Intermittency and number expansions for random interval maps
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CHAPTER 3

Decay of correlations for critically intermittent systems

This chapter is based on: [KZ22].

Abstract

For a family of random intermittent dynamical systems with a superattracting fixed point we prove that a phase transition occurs for the existence of an absolutely continuous invariant probability measure depending on the randomness parameters and the orders of the maps at the superattracting fixed point. In case the systems have an absolutely continuous invariant probability measure, we show that the systems are mixing and that correlations decay polynomially even though some of the deterministic maps present in the system have exponential decay of correlations. This contrasts other known results, where random systems adopt the best decay rate of the deterministic maps in the systems.

§3.1 Introduction

For the random systems from Chapter 2 that have an acs probability measure, one can naturally wonder about the mixing properties and decay of correlations. In this chapter we explore this further, but instead of the random maps from the previous chapter we work with adapted versions. We take a subclass of the maps from Chapter 2 and replace the right branches with the right branch of the doubling map. Under random compositions of these maps orbits then converge superexponentially fast to $\frac{1}{2}$ and diverge exponentially fast from 1, see Figure 3.1(a). The way in which we have adapted the systems from Chapter 2 very much resembles the way in which the LSV maps from (1.11) are adaptations of the standard Manneville-Pomeau maps. We work with these adaptations because they allow us to build a suitable Young tower and use the corresponding results, while preserving the main dynamical properties of the maps from Chapter 2.

We describe the systems we consider in more detail. Just as in the previous chapter we distinguish between so-called good and bad maps. We call a map $T_g : [0,1] \to [0,1]$ *good* if it is given by

$$T_g(x) = \begin{cases} 1 - 2^{r_g}(\frac{1}{2} - x)^{r_g} & \text{if} \quad x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if} \quad x \in [\frac{1}{2}, 1], \end{cases} \tag{3.1}$$

for some $r_g \geq 1$ and denote the class of good maps by $\mathfrak{G}$. A map $T_b : [0,1] \to [0,1]$ is called *bad* if it is given by

$$T_b(x) = \begin{cases} \frac{1}{2} - 2^{\ell_b - 1}(\frac{1}{2} - x)^{\ell_b} & \text{if} \quad x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if} \quad x \in [\frac{1}{2}, 1], \end{cases} \tag{3.2}$$

for some $\ell_b > 1$ and we denote the class of bad maps by $\mathfrak{B}$. The graphs of T_g and T_b are shown in Figure 3.1(b). Note that if $r_g = 1$, then T_g is equal to the doubling map. Furthermore, T_g with $r_g = 2$ and T_b with $\ell_b = 2$ are on $[0, \frac{1}{2})$ equal to the logistic maps $x \mapsto 4x(1-x)$ and $x \mapsto 2x(1-x)$, respectively. One easily computes that each good map T_g and each bad map T_b have non-positive Schwarzian derivative when restricted to $[0, \frac{1}{2})$ or $[\frac{1}{2}, 1]$. Just like in the previous chapter, let $\{T_1, \ldots, T_N\} \subseteq \mathfrak{G} \cup \mathfrak{B}$ be a finite collection of good and bad maps, write $\Sigma_G = \{1 \leq j \leq N : T_j \in \mathfrak{G}\}$ and $\Sigma_B = \{1 \leq j \leq N : T_j \in \mathfrak{B}\}$ for the index sets of the good and bad maps, respectively. We assume that $\Sigma_G, \Sigma_B \neq \emptyset$. We write $\Sigma = \{1, \ldots, N\} = \Sigma_G \cup \Sigma_B$ and let F be the skew product associated to $\{T_j\}_{j \in \Sigma}$ given by

$$F : \Sigma^{\mathbb{N}} \times [0,1] \to \Sigma^{\mathbb{N}} \times [0,1], \, (\omega, x) \mapsto (\tau\omega, T_{\omega_1}(x)),$$

where again τ denotes the left shift on sequences in $\Sigma^{\mathbb{N}}$.

Let $\boldsymbol{p} = (p_j)_{j \in \Sigma}$ be a probability vector with strictly positive entries representing the probabilities with which we choose the maps from $\mathcal{T} = \{T_j\}_{j \in \Sigma}$. Furthermore, let $m_{\boldsymbol{p}}$ be the $\boldsymbol{p}$-Bernoulli measure on $\Sigma^{\mathbb{N}}$. Our first two main results establish that there exists a phase transition for the existence of an acs probability measure for $(\mathcal{T}, \boldsymbol{p})$ that is similar to the phase transition found in Chapter 2. Set $\theta = \sum_{b \in \Sigma_B} p_b \ell_b$.

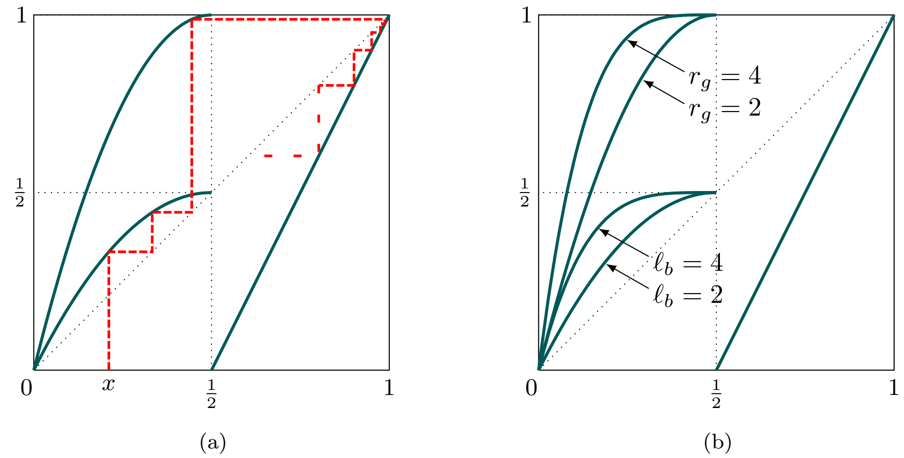


Figure 3.1: In (a) *we see the critically intermittent system consisting of the maps given by* (3.1) *with* $r_g = 2$ *and* (3.2) *with* $\ell_b = 2$. *The dashed lines indicate part of a random orbit of* x. *In* (b) *the graphs of* (3.1) *and* (3.2) *are depicted for several values of* r_g *and* ℓ_b.

Theorem 3.1.1. *If* $\theta \geq 1$, *then no acs probability measure exists for* $(\mathcal{T}, \boldsymbol{p})$.

