee \mathcal{A}\Big).$$

This leads to the following expression for the fiber entropy of $\mathcal{T}$:

$$h^{\text{fib}}(\mathcal{T}) = \lim_{n\to\infty} H_{\hat{\mu}}\Big(\hat{\Delta}|\sigma\Big(\bigvee_{k=1}^{n-1} \hat{F}^{-k}\hat{\Delta}\Big) \vee \mathcal{A}\Big).$$

For a partition $\mathcal{P}$ of $I^{\mathbb{Z}} \times [0,1)$ we can for each $\hat{\omega} \in I^{\mathbb{Z}}$ obtain a partition $\mathcal{P}_{\hat{\omega}}$ of $[0,1)$ by intersecting it with the fiber $\{\hat{\omega}\} \times [0,1)$, i.e. $\mathcal{P}_{\hat{\omega}} = \{Z_{\hat{\omega}} : Z \in \mathcal{P}\}$ where $Z_{\hat{\omega}} = \{x \in [0,1) : (\hat{\omega}, x) \in Z\}$. With this notation, note that

$$\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\hat{\Delta}\Big)_{\hat{\omega}} = \alpha_{\hat{\omega},n} \quad \text{and} \quad \Big(\bigvee_{k=1}^{n-1} \hat{F}^{-k}\hat{\Delta}\Big)_{\hat{\omega}} = T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}.$$

It now follows from [B93, Lemma 2.2.3] that

$$H_{\hat{\mu}}\Big(\bigvee_{k=0}^{n-1} \hat{F}^{-k}\hat{\Delta}|\mathcal{A}\Big) = \int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}(\alpha_{\hat{\omega},n})\, d\hat{\nu}(\hat{\omega}) \tag{5.27}$$

and

$$H_{\hat{\mu}}\Big(\hat{\Delta}|\sigma\Big(\bigvee_{k=1}^{n-1} \hat{F}^{-k}\hat{\Delta}\Big) \vee \mathcal{A}\Big) = \int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}\big(\alpha_{\hat{\omega}_1}\big|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n})\big)d\hat{\nu}(\hat{\omega}).$$

Hence, $h^{\mathrm{fib}}(\mathcal{T})$ can be rewritten as

$$h^{\mathrm{fib}}(\mathcal{T}) = \lim_{n\to\infty} \frac{1}{n} \int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}(\alpha_{\hat{\omega},n})\, d\hat{\nu}(\hat{\omega}) \tag{5.28}$$

$$= \lim_{n\to\infty} \int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}\big(\alpha_{\hat{\omega}_1} \big| \sigma(T_{\hat{\omega}_1}^{-1} \alpha_{\hat{\tau}\hat{\omega},n})\big) d\hat{\nu}(\hat{\omega}). \tag{5.29}$$

Remark 5.5.1. Condition (r7) ensures that $H_{\hat{\mu}}(\hat{\Delta}|\mathcal{A}) < \infty$ as follows from (5.25), which enables us to rewrite $h^{\mathrm{fib}}(\mathcal{T})$ as in (5.26). It follows from (5.27) that $H_{\hat{\mu}}(\hat{\Delta}|\mathcal{A}) < \infty$ is also ensured by requiring

$$\int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}(\alpha_{\hat{\omega}_1})\, d\hat{\nu}(\hat{\omega}) < \infty, \tag{5.30}$$

which is in general weaker than condition (r7). In view of this, we will call a collection $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ on $[0,1)$ satisfying (r1)-(r6) together with (5.30) a random number system as well. All our results also hold in this case.

§5.6 Proofs of the main results

§5.6.1 Asymptotic size of cylinder sets

In this and the next subsection we prove the three statements of Theorem 5.1.6 and we obtain Theorem 5.1.4 as a corollary. We start with Theorem 5.1.6(i) on the asymptotic size of cylinder sets. Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system.

Theorem 5.6.1. *For $\mathbb{P}$-a.a. $\omega \in I^{\mathbb{N}}$ we have*

$$\lim_{n\to\infty} -\frac{\log \lambda\big(\alpha_{\omega,n}(x)\big)}{n} = h^{\mathrm{fib}}(\mathcal{T}) \qquad \lambda\text{-a.e.}$$

Proof. Since $I^{\mathbb{Z}}$ is a Polish space on which $\hat{\tau}$ is invertible and the measure $\hat{\mu}$ is $\hat{F}$-invariant and ergodic, we are in the position to apply [Z08, Proposition 2.2(3)], which gives the Shannon-McMillan-Breiman Theorem for random dynamical systems. We apply it to the partition $\hat{\Delta}$ of $I^{\mathbb{Z}} \times [0,1)$, which is an at most countable partition satisfying $H_{\hat{\mu}}(\hat{\Delta}|\mathcal{A}) < \infty$. Note that [Z08, Proposition 2.2] considers random compositions of continuous transformations and requires the partition under consideration to be finite. However, the continuity of the transformations is not used in the proof, and the necessary condition on the partition is that it is at most countable and satisfies

$$\int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}(\hat{\Delta}_{\hat{\omega}})\, d\hat{\nu}(\hat{\omega}) < \infty. \tag{5.31}$$

As we already observed in Remark 5.5.1, we obtain $\int_{I^{\mathbb{Z}}} H_{\hat{\mu}_{\hat{\omega}}}(\hat{\Delta}_{\hat{\omega}})\, d\hat{\nu}(\hat{\omega}) = H_{\hat{\mu}}(\hat{\Delta}|\mathcal{A})$ from (5.27), so the condition from (5.31) is satisfied using (5.25). Thus, using the formula for $h^{\mathrm{fib}}(\mathcal{T})$ from (5.28), [Z08, Proposition 2.2(3)] gives that

$$\lim_{n\to\infty} -\frac{\log \hat{\mu}_{\hat{\omega}}\big(\alpha_{\hat{\omega},n}(x)\big)}{n} = h^{\mathrm{fib}}(\mathcal{T}), \qquad \hat{\mu}\text{-a.e. } (\hat{\omega}, x) \in I^{\mathbb{Z}} \times [0,1). \tag{5.32}$$

Let $C \in \mathcal{B}_I^{\mathbb{Z}} \times \mathcal{B}$ be the set of points $(\hat{\omega}, x)$ with the following three properties:

(i) the limit statement from (5.32) holds;

(ii) $\frac{d\hat{\mu}}{d\hat{\nu}\times\lambda}(\hat{\omega},x) = \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x)$;

(iii) x is a Lebesgue density point for $\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}$.

It follows from the Lebesgue Differentiation Theorem that for each $\hat{\omega} \in I^{\mathbb{Z}}$ such that $\hat{\mu}_{\hat{\omega}} \ll \lambda$ and thus $\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda} \in L^1([0,1),\lambda)$ we have that λ-a.e. $x \in [0,1)$ is a Lebesgue density point of $\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}$. Together with (5.32) and Lemma 5.4.2 we deduce that $\hat{\mu}(C) = 1$. Define

$$A = \Big\{(\omega,x) \in I^{\mathbb{N}} \times [0,1) : \frac{d\hat{\mu}}{d\hat{\nu}\times\lambda}\Big|_{\Pi^{-1}\{(\omega,x)\}\cap C} = 0\Big\}.$$

Then by Proposition 5.4.1,

$$\mu(A) = \hat{\mu}\big(\Pi^{-1}A \cap C\big) = \int_{\Pi^{-1}A\cap C} \frac{d\hat{\mu}}{d\hat{\nu}\times\lambda}\, d\hat{\nu}\times\lambda = 0,$$

thus $\mathbb{P}\times\lambda(A) = 0$ by (r6). Hence, for $\mathbb{P}\times\lambda$-a.e. (ω,x) there exists an $\hat{\omega} \in I^{\mathbb{Z}}$ such that $(\hat{\omega},x) \in C$ and $\pi(\hat{\omega}) = \omega$ and $\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x) > 0$, where this last part follows from property (ii) of C. For each such (ω,x) and $\hat{\omega}$ it follows from property (iii) of elements in C that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for each integer $n \geq N$,

$$\Big(\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x) - \varepsilon\Big)\lambda\big(\alpha_{\hat{\omega},n}(x)\big) \leq \hat{\mu}_{\hat{\omega}}\big(\alpha_{\hat{\omega},n}(x)\big) \leq \Big(\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x) + \varepsilon\Big)\lambda\big(\alpha_{\hat{\omega},n}(x)\big).$$

Here we used (r5). Combining this with (5.32) yields the result. □

From this result the Random Lochs' Theorem from Theorem 5.1.4 immediately follows.

Proof of Theorem 5.1.4. We have $h^{\text{fib}}(\mathcal{T}), h^{\text{fib}}(\mathcal{S}) \in (0,\infty)$, so Theorem 5.1.5 and Theorem 5.1.6(i) together yield the desired result. □

§5.6.2 The Random Rokhlin Formula

The last item of Theorem 5.1.6 is the Random Rokhlin Formula relating fiber entropy to the Jacobian of the transformations. We first prove an auxiliary lemma, for which we introduce some notation.

Assume as in Theorem 5.1.6(iii) that for each $i \in I$ and $A \in \alpha_i$ the restriction $T_i|_A$ is differentiable. Then for each $i \in I$ the Jacobian JT_i of T_i with respect to λ exists and is equal to $JT_i = |DT_i(x)|$ for λ-a.e. $x \in [0,1)$. From Lemma 5.4.2 together with the $\hat{\tau}$-invariance of $\hat{\nu}$ it follows that for $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$, $\hat{\mu}_{\hat{\tau}\hat{\omega}} \ll \lambda$. By (r5) we obtain that for $\hat{\nu}$-a.e. $\hat{\omega} \in I^{\mathbb{Z}}$,

$$\sigma\big(\lim_{n\to\infty} \alpha_{\hat{\tau}\hat{\omega},n}\big) = \mathcal{B} \text{ up to sets of } \hat{\mu}_{\hat{\tau}\hat{\omega}}\text{-measure zero.} \tag{5.33}$$

Since the standing assumptions of [B93] are satisfied, [B93, Lemma 1.1.2] gives that

$$\hat{\mu}_{\hat{\omega}} \circ T_{\hat{\omega}_1}^{-1} = \hat{\mu}_{\hat{\tau}\hat{\omega}}, \qquad \hat{\nu}\text{-a.a. } \hat{\omega} \in I^{\mathbb{Z}}. \tag{5.34}$$

Let

$$E = \{\hat{\omega} \in I^{\mathbb{Z}} \,:\, \hat{\mu}_{\hat{\tau}\hat{\omega}} \ll \lambda \text{ and } (5.33) \text{ and } (5.34) \text{ hold}\}.$$

Then $\hat{\nu}(E) = 1$. For $\hat{\omega} \in E$ let $C_{\hat{\omega}} = \{x \in [0,1) : \frac{d\hat{\mu}_{\hat{\tau}\hat{\omega}}}{d\lambda}(x) > 0\}$, define $h_{\hat{\omega}} : [0,1) \to [0,\infty)$ by

$$h_{\hat{\omega}}(y) = \begin{cases} \left(\frac{d\hat{\mu}_{\hat{\tau}\hat{\omega}}}{d\lambda}(y)\right)^{-1}, & \text{if } y \in C_{\hat{\omega}}, \\ 1, & \text{if } y \in [0,1)\backslash C_{\hat{\omega}}, \end{cases}$$

and define

$$\mathcal{L}_{\hat{\omega}}\psi(y) = h_{\hat{\omega}}(y) \sum_{z \in T_{\hat{\omega}_1}^{-1}\{y\}} \frac{\psi(z)}{JT_{\hat{\omega}_1}(z)} \cdot \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(z),$$

which, as we will see in the proof of Lemma 5.6.2 below, is well defined as an operator from $L^1(\hat{\mu}_{\hat{\omega}}) = L^1([0,1), \hat{\mu}_{\hat{\omega}})$ to $L^1(\hat{\mu}_{\hat{\tau}\hat{\omega}})$.

