Intermittency and number expansions for random interval maps
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CHAPTER 4

Intermittency generated by attracting and weakly repelling fixed points

This chapter is based on: [Z].

Abstract

In Chapter 2 for a class of critically intermittent random systems a phase transition was found for the finiteness of the absolutely continuous invariant measure. The systems for which this result holds are characterised by the interplay between a superexponentially attracting fixed point and an exponentially repelling fixed point. In this chapter we consider a closely related family of random systems with instead exponentially fast attraction to and polynomially fast repulsion from two fixed points, and show that such a phase transition still exists. The method of the proof however is different and relies on the construction of a suitable invariant set for the Perron-Frobenius operator.

§4.1 Introduction

For the critically intermittent random systems studied in Chapter 2 we asked in Subsection 2.4.3 the question what happens to the absolutely continuous invariant measure, if it exists, when the superexponential convergence to c is replaced by exponential convergence to c and the exponential divergence from 0 and 1 is replaced by polynomial divergence from 0 and 1. In this chapter we investigate this by considering a random system that generates i.i.d. random compositions of a finite fixed number of maps of two types: Type 1 consists of the LSV maps from (1.11) and type 2 consists of LSV maps where the right branch is replaced by increasing branches that map $(\frac{1}{2}, 1]$ to itself and for which the derivative close to $\frac{1}{2}$ is smaller than 1. The random orbits then converge exponentially fast to $\frac{1}{2}$ under applications of maps of type 2, and as soon as a map of type 1 is applied then diverge polynomially fast from 0, see Figure 4.1(a). We will show that such random systems exhibit a phase transition similar to the ones found in Chapters 2 and 3 in the sense that it depends on the features of the maps as well as on the probabilities of choosing the maps whether the system admits a finite absolutely continuous invariant measure or not.

We define the class $\mathfrak{S} = \{S_\alpha : \alpha \in (0,\infty)\}$ where S_α is the LSV map from (1.11), and the class $\mathfrak{R} = \{R_{\alpha,K} : \alpha \in (0,\infty), K \in (0,1)\}$ where

$$R_{\alpha,K}(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if} \quad x \in [0,\frac{1}{2}], \\ \frac{1}{2} + K(x-\frac{1}{2}) + 2(1-K)(x-\frac{1}{2})^2 & \text{if} \quad x \in (\frac{1}{2},1]. \end{cases} \tag{4.1}$$

The graph of $R_{\alpha,K}$ is shown in Figure 4.1(b). The right branch of $R_{\alpha,K}$ is defined in such a way that $\frac{1}{2}$ and 1 are fixed points for $R_{\alpha,K}$ and that under $R_{\alpha,K}$ orbits eventually approach $\frac{1}{2}$ from above. The rate of this convergence to $\frac{1}{2}$ is determined by K. Let $T_1,\dots,T_N \in \mathfrak{S} \cup \mathfrak{R}$ be a finite collection. Similar as in the previous chapters we write

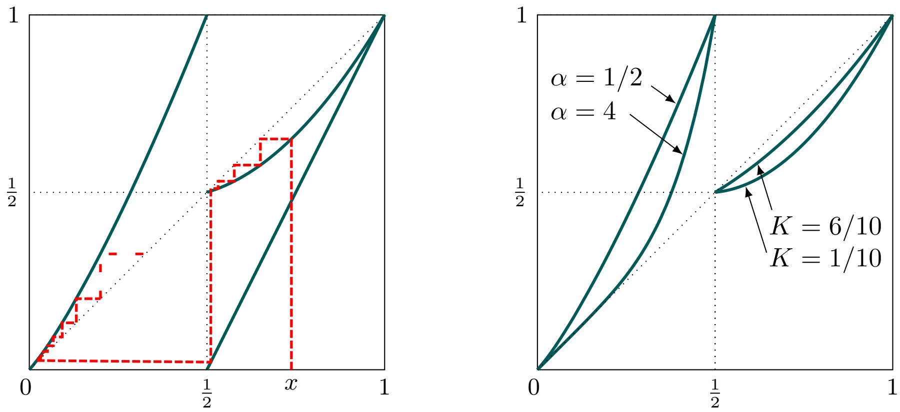
$$\begin{aligned} \Sigma_S &= \{1 \le j \le N : T_j \in \mathfrak{S}\}, \\ \Sigma_R &= \{1 \le j \le N : T_j \in \mathfrak{R}\}, \\ \Sigma &= \{1,\dots,N\} = \Sigma_S \cup \Sigma_R. \end{aligned}$$

We assume that $\Sigma_S, \Sigma_R \neq \emptyset$. For each $j \in \Sigma$ we write $\alpha_j \in (0,\infty)$ if $T_j(x) = x(1+2^{\alpha_j}x^{\alpha_j})$ for $x \in [0,\frac{1}{2}]$. For $j \in \Sigma_R$ we moreover write $K_j \in (0,1)$ if $T_j(x) = \frac{1}{2} + K_j(x-\frac{1}{2}) + 2(1-K_j)(x-\frac{1}{2})^2$ for $x \in (\frac{1}{2},1]$.

Let F be the skew product associated to $\{T_j\}_{j\in\Sigma}$, i.e.

$$F : \Sigma^{\mathbb{N}} \times [0,1] \to \Sigma^{\mathbb{N}} \times [0,1],\ (\omega,x) \mapsto (\tau\omega, T_{\omega_1}(x)), \tag{4.2}$$

where τ denotes the left shift on sequences in $\Sigma^{\mathbb{N}}$. Let $\boldsymbol{p} = (p_j)_{j\in\Sigma}$ be a probability vector with strictly positive entries representing the probabilities with which we choose the maps from $\mathcal{T} = \{T_j\}_{j\in\Sigma}$. Let $m_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-Bernoulli measure on $\Sigma^{\mathbb{N}}$. Since each of the maps T_j $(j \in \Sigma)$ has zero as a neutral fixed point, orbits under $(\mathcal{T},\boldsymbol{p})$ exhibit intermittent behaviour in the sense that periods of chaotic behaviour are followed by



(a) (b)

Figure 4.1: In (a) *we see the critically intermittent system consisting of the maps* S_α *and* $R_{\alpha,K}$ *given by* (1.11) *and* (4.1)*, respectively. The dashed lines indicate part of a random orbit of* x*. In* (b) *the graph of* $R_{\alpha,K}$ *is depicted for several values of* α *and* K*.*

periods of spending time near zero. The amount of time spent near zero generally increases for larger values of p_j ($j \in \Sigma_R$), smaller values of K_j ($j \in \Sigma_R$) and larger values of α_j ($j \in \Sigma$).

We set $\alpha_{\min} = \min\{\alpha_j : j \in \Sigma\}$. Throughout this chapter we assume the following:

Assumption: $\alpha_{\min} < 1$.

Furthermore, we set

$$\eta = \sum_{r\in\Sigma_R} p_r K_r^{-\alpha_{\min}},$$

$$\gamma = \sup\{\delta \geq 0 : \sum_{r\in\Sigma_R} p_r K_r^{-\delta} < 1\}.$$

Note that if $\eta < 1$, then $\gamma > \alpha_{\min}$. We have the following main results.

Theorem 4.1.1. *Suppose* $\eta > 1$*. Then no acs probability measure exists for* $(\mathcal{T}, \boldsymbol{p})$*.*

Theorem 4.1.2. *Suppose* $\eta < 1$*.*

(1) There exists a unique acs probability measure μ *for* $(\mathcal{T}, \boldsymbol{p})$*. Moreover,* F *is ergodic with respect to* $m_{\boldsymbol{p}} \times \mu$*.*

(2) The density $\frac{d\mu}{d\lambda}$ *with respect to the Lebesgue measure* λ *is bounded away from zero and on the intervals* $(0, \frac{1}{2}]$ *and* $(\frac{1}{2}, 1]$ *is decreasing and locally Lipschitz. Furthermore, for each* $\beta \in (\alpha_{\min}, \gamma) \cap (0, 1]$ *there exist* $a_1, a_2 > 0$ *such that*

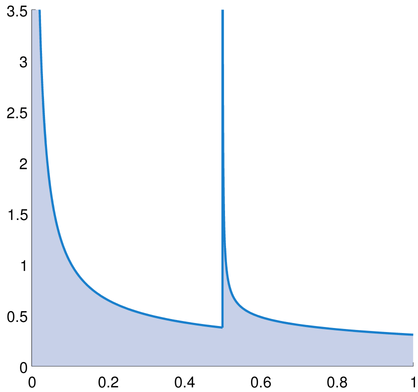
$$\frac{d\mu}{d\lambda}(x) \leq a_1 \cdot x^{-\alpha_{\min}-1+\beta}, \qquad x \in \Big(0, \frac{1}{2}\Big], \tag{4.3}$$

$$\frac{d\mu}{d\lambda}(x) \leq a_2 \cdot \Big(x - \frac{1}{2}\Big)^{-1+\beta}, \qquad x \in \Big(\frac{1}{2}, 1\Big]. \tag{4.4}$$