Theorem 3.1.2. *If* $\theta < 1$, *then there exists a unique acs probability measure* μ *for* $(\mathcal{T}, \boldsymbol{p})$. *Moreover,* F *is mixing with respect to* $m_{\boldsymbol{p}} \times \mu$ *and the density* $\frac{d\mu}{d\lambda}$ *is bounded away from zero.*

Since μ is an acs measure, the density $\frac{d\mu}{d\lambda}$ is a fixed point of the associated Perron-Frobenius operator $P_{\mathcal{T},\boldsymbol{p}}$ being of the form as in (1.20). Moreover, Theorem 3.1.2 tells that this density $\frac{d\mu}{d\lambda}$ is bounded away from zero. Using these two statements it is easy to see that $\frac{d\mu}{d\lambda}$ blows up to infinity when approaching the points $\frac{1}{2}$ and 1 from below. See Figure 3.2 for an example.

Our second set of main results involves the decay of correlations in case $\theta < 1$. Equip $\Sigma^{\mathbb{N}} \times [0,1]$ with the metric

$$d\big((\omega, x), (\omega', y)\big) = 2^{-\min\{i \in \mathbb{N} \,:\, \omega_i \neq \omega'_i\}} + |x - y|. \tag{3.3}$$

For $\alpha \in (0,1)$, let $\mathcal{H}_\alpha$ be the class of α-Hölder continuous functions on $\Sigma^{\mathbb{N}} \times [0,1]$, i.e.

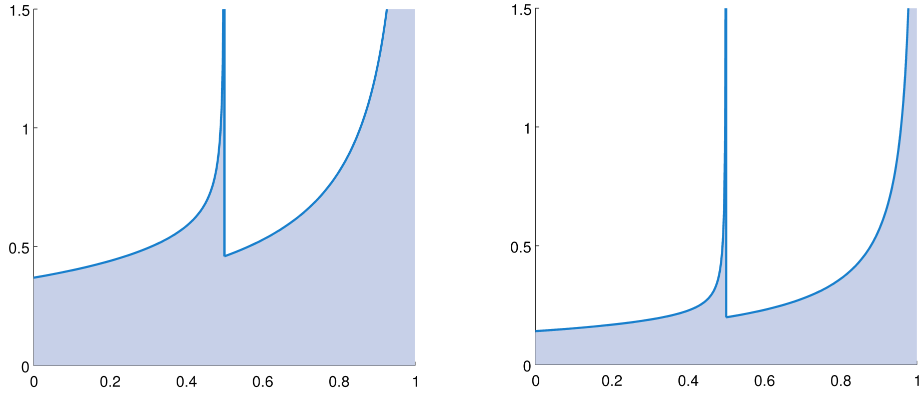
$$\mathcal{H}_\alpha = \left\{ h : \Sigma^{\mathbb{N}} \times [0,1] \to \mathbb{R} \,\middle|\, \sup \left\{ \frac{|h(z_1) - h(z_2)|}{d(z_1, z_2)^\alpha} : z_1, z_2 \in \Sigma^{\mathbb{N}} \times [0,1], z_1 \neq z_2 \right\} < \infty \right\},$$

and set

$$\mathcal{H} = \bigcup_{\alpha \in (0,1)} \mathcal{H}_\alpha.$$

For $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ and $h \in \mathcal{H}$ the *correlations* are defined by

$$\mathrm{Cor}_n(f, h) = \int_{\Sigma^{\mathbb{N}} \times [0,1]} f \circ F^n \cdot h \, dm_{\boldsymbol{p}} \times \mu - \int_{\Sigma^{\mathbb{N}} \times [0,1]} f \, dm_{\boldsymbol{p}} \times \mu \int_{\Sigma^{\mathbb{N}} \times [0,1]} h \, dm_{\boldsymbol{p}} \times \mu.$$



(a) (b)

Figure 3.2: Approximation of $\frac{d\mu}{d\lambda}$ *in case* $\Sigma_G = \{1\}$, $\Sigma_B = \{2\}$, $p_1 = \frac{7}{10}$ *and* $r_1 = 2$ *for two different values of* ℓ_2. *Both pictures depict* $P^{50}_{\mathcal{T},\boldsymbol{p}}(1)$ *with Perron-Frobenius operator* $P_{\mathcal{T},\boldsymbol{p}}$, *where in* (a) *we have taken* $\ell_2 = \frac{3}{2}$ *and in* (b) $\ell_2 = 3$.

Set $\ell_{\max} = \max\{\ell_b : b \in \Sigma_B\}$ and

$$\gamma_1 = \frac{\log \theta}{\log \ell_{\max}}. \tag{3.4}$$

The following result says that correlations decay at least polynomially fast with degree arbitrarily close to γ_1. Here and in the rest of this chapter the notation $f(n) = O(g(n))$ means that there exists a constant $C > 0$ and integer $N \in \mathbb{N}$ such that for each integer $n \geq N$ we have $f(n) \leq C \cdot g(n)$.

Theorem 3.1.3. *Assume that* $\theta < 1$. *If* $\gamma \in (\gamma_1, 0)$, $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ *and* $h \in \mathcal{H}$, *then* $|\mathrm{Cor}_n(f,h)| = O(n^\gamma)$.

For each $b \in \Sigma_B$ set

$$\pi_b = \sum_{j \in \Sigma_B : \ell_j \geq \ell_b} p_j \tag{3.5}$$

and let

$$\gamma_2 = 1 + \max\left\{\frac{\log \pi_b}{\log \ell_b} : b \in \Sigma_B\right\}. \tag{3.6}$$

Note that

$$\gamma_2 = \max\left\{\frac{\log(\pi_b \cdot \ell_b)}{\log \ell_b} : b \in \Sigma_B\right\} \leq \max\left\{\frac{\log \theta}{\log \ell_b} : b \in \Sigma_B\right\} = \gamma_1.$$

In our final result we show that, under additional assumptions on the parameters of the random systems, the class of observables $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ and $h \in \mathcal{H}$

contains functions for which the correlation decay is not faster than polynomially with degree γ_2. The notation $f(n) = \Omega(g(n))$ means that there exists a constant $C > 0$ and integer $N \in \mathbb{N}$ such that for each integer $n \geq N$ we have $f(n) \geq C \cdot g(n)$.

Theorem 3.1.4. *Assume that $\theta < 1$. Furthermore, assume that $\gamma_2 > \gamma_1 - 1$ if $\gamma_1 < -1$ and $\gamma_2 > 2\gamma_1$ if $-1 \leq \gamma_1 < 0$. Let $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ and $h \in \mathcal{H}$ be such that both f and h are identically zero on $\Sigma^{\mathbb{N}} \times \big([0, \frac{1}{2}] \cup [\frac{3}{4}, 1]\big)$ and such that*

$$\int_{\Sigma^{\mathbb{N}} \times [0,1]} f \, dm_{\boldsymbol{p}} \times \mu \cdot \int_{\Sigma^{\mathbb{N}} \times [0,1]} h \, dm_{\boldsymbol{p}} \times \mu > 0.$$

Then

$$|\mathrm{Cor}_n(f,h)| = \Omega(n^{\gamma_2}).$$

In Section 3.4 we provide examples of values of ℓ_b and probability vectors $\boldsymbol{p}$ that satisfy the conditions of Theorem 3.1.4.