Lemma 5.6.2. *For $\hat{\omega} \in E$, $\psi \in L^1(\hat{\mu}_{\hat{\omega}})$ and $n \geq 1$ the following hold.*

(i) $\mathbb{E}_{\hat{\mu}_{\hat{\omega}}}(\psi|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}))(x) = \mathbb{E}_{\hat{\mu}_{\hat{\tau}\hat{\omega}}}(\mathcal{L}_{\hat{\omega}}\psi|\sigma(\alpha_{\hat{\tau}\hat{\omega},n}))(T_{\hat{\omega}_1}x)$ *for $\hat{\mu}_{\hat{\omega}}$-a.e. x.*

(ii) $\lim_{n\to\infty} \mathbb{E}_{\hat{\mu}_{\hat{\tau}\hat{\omega}}}(\mathcal{L}_{\hat{\omega}}\psi|\sigma(\alpha_{\hat{\tau}\hat{\omega},n})) = \mathbb{E}_{\hat{\mu}_{\hat{\tau}\hat{\omega}}}(\mathcal{L}_{\hat{\omega}}\psi|\mathcal{B}) = \mathcal{L}_{\hat{\omega}}\psi$ *for $\hat{\mu}_{\hat{\tau}\hat{\omega}}$-a.e. x.*

Proof. By (5.34) and the definition of $C_{\hat{\omega}}$, $\hat{\mu}_{\hat{\omega}}(T_{\hat{\omega}_1}^{-1}C_{\hat{\omega}}) = \hat{\mu}_{\hat{\tau}\hat{\omega}}(C_{\hat{\omega}}) = 1$. Using the change of variables formula from Lemma 1.2.20(b) we obtain for all $B \in \alpha_{\hat{\tau}\hat{\omega},n}$ that

$$\begin{aligned} \int_B \mathcal{L}_{\hat{\omega}}\psi \, d\hat{\mu}_{\hat{\tau}\hat{\omega}} &= \int_{B\cap C_{\hat{\omega}}} \mathcal{L}_{\hat{\omega}}\psi \frac{d\hat{\mu}_{\hat{\tau}\hat{\omega}}}{d\lambda}\, d\lambda = \int_{B\cap C_{\hat{\omega}}} \sum_{z \in T_{\hat{\omega}_1}^{-1}\{y\}} \frac{\psi(z)}{JT_{\hat{\omega}_1}(z)} \cdot \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(z)\, d\lambda(y) \\ &= \sum_{A\in\alpha_{\hat{\omega}_1}} \int_{T_{\hat{\omega}_1}(A)\cap B\cap C_{\hat{\omega}}} \left(\frac{\psi \cdot \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}}{JT_{\hat{\omega}_1}}\right) \circ (T_{\hat{\omega}_1}|_A)^{-1}(y)\, d\lambda(y) \\ &= \sum_{A\in\alpha_{\hat{\omega}_1}} \int_{A\cap T_{\hat{\omega}_1}^{-1}B\cap T_{\hat{\omega}_1}^{-1}C_{\hat{\omega}}} \psi \cdot \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}\, d\lambda \\ &= \int_{T_{\hat{\omega}_1}^{-1}B} \psi \, d\hat{\mu}_{\hat{\omega}}. \end{aligned}$$

In particular, $\mathcal{L}_{\hat{\omega}} : L^1(\hat{\mu}_{\hat{\omega}}) \to L^1(\hat{\mu}_{\hat{\tau}\hat{\omega}})$ is well defined. We obtain that

$$\begin{aligned} \mathbb{E}_{\hat{\mu}_{\hat{\tau}\hat{\omega}}}(\mathcal{L}_{\hat{\omega}}\psi|\sigma(\alpha_{\hat{\tau}\hat{\omega},n}))(T_{\hat{\omega}_1}x) &= \sum_{B\in\alpha_{\hat{\tau}\hat{\omega},n}} 1_B(T_{\hat{\omega}_1}x) \frac{\int_B \mathcal{L}_{\hat{\omega}}\psi\, d\hat{\mu}_{\hat{\tau}\hat{\omega}}}{\hat{\mu}_{\hat{\tau}\hat{\omega}}(B)} \\ &= \sum_{B\in\alpha_{\hat{\tau}\hat{\omega},n}} 1_{T_{\hat{\omega}_1}^{-1}B}(x) \frac{\int_{T_{\hat{\omega}_1}^{-1}B} \psi\, d\hat{\mu}_{\hat{\omega}}}{\hat{\mu}_{\hat{\omega}}(T_{\hat{\omega}_1}^{-1}B)} \\ &= \sum_{A\in T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}} 1_A(x) \frac{\int_A \psi\, d\hat{\mu}_{\hat{\omega}}}{\hat{\mu}_{\hat{\omega}}(A)} \\ &= \mathbb{E}_{\hat{\mu}_{\hat{\omega}}}(\psi|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}))(x) \end{aligned}$$

for $\hat{\mu}_{\hat{\tau}\hat{\omega}}$-a.e. x. This gives (i). Part (ii) follows from combining (5.33) and Lévy's Upward Theorem. □

This lemma is enough to prove the following Random Rokhlin's Formula.

Theorem 5.6.3. *If for each $i \in I$ and $A \in \alpha_i$ the restriction $T_i|_A$ is differentiable, then*

$$h^{\text{fib}}(\mathcal{T}) = \int_{I^{\mathbb{N}}\times[0,1)} \log|DT_{\omega_1}(x)|\,d\mu(\omega,x).$$

Proof. We follow the reasoning of the proof of [VO16, Theorem 9.7.3]. Write $\phi(x) = -x\log x$. As before, set $\mathcal{A} = \mathcal{B}_I^{\mathbb{Z}} \times [0,1)$. Then using (5.31) and the Dominated Convergence Theorem, we see from (5.29) that

$$\begin{aligned} h^{\text{fib}}(\mathcal{T}) &= \int_{I^{\mathbb{Z}}} \lim_{n\to\infty} H_{\hat{\mu}_{\hat{\omega}}}\left(\alpha_{\hat{\omega}_1}\middle|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n})\right)d\hat{\nu}(\hat{\omega}) \\ &= \int_{I^{\mathbb{Z}}} \lim_{n\to\infty} \int_{[0,1)} \sum_{A\in\alpha_{\hat{\omega}_1}} \phi\big(\mathbb{E}_{\hat{\mu}_{\hat{\omega}}}(1_A|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}))\big)(x)\,d\hat{\mu}_{\hat{\omega}}(x)d\hat{\nu}(\hat{\omega}) \\ &= \int_{I^{\mathbb{Z}}} \sum_{A\in\alpha_{\hat{\omega}_1}} \int_{[0,1)} \phi\big(\lim_{n\to\infty}\mathbb{E}_{\hat{\mu}_{\hat{\omega}}}(1_A|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}))\big)(x)\,d\hat{\mu}_{\hat{\omega}}(x)d\hat{\nu}(\hat{\omega}), \end{aligned} \tag{5.35}$$

where for the last equality we used the Dominated Convergence Theorem again as well as the continuity of ϕ. From Lemma 5.6.2 with $\psi = 1_A$ we get for each $\hat{\omega} \in E$ from (i) that

$$\mathbb{E}_{\hat{\mu}_{\hat{\omega}}}(1_A|\sigma(T_{\hat{\omega}_1}^{-1}\alpha_{\hat{\tau}\hat{\omega},n}))(x) = \mathbb{E}_{\hat{\mu}_{\hat{\tau}\hat{\omega}}}(\mathcal{L}_{\hat{\omega}}1_A|\sigma(\alpha_{\hat{\tau}\hat{\omega},n}))(T_{\hat{\omega}_1}x) \qquad \hat{\mu}_{\hat{\omega}}\text{-a.e.} \tag{5.36}$$

and from (ii) that

$$\lim_{n\to\infty}\mathbb{E}_{\hat{\mu}_{\hat{\tau}\hat{\omega}}}(\mathcal{L}_{\hat{\omega}}1_A|\sigma(\alpha_{\hat{\tau}\hat{\omega},n})) = \mathcal{L}_{\hat{\omega}}1_A \qquad \hat{\mu}_{\hat{\tau}\hat{\omega}}\text{-a.e.} \tag{5.37}$$

Recall that $C_{\hat{\omega}} = \{x \in [0,1) : \frac{d\hat{\mu}_{\hat{\tau}\hat{\omega}}}{d\lambda}(x) > 0\}$ satisfies $\hat{\mu}_{\hat{\tau}\hat{\omega}}(C_{\hat{\omega}}) = 1$. Combining (5.36) and (5.37) with (5.35) and using $\hat{\mu}_{\hat{\tau}\hat{\omega}} = \hat{\mu}_{\hat{\omega}} \circ T_{\hat{\omega}_1}^{-1}$ and the change of variables formula from Lemma 1.2.20(b) we conclude that

$$\begin{aligned} h^{\text{fib}}(\mathcal{T}) &= \int_{I^{\mathbb{Z}}} \sum_{A\in\alpha_{\hat{\omega}_1}} \int_{[0,1)} \phi\big(\mathcal{L}_{\hat{\omega}}1_A(x)\big)\,d\hat{\mu}_{\hat{\tau}\hat{\omega}}(x)\,d\hat{\nu}(\hat{\omega}) \\ &= -\int_{I^{\mathbb{Z}}} \sum_{A\in\alpha_{\hat{\omega}_1}} \left[\int_{T_{\hat{\omega}_1}(A)\cap C_{\hat{\omega}}} \left(\frac{1}{JT_{\hat{\omega}_1}}\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}\log\left(\frac{h_{\hat{\omega}}\circ T_{\hat{\omega}_1}}{JT_{\hat{\omega}_1}}\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}\right)\right)\circ(T_{\hat{\omega}_1}|A)^{-1}\,d\lambda\right]d\hat{\nu}(\hat{\omega}) \\ &= -\int_{I^{\mathbb{Z}}} \sum_{A\in\alpha_{\hat{\omega}_1}} \int_{A\cap T_{\hat{\omega}_1}^{-1}C_{\hat{\omega}}} \log\left(\frac{h_{\hat{\omega}}\circ T_{\hat{\omega}_1}}{JT_{\hat{\omega}_1}}\frac{d\hat{\mu}_{\omega}}{d\lambda}\right)(x)\,d\hat{\mu}_{\hat{\omega}}(x)d\hat{\nu}(\hat{\omega}) \\ &= \int_{I^{\mathbb{Z}}\times[0,1)} \log JT_{\hat{\omega}_1}(x)\,d\hat{\mu}(\hat{\omega},x) - \int_{I^{\mathbb{Z}}\times[0,1)} \log h_{\hat{\omega}}\circ T_{\hat{\omega}_1}(x)\,d\hat{\mu}(\hat{\omega},x) \\ &\quad - \int_{I^{\mathbb{Z}}\times[0,1)} \log\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x)\,d\hat{\mu}(\omega,x). \end{aligned}$$