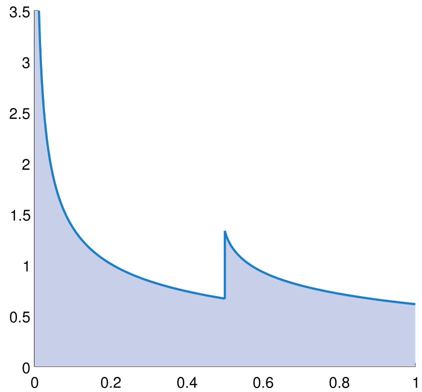
See Figure 4.2 for a plot of $\frac{d\mu}{d\lambda}$. In addition to Theorem 4.1.1 we argue in Section 4.3 that if $\eta > 1$ then an infinite acs measure exists and no physical measure (see footnote 1 on page 39) for F exists. Together with Theorem 4.1.2 this shows that the random system undergoes a phase transition with threshold $\eta = 1$. It is not clear if an acs probability measure exists when $\eta = 1$. We discuss this in Section 4.3 as well. Note that if $\sum_{r\in\Sigma_R} p_r K_r^{-1} < 1$, then $\gamma > 1$. So in this case we can take $\beta = 1$, and then Theorem 4.1.2(2) says that there exists $a > 0$ such that

$$\frac{d\mu}{d\lambda}(x) \leq a \cdot x^{-\alpha_{\min}}, \qquad x \in (0,1]. \tag{4.5}$$

This bound is also found in [LSV99] where only one LSV map $T_1 \in \mathfrak{G}$ with $\alpha_1 \in (0,1)$ is considered and no maps in $\mathfrak{R}$. This suggest that in case $\sum_{r\in\Sigma_R} p_r K_r^{-1} < 1$ the attraction by the maps $\{T_j\}_{j\in\Sigma_R}$ to $\frac{1}{2}$ does not change the order of the pole of the invariant density at zero. Note however that the density in the setting of [LSV99] is shown to be continuous on $(0,1)$, which in general is not the case for the density in the setting of Theorem 4.1.2. See Figure 4.2(b).



(a)



(b)

Figure 4.2: Approximation of $\frac{d\mu}{d\lambda}$ in case $\Sigma_S = \{1\}$, $\Sigma_R = \{2\}$, $p_1 = \frac{7}{10}$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$ for two different values of K_2. Both pictures depict $P^{100}(1)$ with P as in (4.17)*, where in* (a) *we have taken $K_2 = \frac{1}{10}$ (so $\eta < 1 < p_2 K_2^{-1}$) and in* (b) *$K_2 = \frac{6}{10}$ (so $\eta < p_2 K_2^{-1} < 1$).*

With Theorem 4.1.2 we can derive the following result, which says that the density $\frac{d\mu_{\boldsymbol{p}}}{d\lambda}$ in $L^1(\lambda) = L^1([0,1],\lambda)$ depends continuously on the probability vector $\boldsymbol{p} \in \mathbb{R}^N$ w.r.t. the $L^1(\lambda)$-norm. Here we write $\mu_{\boldsymbol{p}}$ for the acs probability measure that corresponds to the probability vector $\boldsymbol{p}$.

Corollary 4.1.3. *For each $n \in \mathbb{N}$, let $\boldsymbol{p}_n = (p_{n,j})_{j\in\Sigma}$ be a strictly positive probability vector such that $\sup_n \sum_{r\in\Sigma_R} p_{n,r} K_r^{-\alpha_{\min}} < 1$ and assume that $\lim_{n\to\infty} \boldsymbol{p}_n = \boldsymbol{p}$ in $\mathbb{R}^N_+$. Then*

$$\lim_{n\to\infty} \left\| \frac{d\mu_{\boldsymbol{p}_n}}{d\lambda} - \frac{d\mu_{\boldsymbol{p}}}{d\lambda} \right\|_{L^1(\lambda)} = 0.$$

Note that the convergence in Corollary 4.1.3 is stronger than in Corollary 2.1.5 where only weak convergence for the acs measure is derived.

Let us briefly give a heuristic explanation of why the value of η determines whether $(\mathcal{T}, \boldsymbol{p})$ admits an acs probability measure or not. We do this by referring to techniques involving inducing. First of all, as results like Kac's Lemma, Proposition 1.2.12 and the Young tower technique discussed in Chapter 3 indicate, given a subset Y in which orbits stay relatively short, the expected time[1] to first return to Y after leaving Y is finite typically if and only if an acs probability measure exists. Let us now argue for our random systems that for $\eta < 1$ ($\eta > 1$) the expected time to first return to $Y = \Sigma^{\mathbb{N}} \times (\frac{1}{2}, 1)$ after leaving Y is finite (infinite), thus suggesting the result of Theorem 4.1.1 and the first part of Theorem 4.1.2. If $\Sigma_R = \emptyset$, then our system is an i.i.d. random LSV map and as explained in Example 1.4.3 we know that in this case the existence of an acs probability measure depends on how long orbits stick close to zero. As follows from the results in [BBD14, BB16, Z18, BQT21], this stickiness at zero is governed by the LSV map with the fastest relaxation rate, i.e. having parameter $\alpha_{\min}$. In particular, a point $(\omega, x) \in Y$ with x close to $\frac{1}{2}$ typically needs of the order $(x - \frac{1}{2})^{-\alpha_{\min}}$ iterations under F to first return to Y as shown in [BB16, Theorem 1.1]. In this case the expected return time to Y behaves roughly as $\kappa := \int_{\frac{1}{2}}^{1} (x - \frac{1}{2})^{-\alpha_{\min}} dx$, which is finite if and only if $\alpha_{\min} < 1$. If $\Sigma_R \neq \emptyset$, then the influence of the stickiness at zero is enhanced because points in $(\frac{1}{2}, 1)$ close to $\frac{1}{2}$ are sent closer to zero when first a number of times maps from $\mathfrak{R}$ are applied before a map from $\mathfrak{S}$ is applied. In this case the expected return time to Y after leaving Y behaves roughly like

$$\sum_{m=0}^{\infty} \sum_{r_1 \in \Sigma_R} \cdots \sum_{r_m \in \Sigma_R} \Big(\prod_{j=1}^{m} p_{r_j} \Big) \int_{\frac{1}{2}}^{1} \Big(T_{r_m} \circ \cdots \circ T_{r_1}(x) - \frac{1}{2} \Big)^{-\alpha_{\min}} dx. \tag{4.6}$$

First of all, for all $r \in \Sigma_R$ and $x \in (\frac{1}{2}, 1)$ we have $T_r(x) \geq \frac{1}{2} + K_r(x - \frac{1}{2})$ and so the quantity in (4.6) can be bounded from above by $\kappa \sum_{m=0}^{\infty} \eta^m$. Hence, if $\eta < 1$, then it is reasonable to expect that the expected return time to Y after leaving Y is finite. On the other hand, if $\eta > 1$, then there exists $\varepsilon > 0$ small enough such that $\eta_\varepsilon := \sum_{r \in \Sigma_R} p_r (K_r + \varepsilon)^{-\alpha_{\min}} > 1$ as well. Since for $x \in (\frac{1}{2}, 1)$ sufficiently close to $\frac{1}{2}$ we have $T_r(x) \leq \frac{1}{2} + (K_r + \varepsilon)(x - \frac{1}{2})$, the quantity in (4.6) can be bounded from below by $\tilde{\kappa} \sum_{m=0}^{\infty} \eta_\varepsilon^m = \infty$ with $\tilde{\kappa} \in (0, \kappa]$. Hence, if $\eta > 1$, then this suggests that the expected return time to Y after leaving Y is infinite.

For the proof of Theorem 4.1.1 we will work out the above sketch in more detail and obtain the result using Kac's Lemma. On the other hand, for the proof of Theorem 4.1.2 we will not make use of an inducing technique. The first reason is that working out in precise detail the above sketch of bounding the expected return time still requires additional work that is not straightforward. Secondly, as we have seen in Subsection 2.2 and Chapter 3, inducing techniques often require the induced transformation to satisfy certain bounded distortion conditions, which are hard to obtain for the random systems in this chapter since the first branch of the maps T_j

[1] Here we mean with 'expected' that we take the expectation with respect to a reference measure, which in our random system is $m_{\boldsymbol{p}} \times \lambda$.

can have positive Schwarzian derivative (namely if $\alpha_j > 1$). For this reason but also because the maps T_r ($r \in \Sigma_R$) have a discontinuity, we cannot use the method from Subsection 2.3.2 either. Furthermore, we remark that not only in Chapters 2 and 3 but also in this chapter we cannot use the technique from Pelikan in [P84, Section 4] discussed in Section 2.1. The main reason is that the constituent maps S and T from Pelikan have competing behaviour at the same fixed point, whereas our systems are characterised by the interplay between the behaviour at two different fixed points.