The proof of Theorem 3.1.2 we present below also carries over to the case that $\Sigma_B = \emptyset$. Applying Theorem 3.1.2 to the case that $\Sigma_B = \emptyset$ and $\Sigma_G = \{g\}$ contains one element yields together with [LM13, Theorem 1.5] the following result on the good maps $T_g \in \mathfrak{G}$.

Corollary 3.1.5. *For any $T_g : [0,1] \to [0,1] \in \mathfrak{G}$ the following hold.*

(a) *T_g admits an invariant probability measure μ_g that is mixing and absolutely continuous with respect to Lebesgue measure λ.*

(b) *There exists a constant $a > 0$ such that for each $f \in L^\infty([0,1], \mu_g)$ and each function $h : [0,1] \to \mathbb{R}$ of bounded variation we have*

$$|\mathrm{Cor}_{n,T_g,\mu_g}(f,h)| = O(e^{-an}),$$

where

$$\mathrm{Cor}_{n,T_g,\mu_g}(f,h) = \int f \circ T_g^n \cdot h \, d\mu_g - \int f d\mu_g \cdot \int h \, d\mu_g.$$

This result is expected in view of the exponential decay of correlations found for the unimodal maps from [KN92, Y92, Y98]. We see that test functions that fall within the scope of both this corollary and Theorems 3.1.3 and 3.1.4 have exponential decay of correlations under a single good map while under the random system with $\Sigma_B \neq \emptyset$ they have polynomial decay of correlations.[1] This indicates that a system of good maps loses its exponential decay of correlations when mixed with bad maps and instead adopts polynomial mixing rates. This is different from what has been observed for other random systems in e.g. [BBD14, BB16, BQT21], where the random systems of LSV maps under consideration adopt the highest decay rate from the rates of the individual LSV maps present in the system. See Example 1.4.3.

[1] Examples of such test functions are Lipschitz continuous functions that vanish outside of $\Sigma^{\mathbb{N}} \times (\frac{1}{2}, \frac{3}{4})$.

The remainder of this chapter is organised as follows. In Section 3.2 we list some preliminaries. In Section 3.3 we prove Theorem 3.1.1. The method of proof for this part is reminiscent of that in Chapter 2 in the sense that we introduce an induced system and apply Kac's Lemma on the first return times. In Section 3.3 we also give estimates on the first return times and the induced map that we use later in Section 3.4. For Theorem 3.1.2 the approach from Subsection 2.3.2 no longer works, because we have introduced a discontinuity for the bad maps (at $\frac{1}{2}$). Instead we prove Theorem 3.1.2, as well as Theorem 3.1.3 and Theorem 3.1.4, by constructing a Young tower on the inducing domain from Section 3.3 and by applying the general theory from [Y99] and [G04]. This is done in Section 3.4 and is inspired by the methods from [BBD14, BB16]. Section 3.5 contains some further results. More specifically, we show that for a specific class of test functions the upper bound from Theorem 3.1.3 can be improved and we obtain a Central Limit Theorem. We end with some final remarks.

§3.2 Preliminaries

We use this section first of all to give some properties of good and bad maps. Secondly, we give the construction of a Young tower, list some of the results from [Y99, G04] and present this in an adapted form, rephrased to our setting and only referring to the parts that are relevant for our purposes.

§3.2.1 Properties of good and bad maps

As in Section 3.1, let $T_1, \dots, T_N \in \mathfrak{G} \cup \mathfrak{B}$ be a finite collection of good and bad maps, and write $\Sigma_G = \{1 \le j \le N : T_j \in \mathfrak{G}\} \neq \emptyset$, $\Sigma_B = \{1 \le j \le N : T_j \in \mathfrak{B}\} \neq \emptyset$ and $\Sigma = \{1, \dots, N\} = \Sigma_G \cup \Sigma_B$. For ease of notation, we refer to the left branch of a map T_j by L_j, i.e. for $x \in [0, \frac{1}{2})$ we write $L_j(x) = T_j(x)$, $j \in \Sigma$. We will use $R : [\frac{1}{2}, 1] \to [0, 1]$, $x \mapsto 2x - 1$ to denote the right branch of the maps T_j.

Throughout this section and the next ones we use the notations for words and compositions of the maps T_j introduced in Section 1.4. We use the same notations for compositions of the left branches L_j, so for $\omega \in \Sigma^{\mathbb{N}}$, $n \in \mathbb{N}_0$ and $x \in [0, \frac{1}{2})$ such that $L_{\omega_j} \circ \cdots \circ L_{\omega_1}(x) \in [0, \frac{1}{2})$ for each $j = 1, \dots, n-1$ we write

$$L_{\omega_1 \cdots \omega_n}(x) = L_\omega^n(x) = \begin{cases} x, & \text{if} \quad n = 0, \\ L_{\omega_n} \circ L_{\omega_{n-1}} \circ \cdots \circ L_{\omega_1}(x), & \text{for } n \ge 1. \end{cases} \tag{3.7}$$

Also in this situation we use the same notation for finite words $\boldsymbol{u} \in \Sigma^m$, $m \ge 1$, and $n \le m$. Furthermore, for any $\boldsymbol{u} = u_1 \cdots u_k \in \Sigma^k$, $k \ge 0$, recall that we abbreviate $p_{\boldsymbol{u}} = \prod_{i=1}^k p_{u_i}$ and also, as in Chapter 2, let $\ell_{\boldsymbol{u}} = \prod_{i=1}^k \ell_{u_i}$ if $\boldsymbol{u} \in \Sigma_B^k$, where we use $p_{\boldsymbol{u}} = \ell_{\boldsymbol{u}} = 1$ in case $k = 0$. Finally, recall the notation $\Sigma^* = \bigcup_{n \ge 0} \Sigma^n$ for the collection of all finite words (including the empty word) with digits from Σ. Similarly we define Σ_G^* and Σ_B^*.

For future reference we give two lemmas on the properties of good and bad maps. The first one involves compositions of bad maps and implies in particular that orbits starting in $[0, \frac{1}{2})$ converge superexponentially fast to $\frac{1}{2}$ under iterations of bad maps.

Lemma 3.2.1. *For each* $x \in [0, \frac{1}{2})$ *and* $\boldsymbol{b} \in \Sigma_B^*$ *we have*

(i) $L_{\boldsymbol{b}}(x) = \frac{1}{2}\big(1 - (1-2x)^{\ell_{\boldsymbol{b}}}\big)$,

(ii) $L_{\boldsymbol{b}}^{-1}(x) = \frac{1}{2}\big(1 - (1-2x)^{\ell_{\boldsymbol{b}}^{-1}}\big)$.