For each $(\hat{\omega},x) \in I^{\mathbb{Z}} \times [0,1)$, set

$$\eta(\hat{\omega},x) = \begin{cases} \left(\frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x)\right)^{-1}, & \text{if } \hat{\mu}_{\hat{\omega}} \ll \lambda \text{ and } \frac{d\hat{\mu}_{\hat{\omega}}}{d\lambda}(x) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then for each $\hat{\omega} \in E$ and $x \in T_{\hat{\omega}_1}^{-1}(C_{\hat{\omega}})$,

$$h_{\hat{\omega}} \circ T_{\hat{\omega}_1}(x) = \eta \circ \hat{F}(\hat{\omega}, x).$$

With the $\hat{F}$-invariance of $\hat{\mu}$ this yields

$$\begin{aligned}\int_{I^{\mathbb{Z}}\times[0,1)} \log h_{\hat{\omega}} \circ T_{\hat{\omega}_1}(x)\, d\hat{\mu}(\hat{\omega}, x) &= \int_E \int_{T_{\hat{\omega}_1}^{-1} C_{\hat{\omega}}} \log \eta \circ \hat{F}(\hat{\omega}, x)\, d\hat{\mu}_{\hat{\omega}}(x) d\hat{\nu}(\hat{\omega}) \\ &= \int_{I^{\mathbb{Z}}\times[0,1)} \log \eta \, d\hat{\mu}.\end{aligned}$$

The result now follows. □

Proof of Theorem 5.1.6. Parts (i) and (iii) are given by Theorem 5.6.1 and Theorem 5.6.3, respectively. By (5.23) for part (ii) it is enough to show that $h_{\hat{\mu}}(\hat{F}) = h_\mu(F)$ and $h_{\hat{\nu}}(\hat{\tau}) = h_\nu(\tau)$. The latter follows immediately, because $(I^{\mathbb{Z}}, \mathcal{B}_I^{\mathbb{Z}}, \hat{\nu}, \hat{\tau})$ is the *natural extension* of the system $(I^{\mathbb{N}}, \mathcal{B}_I^{\mathbb{N}}, \nu, \tau)$, i.e. $(I^{\mathbb{Z}}, \mathcal{B}_I^{\mathbb{Z}}, \hat{\nu}, \hat{\tau})$ is the smallest (in the sense of the σ-algebra) invertible system that has $(I^{\mathbb{N}}, \mathcal{B}_I^{\mathbb{N}}, \nu, \tau)$ as a factor, and entropy is preserved under this construction (see e.g. [DK21, Chapter 5]). For the first part, note that $(I^{\mathbb{N}} \times [0,1), \mathcal{B}_I^{\mathbb{N}} \times \mathcal{B}, \mu, F)$ is a factor of $(I^{\mathbb{Z}} \times [0,1), \mathcal{B}_I^{\mathbb{Z}} \times \mathcal{B}, \hat{\mu}, \hat{F})$. Hence, $h_{\hat{\mu}}(\hat{F}) \geq h_\mu(F)$. Conversely, let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a countable set that generates $\mathcal{B}_I$, so $\sigma(\mathcal{C}) = \mathcal{B}_I$. Define for each $n \in \mathbb{N}$ the partition γ_n of I given by $\gamma_n = \{C_i \backslash \bigcup_{j=1}^{i-1} C_j \, : \, i = 1, \ldots, n\} \cup \{X \backslash \bigcup_{i=1}^n C_i\}$. Then $\sigma(\gamma_n) \subseteq \sigma(\gamma_{n+1})$ for each $n \geq 1$ and $\sigma(\lim_{n\to\infty} \gamma_n) = \mathcal{B}_I$. Similarly, since $\mathcal{B}$ is countably generated there exists a sequence of finite partitions $(\beta_n)_{n\geq 1}$ of $[0,1)$ such that $\sigma(\lim_{n\to\infty} \beta_n) = \mathcal{B}$. Then

$$\xi_n = \{(\cdots \times I \times A_{-n} \times A_{-n+1} \times \cdots \times A_n \times I \times \cdots) \times B : A_i \in \gamma_n, i = -n, \ldots, n, B \in \beta_n\}$$

(where A_0 is on the 0-th position) is a finite partition of $I^{\mathbb{Z}} \times [0,1)$ for each $n \in \mathbb{N}$ such that $\sigma(\lim_{n\to\infty} \xi_n) = \mathcal{B}_I^{\mathbb{Z}} \times \mathcal{B}$. Note that $\hat{F}^{-n-1}\xi_n$ specifies sets in positions 1 to $2n+1$ in the first coordinate. Recall that $\Pi : I^{\mathbb{Z}} \times [0,1) \to I^{\mathbb{N}} \times [0,1)$ denotes the canonical projection. We conclude using Lemma 1.2.18 that

$$\begin{aligned}h_{\hat{\mu}}(\hat{F}) &= \lim_{n\to\infty} h_{\hat{\mu}}(\hat{F}, \xi_n) = \lim_{n\to\infty} h_{\hat{\mu}}(\hat{F}, \hat{F}^{-n-1}\xi_n) \\ &= \lim_{n\to\infty} h_\mu(F, \Pi(\hat{F}^{-n-1}\xi_n)) \leq h_\mu(F).\end{aligned}$$

This finishes the proof. □

§5.6.3 The Central Limit Theorem

In case we compare a random number system to an NTFM, then under additional assumptions on both systems we can obtain a Central Limit Theorem for Theorem 5.1.4 in a way comparable to what has been done in [H09] for two NTFM's. Given a random number system $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ and an NTFM $(S, \tilde{\mu}, \gamma)$, for each $n \in \mathbb{N}$, $\omega \in I^{\mathbb{N}}$ and $x \in [0,1)$, let

$$m_{\mathcal{T},S}(n, \omega, x) = \sup\{m \in \mathbb{N} : \alpha_{\omega,n}(x) \subseteq \gamma_m(x)\}.$$

This is the analog of (5.13) for comparing two random number systems. We first introduce additional assumptions that we put on the systems. The following property can be found in [H09, Property 2.1].

Definition 5.6.4. Let $(S,\tilde{\mu},\gamma)$ be an NTFM. We say that S satisfies the *zero-property* if

$$\lim_{n\to\infty} \frac{-\log\lambda(\gamma_n(x)) - nh_{\tilde{\mu}}(S)}{\sqrt{n}} = 0 \qquad \lambda\text{-a.e.}$$

The zero-property is rather strong, but [H09, Section 3.2] contains several examples of NTFM's that satisfy it, including the N-adic transformations from Example 5.1.1 and β-transformations $S(x) = \beta x \bmod 1$ with $\beta > 1$ a so-called *Parry number*, i.e. a number β for which the set $\{S^n(\beta-1) : n \geq 0\}$ is finite. The following lemma, which compares to [H09, Lemma 2.1], yields a consequence of the zero-property.

Lemma 5.6.5 (cf. Lemma 2.1 in [H09]). *Let $\mathcal{T} = (I,\mathbb{P},\{T_i\}_{i\in I},\mu,\{\alpha_i\}_{i\in I})$ be a random number system and let $(S,\tilde{\mu},\gamma)$ be an NTFM that satisfies the zero-property. Assume that $h^{\mathrm{fib}}(\mathcal{T}), h_{\tilde{\mu}}(S) \in (0,\infty)$. Then*

$$\log\left(\frac{\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x))}{\lambda(\alpha_{\omega,n}(x))}\right) = o(\sqrt{n}) \quad \textit{in } \mu\textit{-probability.}$$

That is, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\big(\big\{(\omega,x) : \big|\log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) - \log\lambda(\alpha_{\omega,n}(x))\big| > \varepsilon\sqrt{n}\big\}\big) = 0.$$

Proof. By definition of $m_{\mathcal{T},S}(n,\omega,x)$ we see that $\alpha_{\omega,n}(x) \subseteq \gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)$. Since $\mu \ll \mathbb{P}\times\lambda$, it suffices to show that for all $\varepsilon,\tilde{\varepsilon} > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\mathbb{P}\times\lambda\big(\big\{(\omega,x) \,:\, \log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) - \log\lambda(\alpha_{\omega,n}(x)) > \varepsilon\sqrt{n}\big\}\big) < \tilde{\varepsilon}.$$

Fix $\varepsilon,\tilde{\varepsilon} > 0$ and set $\eta = \varepsilon\sqrt{\frac{h_{\tilde{\mu}}(S)}{3h^{\mathrm{fib}}(\mathcal{T})}}$. For each $n \in \mathbb{N}$, we put

$$A_n = I^{\mathbb{N}} \times \big\{x \in [0,1] \,:\, \exists\, k \geq n \text{ s.t. } |\log\lambda(\gamma_k(x)) + kh_{\tilde{\mu}}(S)| > \frac{1}{2}\eta\sqrt{k}\,\big\}.$$

Because S satisfies the zero-property, we know that there exists an $n_0 \in \mathbb{N}$ such that $\mathbb{P}\times\lambda(A_{n_0}) \leq \frac{\tilde{\varepsilon}}{3}$. Note that if $(\omega,x) \notin A_{n_0}$, then for each $n > n_0$ we have

$$\log\lambda(\gamma_{n-1}(x)) \leq \frac{1}{2}\eta\sqrt{n} - (n-1)h_{\tilde{\mu}}(S),$$
$$\log\lambda(\gamma_n(x)) \geq -\frac{1}{2}\eta\sqrt{n} - nh_{\tilde{\mu}}(S),$$

which when combined leads to

$$\log\lambda(\gamma_{n-1}(x)) - \log\lambda(\gamma_n(x)) \leq h_{\tilde{\mu}}(S) + \eta\sqrt{n}. \tag{5.38}$$

For each $n \geq 1$ put

$$B_n = \Big\{(\omega, x) \in I^{\mathbb{N}} \times [0,1) \,:\, \frac{1}{2} < \frac{m_{\mathcal{T},S}(k,\omega,x)}{k}\frac{h_{\tilde{\mu}}(S)}{h^{\mathrm{fib}}(\mathcal{T})} < 2 \quad \forall\, k \geq n\Big\}. \tag{5.39}$$

From Theorem 5.1.4 it follows that there exists an integer $n_1 > 2n_0\frac{h_{\tilde{\mu}}(S)}{h^{\mathrm{fib}}(\mathcal{T})}$ such that $\mathbb{P} \times \lambda(B_{n_1}) > 1 - \frac{\tilde{\varepsilon}}{3}$. Note from (5.39) that for each $(\omega, x) \in B_{n_1}$ we have $n_0 < \frac{1}{2}n_1\frac{h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)} < m_{\mathcal{T},S}(n_1, \omega, x)$. Therefore, it follows from (5.38) that, for each $(\omega, x) \in B_{n_1} \setminus A_{n_0}$ and for each $n > n_1$,

$$\log \lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) \leq h_{\tilde{\mu}}(S) + \eta\sqrt{m_{\mathcal{T},S}(n,\omega,x)+1} + \log \lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x)). \tag{5.40}$$