Instead, for the case $\eta < 1$ we show the existence of an acs probability measure by considering a suitable set of functions that is invariant with respect to the Perron-Frobenius operator of the random system. We will then apply the Arzelà-Ascoli Theorem to prove that this set has a fixed point. This approach is similar to the one in Section 2 of [LSV99] where only one LSV map is considered.

The remainder of this chapter is organised as follows. Section 4.2 concentrates on proving Theorems 4.1.1 and 4.1.2 and Corollary 4.1.3. This chapter will be concluded in Section 4.3 with some final remarks.

§4.2 Phase transition for the acs measure

As in Section 4.1, let $T_1, \dots, T_N \in \mathfrak{S} \cup \mathfrak{R}$ be a finite collection, write $\Sigma_S = \{1 \le j \le N : T_j \in \mathfrak{S}\}$, $\Sigma_R = \{1 \le j \le N : T_j \in \mathfrak{R}\}$ and $\Sigma = \{1, \dots, N\} = \Sigma_S \cup \Sigma_R$ and assume that $\Sigma_S, \Sigma_R \neq \emptyset$ and $\alpha_{\min} < 1$. Furthermore, we again denote by F the skew product associated to $\mathcal{T} = \{T_j\}_{j\in\Sigma}$ given by (4.2), let $\boldsymbol{p} = (p_j)_{j\in\Sigma}$ be a probability vector with strictly positive entries and let $m_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-Bernoulli measure on $\Sigma^{\mathbb{N}}$. Also, recall that

$$\eta = \sum_{r\in\Sigma_R} p_r K_r^{-\alpha_{\min}}.$$

§4.2.1 The case $\eta > 1$

In this subsection we prove Theorem 4.1.1, namely that any acs measure for $(\mathcal{T}, \boldsymbol{p})$ must be infinite if $\eta > 1$. Throughout this subsection we use the notations for words and compositions of the maps T_j introduced in Section 1.4. Furthermore, we will use the following well-known results.

Let $j \in \Sigma$ and define the sequence $\{x_n(j)\}$ in $(0, \frac{1}{2}]$ by

$$x_1(j) = \frac{1}{2} \quad \text{and} \quad x_n(j) = T_j|_{[0,\frac{1}{2}]}^{-1}\big(x_{n-1}(j)\big) \text{ for each integer } n \ge 2.$$

As explained in e.g. the beginning of Section 6.2 of [Y99] there exists a constant $C_j > 1$ such that for each $n \in \mathbb{N}$

$$C_j^{-1} n^{-\frac{1}{\alpha_j}} \le x_n(j) \le C_j n^{-\frac{1}{\alpha_j}}. \tag{4.7}$$

Furthermore, we define for each $\omega \in \Sigma^{\mathbb{N}}$ the random sequence $\{x_n(\omega)\}$ in $(0, \frac{1}{2}]$ by

$$x_1(\omega) = \frac{1}{2} \quad \text{and} \quad x_n(\omega) = T_{\omega_1}|^{-1}_{[0,\frac{1}{2}]}(x_{n-1}(\tau\omega)) \text{ for each integer } n \geq 2.$$

Then, for each $\omega \in \Sigma^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$$T^{n-1}_{\omega}((x_{n+1}(\omega), x_n(\omega)]) = \Big(x_2(\tau^{n-1}\omega), \frac{1}{2}\Big]. \tag{4.8}$$

Letting $i \in \Sigma$ be such that $\alpha_i = \alpha_{\min}$, it has been shown in [BBD14, Lemma 4.4] that for each $\omega \in \Sigma^{\mathbb{N}}$ and $n \in \mathbb{N}$ we have

$$x_n(i) \leq x_n(\omega). \tag{4.9}$$

Proof of Theorem 4.1.1. Suppose that $\eta > 1$ and that μ is an acs probability measure for $(\mathcal{T}, \boldsymbol{p})$. We will use Kac's Lemma to arrive at a contradiction. Define

$$\begin{aligned} A_j &= \Big(x_2(j), T_j|^{-1}_{[0,\frac{1}{2}]}\Big(\frac{3}{4}\Big)\Big), \qquad j \in \Sigma, \\ B_j &= \Big(\frac{3}{4}, T_j|^{-1}_{(\frac{1}{2},1]}\Big(\frac{3}{4}\Big)\Big), \qquad j \in \Sigma, \\ Y &= \bigcup_{j\in\Sigma} [j] \times (A_j \cup B_j). \end{aligned}$$

We consider the first return time map φ_Y to Y under F as defined in (1.6). Since $\eta > 1$, there exists $\delta > 0$ small enough such that

$$\zeta := \sum_{r\in\Sigma_R} p_r M_r^{-\alpha_{\min}} \geq 1, \qquad \text{where } M_r := K_r + 2(1-K_r)\cdot\delta \ \ \forall r \in \Sigma_R. \tag{4.10}$$

For each $x \in (\frac{1}{2}, \frac{1}{2} + \delta)$ we have

$$T_r(x) = \frac{1}{2} + \Big(K_r + 2(1-K_r)\Big(x - \frac{1}{2}\Big)\Big)\Big(x - \frac{1}{2}\Big) \leq \frac{1}{2} + M_r\Big(x - \frac{1}{2}\Big). \tag{4.11}$$

For $\boldsymbol{r} = (r_1, \ldots, r_n) \in \Sigma_R^n$ we write $M_{\boldsymbol{r}} = \prod_{l=1}^n M_{r_l}$ with $M_{\boldsymbol{r}} = 1$ if $n = 0$. Furthermore, fix $t \in \Sigma_R$. It is easy to see that $\lim_{n\to\infty} T_t^n(\frac{3}{4}) = \frac{1}{2}$, so there exists an integer $k \geq 0$ such that $T_t^k(\frac{3}{4}) \in (\frac{1}{2}, \frac{1}{2} + \delta)$ holds.

Let $(\omega, x) \in Y$ and t and k be as above. Furthermore, fix $s \in \Sigma_S$. Suppose that

$$\omega \in [u\underbrace{t\cdots t}_{k \text{ times}}\boldsymbol{r}s] = [ut^k\boldsymbol{r}s], \quad \text{for some } u \in \Sigma,\ \boldsymbol{r} \in \Sigma_R^n, n \geq 0.$$

We then have $T^l_\omega(x) \in (\frac{1}{2}, \frac{3}{4})$ for all $1 \leq l \leq 1 + k + n$. It follows from $T_{\omega_1}(x) \leq \frac{3}{4}$, $T_t^k(\frac{3}{4}) \in (\frac{1}{2}, \frac{1}{2} + \delta)$ and (4.11) that

$$T^{1+k+n}_\omega(x) \leq T^{k+n}_{\tau\omega}\Big(\frac{3}{4}\Big) \leq \frac{1}{2} + M_{\boldsymbol{r}}\Big(T_t^k\Big(\frac{3}{4}\Big) - \frac{1}{2}\Big),$$

which gives

$$T_\omega^{2+k+n}(x) \le M_{\boldsymbol{r}} \Big(2T_t^k\Big(\frac{3}{4}\Big) - 1 \Big) \tag{4.12}$$

Fix $i \in \Sigma$ such that $\alpha_i = \alpha_{\min}$. There exists an $m \in \mathbb{N}$ such that $T_\omega^{2+k+n}(x) \in (x_{m+1}(i), x_m(i)]$. It follows from (4.8) and (4.9) that

$$\varphi_Y(\omega, x) \ge 2 + k + n + m. \tag{4.13}$$

We give a lower bound for m in terms of $\boldsymbol{r}$. It follows from (4.7) and (4.12) that

$$C_i^{-1}(m+1)^{-\frac{1}{\alpha_i}} \le M_{\boldsymbol{r}} \Big(2T_t^k\Big(\frac{3}{4}\Big) - 1 \Big).$$

Solving for m yields

$$m \ge D_1 \cdot M_{\boldsymbol{r}}^{-\alpha_i} - 1, \tag{4.14}$$

where we defined $D_1 = C_i^{-\alpha_i} \cdot (2T_t^k(\frac{3}{4}) - 1)^{-\alpha_i}$. Combining (4.13) and (4.14) yields

$$\begin{aligned} \int_Y \varphi_Y \, dm_{\boldsymbol{p}} \times \mu &\ge \sum_{u\in\Sigma} \sum_{n=0}^{\infty} \sum_{\boldsymbol{r}\in\Sigma_R^n} \int_{[ut^k\boldsymbol{r}s]\times(A_u\cup B_u)} \varphi_Y \, dm_{\boldsymbol{p}} \times \mu \\ &\ge \sum_{u\in\Sigma} \sum_{n=0}^{\infty} \sum_{\boldsymbol{r}\in\Sigma_R^n} m_{\boldsymbol{p}}\big([ut^k\boldsymbol{r}s]\big) \int_{A_u\cup B_u} D_1 \cdot M_{\boldsymbol{r}}^{-\alpha_i} d\mu(x) \\ &= D_2 \cdot \sum_{n=0}^{\infty} \zeta^n, \end{aligned} \tag{4.15}$$

where

$$D_2 = D_1 \cdot p_t^k p_s \cdot \sum_{u\in\Sigma} p_u \mu(A_u \cup B_u) = D_1 \cdot p_t^k p_s \cdot m_{\boldsymbol{p}} \times \mu(Y).$$