Proof. Part (i) holds trivially if $|\boldsymbol{b}| = 0$. Now suppose for some $k \geq 0$ that (i) holds for all $\boldsymbol{b} \in \Sigma_B^*$ with $|\boldsymbol{b}| \leq k$. Let $\boldsymbol{b}b_{k+1} \in \Sigma_B^{k+1}$. Then $L_{\boldsymbol{b}}(x) \in [0, \frac{1}{2})$ and

$$\begin{aligned} L_{\boldsymbol{b}b_{k+1}}(x) &= \frac{1}{2}\Big(1 - (1 - 2L_{\boldsymbol{b}}(x))^{\ell_{b_{k+1}}}\Big) \\ &= \frac{1}{2}\Big(1 - \big((1-2x)^{\ell_{\boldsymbol{b}}}\big)^{\ell_{b_{k+1}}}\Big) = \frac{1}{2}\Big(1 - (1-2x)^{\ell_{\boldsymbol{b}b_{k+1}}}\Big). \end{aligned}$$

This proves (i), and (ii) follows easily from (i). □

For $g \in \Sigma_G$ the map $L_g : [0, \frac{1}{2}) \to [0, 1)$ is invertible and for $b \in \Sigma_B$ the map $L_b : [0, \frac{1}{2}) \to [0, \frac{1}{2})$ is invertible. For all $j \in \Sigma$ the map L_j is strictly increasing with continuous and decreasing derivative and, if T_j is not the doubling map (i.e. $j \in \Sigma_B$ or $j \in \Sigma_G$ with $r_j > 1$),

$$\max_{x \in [0, \frac{1}{2})} DL_j(x) = DT_j(0) > 1 \quad \text{and} \quad \lim_{x \uparrow \frac{1}{2}} DL_j(x) = 0.$$

This allows us to define for each $g \in \Sigma_G$ with $r_g > 1$ the point x_g as the point in $(0, \frac{1}{2})$ for which $DL_g(x_g) = 1$ and for each $b \in \Sigma_B$ the point x_b as the point in $(0, \frac{1}{2})$ for which $DL_b(x_b) = 1$.

Lemma 3.2.2. *The following hold.*

(i) For each $g \in \Sigma_G$ *it holds that* $L_g^{-1}\big(\frac{1}{2}\big) \leq \frac{1}{4}$.

(ii) For each $g \in \Sigma_G$ *with* $r_g > 1$ *it holds that* $L_g^{-1}\big(\frac{1}{2}\big) < x_g$.

(iii) For each $b \in \Sigma_B$ *it holds that* $L_b^{-1}\big(\frac{1}{4}\big) < x_b$.

Proof. One can compute that

$$L_g^{-1}\Big(\frac{1}{2}\Big) = \frac{1}{2}\big(1 - 2^{-1/r_g}\big) \tag{3.8}$$

for all $g \in \Sigma_G$. If $r_g > 1$, then

$$x_g = \frac{1}{2} - \big(r_g \cdot 2^{r_g}\big)^{1/(1-r_g)}.$$

For each $b \in \Sigma_B$ we have

$$L_b^{-1}\Big(\frac{1}{4}\Big) = \frac{1}{2}\big(1 - 2^{-1/\ell_b}\big) \quad \text{and} \quad x_b = \frac{1}{2}\big(1 - \ell_b^{1/(1-\ell_b)}\big). \tag{3.9}$$

Since $r_g \geq 1$ and thus $2^{-1/r_g} \geq \frac{1}{2}$ for each $g \in \Sigma_G$, (i) follows. Furthermore, one can show that $2^{-1/x-1} > (x \cdot 2^x)^{1/(1-x)}$ and $2^{-1/x} > x^{1/(1-x)}$ hold for all $x > 1$, which give (ii) and (iii), respectively. □

§3.2.2 Young towers

Let $\mathcal{F}$ be the product σ-algebra on $\Sigma^{\mathbb{N}} \times [0,1]$ given by the Borel σ-algebra's on both coordinates. We call the set on which we will induce $Y \in \mathcal{F}$. It will be defined later and will be such that the first return time map φ on Y associated to the skew product F given by

$$\varphi(\omega, x) = \inf\{n \geq 1 : F^n(\omega, x) \in Y\}$$

satisfies $\varphi(\omega, x) < \infty$ for all $(\omega, x) \in Y$. The induced transformation $F_Y : Y \to Y$ as given in Subsection 1.2.1 by $F_Y(\omega, x) = F^{\varphi(\omega,x)}(\omega, x)$ is then well defined.

On the inducing domain Y one can construct a Young tower for F. For the convenience of the reader we briefly give this construction, which is outlined in a more general fashion and in more detail in [Y99]. Let $\mathcal{P}$ be a countable partition of Y into measurable sets on which the first return time function φ is constant, called a *first return time partition*. Let φ_P be the constant value that φ assumes on P. The tower is defined by

$$\Delta = \{(z, n) \in Y \times \{0, 1, 2 \ldots\} : n < \varphi(z)\}.$$

The l-th level of the tower is $\Delta_l := \Delta \cap \{n = l\}$ and for each $P \in \mathcal{P}$ let $\Delta_{l,P} = \Delta_l \cap \{z \in P\}$. We equip Δ with the Borel σ-algebra $\mathcal{B}$ and let m be the unique measure on Δ that corresponds to $m_{\boldsymbol{p}} \times \lambda$ on each level of the tower. On Δ define the function

$$G(z, n) = \begin{cases} (z, n+1), & \text{if } n + 1 < \varphi_Y(z), \\ (F_Y(z), 0), & \text{otherwise.} \end{cases}$$

Let the induced map $G^{\varphi} : \Delta_0 \to \Delta_0$ be defined by $G^{\varphi}(v) = G^{\varphi(z)}(v)$, where $z \in Y$ is such that $v = (z, 0)$. We identify G^{φ} with F^{φ} by identifying Δ_0 with Y and using the correspondence $G^{\varphi}(z, 0) = (F^{\varphi}(z), 0)$. The *separation time* is defined for each $(z_1, l_1), (z_2, l_2) \in \Delta$ by $s((z_1, l_1), (z_2, l_2)) = 0$ if $l_1 \neq l_2$ and otherwise letting $s((z_1, l_1), (z_2, l_2)) = s(z_1, z_2)$ be given by

$$s(z_1, z_2) = \inf\{n \geq 0 : (G^{\varphi})^n(z_1, 0),\ (G^{\varphi})^n(z_2, 0) \text{ lie in distinct } \Delta_{0,P},\ P \in \mathcal{P}\}. \tag{3.10}$$

The setup from [Y99] assumes the following conditions on Δ and G:

(t1) $gcd\{\varphi_P : P \in \mathcal{P}\} = 1$;

(t2) All the sets in the construction above are measurable and $m(Y) < \infty$.

(t3) For each $P \in \mathcal{P}$ the top level $\Delta_{\varphi_P - 1, P}$ above P is mapped bijectively onto $\Delta_0 = Y \times \{0\}$ under the map G;

(t4) The partition $\eta = \{\Delta_{l,P} : P \in \mathcal{P},\ 0 \leq l \leq \varphi_P - 1\}$ generates $\mathcal{B}$.