For each $n \geq 1$ and interval $E \in \gamma_n$ use ∂E to denote the boundary of E, i.e. the collection of its two endpoints, and use $\mathrm{dist}(x, \partial E) = \inf\{|x - a| \,:\, a \in \partial E\}$ to denote the distance from x to the nearest boundary point of E. For each $n \in \mathbb{N}$ and $(\omega, x) \in I^{\mathbb{N}} \times [0,1)$ we have that $\alpha_{\omega,n}(x) \nsubseteq \gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x)$, so $\mathrm{dist}(x, \partial\gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x)) \leq \lambda(\alpha_{\omega,n}(x))$ and with (5.40) we obtain for each $(\omega, x) \in B_{n_1} \setminus A_{n_0}$ and each $n > n_1$ that

$$\begin{aligned} &\frac{\log \lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) - \log \lambda(\alpha_{\omega,n}(x))}{\sqrt{n}} \\ &\quad\leq \frac{\log \lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x)) - \log \mathrm{dist}(x, \partial\gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x))}{\sqrt{n}} + \frac{h_{\tilde{\mu}}(S)}{\sqrt{n}} \\ &\qquad + \eta\sqrt{\frac{m_{\mathcal{T},S}(n,\omega,x)+1}{n}}. \end{aligned} \tag{5.41}$$

For each $n \in \mathbb{N}$ and interval $E \in \gamma_n$, we define a new interval E' by removing from both ends of E an interval of length $\frac{\tilde{\varepsilon}}{\pi^2 n^2}\lambda(E)$. Furthermore, we define

$$C = [0,1)\setminus\Big(\bigcup_{n\in\mathbb{N}}\bigcup_{E\in\gamma_n} E\setminus E'\Big).$$

Then

$$\lambda(C) \geq 1 - \sum_{n\in\mathbb{N}}\sum_{E\in\gamma_n}\frac{2\tilde{\varepsilon}}{\pi^2 n^2}\lambda(E) = 1 - \frac{\tilde{\varepsilon}}{3},$$

so $\mathbb{P} \times \lambda\big(\big(B_{n_1} \cap (I^{\mathbb{N}} \times C)\big) \setminus A_{n_0}\big) \geq 1 - \tilde{\varepsilon}$. For each $(\omega, x) \in I^{\mathbb{N}} \times C$ and $n \in \mathbb{N}$ we have the bound

$$\mathrm{dist}(x, \partial\gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x)) \geq \frac{\tilde{\varepsilon}}{\pi^2 n^2}\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)+1}(x)).$$

Combining this with (5.39) and (5.41) gives for each integer $n > n_1$ and each $(\omega, x) \in \big(B_{n_1} \cap (I^{\mathbb{N}} \times C)\big) \setminus A_{n_0}$ that

$$\frac{\log \lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) - \log \lambda(\alpha_{\omega,n}(x))}{\sqrt{n}} \leq \frac{|\log\frac{\pi^2 n^2}{\tilde{\varepsilon}}| + h_{\tilde{\mu}}(S)}{\sqrt{n}} + \eta\sqrt{\frac{2h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)} + \frac{1}{n}}. \tag{5.42}$$

We now take an integer $N > n_1$ large enough such that for each integer $n \geq N$ the right-hand side of (5.42) is bounded by $\eta\sqrt{\frac{3h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)}} = \varepsilon$. This yields the lemma. $\square$

Furthermore, we consider random number systems with the following property.

Definition 5.6.6. Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system. We say that $\mathcal{T}$ satisfies the *CLT-property* if there is some $\sigma > 0$ such that

$$\lim_{n\to\infty} \mu\Big(\Big\{(\omega, x) \in I^{\mathbb{N}}\times[0,1) \colon \frac{-\log\lambda(\alpha_{\omega,n}(x)) - nh^{\mathrm{fib}}(\mathcal{T})}{\sigma\sqrt{n}} \leq u\Big\}\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}dt$$

holds for each $u \in \mathbb{R}$.

With the properties from Definitions 5.6.4 and 5.6.6 we obtain the following result.

Theorem 5.6.7. *Let $\mathcal{T} = (I, \mathbb{P}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ be a random number system satisfying the CLT-property with constant $\sigma > 0$ and let $(S, \tilde{\mu}, \gamma)$ be an NTFM that satisfies the zero-property. Assume that $h^{\mathrm{fib}}(\mathcal{T}), h_{\tilde{\mu}}(S) \in (0,\infty)$. Then for all $u \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mu\Big(\Big\{(\omega, x) \in I^{\mathbb{N}} \times [0,1) : \frac{m_{\mathcal{T},S}(n,\omega,x) - n\frac{h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)}}{\kappa\sqrt{n}} \leq u\Big\}\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}\,dt,$$

where $\kappa = \frac{\sigma}{h_{\tilde{\mu}}(S)}$.

Proof. Fix some $u \in \mathbb{R}$. We rewrite

$$\begin{aligned}\frac{m_{\mathcal{T},S}(n,\omega,x) - n\frac{h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)}}{\kappa\sqrt{n}} &= \frac{-\log\lambda(\alpha_{\omega,n}(x)) - nh^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}} \\ &\quad+ \frac{\log\lambda(\alpha_{\omega,n}(x)) - \log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x))}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}} \\ &\quad+ \frac{\log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) + h_{\tilde{\mu}}(S)m_{\mathcal{T},S}(n,\omega,x)}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}}.\end{aligned} \tag{5.43}$$

The last term can be written as

$$\begin{aligned}&\frac{\log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) + h_{\tilde{\mu}}(S)m_{\mathcal{T},S}(n,\omega,x)}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}} \\ &\quad= \frac{1}{h_{\tilde{\mu}}(S)\kappa}\sqrt{\frac{m_{\mathcal{T},S}(n,\omega,x)}{n}}\,\frac{\log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) + h_{\tilde{\mu}}(S)m_{\mathcal{T},S}(n,\omega,x)}{\sqrt{m_{\mathcal{T},S}(n,\omega,x)}}.\end{aligned} \tag{5.44}$$

From Theorem 5.1.4 we know that $\lim_{n\to\infty}\frac{m_{\mathcal{T},S}(n,\omega,x)}{n} = \frac{h^{\mathrm{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)} < \infty$ for $\mathbb{P}\times\lambda$-a.e. $(\omega,x) \in I^{\mathbb{N}} \times [0,1)$. Since $\lim_{n\to\infty} m_{\mathcal{T},S}(n,\omega,x) = \infty$ for $\mathbb{P}\times\lambda$-a.e. $(\omega,x) \in I^{\mathbb{N}} \times [0,1)$, it then follows from the zero-property of S that (5.44) converges to 0 as $n \to \infty$ $\mathbb{P}\times\lambda$-a.e. and thus also μ-a.e. Hence it converges in μ-probability as well. Furthermore, we know from Lemma 5.6.5 that the second term on the right-hand side of (5.43) converges to 0 as $n \to \infty$ in μ-probability.

Define three sequences of random variables $(X_n)_{n\geq 1}, (Y_n)_{n\geq 1}$ and $(Z_n)_{n\geq 1}$ on $I^{\mathbb{N}} \times [0,1)$ by setting

$$\begin{aligned} X_n &= \frac{m_{\mathcal{T},S}(n,\omega,x) - n\frac{h^{\text{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)}}{\kappa\sqrt{n}}, \\ Y_n &= \frac{-\log\lambda(\alpha_{\omega,n}(x)) - nh^{\text{fib}}(\mathcal{T})}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}}, \\ Z_n &= \frac{\log\lambda(\alpha_{\omega,n}(x)) - \log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x))}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}} \\ &\quad + \frac{\log\lambda(\gamma_{m_{\mathcal{T},S}(n,\omega,x)}(x)) + h_{\tilde{\mu}}(S) m_{\mathcal{T},S}(n,\omega,x)}{h_{\tilde{\mu}}(S)\kappa\sqrt{n}}. \end{aligned}$$

Then by the above for each $\varepsilon > 0$, $\lim_{n\to\infty}\mu(|Z_n| > \varepsilon) = 0$ and since $\mathcal{T}$ satisfies the CLT-property for each $u \in \mathbb{R}$ it holds that

$$\lim_{n\to\infty}\mu(Y_n \leq u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}dt.$$

Fix some $u \in \mathbb{R}$ and some $\varepsilon > 0$. We are interested in $\lim_{n\to\infty}\mu(X_n \leq u)$. From (5.43) we see that

$$\begin{aligned} \mu(X_n \leq u) &= \mu(Y_n \leq u - Z_n) \\ &= \mu(Y_n \leq u - Z_n,\, |Z_n| \leq \varepsilon) + \mu(Y_n \leq u - Z_n,\, |Z_n| > \varepsilon). \end{aligned}$$

Since $\lim_{n\to\infty}\mu(Y_n \leq u - Z_n,\, |Z_n| > \varepsilon) = 0$ we get

$$\begin{aligned} \limsup_{n\to\infty}\mu(X_n \leq u) &= \limsup_{n\to\infty}\mu(Y_n \leq u - Z_n,\, |Z_n| \leq \varepsilon) \\ &\leq \limsup_{n\to\infty}\mu(Y_n \leq u + \varepsilon) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u+\varepsilon} e^{-t^2/2}dt. \end{aligned}$$

On the other hand,

$$\begin{aligned} \liminf_{n\to\infty}\mu(X_n \leq u) &= \liminf_{n\to\infty}\mu(Y_n \leq u - Z_n,\, |Z_n| \leq \varepsilon) \\ &\geq \liminf_{n\to\infty}\mu(Y_n \leq u - \varepsilon,\, |Z_n| \leq \varepsilon) \\ &\geq \liminf_{n\to\infty}\big(\mu(Y_n \leq u - \varepsilon) - \mu(|Z_n| > \varepsilon)\big) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u-\varepsilon} e^{-t^2/2}dt. \end{aligned}$$

Since this holds for all $\varepsilon > 0$ we get the result. □

We will now identify a class of random number systems that satisfies the CLT property. Let $\{T_i : [0,1) \to [0,1)\}_{i\in I}$ be a countable collection of transformations that satisfy the following two properties:

(p1) For each $i \in I$ there exists an interval partition α_i of $[0,1)$, such that for each $A \in \alpha_i$, $T_i|_A$ has non-positive Schwarzian derivative and $T_i(A) = [0,1)$.

(p2) There exist $1 < K \leq M < \infty$ such that, for all $x \in [0,1)$ and all $i \in I$,

$$K \leq |DT_i(x)| \leq M. \tag{5.45}$$

In particular, each T_i is expanding and has finitely many onto branches (so each partition α_i has at most finitely many non-empty intervals) and $T_i|_A$ is C^3 for each $A \in \alpha_i$. We let $\mathcal{D}$ denote the set of all collections $\{T_i\}_{i\in I}$ satisfying conditions (p1) and (p2) for some countable index set I and show that each element of $\mathcal{D}$ gives a random number system, that under some additional assumptions satisfies the CLT-property.