Almost every orbit that starts in $\Sigma^{\mathbb{N}} \times [0,1]$ will eventually enter $\Sigma^{\mathbb{N}} \times (\frac{1}{2}, \frac{3}{4})$, either via $\bigcup_{j\in\Sigma}[j] \times A_j$ or via $\bigcup_{j\in\Sigma}[j] \times B_j$. Hence, we have $\bigcup_{n=0}^{\infty} F^{-n}Y = \Sigma^{\mathbb{N}} \times [0,1]$ up to some set of measure zero, i.e. Y is a sweep-out set. This together with the F-invariance of $m_{\boldsymbol{p}} \times \mu$ yields

$$1 = m_{\boldsymbol{p}} \times \mu(\Sigma^{\mathbb{N}} \times [0,1]) \le \sum_{n=0}^{\infty} m_{\boldsymbol{p}} \times \mu(F^{-n}Y) = \sum_{n=0}^{\infty} m_{\boldsymbol{p}} \times \mu(Y).$$

This gives $m_{\boldsymbol{p}} \times \mu(Y) > 0$ and so $D_2 > 0$. Hence, from (4.15) and $\zeta \ge 1$ it now follows that

$$\int_Y \varphi_Y \, dm_{\boldsymbol{p}} \times \mu = \infty. \tag{4.16}$$

On the other hand, since μ is a probability measure by assumption, we obtain from the Ergodic Decomposition Theorem and Kac's Lemma in a similar way as in Subsection 3.3.1 that

$$\int_Y \varphi_Y \, dm_{\boldsymbol{p}} \times \mu \le 1,$$

which is in contradiction with (4.16). □

§4.2.2 The case $\eta < 1$

In this subsection we will prove Theorem 4.1.2 and Corollary 4.1.3. For this we will identify a suitable set of functions which is preserved by the Perron-Frobenius operator $P = P_{\mathcal{T},\boldsymbol{p}}$ associated to $(\mathcal{T}, \boldsymbol{p})$ being of the form as in (1.20). We will do this in a number of steps in a way that is similar to the approach of Section 2 in [LSV99].

Suppose $\eta < 1$. On $[0,1]$ we define for each $j \in \Sigma$ the functions $x \mapsto y_j(x)$ and $x \mapsto \xi_j(x)$ by $y_j(x) = (T_j|_{[0,\frac{1}{2}]})^{-1}(x)$ and $\xi_j(x) = (2y_j(x))^{\alpha_j}$. Furthermore, we define on $[0,1]$ the function $z(x) = \frac{x+1}{2}$ and on $(\frac{1}{2}, 1]$ we define for each $r \in \Sigma_R$ the function $z_r(x) = (T_r|_{(\frac{1}{2},1]})^{-1}(x)$. Whenever convenient, we will just write y_j for $y_j(x)$ and similarly for ξ_j, z and z_r. Writing $p_S = \sum_{s\in\Sigma_S} p_s$, we then have

$$Pf(x) = \begin{cases} \sum_{j\in\Sigma} p_j \frac{f(y_j)}{1+(\alpha_j+1)\xi_j} + p_S \frac{f(z)}{2}, & x \in [0, \frac{1}{2}] \\ \sum_{j\in\Sigma} p_j \frac{f(y_j)}{1+(\alpha_j+1)\xi_j} + p_S \frac{f(z)}{2} + \sum_{r\in\Sigma_R} p_r \frac{f(z_r)}{DR_{\alpha,K}(z_r)}, & x \in (\frac{1}{2}, 1]. \end{cases} \tag{4.17}$$

Note that $x \mapsto y_j(x)$, $x \mapsto \xi_j(x)$, $x \mapsto z(x)$ and $x \mapsto z_r(x)$ are increasing and continuous on $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$. This in combination with the fact that $R_{\alpha,K}$ is C^1 on $(\frac{1}{2}, 1]$ with increasing derivative gives that the set

$$\mathcal{C}_0 = \left\{ f \in L^1(\lambda) : f \geq 0, f \text{ decreasing and continuous on } \left(0, \frac{1}{2}\right] \text{ and } \left(\frac{1}{2}, 1\right] \right\}$$

is preserved by P, i.e. $P\mathcal{C}_0 \subseteq \mathcal{C}_0$.

Since $\eta < 1$, we have $\gamma = \sup\{\delta \geq 0 : \sum_{r\in\Sigma_R} p_r K_r^{-\delta} < 1\} > \alpha_{\min}$, so $(\alpha_{\min}, \gamma)$ is non-empty. In the remainder of this subsection we fix a $\beta \in (\alpha_{\min}, \gamma) \cap (0, 1]$. We set $\alpha_{\max} = \max\{\alpha_j : j \in \Sigma\}$ and $d = \alpha_{\max} + 2$. We need the following two lemmas.

Lemma 4.2.1. *For each $\alpha > 0$ the function $x \mapsto \frac{(1+x)^d}{1+(\alpha+1)x}$ is increasing on $[0,1]$.*

Proof. Set

$$f_\alpha(x) = \frac{(1+x)^d}{1+(\alpha+1)x}, \qquad x \in [0,1].$$

Furthermore, set $g(x) = (1+x)^d$ and $h_\alpha(x) = 1 + (\alpha+1)x$ where $x \in [0,1]$. Then

$$f'_\alpha(x) = \frac{h_\alpha(x)g'(x) - g(x)h'_\alpha(x)}{h_\alpha(x)^2}.$$

We have

$$\begin{aligned} h_\alpha(x)g'(x) &= (1 + (\alpha+1)x) \cdot d(1+x)^{d-1} \\ &\geq (1+x)^d \cdot d \geq (1+x)^d \cdot (\alpha+1) \\ &= g(x)h'_\alpha(x), \end{aligned}$$

so $f'_\alpha(x) \geq 0$ holds for all $x \in [0,1]$. □

CHAPTER 4

Define for each $K > 0$ and $b \geq 0$ the function $H_{K,b} : (\frac{1}{2}, 1] \to \mathbb{R}$ by

$$H_{K,b}(x) = \frac{(K + 2(1-K)(x-\frac{1}{2}))^b}{K + 4(1-K)(x-\frac{1}{2})}, \qquad x \in \Big(\frac{1}{2}, 1\Big].$$

Lemma 4.2.2. *Let $K > 0$ and $b \geq 0$.*

(i) If $b \geq 2$, then $H_{K,b}$ is increasing.

(ii) If $b \leq 1$, then $H_{K,b}$ is decreasing.

Proof. Set $f_K(x) = K + 2(1-K)(x - \frac{1}{2})$ and $g_K(x) = K + 4(1-K)(x-\frac{1}{2})$ where $x \in (\frac{1}{2}, 1]$. Note that $g_K'(x) = 2f_K'(x)$. Then for $x \in (\frac{1}{2}, 1)$

$$\begin{aligned} H_{K,b}'(x) &= \frac{g_K(x) \cdot b \cdot f_K(x)^{b-1} f_K'(x) - f_K(x)^b \cdot g_K'(x)}{g_K(x)^2} \\ &= \frac{f_K(x)^b \cdot f_K'(x)\big(b \cdot \frac{g_K(x)}{f_K(x)} - 2\big)}{g_K(x)^2}. \end{aligned}$$

If $b \geq 2$, then

$$b \cdot \frac{g_K(x)}{f_K(x)} - 2 \geq 2 \cdot \frac{g_K(x)}{f_K(x)} - 2 \geq 2 \cdot \frac{f_K(x)}{f_K(x)} - 2 = 0$$

and thus $H_{K,b}'(x) \geq 0$. This proves (i). If $b \leq 1$, then

$$b \cdot \frac{g_K(x)}{f_K(x)} - 2 \leq \frac{g_K(x)}{f_K(x)} - 2 \leq \frac{2f_K(x)}{f_K(x)} - 2 = 0$$

and thus $H_{K,b}'(x) \leq 0$. This proves (ii). □

We can now prove the following lemma.

Lemma 4.2.3. *The set*

$$\mathcal{C}_1 = \Big\{ f \in \mathcal{C}_0 : x \mapsto x^d f(x) \text{ incr. on } \Big(0, \frac{1}{2}\Big],\ x \mapsto \Big(x - \frac{1}{2}\Big)^d f(x) \text{ incr. on } \Big(\frac{1}{2}, 1\Big] \Big\}$$

is preserved by P.