(t5) The restrictions $G^{\varphi}|_{\Delta_{0,P}} : \Delta_{0,P} \to \Delta_0$ and their inverses are non-singular with respect to m, so that the Jacobian $J_m G^{\varphi}$ with respect to m exists and is > 0 m-a.e.

(t6) There are constants $C > 0$ and $\beta \in (0,1)$ such that for each $P \in \mathcal{P}$ and all $(z_1, 0), (z_2, 0) \in \Delta_{0,P}$,

$$\left| \frac{J_m G^{\varphi}(z_1, 0)}{J_m G^{\varphi}(z_2, 0)} - 1 \right| \leq C \beta^{s(G^{\varphi}(z_1,0), G^{\varphi}(z_2,0))}.$$

Under these conditions [Y99] yields the following results on the existence of invariant measures for the map G on the tower Δ and decay of correlations. Define for all $\delta \in (0,1)$ the following function space on Δ:

$$\mathcal{C}_\delta = \left\{ \hat{f} : \Delta \to \mathbb{R} \,\middle|\, \sup \left\{ \frac{|\hat{f}(\upsilon_1) - \hat{f}(\upsilon_2)|}{\delta^{s(\upsilon_1, \upsilon_2)}} : \upsilon_1, \upsilon_2 \in \Delta, \upsilon_1 \neq \upsilon_2 \right\} < \infty \right\}. \tag{3.11}$$

For $\upsilon \in \Delta$ let $\hat{\varphi}(\upsilon) := \inf\{n \geq 0 : G^n(\upsilon) \in \Delta_0\}$.

Theorem 3.2.3 (Theorem 1 and 3 from [Y99]). *If (t1)–(t6) hold and we have $\int_Y \varphi \, dm < \infty$, then the following statements hold.*[2]

(i) $G : \Delta \to \Delta$ admits an invariant probability measure ν that is absolutely continuous w.r.t. m;

(ii) The density $\frac{d\nu}{dm}$ is bounded away from zero and is in $\mathcal{C}_\beta$ with β as in (t6). Moreover, there is a constant $C^+ > 0$ such that for each $\Delta_{l,P}$ and each $\upsilon_1, \upsilon_2 \in \Delta_{l,P}$

$$\left| \frac{\frac{d\nu}{dm}(\upsilon_1)}{\frac{d\nu}{dm}(\upsilon_2)} - 1 \right| \leq C^+ \, \beta^{s(\upsilon_1, \upsilon_2)}. \tag{3.12}$$

(iii) G is exact, hence ergodic and mixing.

(iv) (Polynomial decay of correlations) If, moreover, $m(\{\upsilon \in \Delta : \hat{\varphi}(\upsilon) > n\}) = O(n^{-\alpha})$ for some $\alpha > 0$, then for all $\hat{f} \in L^\infty(\Delta, m)$, all $\delta \in (0,1)$ and all $\hat{h} \in \mathcal{C}_\delta$,

$$\left| \int_\Delta \hat{f} \circ G^n \cdot \hat{h} \, d\nu - \int_\Delta \hat{f} \, d\nu \int_\Delta \hat{h} \, d\nu \right| = O(n^{-\alpha}).$$

We will also use [G04, Theorem 6.3] by Gouëzel which, when adapted to our setting, says the following.

Theorem 3.2.4 (Theorem 6.3 from [G04]). *Let ρ be an invariant and mixing probability measure for F on $\Sigma^{\mathbb{N}} \times [0,1]$. Let $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], \rho)$ and $h \in L^1(\Sigma^{\mathbb{N}} \times [0,1], \rho)$ be such that both f and h are identically zero on $(\Sigma^{\mathbb{N}} \times [0,1]) \backslash Y$. Assume that there is a $\delta \in (0,1)$ such that the following three conditions hold.*

[2] The version in [Y99, Theorem 3] of statement (iv) above only states the result for $\delta = \beta$. Note however that statement (iv) for the case $0 < \delta < \beta$ follows from $\mathcal{C}_\delta \subseteq \mathcal{C}_\beta$ and that if $\beta < \delta < 1$, then the bound in (t6) where β is replaced by δ also holds and therefore [Y99, Theorem 3] can be applied with δ taking the role of β.

(g1) There is a constant $C^* > 0$ *such that for each* $n \geq 0$ *and* $z_1, z_2 \in \bigvee_{k=0}^{n-1} F_Y^{-n}\mathcal{P}$,

$$\left|\log \frac{J_\rho F^\varphi(z_1)}{J_\rho F^\varphi(z_2)}\right| \leq C^* \cdot \delta^n.$$

(g2) There is a $\zeta > 1$ *such that* $\rho(\varphi > n) = O(n^{-\zeta})$.

(g3) $\sup\left\{\frac{|h(z_1)-h(z_2)|}{\delta^{s(z_1,z_2)}} \,:\, z_1, z_2 \in Y,\, z_1 \neq z_2\right\} < \infty$.

Then there is a constant $\tilde{C} > 0$ *such that*

$$\left|\mathrm{Cor}_{n,F,\rho}(f,h) - \Big(\sum_{k>n} \rho(\varphi > k)\Big) \int f\, d\rho \int h\, d\rho\right| \leq \tilde{C} \cdot K_\zeta(n),$$

where

$$K_\zeta(n) = \begin{cases} n^{-\zeta}, & \text{if } \zeta > 2, \\ \frac{\log n}{n^2}, & \text{if } \zeta = 2, \\ n^{2-2\zeta}, & \text{if } \zeta \in (1,2). \end{cases}$$

§3.3 Inducing the random map on $(\frac{1}{2}, \frac{3}{4})$

§3.3.1 The induced system

Define

$$\tilde{\Omega} = \{\omega \in \Sigma^{\mathbb{N}} : \omega_i \in \Sigma_G \text{ for infinitely many } i \in \mathbb{N}\},$$
$$Y = \left\{(\omega, x) \in \tilde{\Omega} \times \left(\frac{1}{2}, \frac{3}{4}\right) : T_\omega^n(x) \neq \frac{1}{2} \text{ for all } n \in \mathbb{N}\right\}.$$

The set Y, which equals $\Sigma^{\mathbb{N}} \times (\frac{1}{2}, \frac{3}{4})$ up to a set of measure zero, will be our inducing domain. Recall the definition of the first return time function

$$\varphi : Y \to \mathbb{N},\ (\omega, x) \mapsto \inf\{n \geq 1 \,:\, F^n(\omega, x) \in Y\}.$$