Proposition 5.6.8. *Let $\{T_i\}_{i\in I} \in \mathcal{D}$, $\{\alpha_i\}_{i\in I}$ the set of partitions given by Property (p1), let $\boldsymbol{p} = (p_i)_{i\in I}$ be a strictly positive probability vector and let $m_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-Bernoulli measure on $I^{\mathbb{N}}$. Then there exists a unique measure μ such that $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ is a random number system. Moreover, $\mathcal{T}$ satisfies the CLT-property if and only if for each measurable function $\psi : I^{\mathbb{N}} \times [0,1) \to \mathbb{R}$ we have $\varphi \circ F \neq \psi \circ F - \psi$, where F is the skew product associated to $\{T_i\}_{i\in I}$ and $\varphi : I^{\mathbb{N}} \times [0,1) \to \mathbb{R}$ is given by*

$$\varphi(\omega, x) = \log |DT_{\omega_1}(x)| - h^{\text{fib}}(\mathcal{T}). \tag{5.46}$$

To prove Proposition 5.6.8 we use two theorems by Young [Y99], both of which we have already used in Chapter 3. The results from [Y99] are formulated for Young towers, i.e. extensions of induced systems for suitable return time maps. We will, however, apply them to the system itself. That is, we will take the whole space $I^{\mathbb{N}} \times [0,1)$ as the inducing domain and as a consequence the return time function $R : I^{\mathbb{N}} \times [0,1) \to \mathbb{N}$ will have $R(\omega, x) = 1$ for each (ω, x). In particular $\int_{I^{\mathbb{N}} \times [0,1)} R \, dm_{\boldsymbol{p}} \times \lambda = 1$. For the convenience of the reader, we will reformulate the results from [Y99] that are relevant for the proof of Proposition 5.6.8 here for our setting, together with the necessary conditions.

The skew product F maps each element $[i] \times A$, $i \in I$ and $A \in \alpha_i$, bijectively onto $I^{\mathbb{N}} \times [0,1)$. Moreover, both $F|_{[i]\times A}$ and its inverse are non-singular with respect to $m_{\boldsymbol{p}} \times \lambda$ (giving (r4)). Hence, the Jacobian $J_{m_{\boldsymbol{p}}\times\lambda} F$ exists and is positive $m_{\boldsymbol{p}} \times \lambda$-a.e. By condition (p2) the collection $\{[i] \times A : i \in I, A \in \alpha_i\}$ generates the σ-algebra $\mathcal{B}_I^{\mathbb{N}} \times \mathcal{B}$ (giving (r5)). For each $(\omega, x), (\tilde{\omega}, y) \in I^{\mathbb{N}} \times [0,1)$ write $s\big((\omega, x), (\tilde{\omega}, y)\big)$ for the smallest $n \geq 0$ such that $F^n(\omega, x)$ and $F^n(\tilde{\omega}, y)$ lie in distinct sets $[i] \times A$. The results from [Y99, Theorem 1] then imply, among other things, the following: if there are $C_1 > 0$, $\eta \in (0,1)$ such that for all $[i] \times A$ and all $(\omega, x), (\tilde{\omega}, y) \in [i] \times A$ it holds that

$$\left| \frac{J_{m_{\boldsymbol{p}}\times\lambda} F(\omega, x)}{J_{m_{\boldsymbol{p}}\times\lambda} F(\tilde{\omega}, y)} - 1 \right| \leq C_1 \eta^{s(F(\omega,x), F(\tilde{\omega},y))}, \tag{5.47}$$

then F admits an invariant and ergodic invariant probability measure μ that is absolutely continuous with respect to $m_{\boldsymbol{p}} \times \lambda$ with a density that is bounded away from 0. We will use this to show that each $\{T_i\}_{i\in I} \in \mathcal{D}$ yields a random number system.

For the statement about the CLT-property in Proposition 5.6.8 we apply [Y99, Theorem 4] to φ from (5.46). For this we need to verify that $\int_{I^{\mathbb{N}}\times[0,1)} \varphi \, d\mu = 0$ and that there is a constant $C_2 > 0$ such that

$$|\varphi(\omega, x) - \varphi(\tilde{\omega}, y)| \le C_2 \eta^{s((\omega,x),(\tilde{\omega},y))} \tag{5.48}$$

for all $(\omega, x), (\tilde{\omega}, y) \in I^{\mathbb{N}} \times [0, 1)$, where η is the constant from (5.47). Under these conditions [Y99, Theorem 4] states that the sequence $\left(\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ F^i\right)_n$ converges in distribution with respect to μ to a normal distribution with mean 0 and variance σ^2 for some $\sigma > 0$ if and only if $\varphi \circ F \neq \psi \circ F - \psi$ for any measurable function $\psi : I^{\mathbb{N}} \times [0, 1) \to \mathbb{R}$.

Proof of Proposition 5.6.8. Let $\{T_i\}_{i\in I} \in \mathcal{D}$, $\{\alpha_i\}_{i\in I}$ the set of partitions from Property (p1), $\boldsymbol{p} = (p_i)_{i\in I}$ be a strictly positive probability vector and $m_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-Bernoulli measure on $I^{\mathbb{N}}$. A suitable invariant measure μ for the skew product F is obtained from [Y99, Theorem 1] once we show that (5.47) holds. First note that (r1), (r2), (r3) follow straightforwardly and (r4), (r5) were already addressed above. By Property (p2) the partition Δ is finite, yielding (r7) once we have μ. Hence, we focus on (r6).

Since each branch of each T_i has non-positive Schwarzian derivative and we have $\inf_{(i,x)} |DT_i(x)| > 1$, it follows from the Koebe Principle, i.e. (1.14) and (1.15), that there exists $K_1, K_2 > 0$ such that for each $\omega \in I^{\mathbb{N}}$, $n \in \mathbb{N}$, $A \in \alpha_{\omega,n}$ and $x, y \in A$ we have

$$\frac{1}{K_1} \le \frac{DT_\omega^n(x)}{DT_\omega^n(y)} \le K_1 \tag{5.49}$$

and

$$\left|\frac{DT_\omega^n(x)}{DT_\omega^n(y)} - 1\right| \le K_2 \cdot \frac{|T_\omega^n(x) - T_\omega^n(y)|}{\lambda(T_\omega^n(A))} = K_2 \cdot |T_\omega^n(x) - T_\omega^n(y)|. \tag{5.50}$$

For each $i \in I$ and $A \in \alpha_i$ we have for all measurable sets $E \subseteq [i]$ and $B \subseteq A$ that

$$m_{\boldsymbol{p}} \times \lambda\big(F(E \times B)\big) = \int_{E\times B} \frac{1}{p_{\omega_1}} DT_{\omega_1}(x) \, dm_{\boldsymbol{p}} \times \lambda(\omega, x),$$

from which it follows that

$$J_{m_{\boldsymbol{p}}\times\lambda} F(\omega, x) = \frac{1}{p_{\omega_1}} DT_{\omega_1}(x). \tag{5.51}$$

Combining this with (5.50) yields for i and A and $(\omega, x), (\tilde{\omega}, y) \in [i] \times A$ that

$$\left|\frac{J_{m_{\boldsymbol{p}}\times\lambda} F(\omega, x)}{J_{m_{\boldsymbol{p}}\times\lambda} F(\tilde{\omega}, y)} - 1\right| = \left|\frac{DT_i(x)}{DT_i(y)} - 1\right| \le K_2 \cdot |T_i(x) - T_i(y)|. \tag{5.52}$$

Assume $s\big(F(\omega, x), F(\tilde{\omega}, y)\big) = n$. Then for each $2 \le k \le n+1$ we have $\omega_k = \tilde{\omega}_k$ and that $T_\omega^{k-1}(x)$ and $T_{\tilde{\omega}}^{k-1}(y) = T_\omega^{k-1}(y)$ are in the same interval of the partition α_{ω_k}, and thus from the Mean Value Theorem and property (p2) that

$$K \le \min |DT_{\omega_k}| \le \left|\frac{T_{\omega_k}(T_\omega^{k-1}(x)) - T_{\omega_k}(T_\omega^{k-1}(y))}{T_\omega^{k-1}(x) - T_\omega^{k-1}(y)}\right| = \left|\frac{T_\omega^k(x) - T_\omega^k(y)}{T_\omega^{k-1}(x) - T_\omega^{k-1}(y)}\right|.$$

We conclude that

$$|T_i(x) - T_i(y)| \leq K^{-1}|T_\omega^2(x) - T_\omega^2(y)| \leq \cdots \leq K^{-n}|T_\omega^{n+1}(x) - T_\omega^{n+1}(y)| \leq K^{-n}.$$

Together with (5.52) this shows that (5.47) holds with $\eta = K^{-1}$. Hence, we obtain an F-invariant and F-ergodic measure μ that is equivalent to $m_{\boldsymbol{p}} \times \lambda$. This implies that $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ is a random number system.

What is left is to prove the statement on the CLT-property. Note that

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ F^i(\omega, x) = \frac{\log |DT_\omega^n(x)| - nh^{\text{fib}}(\mathcal{T})}{\sqrt{n}}.$$

Since $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i\in I}, \mu, \{\alpha_i\}_{i\in I})$ is a random number system, Theorem 5.1.6(iii) implies that $\int_{I^{\mathbb{N}}\times[0,1)} \varphi \, d\mu = 0$. Assume for a moment that also condition (5.48) holds, i.e. that we satisfy the conditions of [Y99, Theorem 4]. It then follows that there is some $\sigma > 0$ such that

$$\lim_{n\to\infty} \mu\Big(\Big\{(\omega, x) \in I^{\mathbb{N}} \times [0, 1) \, : \, \frac{\log |DT_\omega^n(x)| - nh^{\text{fib}}(\mathcal{T})}{\sigma\sqrt{n}} \leq u\Big\}\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} \, dt$$

if and only if $\varphi \circ F \neq \psi \circ F - \psi$ for any measurable function $\psi : I^{\mathbb{N}} \times [0, 1) \to \mathbb{R}$. From (5.49) it follows that for each $\omega \in I^{\mathbb{N}}$, $n \in \mathbb{N}$ and $x \in [0, 1)$,

$$\begin{aligned}\lambda(\alpha_{\omega,n}(x)) &\leq \frac{1}{\inf_{y\in\alpha_{\omega,n}(x)} |DT_\omega^n(y)|} = \frac{|DT_\omega^n(x)|}{\inf_{y\in\alpha_{\omega,n}(x)} |DT_\omega^n(y)|} \cdot |DT_\omega^n(x)|^{-1} \\ &\leq K_1 \cdot |DT_\omega^n(x)|^{-1}\end{aligned}$$

and similarly

$$\lambda(\alpha_{\omega,n}(x)) \geq \frac{1}{\sup_{y\in\alpha_{\omega,n}(x)} |DT_\omega^n(y)|} \geq \frac{1}{K_1} \cdot |DT_\omega^n(x)|^{-1}.$$

Hence, if for each $n \geq 1$ we write

$$X_n(\omega, x) = \frac{-\log \lambda(\alpha_{\omega,n}(x)) - nh^{\text{fib}}(\mathcal{T})}{\sigma\sqrt{n}},$$

then

$$\frac{-\log K_1}{\sigma\sqrt{n}} + \frac{\log |DT_\omega^n(x)| - nh^{\text{fib}}(\mathcal{T})}{\sigma\sqrt{n}} \leq X_n(\omega, x) \leq \frac{\log K_1}{\sigma\sqrt{n}} + \frac{\log |DT_\omega^n(x)| - nh^{\text{fib}}(\mathcal{T})}{\sigma\sqrt{n}},$$

and we see that to prove the last part of the proposition, it is enough to show that (5.48) with $\eta = K^{-1}$ holds.