Proof. Let $f \in \mathcal{C}_1$. Let $x \in (0, \frac{1}{2}]$. Using that for each $j \in \Sigma$ we have $x = y_j(1+\xi_j)$ and that $z(x) - \frac{1}{2} = \frac{x}{2}$, we obtain

$$\begin{aligned} x^d Pf(x) &= \sum_{j \in \Sigma} p_j \Big(\frac{x}{y_j}\Big)^d \frac{y_j^d f(y_j)}{1 + (\alpha_j + 1)\xi_j} + \frac{p_S}{2} \Big(\frac{x}{z - \frac{1}{2}}\Big)^d \Big(z - \frac{1}{2}\Big)^d f(z) \\ &= \sum_{j \in \Sigma} p_j \frac{(1+\xi_j)^d}{1 + (\alpha_j+1)\xi_j} \cdot y_j^d f(y_j) + p_S \cdot 2^{d-1} \cdot \Big(z - \frac{1}{2}\Big)^d f(z). \end{aligned}$$

Because $x \mapsto \xi_j(x)$ is increasing for each $j \in \Sigma$ it follows from Lemma 4.2.1 that $x \mapsto \frac{(1+\xi_j(x))^d}{1+(\alpha_j+1)\xi_j(x)}$ is increasing for each $j \in \Sigma$. Combining this with the fact that $f \in \mathcal{C}_1$, that $y_j \in (0, \frac{1}{2}]$ for each $j \in \Sigma$ and that $z \in (\frac{1}{2}, 1]$ we conclude that $x \mapsto x^d Pf(x)$ is increasing on $(0, \frac{1}{2}]$.

Now let $x \in (\frac{1}{2}, 1]$. Then

$$\begin{aligned}\Big(x-\frac{1}{2}\Big)^d Pf(x) &= \Big(\frac{x-\frac{1}{2}}{x}\Big)^d \sum_{j\in\Sigma} p_j \Big(\frac{x}{y_j}\Big)^d \frac{y_j^d f(y_j)}{1+(\alpha_j+1)\xi_j} \\ &+ \frac{p_S}{2}\Big(\frac{x-\frac{1}{2}}{z-\frac{1}{2}}\Big)^d \Big(z-\frac{1}{2}\Big)^d f(z) \\ &+ \sum_{r\in\Sigma_R} \frac{p_r}{DR_{\alpha_r,K_r}(z_r)} \Big(\frac{x-\frac{1}{2}}{z_r-\frac{1}{2}}\Big)^d \Big(z_r-\frac{1}{2}\Big)^d f(z_r).\end{aligned}$$

Using again that for each $j \in \Sigma$ we have $x = y_j(1+\xi_j)$, that $z - \frac{1}{2} = \frac{x}{2}$ and also that $x - \frac{1}{2} = K_r(z_r - \frac{1}{2}) + 2(1-K_r)(z_r - \frac{1}{2})^2$ for each $r \in \Sigma_R$, we obtain

$$\begin{aligned}\Big(x-\frac{1}{2}\Big)^d Pf(x) &= \Big(1-\frac{1}{2x}\Big)^d \sum_{j\in\Sigma} p_j \frac{(1+\xi_j)^d}{1+(\alpha_j+1)\xi_j} \cdot y_j^d f(y_j) \\ &+ \frac{p_S}{2}\Big(2-\frac{1}{x}\Big)^d \Big(z-\frac{1}{2}\Big)^d f(z) \\ &+ \sum_{r\in\Sigma_R} p_r \frac{(K_r+2(1-K_r)(z_r-\frac{1}{2}))^d}{K_r+4(1-K_r)(z_r-\frac{1}{2})} \Big(z_r-\frac{1}{2}\Big)^d f(z_r).\end{aligned}$$

Note that $x \mapsto (1-\frac{1}{2x})^d$ and $x \mapsto (2-\frac{1}{x})^d$ are positive and increasing on $(\frac{1}{2}, 1]$. Combining this with Lemma 4.2.1 and Lemma 4.2.2(i) and with the fact that $f \in \mathcal{C}_1$ we conclude that $x \mapsto (x-\frac{1}{2})^d Pf(x)$ is increasing on $(\frac{1}{2}, 1]$. □

We set $t_1 = \alpha_{\min} + 1 - \beta$ and $t_2 = 1 - \beta$. It follows from $\beta \in (\alpha_{\min}, 1]$ that $t_1 \in [\alpha_{\min}, 1)$ and $t_2 \in [0, 1-\alpha_{\min})$.

Lemma 4.2.4. *For sufficiently large $a_1, a_2 > 0$, the set*

$$\mathcal{C}_2 = \Big\{ f \in \mathcal{C}_1 : f(x) \le a_1 x^{-t_1} \text{ on } \Big(0, \frac{1}{2}\Big], f(x) \le a_2\Big(x-\frac{1}{2}\Big)^{-t_2} \text{ on } \Big(\frac{1}{2}, 1\Big], \int_0^1 f d\lambda = 1 \Big\}$$

is preserved by P.

Proof. Let $f \in \mathcal{C}_2$. First, let $x \in (\frac{1}{2}, 1]$. For each $j \in \Sigma$ we have $y_j \le \frac{1}{2}$ and thus, using that $f \in \mathcal{C}_1$,

$$y_j^d f(y_j) \le 2^{-d} f\Big(\frac{1}{2}\Big) \le 2^{-d} \cdot 2 \cdot \int_0^{\frac{1}{2}} f(u)du \le 2^{-d+1}.$$

Furthermore, for each $j \in \Sigma$ we have

$$T_j\Big(\frac{1}{4}\Big) = \frac{1}{4}(1+2^{-\alpha_j}) \le \frac{1}{4}(1+1) = \frac{1}{2},$$

which gives $y_j \in (\frac{1}{4}, \frac{1}{2}]$. Setting $M := 2^{d+1}$ we obtain for each $j \in \Sigma$ that

$$\frac{f(y_j)}{1+(\alpha_j+1)\xi_j} = y_j^d f(y_j) \cdot \frac{y_j^{-d}}{1+(\alpha_j+1)\cdot(2y_j)^{\alpha_j}} \le 2^{-d+1}\cdot 4^d = M. \tag{4.18}$$

It also follows from $f \in \mathcal{C}_1$ that

$$\left(z-\frac{1}{2}\right)^d f(z) \le \left(1-\frac{1}{2}\right)^d f(1) \le 2^{-d}\cdot 2\cdot \int_{\frac{1}{2}}^1 f(u)du \le 2^{-d+1}.$$

Using that $z \in (\frac{3}{4}, 1]$, this gives

$$f(z) \le 2^{-d+1}\cdot\left(z-\frac{1}{2}\right)^{-d} \le 2^{-d+1}\cdot\left(\frac{3}{4}-\frac{1}{2}\right)^{-d} = M. \tag{4.19}$$

Combining (4.17), (4.18) and (4.19) and using that $f \in \mathcal{C}_2$ gives

$$Pf(x) \le M + \frac{p_S}{2}\cdot M + \sum_{r\in\Sigma_R} \frac{p_r}{DR_{\alpha_r,K_r}(z_r)}\cdot a_2\left(z_r-\frac{1}{2}\right)^{-t_2}.$$

For each $r \in \Sigma_R$ we have $x-\frac{1}{2} = K_r(z_r-\frac{1}{2}) + 2(1-K_r)(z_r-\frac{1}{2})^2$ and therefore

$$\frac{1}{DR_{\alpha_r,K_r}(z_r)}\left(\frac{x-\frac{1}{2}}{z_r-\frac{1}{2}}\right)^{t_2} = \frac{(K_r+2(1-K_r)(z_r-\frac{1}{2}))^{t_2}}{K_r+4(1-K_r)(z_r-\frac{1}{2})},$$

which by Lemma 4.2.2(ii) can be bounded from above by $H_{K_r,t_2}(\frac{1}{2}) = K_r^{t_2-1}$. Furthermore, since $t_2 \ge 0$ we have $(x-\frac{1}{2})^{t_2} \le 2^{-t_2}$. We obtain

$$Pf(x) \le \left\{\frac{M(1+\frac{p_S}{2})\cdot 2^{-t_2}}{a_2} + \sum_{r\in\Sigma_R} p_r\cdot K_r^{t_2-1}\right\}\cdot a_2\cdot\left(x-\frac{1}{2}\right)^{-t_2}. \tag{4.20}$$

We have $t_2-1=-\beta$ and $\beta<\gamma$, so

$$\sum_{r\in\Sigma_R} p_r\cdot K_r^{t_2-1} = \sum_{r\in\Sigma_R} p_r\cdot K_r^{-\beta} < 1.$$

Hence, there exists an $a_2>0$ sufficiently large such that the term in braces in (4.20) is bounded by 1.