For each $(\omega, x) \in Y$, the following happens under iterations of F. Firstly, $T_\omega(x) = R(x) \in (0, \frac{1}{2})$. By Lemma 3.2.2 there is an $a > 1$ such that $DT_g(y) \geq a$ for all $y \in (0, L_g^{-1}(\frac{1}{2})]$ and $g \in \Sigma_G$. Combining this with Lemma 3.2.1 yields by definition of Y that

$$\kappa(\omega, x) := \inf\left\{n \geq 1 \,:\, T_\omega^n(x) > \frac{1}{2}\right\} < \infty. \tag{3.13}$$

Note that $\omega_{\kappa(\omega,x)} \in \Sigma_G$. Then, again since $(\omega, x) \in Y$, so $T_\omega^n(x) \neq \frac{1}{2}$ for all $n \in \mathbb{N}$,

$$l(\omega, x) := \inf\left\{n \geq 0 \,:\, T_\omega^{\kappa(\omega,x)+n} \in \left(\frac{1}{2}, \frac{3}{4}\right)\right\} < \infty \tag{3.14}$$

and $T_\omega^{\kappa(\omega,x)+l(\omega,x)}(x) = R^{l(\omega,x)} \circ T_\omega^{\kappa(\omega,x)}(x)$. Thus

$$\varphi(\omega, x) = \kappa(\omega, x) + l(\omega, x) < \infty \tag{3.15}$$

for all $(\omega, x) \in Y$. We will first derive an estimate for $l(\omega, x)$.

For each $g \in \Sigma_G$ let $J_g = (L_g^{-1}(\frac{1}{2}), \frac{1}{2})$. The intervals J_g are such that if $T_\omega^n(x) \in J_g$ and $\omega_{n+1} = g$, then $T_\omega^{n+1}(x) > \frac{1}{2}$. For each $b \in \Sigma_B$ let $J_b = [x_b, \frac{1}{2})$ with x_b as defined above Lemma 3.2.2. Define

$$m(\omega, x) := \inf\{n \geq 1 \, : \, T_\omega^n(x) \in J_{\omega_{n+1}}\} < \kappa(\omega, x). \tag{3.16}$$

If $\omega_{m(\omega,x)+1} \in \Sigma_G$, then $T_\omega^{m(\omega,x)+1}(x) > \frac{1}{2}$ by the definition of the intervals J_g. If $\omega_{m(\omega,x)+1} \in \Sigma_B$, then $T_\omega^{m(\omega,x)+1}(x) \in (\frac{1}{4}, \frac{1}{2})$ by Lemma 3.2.2(iii). We then see by Lemma 3.2.2(i) that the number $m(\omega, x)$ is such that after this time any application of a good map will bring the orbit of x in the interval $\left(\frac{1}{2}, 1\right)$. We have the following estimate on l.

Lemma 3.3.1. *Let* $(\omega, x) \in Y$. *Write* $\boldsymbol{d} = \omega_{m(\omega,x)+1} \cdots \omega_{\kappa(\omega,x)-1} \in \Sigma_B^*$ *and* $g = \omega_{\kappa(\omega,x)} \in \Sigma_G$. *Then*

$$l(\omega, x) \geq \ell_{\boldsymbol{d}} r_g \frac{\log\left((1 - 2 \cdot T_\omega^{m(\omega,x)}(x))^{-1}\right)}{\log 2} - 2.$$

Proof. Set $y = T_\omega^{m(\omega,x)}(x)$. It follows from Lemma 3.2.1 that

$$T_\omega^{\kappa(\omega,x)}(x) = L_g \circ L_{\boldsymbol{d}}(y) = 1 - (1 - 2y)^{\ell_{\boldsymbol{d}} r_g}.$$

By definition of R, it follows that $l(\omega, x)$ is equal to the minimal $l \in \mathbb{N}_0$ such that

$$2^l \cdot (1 - 2y)^{\ell_{\boldsymbol{d}} r_g} > \frac{1}{4}. \tag{3.17}$$

Solving for l gives the lemma. □

With this estimate we can now apply Kac's Lemma to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Suppose μ is an acs probability measure for $(\mathcal{T}, \boldsymbol{p})$. We first show that $\mu((\frac{1}{2}, \frac{3}{4})) > 0$. Since $F^2(Y)$ equals $\Sigma^{\mathbb{N}} \times [0, 1]$ up to some set of $m_{\boldsymbol{p}} \times \lambda$-measure zero and by (3.15) all $(\omega, x) \in Y$ have a finite first return time to Y under F, it follows that $\Sigma^{\mathbb{N}} \times [0, 1]$ equals $\bigcup_{n=0}^{\infty} F^{-n}Y$ up to some set of $m_{\boldsymbol{p}} \times \lambda$-measure zero. Since μ is absolutely continuous with respect to λ, we obtain that

$$1 = m_{\boldsymbol{p}} \times \mu\big(\Sigma^{\mathbb{N}} \times [0, 1]\big) \leq \sum_{n=0}^{\infty} m_{\boldsymbol{p}} \times \mu(F^{-n}Y) = \sum_{n=0}^{\infty} \mu\Big(\Big(\frac{1}{2}, \frac{3}{4}\Big)\Big),$$

from which we see that we indeed must have $\mu((\frac{1}{2}, \frac{3}{4})) > 0$. Using the continuity of the measure, we know there exists an $a > \frac{1}{2}$ such that $\mu((a, \frac{3}{4})) > 0$. Note that for all $(\omega, x) \in \Sigma^{\mathbb{N}} \times (a, \frac{3}{4}) \cap Y$ we have

$$T_\omega^{m(\omega,x)}(x) \geq R(a). \tag{3.18}$$

Fix a $g \in \Sigma_G$ and consider the subsets $A_{\boldsymbol{b}} = [\boldsymbol{b}g] \times (a, \frac{3}{4}) \cap Y$, $\boldsymbol{b} \in \Sigma_B^*$, of $\Sigma^{\mathbb{N}} \times (a, \frac{3}{4}) \cap Y$. Set $p_B = \sum_{b \in \Sigma_B} p_b$. Then by (3.15), (3.18) and Lemma 3.3.1 we get

$$\begin{aligned}
\int_Y \varphi \, dm_{\boldsymbol{p}} \times \mu &\geq \sum_{\boldsymbol{b} \in \Sigma_B^*} \int_{A_{\boldsymbol{b}}} l(\omega, x) \, dm_{\boldsymbol{p}} \times \mu \\
&\geq \mu\Big(\Big(a, \frac{3}{4}\Big)\Big) \cdot \sum_{\boldsymbol{b} \in \Sigma_B^*} p_{\boldsymbol{b}g} \cdot \Big(\ell_{\boldsymbol{b}} r_g \cdot \frac{\log\big((1 - 2 \cdot R(a))^{-1}\big)}{\log 2} - 2\Big) \\
&= \mu\Big(\Big(a, \frac{3}{4}\Big)\Big) \cdot p_g \cdot \Big(r_g \cdot \frac{\log\big((1 - 2 \cdot R(a))^{-1}\big)}{\log 2} \sum_{\boldsymbol{b} \in \Sigma_B^*} p_{\boldsymbol{b}} \ell_{\boldsymbol{b}} - 2 \sum_{\boldsymbol{b} \in \Sigma_B^*} p_{\boldsymbol{b}}\Big) \\
&= M_1 \cdot \sum_{k \geq 0} \theta^k - M_2 \cdot \frac{1}{1 - p_B},
\end{aligned}$$