Let $(\omega, x), (\tilde{\omega}, y) \in I^{\mathbb{N}} \times [0, 1)$. We first consider the case that $s\big((\omega, x), (\tilde{\omega}, y)\big) > 0$. Let $i \in I$ and $A \in \alpha_i$ be such that $(\omega, x), (\tilde{\omega}, y) \in [i] \times A$. It then follows from (5.51) that

$$|\varphi(\omega, x) - \varphi(\tilde{\omega}, y)| = \Big| \log \Big| \frac{DT_{\omega_1}(x)}{DT_{\tilde{\omega}_1}(y)} \Big|\Big| = \Big| \log \Big| \frac{J_{m_{\boldsymbol{p}}\times\lambda} F(\omega, x)}{J_{m_{\boldsymbol{p}}\times\lambda} F(\tilde{\omega}, y)} \Big|\Big|.$$

Recall from the first part of the proof that

$$\Big|\frac{J_{m_{\boldsymbol{p}}\times\lambda}F(\omega,x)}{J_{m_{\boldsymbol{p}}\times\lambda}F(\tilde{\omega},y)}-1\Big|\leq K_2\cdot K^{-s(F(\omega,x),F(\tilde{\omega},y))}.$$

Combining this with the fact that for all $x>0$,

$$|\log x|\leq\max\{x-1,x^{-1}-1\}\leq\max\{|x-1|,|x^{-1}-1|\},$$

yields that

$$\Big|\log\Big|\frac{J_{m_{\boldsymbol{p}}\times\lambda}F(\omega,x)}{J_{m_{\boldsymbol{p}}\times\lambda}F(\tilde{\omega},y)}\Big|\Big|\leq K_2\cdot K^{-s(F(\omega,x),F(\tilde{\omega},y))}\leq\tilde{K}_2\cdot K^{-s((\omega,x),(\tilde{\omega},y))},$$

where $\tilde{K}_2=K_2\cdot K$. In case $s\big((\omega,x),(\tilde{\omega},y)\big)=0$ we just notice from (5.45) that

$$\Big|\log\Big|\frac{DT_{\omega_1}(x)}{DT_{\tilde{\omega}_1}(y)}\Big|\Big|\leq\log M-\log K.$$

By taking $C_2=\max\{\tilde{K}_2,\log M-\log K\}$ we obtain the result. □

Remark 5.6.9. (i) A natural question is whether a Central Limit Theorem would hold when comparing two random number systems. For two random number systems $\mathcal{T}=(I,\mathbb{P},\{T_i\}_{i\in I},\mu,\{\alpha_i\}_{i\in I})$ and $\mathcal{S}=(J,\mathbb{Q},\{S_j\}_{j\in J},\rho,\{\gamma_j\}_{j\in J})$ the limit statement for an annealed result with respect to $\mathcal{S}$ would describe a subset of $I^{\mathbb{N}}\times J^{\mathbb{N}}\times[0,1)$. One might expect that a Central Limit Theorem with measure $\mathbb{P}\times\mathbb{Q}\times\lambda$ holds for random number systems $\mathcal{T}$ and $\mathcal{S}$ with invariant measures $\mu=\mathbb{P}\times\lambda$ and $\rho=\mathbb{Q}\times\lambda$, respectively. For a quenched result with respect to $\mathcal{S}$ the arguments for Theorem 5.6.7 might work if $\mathcal{S}$ satisfies a random zero-property, where in Definition 5.6.4 we replace $\gamma_n(x)$ and $h_{\tilde{\mu}}(S)$ by $\gamma_{\hat{\omega},n}(x)$ and $h^{\text{fib}}(\mathcal{S})$, respectively, and ask for the limit to hold for $\mathbb{Q}$-a.e. $\tilde{\omega}$.

(ii) The Central Limit Theorem from [H09, Corollary 2.1] derived for the quantity $m_{T,S}(n,x)$ for two NTFM's T and S asks for T to satisfy the zero-property and for S to satisfy a property called the weak invariance principle, which seems to be quite strong. By asking that S satisfies the zero-property, we have obtained the Central Limit Theorem under the somewhat less restrictive CLT-property on the random number system $\mathcal{T}$. This implies that the Central Limit Theorem from [H09, Corollary 2.1] also holds under the assumptions that the NTFM T has the CLT-property and S has the zero-property.

§5.7 Examples involving well-known number expansions

Below we consider some specific examples of random number systems with relations to number expansions.

Example 5.7.1 (Random integer base expansions). In this example we generalize the setting of Example 5.1.3. Consider a sequence of integers $2 \le N_1 < N_2 < N_3 < \dots$. Set $I = \{N_1, N_2, \dots\}$, let $\boldsymbol{p} = (p_i)_{i \in I}$ be a strictly positive probability vector and let $m_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-Bernoulli measure on $I^{\mathbb{N}}$. Assume that

$$\sum_{i \in I} p_i \log^2 i < \infty. \tag{5.53}$$

For each $i \in I$, let $T_i(x) = ix \bmod 1$. The maps T_i are non-singular and piecewise strictly monotonic and C^1 with respect to the partitions $\alpha_i = \{A_{i,j}\}_{j \ge 0}$ given by

$$A_{i,j} = \begin{cases} \left[\frac{j}{i}, \frac{j+1}{i}\right), & \text{if } 0 \le j \le i-1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus, conditions (r2) and (r4) are satisfied. The countability of I accounts for (r1) and (r3). The invariance of $m_{\boldsymbol{p}} \times \lambda$ for the skew product F follows directly (using that all maps T_i preserve λ) and its ergodicity follows from standard results, such as e.g. [M85b, Theorem 5.1]. Hence, we get (r6). All maps are expanding, since $DT_i(x) \ge 2$ for all $x \in [0,1)$ and $i \in I$, which implies (r5). Finally, the $\hat{F}$-invariant measure that is obtained by applying Proposition 5.4.1 to $m_{\boldsymbol{p}} \times \lambda$ is $\hat{m}_{\boldsymbol{p}} \times \lambda$, where $\hat{m}_{\boldsymbol{p}}$ is the $\boldsymbol{p}$-Bernoulli measure on $I^{\mathbb{Z}}$. Hence, it follows from (5.53) that condition (5.30) is satisfied (because $(\hat{m}_{\boldsymbol{p}} \times \lambda)_{\hat{\omega}} = \lambda$ for $\hat{m}_{\boldsymbol{p}}$-a.e. $\hat{\omega}$ by Fubini's Theorem), so $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i \in I}, m_{\boldsymbol{p}} \times \lambda, \{\alpha_i\}_{i \in I})$ is a random number system and thus Theorem 5.1.6 applies. Combining Theorem 5.1.6(i), (iii) and (5.53) then gives for $m_{\boldsymbol{p}}$-a.e. ω and λ-a.e. x that

$$\begin{aligned} \lim_{n \to \infty} -\frac{\log \lambda\big(\alpha_{\omega,n}(x)\big)}{n} &= h^{\text{fib}}(\mathcal{T}) = \int_{I^{\mathbb{N}}} \int_{[0,1)} \log |DT_{\omega_1}(x)| \, d\lambda(x) dm_{\boldsymbol{p}}(\omega) \\ &= \int_{I^{\mathbb{N}}} \log \omega_1 dm_{\boldsymbol{p}}(\omega) = \sum_{i \in I} p_i \log i < \infty. \end{aligned}$$

Note that the right-hand side is the weighted sum of the entropies $h_\lambda(T_i) = \log i$.

Note also that the collection $\{T_i\}_{i \in I}$ does not necessarily fall into the set $\mathcal{D}$ from the previous section, since there does not need to be a uniform upper bound on the derivatives of the maps T_i. We show that $\mathcal{T}$ satisfies the CLT-property nonetheless. For each $j \in \mathbb{N}$, define the random variable X_j on $I^{\mathbb{N}} \times [0,1)$ as

$$X_j(\omega, x) = -\sum_{A \in \alpha_{\omega_j}} 1_{(T_\omega^{j-1})^{-1} A}(x) \log \lambda(A) = -\log \lambda\big(\alpha_{\omega_j}(T_\omega^{j-1}(x))\big).$$

Then $\{X_j\}_{j \ge 1}$ is an i.i.d. sequence on $(I^{\mathbb{N}} \times [0,1), \mathcal{B}_I^{\mathbb{N}} \times \mathcal{B}, m_{\boldsymbol{p}} \times \lambda)$. Since each map T_i preserves λ, we obtain

$$\begin{aligned} \mathbb{E}_{m_{\boldsymbol{p}} \times \lambda}(X_j) &= -\int_{I^{\mathbb{N}}} \int_{[0,1)} \log \lambda\big(\alpha_{\omega_j}(T_\omega^{j-1}(x))\big) \, d\lambda(x) dm_{\boldsymbol{p}}(\omega) \\ &= -\int_{I^{\mathbb{N}}} \int_{[0,1)} \log \lambda\big(\alpha_{\omega_j}(x)\big) \, d\lambda(x) dm_{\boldsymbol{p}}(\omega) \\ &= \int_{I^{\mathbb{N}}} \log \omega_j \, dm_{\boldsymbol{p}}(\omega) = h^{\text{fib}}(\mathcal{T}). \end{aligned}$$

Similarly,

$$\sigma^2 = \mathrm{Var}(X_j) = \int_{I^{\mathbb{N}}} (\log \omega_j - h^{\mathrm{fib}}(\mathcal{T}))^2 dm_{\boldsymbol{p}}(\omega) = \sum_{i\in I} p_i \log^2 i - \Big(\sum_{i\in I} p_i \log i\Big)^2. \tag{5.54}$$

It follows from (5.53) that $\sigma^2 \in (0,\infty)$. Also $-\log\lambda(\alpha_{\omega,n}(x)) = \sum_{j=1}^n X_j(\omega,x)$, hence from the Central Limit Theorem we get

$$\frac{-\log\lambda\big(\alpha_{\omega,n}(x)\big) - nh^{\mathrm{fib}}(\mathcal{T})}{\sigma\sqrt{n}} = \frac{\sum_{j=1}^n X_j - n\mathbb{E}_{m_{\boldsymbol{p}}\times\lambda}(X_j)}{\sigma\sqrt{n}} \to \mathcal{N}(0,1),$$

where the convergence is in distribution with respect to $m_{\boldsymbol{p}}\times\lambda$. Hence, $\mathcal{T}$ satisfies the CLT-property with variance σ^2 and with respect to $m_{\boldsymbol{p}}\times\lambda$.