Now let $x \in (0, \frac{1}{2}]$. Using that $f \in \mathcal{C}_2$, it follows from (4.17) that

$$Pf(x) \le \sum_{j\in\Sigma} p_j \frac{a_1\cdot y_j^{-t_1}}{1+(\alpha_j+1)\xi_j} + \frac{p_S\cdot a_2}{2}\cdot\left(z-\frac{1}{2}\right)^{-t_2}. \tag{4.21}$$

For each $j \in \Sigma$ we have, using that $x = y_j(1+\xi_j)$ and that $t_1 \in (0,1)$,

$$\frac{y_j^{-t_1}}{1+(\alpha_j+1)\xi_j} = \frac{x^{-t_1}(1+\xi_j)^{t_1}}{1+(\alpha_j+1)\xi_j} \le \frac{x^{-t_1}(1+t_1\xi_j)}{1+(\alpha_j+1)\xi_j} \le x^{-t_1}. \tag{4.22}$$

Fix an $i \in \Sigma$ with $\alpha_i = \alpha_{\min}$. Applying for each $j \in \Sigma \backslash \{i\}$ the bound (4.22) to (4.21) and using that $z - \frac{1}{2} = \frac{x}{2}$ yields

$$Pf(x) \leq \Big\{ p_i \Big(\frac{x}{y_i}\Big)^{t_1} \cdot \frac{1}{1+(\alpha_i+1)\xi_i} + (1-p_i) + \frac{p_S \cdot a_2 \cdot 2^{t_2-1}}{a_1} \cdot x^{t_1-t_2} \Big\} \cdot a_1 \cdot x^{-t_1}. \tag{4.23}$$

It remains to find a_1 sufficiently large such that the term in braces in (4.23) is bounded by 1. First of all, using again that $x = y_i(1+\xi_i)$ and that $t_1 \in (0,1)$ we get

$$\Big(\frac{x}{y_i}\Big)^{t_1} \cdot \frac{1}{1+(\alpha_i+1)\xi_i} = \frac{(1+\xi_i)^{t_1}}{1+(\alpha_i+1)\xi_i} \leq \frac{1+t_1\xi_i}{1+(\alpha_i+1)\xi_i}. \tag{4.24}$$

Furthermore, we have

$$x^{t_1-t_2} = x^{\alpha_{\min}} = y_i^{\alpha_i}(1+\xi_i)^{\alpha_i} \leq y_i^{\alpha_i} \cdot 2^{\alpha_i} = \xi_i. \tag{4.25}$$

It follows from (4.24) and (4.25) that the term in braces in (4.23) is bounded by

$$p_i \frac{1 + t_1\xi_i + \frac{p_i^{-1} \cdot p_S \cdot a_2 \cdot 2^{t_2-1}}{a_1} \cdot \xi_i \cdot (1+(\alpha_i+1)\xi_i)}{1+(\alpha_i+1)\xi_i} + (1-p_i). \tag{4.26}$$

Using that $1+(\alpha_i+1)\xi_i \leq \alpha_i + 2$ we get that the numerator in (4.26) is bounded by

$$1 + \Big(t_1 + \frac{p_i^{-1} \cdot p_S \cdot a_2 \cdot 2^{t_2-1}(\alpha_i+2)}{a_1}\Big)\xi_i.$$

Taking $a_1 > 0$ sufficiently large such that $t_1 + \frac{p_i^{-1} \cdot p_S \cdot a_2 \cdot 2^{t_2-1}(\alpha_i+2)}{a_1} \leq 1 \leq \alpha_i + 1$ now yields the result. □

Lemma 4.2.5. *The set $\mathcal{C}_2$ is compact with respect to the $L^1(\lambda)$-norm.*

Proof. For each $f \in \mathcal{C}_2$ let ϕ_f denote the continuous extension of $(0, \frac{1}{2}] \ni x \mapsto x^d f(x)$ to $[0, \frac{1}{2}]$ and let ψ_f denote the continuous extension of $(\frac{1}{2}, 1] \ni x \mapsto (x - \frac{1}{2})^d f(x)$ to $[\frac{1}{2}, 1]$. Furthermore, we define $\mathcal{A}_1 = \{\phi_f : f \in \mathcal{C}_2\}$ and $\mathcal{A}_2 = \{\psi_f : f \in \mathcal{C}_2\}$. For each $f \in \mathcal{C}_2$ we have, for $x, y \in [0, \frac{1}{2}]$ with $x \geq y$, that

$$\begin{aligned} 0 \leq \phi_f(x) - \phi_f(y) &\leq f(x)(x^d - y^d) \leq a_1 x^{-t_1} \cdot d \int_y^x t^{d-1} dt \\ &\leq a_1 x^{d-1-t_1} \cdot d|x-y| \leq a_1 \cdot 2^{-d+1+t_1} \cdot d|x-y|. \end{aligned} \tag{4.27}$$

and for $x, y \in [\frac{1}{2}, 1]$ with $x \geq y$, that

$$\begin{aligned} 0 \leq \psi_f(x) - \psi_f(y) &\leq f(x)\Big(\Big(x - \frac{1}{2}\Big)^d - \Big(y - \frac{1}{2}\Big)^d\Big) \\ &\leq a_2 \Big(x - \frac{1}{2}\Big)^{-t_2} \cdot d \int_y^x \Big(t - \frac{1}{2}\Big)^{d-1} dt \\ &\leq a_2 \Big(x - \frac{1}{2}\Big)^{d-1-t_2} \cdot d|x-y| \leq a_2 \cdot 2^{-d+1+t_2} \cdot d|x-y|. \end{aligned} \tag{4.28}$$

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Also, from the definition of $\mathcal{C}_2$ in Lemma 4.2.4 and the fact that $d > \max\{t_1, t_2\}$ we see that $\phi_f(0) = \psi_f(\frac{1}{2}) = 0$ holds for each $f \in \mathcal{C}_2$. It follows that $\mathcal{A}_1$ and $\mathcal{A}_2$ are uniformly bounded and equicontinuous, so from the Arzelà-Ascoli Theorem we obtain that $\mathcal{A}_1$ and $\mathcal{A}_2$ are compact in $C([0, \frac{1}{2}])$ and $C([\frac{1}{2}, 1])$, respectively, w.r.t. the supremum norm.

Now let $\{f_n\}$ be a sequence in $\mathcal{C}_2$. It follows from the above that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{\phi_{f_{n_k}}\}$ converges uniformly to some $\phi^* \in C([0, \frac{1}{2}])$ and $\{\psi_{f_{n_k}}\}$ converges uniformly to some $\psi^* \in C([\frac{1}{2}, 1])$ (for this we take a suitable subsequence of a subsequence of $\{f_n\}$). Now define the measurable function f^* on $(0, 1]$ by

$$f^*(x) = \begin{cases} x^{-d}\phi^*(x) & \text{if} \quad x \in (0, \frac{1}{2}], \\ (x - \frac{1}{2})^{-d}\psi^*(x) & \text{if} \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

Then f^* is continuous on $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$. Moreover, $\{f_{n_k}\}$ converges pointwise to f^*. First of all, this gives $f^* \in \mathcal{C}_1$ once we know $f^* \in L^1(\lambda)$. Secondly, this gives combined with

$$\sup_{k\in\mathbb{N}} f_{n_k}(x) \le a_1 x^{-t_1} \text{ for } x \in \Big(0, \frac{1}{2}\Big], \qquad \sup_{k\in\mathbb{N}} f_{n_k}(x) \le a_2\Big(x - \frac{1}{2}\Big)^{-t_2} \text{ for } x \in \Big(\frac{1}{2}, 1\Big]$$

and

$$\int_0^{\frac{1}{2}} x^{-t_1} dx < \infty, \qquad \int_{\frac{1}{2}}^1 \Big(x - \frac{1}{2}\Big)^{-t_2} dx < \infty,$$

that $f^*(x) \le a_1 x^{-t_1}$ for $x \in (0, \frac{1}{2}]$ and $f^*(x) \le a_2(x - \frac{1}{2})^{-t_2}$ for $x \in (\frac{1}{2}, 1]$, and that

$$\lim_{k\to\infty} \|f^* - f_{n_k}\|_1 = 0 \quad \text{and so} \quad \int_0^1 f^* d\lambda = 1$$

using the Dominated Convergence Theorem. We conclude that $f^* \in \mathcal{C}_2$ and that f^* is a limit point of $\{f_n\}$ with respect to the $L^1(\lambda)$-norm. □

Using the previous lemmas we are now ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. (1) Take $f \in \mathcal{C}_2$ and define the sequence of functions $\{f_n\}$ by $f_n = \frac{1}{n}\sum_{i=0}^{n-1} P^i f$. Using that P preserves $\mathcal{C}_2$ and that the average of a finite collection of elements of $\mathcal{C}_2$ is also an element of $\mathcal{C}_2$, we obtain that $\{f_n\}$ is a sequence in $\mathcal{C}_2$. It follows from Lemma 4.2.5 that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ that converges w.r.t. the $L^1(\lambda)$-norm to some $f^* \in \mathcal{C}_2$. As is standard, we then obtain that $Pf^*(x) = f^*(x)$ holds for λ-a.e. $x \in [0, 1]$ by noting that

$$\begin{aligned} \|Pf^* - f^*\|_1 &\le \|Pf^* - Pf_{n_k}\|_1 + \|Pf_{n_k} - f_{n_k}\|_1 + \|f_{n_k} - f^*\|_1 \\ &\le 2\|f_{n_k} - f^*\|_1 + \Big\|\frac{1}{n_k}\sum_{i=0}^{n_k-1} P^{i+1}f - \frac{1}{n_k}\sum_{i=0}^{n_k-1} P^i f\Big\|_1 \\ &\le 2\|f_{n_k} - f^*\|_1 + \frac{1}{n_k}\|P^{n_k}f - f\|_1 \\ &\le 2\|f_{n_k} - f^*\|_1 + \frac{2}{n_k}\|f\|_1 \to 0, \qquad k \to \infty. \end{aligned}$$