with $M_1 = \mu((a, \frac{3}{4})) \cdot p_g r_g \cdot \frac{\log((1-2 \cdot R(a))^{-1})}{\log 2} > 0$ and $M_2 = 2\mu((a, \frac{3}{4})) \cdot p_g$. It now follows from $\theta \geq 1$ that

$$\int_Y \varphi \, dm_{\boldsymbol{p}} \times \mu = \infty. \tag{3.19}$$

On the other hand, since μ is a probability measure by assumption, we obtain from the Ergodic Decomposition Theorem, see e.g. [EW11, Theorem 6.2], that there exists a probability space $(E, \mathcal{E}, \nu)$ and a measurable map $e \mapsto \mu_e$ with μ_e an F-invariant ergodic probability measure for ν-a.e. $e \in E$, such that

$$\int_Y \varphi dm_{\boldsymbol{p}} \times \mu = \int_E \Big(\int_Y \varphi d\mu_e\Big) d\nu(e).$$

For each $e \in E$ for which μ_e is an F-invariant ergodic probability measure we have $\int_Y \varphi d\mu_e = \mu_e(X) = 1$ if $\mu_e(Y) > 0$ by Kac's Lemma, i.e. Lemma 1.2.13, and we have $\int_Y \varphi d\mu_e = 0$ if $\mu_e(Y) = 0$. This gives

$$\int_Y \varphi dm_{\boldsymbol{p}} \times \mu \leq \nu(E) = 1,$$

which is in contradiction with (3.19). □

§3.3.2 Estimates on the first return time

From now on we only consider the case $\theta < 1$. We first define a first return time partition for F to Y. For any $u \in \Sigma$, $g \in \Sigma_G$ and $\boldsymbol{s}, \boldsymbol{w} \in \Sigma^*$ write

$$P_{u\boldsymbol{s}g\boldsymbol{w}} := ([u\boldsymbol{s}g\boldsymbol{w}] \cap \tilde{\Omega}) \times (R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{s}} \circ R)^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big)$$

and define the collection of sets

$$\mathcal{P} = \Big\{P_{u\boldsymbol{s}g\boldsymbol{w}} \, : \, u \in \Sigma, \, g \in \Sigma_G, \, \boldsymbol{s}, \boldsymbol{w} \in \Sigma^*\Big\}. \tag{3.20}$$

See Figure 3.3 for an illustration.

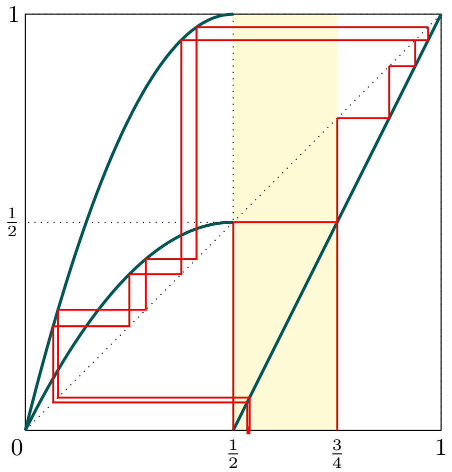


Figure 3.3: Example of a first return time partition element $P_{\boldsymbol{usgw}}$ *with* $|\boldsymbol{s}| = 2$ *and* $|\boldsymbol{w}| = 3$ *projected onto* $[0,1]$*. The yellow area indicates the inducing domain* Y *projected onto* $[0,1]$*.*

Chapter 3

Proposition 3.3.2. *The collection* $\mathcal{P}$ *is a* first return time partition *of* F *to* Y *and for all* (ω, x) *in a set* $P_{\boldsymbol{usgw}}$ *it holds that* $\kappa(\omega, x) = 2 + |\boldsymbol{s}|$ *and* $l(\omega, x) = |\boldsymbol{w}|$*, so*

$$\varphi(\omega, x) = 2 + |\boldsymbol{s}| + |\boldsymbol{w}|. \tag{3.21}$$

Proof. Let $(\omega, x) \in Y$. Since $\kappa(\omega, x), l(\omega, x) < \infty$ it is clear that we can find suitable $u \in \Sigma$, $g \in \Sigma_G$ and $\boldsymbol{s}, \boldsymbol{w} \in \Sigma^*$ so that $(\omega, x) \in P_{\boldsymbol{usgw}}$, so $\mathcal{P}$ covers Y. Now fix a set $P_{\boldsymbol{usgw}} \in \mathcal{P}$. By the definition of the set $(R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{s}} \circ R)^{-1}(\frac{1}{2}, \frac{3}{4})$ one has for any $(\omega, x) \in P_{\boldsymbol{usgw}}$ that

$$\begin{aligned} T_\omega^n(x) &< \tfrac{1}{2}, \quad 1 \le n \le 1 + |\boldsymbol{s}|, \\ T_\omega^n(x) &> \tfrac{3}{4}, \quad 2 + |\boldsymbol{s}| \le n \le 1 + |\boldsymbol{s}| + |\boldsymbol{w}|, \\ T_\omega^{2+|\boldsymbol{s}|+|\boldsymbol{w}|}(x) &\in (\tfrac{1}{2}, \tfrac{3}{4}). \end{aligned}$$

Hence, $\kappa(\omega, x) = 2 + |\boldsymbol{s}|$ and $l(\omega, x) = |\boldsymbol{w}|$ for each $(\omega, x) \in P_{\boldsymbol{usgw}}$, and (3.21) follows from (3.15). From this we immediately obtain that the sets in $\mathcal{P}$ are disjoint. To see this, suppose there are two different sets $P_{\boldsymbol{usgw}}, P_{\tilde{u}\tilde{\boldsymbol{s}}\tilde{g}\tilde{\boldsymbol{w}}} \in \mathcal{P}$ with $P_{\boldsymbol{usgw}} \cap P_{\tilde{u}\tilde{\boldsymbol{s}}\tilde{g}\tilde{\boldsymbol{w}}} \neq \emptyset$ and let $(\omega, x) \in P_{\boldsymbol{usgw}} \cap P_{\tilde{u}\tilde{\boldsymbol{s}}\tilde{g}\tilde{\boldsymbol{w}}}$. Then without loss of generality we can assume that $[\tilde{u}\tilde{\boldsymbol{s}}\tilde{g}\tilde{\boldsymbol{w}}] \subseteq [u\boldsymbol{s}g\boldsymbol{w}]$, where the inclusion is strict. But this would give that