Recall that the digit sequence $(d_n^{\mathcal{T}}(\omega,x))_{n\geq 1}$ was defined in (5.9) by setting $d_n^{\mathcal{T}}(\omega,x) = j_n$ if $T_\omega^{n-1}(x) \in A_{\omega_n,j_n}$. In a similar way as in Example 5.1.3 it can be shown that this yields a number expansion of x given by

$$x = \sum_{n=1}^\infty \frac{d_n^{\mathcal{T}}(\omega,x)}{\prod_{i\in I} i^{c_{n,i}(\omega)}},$$

where $c_{n,i}(\omega) = \#\{1\leq j\leq n : \omega_j = i\}$. Hence, the random number system $\mathcal{T}$ produces number expansions of numbers $x\in[0,1)$ in mixed integer bases $N_1, N_2, \ldots$.

Consider the random number system $\mathcal{T}$ from above and another random number system of this form for $2 \leq M_1 < M_2 < \ldots$ with $J = \{M_1, M_2, \ldots\}$ and a probability vector $\boldsymbol{q} = (q_j)_{j\in J}$ specifying the Bernoulli measure $m_{\boldsymbol{q}}$ on $J^{\mathbb{N}}$. Assume that $\boldsymbol{q}$ satisfies the condition (5.53) as well. Write $\{S_j\}_{j\in J}$ for the maps $S_j(x) = jx \bmod 1$ and let $\{\gamma_j\}_{j\in J}$ be the corresponding partitions into maximal intervals on which the maps S_j are monotone. For the random number systems $\mathcal{T} = (I^{\mathbb{N}}, m_{\boldsymbol{p}}, \{T_i\}_{i\in I}, m_{\boldsymbol{p}}\times\lambda, \{\alpha_i\}_{i\in I})$ and $\mathcal{S} = (J^{\mathbb{N}}, m_{\boldsymbol{q}}, \{S_j\}_{j\in J}, m_{\boldsymbol{q}}\times\lambda, \{\gamma_j\}_{j\in J})$ we obtain

$$h^{\mathrm{fib}}(\mathcal{T}) = \sum_{i\in I} p_i \log i, \quad h^{\mathrm{fib}}(\mathcal{S}) = \sum_{j\in J} q_j \log j.$$

The Random Lochs' Theorem from Theorem 5.1.4 then states that for $m_{\boldsymbol{p}}\times m_{\boldsymbol{q}}$-a.e. $(\omega,\tilde{\omega}) \in I^{\mathbb{N}}\times J^{\mathbb{N}}$ it holds that

$$\lim_{n\to\infty} \frac{m_{\mathcal{T},\mathcal{S}}(n,\omega,\tilde{\omega},x)}{n} = \frac{\sum_{i\in I} p_i \log i}{\sum_{j\in J} q_j \log j} \qquad \lambda\text{-a.e.}$$

In other words, given $\omega, \tilde{\omega}$ and the first n digits $d_1^{\mathcal{T}}(\omega,x), \ldots, d_n^{\mathcal{T}}(\omega,x)$ of an unknown x, then typically we can determine approximately the first $n\frac{\sum_{i\in I} p_i \log i}{\sum_{j\in J} q_j \log j}$ digits of x in mixed integer bases $M_1, M_2, \ldots$ generated by the random system $\mathcal{S}$.

Moreover, if we take the NTFM $S(x) = Mx \bmod 1$ for some integer $M \geq 2$, then from Theorem 5.1.4 we obtain for $m_{\boldsymbol{p}}$-a.e. $\omega\in I^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{m_{\mathcal{T},S}(n,\omega,x)}{n} = \frac{\sum_{i\in I} p_i \log i}{\log M} \qquad \lambda\text{-a.e.}$$

From [H09, Section 3.2] we know that S has the zero-property, so that Theorem 5.6.7 gives for each $u \in \mathbb{R}$ that

$$\lim_{n\to\infty} m_{\boldsymbol{p}} \times \lambda\Big(\Big\{(\omega, x) \in I^{\mathbb{N}} \times [0,1) \, : \, \frac{m_{\mathcal{T},S}(n,\omega,x) - n\frac{\sum_{i\in I} p_i \log i}{\log M}}{\frac{\sigma}{\log M}\sqrt{n}} \le u\Big\}\Big)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt,$$

where σ is as in (5.54).

Remark 5.7.2. Arguments almost identical to the ones in Example 5.7.1 hold for any random system consisting of GLS-transformations. A GLS-transformation, see [BBDK96], is a piecewise linear map $T : [0,1) \to [0,1)$ specified by an at most countable interval partition of $[0,1)$ and for each of these intervals an orientation, such that T maps each interval linearly onto $[0,1)$ with the specified orientation. For example, we can fix some $N \in \{2,3,\dots\} \cup \{\infty\}$ (the number of branches of the map), $\eta \in (0,1)$ (providing a lower bound on the slope of the branches) and $q = (q_j)_{j=0}^{N-1} \in (0,\eta)^N$ with $\sum_{j=0}^{N-1} q_j = 1$ (the sizes of the intervals). The transformation $T_q : [0,1) \to [0,1)$ given by

$$T_q(x) = \sum_{j=0}^{N-1} \frac{1}{q_j}\Big(x - \sum_{k=0}^{j-1} q_k\Big) 1_{\left[\sum_{k=0}^{j-1} q_k, \sum_{k=0}^{j} q_k\right)}(x)$$

is a GLS-transformation mapping each interval $\left[\sum_{k=0}^{j-1} q_k, \sum_{k=0}^{j} q_k\right)$ linearly and orientation preservingly onto $[0,1)$. We can set $I = \{q = (q_j)_{j=0}^{N-1} \in (0,\eta)^N \, : \, \sum_{j=0}^{N-1} q_j = 1\}$, let $\mathbb{P}$ be a τ-invariant probability measure on $I^{\mathbb{N}}$ and consider the family $\{T_q\}_{q\in I}$. (Note that contrary to in Example 5.7.1 this I is not countable.) We assume that

$$-\int_{I^{\mathbb{N}}} \int_{[0,1)} \log \lambda\big(\alpha_{\omega_1}(x)\big) d\lambda(x) d\mathbb{P}(\omega) \in (0,\infty)$$

and that the skew product F associated to $\{T_q\}_{q\in I}$ is ergodic. Then it can in a similar way as in Example 5.7.1 be shown that $\mathcal{T} = (I^{\mathbb{N}}, \mathbb{P}, \{T_q\}_{q\in I}, \mathbb{P}\times\lambda, \{\alpha_q\}_{q\in I})$ is a random number system and that

$$h^{\text{fib}}(\mathcal{T}) = \int_{I^{\mathbb{N}}} h_\lambda(T_{\omega_1})\, d\mathbb{P}(\omega).$$

If we furthermore assume that $\mathbb{P} = \pi^{\mathbb{N}}$ with π a non-trivial probability measure on I and that

$$\int_I \int_{[0,1)} \log^2 \lambda\big(\alpha_i(x)\big)\, d\lambda(x) d\pi(i) < \infty,$$

then $\mathcal{T}$ satisfies the CLT-property with variance

$$\sigma^2 = \int_I \int_{[0,1)} \Big(\log \lambda\big(\alpha_i(x)\big) + h^{\text{fib}}(\mathcal{T})\Big)^2 d\lambda(x) d\pi(i) \in (0,\infty).$$

Number expansions obtained from this system are random versions of what are called *generalised Lüroth series expansions.* A particular instance of this class was studied in [KM22b].

CHAPTER 5

Remark 5.7.3. The family of GLS-transformations from Remark 5.7.2 provides examples of random number systems that do not satisfy (s1) from Section 5.3 by having an uncountable index set. We can also construct from these transformations random number systems that satisfy (s1) but not (s2). As an easy example, set $I = \{2, 3, \ldots\}$, define the probability vector $\boldsymbol{p} = (p_i)_{i\in I}$ by

$$p_i = C \cdot \frac{1}{i \cdot (\log i)^2}$$

with normalisation $C^{-1} = \sum_{n=2}^{\infty} \frac{1}{n\cdot(\log n)^2}$, and again write $m_{\boldsymbol{p}}$ for the $\boldsymbol{p}$-Bernoulli measure on $I^{\mathbb{N}}$. Furthermore, for each $i \in I$, let

$$T_i : [0,1) \to [0,1), \qquad T_i(x) = \begin{cases} \frac{1}{\frac{1}{2}-\frac{1}{i+1}} x, & \text{if } x \in [0, \frac{1}{2} - \frac{1}{i+1}), \\ \frac{1}{\frac{1}{2}+\frac{1}{i+1}} \left(x - \frac{1}{2} + \frac{1}{i+1}\right), & \text{if } x \in [\frac{1}{2} - \frac{1}{i+1}, 1) \end{cases}$$

and $\alpha_i = \{A_{i,0}, A_{i,1}\}$ with $A_{i,0} = [0, \frac{1}{2} - \frac{1}{i+1})$ and $A_{i,1} = [\frac{1}{2} - \frac{1}{i+1}, 1)$. Then $\mathcal{T} = (I^{\mathbb{N}}, m_{\boldsymbol{p}}, \{T_i\}_{i\in I}, m_{\boldsymbol{p}} \times \lambda, \{\alpha_i\}_{i\in I})$ is a random number system that meets (r1)-(r7) and (s1). However, (s2) is not satisfied, since the entropy $H_{m_{\boldsymbol{p}}}(\iota)$ of the partition $\iota = \{[i] : i \in I\}$ satisfies

$$H_{m_{\boldsymbol{p}}}(\iota) = -\sum_{i\in I} p_i \log p_i = C \cdot \sum_{i=2}^{\infty} \frac{1}{i\cdot(\log i)^2} \cdot \log(i \cdot (\log i)^2) - \tilde{C}$$

with $\tilde{C} = C \cdot \log(C) \cdot \sum_{i=2}^{\infty} \frac{1}{i\cdot(\log i)^2} = \log(C) \in \mathbb{R}$, and thus

$$H_{m_{\boldsymbol{p}}}(\iota) \geq C \cdot \sum_{i=2}^{\infty} \frac{1}{i\cdot(\log i)^2} \cdot \log i - \tilde{C} = C \cdot \sum_{i=2}^{\infty} \frac{1}{i \cdot \log i} - \tilde{C} = \infty.$$

This indicates that the class of random systems is considerably bigger than the class of random number systems that meets the additional conditions (s1)-(s4).