Hence, $(\mathcal{T}, \boldsymbol{p})$ admits an acs probability measure μ with $\frac{d\mu}{d\lambda} \in \mathcal{C}_2$. It follows from the properties of $\mathcal{C}_2$ that $\frac{d\mu}{d\lambda}$ has full support on $[0,1]$, i.e. there is a version such that $\frac{d\mu}{d\lambda}(x) > 0$ for all $x \in [0,1]$, so we obtain from Theorem 1.2.6 that μ is the only acs probability measure once we know that F is ergodic with respect to $m_{\boldsymbol{p}} \times \mu$. So let $A \subseteq \Sigma^{\mathbb{N}} \times [0,1]$ be Borel measurable such that $F^{-1}A = A$. Suppose $m_{\boldsymbol{p}} \times \mu(A) > 0$. The probability measure ρ on $\Sigma^{\mathbb{N}} \times [0,1]$ given by

$$\rho(B) = \frac{m_{\boldsymbol{p}} \times \mu(A \cap B)}{m_{\boldsymbol{p}} \times \mu(A)}$$

for Borel measurable sets $B \subseteq \Sigma^{\mathbb{N}} \times [0,1]$ is F-invariant and absolutely continuous with respect to $m_{\boldsymbol{p}} \times \lambda$ with density

$$\frac{d\rho}{dm_{\boldsymbol{p}} \times \lambda}(\omega, x) = \frac{1}{m_{\boldsymbol{p}} \times \mu(A)} 1_A(\omega, x) \frac{d\mu}{d\lambda}(x), \qquad m_{\boldsymbol{p}} \times \lambda\text{-a.e.} \tag{4.29}$$

According to Lemma 1.4.1 this yields an acs measure $\tilde{\mu}$ for $(\mathcal{T}, \boldsymbol{p})$ such that $\rho = m_{\boldsymbol{p}} \times \tilde{\mu}$. From this we see that also

$$\frac{d\rho}{dm_{\boldsymbol{p}} \times \lambda}(\omega, x) = \frac{d\tilde{\mu}}{d\lambda}(x), \qquad m_{\boldsymbol{p}} \times \lambda\text{-a.e.} \tag{4.30}$$

Write L for the support of $\frac{d\tilde{\mu}}{d\lambda}$, i.e. $L := \{x \in [0,1] : \frac{d\tilde{\mu}}{d\lambda}(x) > 0\}$. Combining (4.29) and (4.30) and using that $\frac{d\mu}{d\lambda}$ has full support on $[0,1]$, we obtain

$$A = \Sigma^{\mathbb{N}} \times L \mod m_{\boldsymbol{p}} \times \lambda. \tag{4.31}$$

Using the non-singularity of F with respect to $m_{\boldsymbol{p}} \times \lambda$, we also obtain from this that

$$F^{-1}A = F^{-1}(\Sigma^{\mathbb{N}} \times L) \mod m_{\boldsymbol{p}} \times \lambda. \tag{4.32}$$

Combining (4.31) and (4.32) with

$$\Sigma^{\mathbb{N}} \times L = \bigcup_{j \in \Sigma} [j] \times L \quad \text{and} \quad F^{-1}(\Sigma^{\mathbb{N}} \times L) = \bigcup_{j \in \Sigma} [j] \times T_j^{-1} L$$

yields

$$L = T_j^{-1} L \mod \lambda$$

for each $j \in \Sigma$. For all $i \in \Sigma$ with $\alpha_i < 1$, in particular for $i \in \Sigma$ with $\alpha_i = \alpha_{\min}$, we have that T_i is ergodic with respect to λ, see e.g. [Y99, Theorem 5]. In particular we have $\lambda(L) \in \{0,1\}$. Together with (4.31) this shows that $m_{\boldsymbol{p}} \times \lambda(A) \in \{0,1\}$. Since $\mu \ll \lambda$, it follows from the assumption $m_{\boldsymbol{p}} \times \mu(A) > 0$ that $m_{\boldsymbol{p}} \times \mu(A) = 1$. We conclude that F is ergodic with respect to $m_{\boldsymbol{p}} \times \mu$.

(2) Since $\frac{d\mu}{d\lambda} \in \mathcal{C}_2$, it follows that $\frac{d\mu}{d\lambda}$ is bounded away from zero, is decreasing on the intervals $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$, and satisfies (4.3) and (4.4) with $a_1, a_2 > 0$ as in

Lemma 4.2.4. Furthermore, applying the last three inequalities in (4.27) with $f = \frac{d\mu}{d\lambda}$ yields, for $x, y \in (0, \frac{1}{2}]$ with $x \geq y$,

$$
\begin{aligned}
0 \leq \frac{d\mu}{d\lambda}(y) - \frac{d\mu}{d\lambda}(x) &= y^{-d}\Big(y^d \frac{d\mu}{d\lambda}(y) - y^d \frac{d\mu}{d\lambda}(x)\Big) \\
&\leq y^{-d} \cdot \frac{d\mu}{d\lambda}(x)(x^d - y^d) \\
&\leq y^{-d} \cdot a_1 \cdot 2^{-d+1+t_1} \cdot d|x-y|
\end{aligned}
$$

and likewise applying the last three inequalities in (4.28) with $f = \frac{d\mu}{d\lambda}$ yields for $x, y \in (\frac{1}{2}, 1]$ with $x \geq y$,

$$
\begin{aligned}
0 \leq \frac{d\mu}{d\lambda}(y) - \frac{d\mu}{d\lambda}(x) &= \Big(y - \frac{1}{2}\Big)^{-d}\Big(\Big(y - \frac{1}{2}\Big)^d \frac{d\mu}{d\lambda}(y) - \Big(y - \frac{1}{2}\Big)^d \frac{d\mu}{d\lambda}(x)\Big) \\
&\leq \Big(y - \frac{1}{2}\Big)^{-d} \cdot \frac{d\mu}{d\lambda}(x)\Big(\Big(x - \frac{1}{2}\Big)^d - \Big(y - \frac{1}{2}\Big)^d\Big) \\
&\leq \Big(y - \frac{1}{2}\Big)^{-d} \cdot a_2 \cdot 2^{-d+1+t_2} \cdot d|x-y|
\end{aligned}
$$

Hence, $\frac{d\mu}{d\lambda}$ is locally Lipschitz on the intervals $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$. □

We conclude this subsection with the proof of Corollary 4.1.3.

Proof of Corollary 4.1.3. For each $n \in \mathbb{N}$, let $\boldsymbol{p}_n = (p_{n,j})_{j\in\Sigma}$ be a strictly positive probability vector such that $\sup_n \sum_{r\in\Sigma_R} p_{n,r} K_r^{-\alpha_{\min}} < 1$ and assume that $\lim_{n\to\infty} \boldsymbol{p}_n = \boldsymbol{p}$ in $\mathbb{R}_+^N$. In order to conclude that $\frac{d\mu_{\boldsymbol{p}_n}}{d\lambda}$ converges in $L^1(\lambda)$ to $\frac{d\mu_{\boldsymbol{p}}}{d\lambda}$ we will show that each subsequence of $\{\frac{d\mu_{\boldsymbol{p}_n}}{d\lambda}\}$ has a further subsequence that converges in $L^1(\lambda)$ to $\frac{d\mu_{\boldsymbol{p}}}{d\lambda}$.

Let $\{\boldsymbol{q}_k\}$ be a subsequence of $\{\boldsymbol{p}_n\}$, and for convenience write $f_k = \frac{d\mu_{\boldsymbol{q}_k}}{d\lambda}$ for each $k \in \mathbb{N}$. First of all, observe that from $\sup_n \sum_{r\in\Sigma_R} p_{n,r} K_r^{-\alpha_{\min}} < 1$ and $\lim_{n\to\infty} \boldsymbol{p}_n = \boldsymbol{p}$ it follows from the proof of Lemma 4.2.4 that there exist sufficiently large $a_1, a_2 > 0$ and $\beta \in (\alpha_{\min}, \gamma)$ sufficiently close to $\alpha_{\min}$ such that $\mathcal{C}_2 = \mathcal{C}_2(a_1, a_2, \beta)$ from Lemma 4.2.4 contains the sequence $\{f_k\}$. Hence, it follows from Lemma 4.2.5 that $\{f_k\}$ has a subsequence $\{f_{k_m}\}$ that converges with respect to the $L^1(\lambda)$-norm to some $\tilde{f} \in \mathcal{C}_2$. We have

$$
\begin{aligned}
\|P_{F,\boldsymbol{p}}\tilde{f} - \tilde{f}\|_1 &\leq \|P_{F,\boldsymbol{p}}\tilde{f} - P_{F,\boldsymbol{q}_{k_m}}\tilde{f}\|_1 + \|P_{F,\boldsymbol{q}_{k_m}}\tilde{f} - f_{k_m}\|_1 + \|f_{k_m} - \tilde{f}\|_1 \\
&\leq \sum_{j\in\Sigma} |p_j - q_{k_m,j}| \cdot \|P_{T_j}\tilde{f}\|_1 + \|P_{F,\boldsymbol{q}_{k_m}}\tilde{f} - P_{F,\boldsymbol{q}_{k_m}} f_{k_m}\|_1 + \|f_{k_m} - \tilde{f}\|_1 \\
&\leq \sum_{j\in\Sigma} |p_j - q_{k_m,j}| \cdot \|\tilde{f}\|_1 + 2\|f_{k_m} - \tilde{f}\|_1.
\end{aligned}
$$

Since we have $\lim_{m\to\infty} \boldsymbol{q}_{k_m} = \boldsymbol{p}$ in $\mathbb{R}_+^N$ and $\lim_{m\to\infty} \|f_{k_m} - \tilde{f}\|_1 = 0$ we obtain that $P_{F,\boldsymbol{p}}\tilde{f}(x) = \tilde{f}(x)$ holds for λ-a.e. $x \in [0,1]$. It follows from Theorem 4.1.2 that $(\mathcal{T}, \boldsymbol{p})$ admits only one acs probability measure, so we conclude that $\tilde{f} = \frac{d\mu_{\boldsymbol{p}}}{d\lambda}$ holds λ-a.e. Hence, $\{f_{k_m}\}$ converges in $L^1(\lambda)$ to $\frac{d\mu_{\boldsymbol{p}}}{d\lambda}$. □

§4.3 Final remarks

Suppose $\eta > 1$. Then Theorem 4.1.1 says that no acs probability measure exists. We claim that in fact no physical measure for F can exist in this case but that an infinite acs measure exists with a density that has full support on $[0,1]$. Indeed, for the case that $\alpha_j \leq 1$ holds for each $j \in \Sigma$, the existence of a σ-finite acs measure can, regardless of the value of η, be proven by applying the same steps of the inducing technique as in Section 2.2 for the inducing domain $Y = \bigcup_{j\in\Sigma}[j] \times (x_2(j), \frac{1}{2})$ with $x_2(j)$ as in Subsection 4.2.1 and using that in this case the maps T_j have non-positive Schwarzian derivative. If $\eta > 1$, this acs measure then must be infinite by Theorem 4.1.1. Furthermore, it is clear that the corresponding density must have full support because Y is a sweep-out set. Since for an i.i.d. random LSV map the dynamics are dominated by the LSV map with the fastest relaxation rate, the existence of such an infinite acs measure can therefore also be expected under the conditions $\eta > 1$ and $\alpha_{\min} < 1$ without assuming $\alpha_j \leq 1$ for each $j \in \Sigma$. Similar as in Chapters 2 and 3, Aaronson's Ergodic Theorem [A97, Theorem 2.4.2] applied to this infinite acs measure then yields that no physical measure for F exists if $\eta > 1$.

It does not become clear from the results of Theorems 4.1.1 and 4.1.2 if an acs probability measure exists if $\eta = 1$. The proof of Theorem 4.1.1 does not work for $\eta = 1$ because in this case there exists no $\delta > 0$ such that ζ from (4.10) is at least 1, and the proof of Theorem 4.1.2 fails for $\eta = 1$ because in this case we have $\gamma = \alpha_{\min}$ and therefore the set $(\alpha_{\min}, \gamma) \cap (0,1]$ from which we pick β is empty. For each $\delta > 0$ the result and proof of Theorem 4.1.1 do however carry over if $\eta = 1$ and the maps T_r $(r \in \Sigma_R)$ are slightly adapted such that on $(\frac{1}{2}, \frac{1}{2} + \delta)$ they would be linear with derivative K_r. Indeed, in that case the bound in (4.11) can be replaced with $T_r(x) = \frac{1}{2} + K_r(x - \frac{1}{2})$ where $x \in (\frac{1}{2}, \frac{1}{2} + \delta)$. We therefore conjecture that if $\eta = 1$ then no acs probability measure exists and a possible approach is to work with a sharper bound on the term $K_r + 2(1 - K_r)(x - \frac{1}{2})$ in (4.11) that is not uniform in $x \in (\frac{1}{2}, \frac{1}{2} + \delta)$ as opposed to the upper bound M_r in (4.11).

The proof of Theorem 4.1.2 immediately carries over to the case that $\Sigma_R = \emptyset$ by taking $\beta = 1$, thus recovering the result from [Z18] that a random system generated by i.i.d. random compositions of finitely many LSV maps admits a unique absolutely continuous invariant probability measure if $\alpha_{\min} < 1$ with density as in (4.5) for some $a > 0$. To show that in case $\Sigma_R = \emptyset$ this density is decreasing and continuous on the whole interval $(0,1]$ similar arguments as in Subsection 4.2.2 can be used with the sets $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ replaced by

$$\begin{aligned}
\mathcal{K}_0 &= \Big\{ f \in L^1(\lambda) : f \geq 0, f \text{ decreasing and continuous on } (0,1] \Big\}, \\
\mathcal{K}_1 &= \Big\{ f \in \mathcal{K}_0 : x \mapsto x^{\alpha_{\max}+1} f(x) \text{ increasing on } (0,1] \Big\}, \\
\mathcal{K}_2 &= \Big\{ f \in \mathcal{K}_1 : f(x) \leq a x^{-\alpha_{\min}} \text{ on } (0,1], \int_0^1 f d\lambda = 1 \Big\} \quad \text{with } a > 0 \text{ large enough.}
\end{aligned}$$

This has been done in [Z18].

It would be interesting to study further statistical properties of the random systems from Theorem 4.1.2. As discussed in Example 1.4.3 the annealed dynamics of random systems of LSV maps are dominated by the LSV map with the fastest relaxation rate, namely $S_{\alpha_{\min}}$, and annealed correlations decay as fast as $n^{1-1/\alpha_{\min}}$. This behaviour is significantly different from the behaviour of the random systems from Theorem 4.1.2 where the annealed dynamics are determined by the interplay between the exponentially fast attraction to $\frac{1}{2}$ and polynomially fast repulsion from zero. We conjecture that the random systems from Theorem 4.1.2 are mixing and that in case $\Sigma_R = \{1\}$ the decay of annealed correlations is at least polynomially fast with degree $1 - \frac{1}{\alpha_{\min}} \cdot \min\{\frac{\log p_1}{\log K_1}, 1\}$.

Finally, a natural question is whether the results of Theorems 4.1.1 and 4.1.2 can be extended to a more general class of one-dimensional random dynamical systems that exhibit this interplay between two fixed points, one to which orbits converge exponentially fast and one from which orbits diverge polynomially fast. First of all, if being C^1 and having $\frac{1}{2}$ as attracting fixed point are the only conditions we put on the right branches of the maps in $\mathfrak{R}$, then it can be shown in a similar way as in the proof of Theorem 4.1.1 that $(\mathcal{T}, \boldsymbol{p})$ admits no acs probability measure if

$$\sum_{r \in \Sigma_R} p_r \cdot \Big(\lim_{x \downarrow \frac{1}{2}} |DT_r(x)| \Big)^{-\alpha_{\min}} > 1$$

by applying Kac's Lemma. Secondly, we used in the proofs of Lemma 4.2.3 and Lemma 4.2.4 that $\frac{1}{DR_{\alpha_r,K_r}(z_r)} \big(\frac{x-\frac{1}{2}}{z_r - \frac{1}{2}} \big)^d$ is increasing and $\frac{1}{DR_{\alpha_r,K_r}(z_r)} \big(\frac{x-\frac{1}{2}}{z_r - \frac{1}{2}} \big)^{t_2}$ is decreasing, respectively, by means of the results on $H_{K,b}$ in Lemma 4.2.2. However, for other maps that have the property that $\frac{1}{2}$ and 1 are fixed points and that orbits are attracted to $\frac{1}{2}$ exponentially fast this is not true in general. Still a phase transition is to be expected, but different techniques are needed to prove this. This is also the case when we drop the condition that 1 is a fixed point of the maps in $\mathfrak{R}$, for instance by taking $R_{\alpha,K}(x) = \frac{1}{2} + K(x - \frac{1}{2})$ if $x \in (\frac{1}{2}, 1]$, in which case Lemma 4.2.3 would not hold. Thirdly, the results of Theorems 4.1.1 and 4.1.2 might carry over if we allow the left branches to only satisfy the conditions on the left branch of the maps $\{T_\alpha : [0,1] \to [0,1]\}_{\alpha \in (0,1)}$ considered in [M05] or Section 5 of [LSV99]. Each map T_α then satisfies $T_\alpha(0) = 0$ and $DT_\alpha(x) = 1 + Cx^\alpha + o(x^\alpha)$ for x close to zero and where $C > 0$ is some constant.