Example 5.7.4 (Random continued fraction expansions). Let $(\{T_0, T_1\}, m_{\boldsymbol{p}})$ be the random Gauss-Rényi map from Example 1.4.2, where $T_0 = G$ is the Gauss map from Example 1.3.2, $T_1 = R$ is the Rényi map from Example 1.3.3 and $\boldsymbol{p} = \{p_0, p_1\}$ with $p_0 = p \in (0,1)$ and $p_1 = 1-p$ gives the probabilities of choosing G and R, respectively. The partitions $\alpha_0 = \{A_{0,j}\}_{j\geq 0}$ and $\alpha_1 = \{A_{1,j}\}_{j\geq 0}$ are given by

$$A_{0,j} = \left(\frac{1}{j+2}, \frac{1}{j+1}\right] \quad \text{and} \quad A_{1,j} = \left[\frac{j}{j+1}, \frac{j+1}{j+2}\right), \quad j \geq 0.$$

In [KKV17] it was proven that there exists a measure ρ_p equivalent to λ such that $m_{\boldsymbol{p}} \times \rho_p$ is invariant and ergodic with respect to the skew product F with a density $\frac{d\rho_p}{d\lambda}$ that is bounded away from zero and is of bounded variation, so in particular bounded. For the system $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i=0,1}, m_{\boldsymbol{p}} \times \rho_p, \{\alpha_i\}_{i=0,1})$ conditions (r1)–(r6) follow straightforwardly. For (r7), let $M > 1$ be such that $M^{-1} \leq \frac{d\rho_p}{d\lambda} \leq M$ and note that

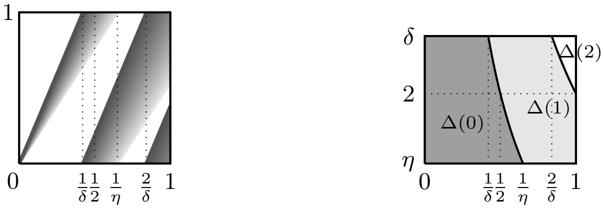
$$H_{m_{\boldsymbol{p}}\times\rho_p}(\Delta) \leq M \log M + M \sum_{j\geq 0} \frac{\log((j+1)(j+2))}{(j+1)(j+2)}.$$

It follows that $H_{m_{\boldsymbol{p}}\times\rho_p}(\Delta) < \infty$. Thus (r7) holds and $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i=0,1}, m_{\boldsymbol{p}} \times \rho_p, \{\alpha_i\}_{i=0,1})$ is a random number system. For the fiber entropy we obtain from Theorem 5.1.6(iii) that

$$\begin{aligned} h^{\text{fib}}(\mathcal{T}) &= \int_{I^{\mathbb{N}}} \int_{[0,1)} \log |DT_{\omega_1}(x)|\, d\rho_p(x) dm_{\boldsymbol{p}}(\omega) \\ &= -2 \int_0^1 p \log x + (1-p) \log(1-x)\, d\rho_p(x). \end{aligned}$$

It follows from equation (8) in [KKV17] that the digit sequences $(d_n^{\mathcal{T}}(\omega, x))_{n\geq 1}$ from this random number system give the semi-regular continued fraction expansions of real numbers $x \in [0,1]$ by

$$x = \omega_1 + \cfrac{(-1)^{\omega_1}}{d_1^{\mathcal{T}}(\omega,x) + 1 + \omega_2 + \cfrac{(-1)^{\omega_2}}{d_2^{\mathcal{T}}(\omega,x) + 1 + \omega_3 + \ddots}}.$$



(a) S_β for all $\eta \leq \beta \leq \delta$. (b) The partition Δ.

Figure 5.3: In (a) we see the graphs of all the maps S_β for $\beta \in [\eta, \delta]$ from Example 5.7.5 for some $1 < \eta < 2 < \delta < 3$. Each shade of grey corresponds to one graph. In (b) we see the elements of the partition Δ for values η and δ as in (a).

Example 5.7.5. (Random β-expansions in alternate base) Fix two constants $1 < \eta < \delta$ and let $J = [\eta, \delta]$. For each $\beta \in J$, let $S_\beta : [0,1) \to [0,1)$, $x \mapsto \beta x \bmod 1$ be the β-transformation, which is the piecewise linear map with slope β on the partition $\gamma_\beta = \{C_{\beta,j}\}_{j\geq 0}$ given by

$$C_{\beta,j} = \begin{cases} \left[\frac{j}{\beta}, \frac{j+1}{\beta}\right), & \text{if } 0 \leq j < \lceil\beta\rceil - 1 \\ \left[\frac{\lceil\beta\rceil-1}{\beta}, 1\right), & \text{if } j = \lceil\beta\rceil - 1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\lceil\beta\rceil$ indicates the smallest integer not smaller than β. See Figure 5.3(a) for some graphs. The study of β-transformations was initiated by Rényi in [R57b]. The β-transformations are related to β-expansions of real numbers, which are expressions of the form

$$x = \sum_{n\geq 1} \frac{b_n}{\beta^n}, \quad b_n \in \{0, 1, \ldots, \lceil\beta\rceil - 1\}.$$

Fix some periodic sequence

$$u = (u_1, u_2, \dots, u_m, u_1, u_2, \dots, u_m, u_1, \dots) \in J^{\mathbb{N}}$$

of period length $m \geq 2$ and consider the shift invariant measure $\mathbb{Q}$ on $J^{\mathbb{N}}$ defined by

$$\mathbb{Q} = \frac{1}{m} \sum_{i=1}^{m} \delta_{\tau^{i-1} u}, \tag{5.55}$$

where δ_y denotes the Dirac measure at the point $y \in J^{\mathbb{N}}$. We define the map $\psi : \{1, 2, \dots, m\} \times [0,1) \to J^{\mathbb{N}} \times [0,1)$ by

$$\psi(i, x) = \big((u_i, u_{i+1}, \dots, u_m, u_1, u_2, \dots, u_m, u_1, \dots), x\big),$$

which is measurable if we put on $\{1, 2, \dots, m\} \times [0,1)$ the σ-algebra

$$\mathcal{A} = \Big\{ \bigcup_{i=1}^{m} \{i\} \times B_i \, : \, B_1, \dots, B_m \in \mathcal{B} \Big\}.$$

Defining the transformation $T_u : \{1, 2, \dots, m\} \times [0,1) \to \{1, 2, \dots, m\} \times [0,1)$ by

$$T_u(i, x) = \big((i+1) \bmod m, T_{u_i}(x)\big),$$

it follows from [CCD21] that there are probability measures $\mu_{u,1}, \dots, \mu_{u,m}$ on $([0,1), \mathcal{B})$ such that the probability measure μ_u on $(\{1, \dots, m\} \times [0,1), \mathcal{A})$ given by

$$\mu_u\Big(\bigcup_{i=1}^{m} \{i\} \times B_i\Big) = \frac{1}{m} \sum_{i=1}^{m} \mu_{u,i}(B_i), \qquad B_1, \dots, B_m \in \mathcal{B},$$

is an ergodic invariant measure for T_u that is equivalent to the probability measure λ_u on $(\{1, \dots, m\} \times [0,1), \mathcal{A})$ given by

$$\lambda_u\Big(\bigcup_{i=1}^{m} \{i\} \times B_i\Big) = \frac{1}{m} \sum_{i=1}^{m} \lambda(B_i), \qquad B_1, \dots, B_m \in \mathcal{B}.$$

Define the probability measure ρ on $J^{\mathbb{N}} \times [0,1)$ by

$$\rho = \frac{1}{m} \sum_{i=1}^{m} \delta_{\tau^{i-1} u} \times \mu_{u,i}.$$

Since $\mathbb{Q} \times \lambda = \lambda_u \circ \psi^{-1}$ and ψ is an isomorphism between the dynamical systems $(\{1, \dots, m\} \times [0,1), \mathcal{A}, \mu_u, T_u)$ and $(J^{\mathbb{N}} \times [0,1), \mathcal{B}_J^{\mathbb{N}} \times \mathcal{B}, \rho, F)$ with F the skew product associated to $\{S_\beta\}_{\beta \in J}$, it follows that ρ is an ergodic invariant measure for F and that ρ is equivalent to $\mathbb{Q} \times \lambda$. Since δ is an upper bound for J, the collection Δ is finite and thus $H_\rho(\Delta) < \infty$. Figure 5.3(b) illustrates the sets $\Delta(j)$ for $\eta \in (1,2)$ and $\delta \in (2,3)$. It follows that $\mathcal{S} = (J, \mathbb{Q}, \{S_\beta\}_{\beta \in J}, \rho, \{\gamma_\beta\}_{\beta \in J})$ is a random number system. From Theorem 5.1.6(iii) we get

$$h^{\text{fib}}(\mathcal{S}) = \int_{J^{\mathbb{N}}} \int_{[0,1)} \log \omega_1 \, d\rho(\omega, x) = \frac{1}{m} \sum_{i=1}^{m} \log u_i.$$

By the definition of the digit sequence $(d_n^{\mathcal{S}}(\omega, x))_{n\geq 1}$ we can write for each $\omega \in J^{\mathbb{N}}$, $x \in [0,1)$ and $n \geq 1$ that

$$S_\omega^n(x) = \omega_n S_\omega^{n-1}(x) - d_n^{\mathcal{S}}(\omega, x),$$

so that

$$x = \frac{d_1^{\mathcal{S}}(\omega, x)}{\omega_1} + \frac{d_2^{\mathcal{S}}(\omega, x)}{\omega_1\omega_2} + \cdots + \frac{d_n^{\mathcal{S}}(\omega, x)}{\omega_1 \cdots \omega_n} + \frac{S_\omega^n(x)}{\omega_1 \cdots \omega_n}.$$

Since $\lim_{n\to\infty} \frac{S_\omega^n(x)}{\omega_1\cdots\omega_n} \leq \lim_{n\to\infty} \frac{1}{\eta^n} = 0$, for each $x \in [0,1)$ and $\omega \in J^{\mathbb{N}}$ we obtain the *random mixed β-expansion*

$$x = \sum_{n\geq 1} \frac{d_n^{\mathcal{S}}(\omega, x)}{\omega_1 \cdots \omega_n}.$$

With our choice of $\mathbb{Q}$ from (5.55) it holds that for $\mathbb{Q}$-a.e. $\omega \in J^{\mathbb{N}}$ with $\omega_1 = u_1$ the random mixed β-expansions produced by the system $\mathcal{S}$ are the *greedy* $(u_1, \ldots, u_m)$-*expansions in alternate base* that are the object of study in [CCD21].

With Theorem 5.1.4 we can compare the semi-regular continued fraction digits from the random continued fraction map from Example 5.7.4 with the alternate base greedy β-expansions. If we let $\mathcal{T} = (I, m_{\boldsymbol{p}}, \{T_i\}_{i=0,1}, m_{\boldsymbol{p}} \times \rho_p, \{\alpha_i\}_{i=0,1})$ be the system from Example 5.7.4 and let $(J, \mathbb{Q}, \{S_\beta\}_{\beta\in J}, \rho, \{\gamma_\beta\}_{\beta\in J})$ be the system from Example 5.7.5, then Theorem 5.1.4 tells us that for $m_{\boldsymbol{p}} \times \mathbb{Q}$-a.e. $(\omega, \tilde{\omega}) \in I^{\mathbb{N}} \times J^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{m_{\mathcal{T},\mathcal{S}}(n, \omega, \tilde{\omega}, x)}{n} = \frac{-2\int_{[0,1)} p \log x + (1-p)\log(1-x)\, d\rho_p(x)}{\frac{1}{m}\sum_{i=1}^m \log u_i} \quad \lambda\text{-a.e.}$$