$$\varphi(\omega, x) = 2 + |\boldsymbol{s}| + |\boldsymbol{w}| < 2 + |\tilde{\boldsymbol{s}}| + |\tilde{\boldsymbol{w}}| = \varphi(\omega, x),$$

a contradiction. □

For the estimates we give below, we split the word $\boldsymbol{s}$ into two parts $\boldsymbol{s} = \boldsymbol{vb}$, where $\boldsymbol{b}$ specifies the string of bad digits that immediately precedes $\omega_{\kappa(\omega,x)}$. In other words, if we write

$$A_G = \{\boldsymbol{v} \in \Sigma^* \,:\, v_{|\boldsymbol{v}|} \in \Sigma_G\}$$

for the set of words that end with a good digit, then for any $\boldsymbol{s} \in \Sigma^*$ there are unique $\boldsymbol{v} \in A_G$ and $\boldsymbol{b} \in \Sigma_B^*$ such that $\boldsymbol{s} = \boldsymbol{vb}$. Recall γ_1 and γ_2 from (3.4) and (3.6), respectively. In the remainder of this subsection we prove the following result.

Proposition 3.3.3. *Suppose $\theta < 1$. Then the following statements hold.*

(i) $\int_Y \varphi \, dm_{\boldsymbol{p}} \times \lambda < \infty$.

(ii) $m_{\boldsymbol{p}} \times \lambda(\varphi > n) = O(n^{\gamma - 1})$ *for any* $\gamma \in (\gamma_1, 0)$.

(iii) $m_{\boldsymbol{p}} \times \lambda(\varphi > n) = \Omega(n^{\gamma_2 - 1})$.

For the proof of Proposition 3.3.3 we will first prove three lemmas. Write

$$s = \Big(\min \Big\{ \Big\{ DL_g\Big(L_g^{-1}\Big(\frac{1}{2}\Big)\Big) \, : \, g \in \Sigma_G \Big\} \cup \Big\{ DL_b\Big(L_b^{-1}\Big(\frac{1}{4}\Big)\Big) \, : \, b \in \Sigma_B \Big\} \Big)^{-1}. \quad (3.22)$$

The number $\frac{1}{s}$ will serve below as a lower bound on the derivative of the maps T_j in some situation. Using Lemma 3.2.2, we see that $s \in (0,1)$.

Lemma 3.3.4. *For each $n \in \mathbb{N}$ we have*

$$m_{\boldsymbol{p}} \times \lambda(\varphi > n) \leq \frac{1}{4} \cdot \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-j-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}},$$

$$m_{\boldsymbol{p}} \times \lambda(\varphi > n) \geq \frac{1}{4} \cdot \min\{p_g \, : \, g \in \Sigma_G\} \cdot \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\min}^{-1}}}.$$

Proof. For $P = P_{\boldsymbol{uvbgw}} \in \mathcal{P}$ we know from Proposition 3.3.2 that the first return time is constant on P and equal to $\varphi_P = 2 + |\boldsymbol{v}| + |\boldsymbol{b}| + |\boldsymbol{w}|$. Let $n \in \mathbb{N}$. Then

$$m_{\boldsymbol{p}} \times \lambda(\varphi > n) = \sum_{P : \varphi_P > n} m_{\boldsymbol{p}} \times \lambda(P).$$

To obtain the desired lower bound on $m_{\boldsymbol{p}} \times \lambda(\varphi > n)$ we only consider those $P = P_{\boldsymbol{uvbgw}} \in \mathcal{P}$ where $\boldsymbol{v} = \epsilon$ is the empty word. From Lemma 3.2.1 we get

$$(R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}})^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big) = \Big(\frac{1}{2}\Big(1 - \frac{1}{2^{(|\boldsymbol{w}|+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}}\Big), \frac{1}{2}\Big(1 - \frac{1}{2^{(|\boldsymbol{w}|+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}}\Big)\Big). \quad (3.23)$$

Since R has derivative 2, we then have

$$\lambda\Big((R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}} \circ R)^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big)\Big) = \frac{1}{4}\Big(\frac{1}{2^{(|\boldsymbol{w}|+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(|\boldsymbol{w}|+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}}\Big)$$

and thus

$$
\begin{aligned}
&m_{\boldsymbol{p}} \times \lambda(\varphi > n) \\
&\geq \sum_{u \in \Sigma} \sum_{g \in \Sigma_G} \sum_{\boldsymbol{b} \in \Sigma_B^*} \sum_{\substack{\boldsymbol{w} \in \Sigma^*: \\ |\boldsymbol{w}| \geq \max\{n-2-|\boldsymbol{b}|,0\}}} \frac{m_{\boldsymbol{p}}([u\boldsymbol{b}g\boldsymbol{w}])}{4} \Big(\frac{1}{2^{(|\boldsymbol{w}|+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(|\boldsymbol{w}|+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} \Big) \\
&= \sum_{g \in \Sigma_G} \frac{p_g}{4} \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} p_{\boldsymbol{b}} \sum_{l=\max\{n-2-k,0\}}^{\infty} \Big(\frac{1}{2^{(l+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(l+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} \Big) \qquad (3.24) \\
&\geq \frac{1}{4} \cdot \min\{p_g \,:\, g \in \Sigma_G\} \cdot \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max\{n-1-k,1\}\ell_{\boldsymbol{b}}^{-1} r_{\min}^{-1}}}.
\end{aligned}
$$

For the upper bound, we look for the smallest derivative to bound the length of $(R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}} \circ L_{\boldsymbol{v}} \circ R)^{-1}(\frac{1}{2}, \frac{3}{4})$. If $\boldsymbol{v} = \epsilon$, then from above we see

$$
\lambda\Big((R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}} \circ R)^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big)\Big) = \frac{s^{|\boldsymbol{v}|}}{4} \Big(\frac{1}{2^{(|\boldsymbol{w}|+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(|\boldsymbol{w}|+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} \Big).
$$

On the other hand, if $\boldsymbol{v} = v_1 \cdots v_j$ with $j \geq 1$, then $v_j \in \Sigma_G$. We have

$$
(R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}})^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big) \subseteq \Big(0, \frac{1}{2}\Big).
$$

As follows from before s from (3.22) represents the smallest possible shrinkage factor when applying $L_{v_j}^{-1}$. If $j \geq 2$, then by Lemma 3.2.2(i) we have $L_{v_{j-1}}^{-1}(L_{v_j}^{-1}(\frac{1}{2})) \leq L_{v_{j-1}}^{-1}(\frac{1}{4})$. Hence, $s^{-|\boldsymbol{v}|}$ is a lower bound for the derivative of $L_{\boldsymbol{v}}$ for any $\boldsymbol{v} \in A_G$ on $(R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}})^{-1}(\frac{1}{2}, \frac{3}{4})$. It then follows from (3.23) that

$$
\lambda\Big((R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{b}} \circ L_{\boldsymbol{v}} \circ R)^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big)\Big) \leq \frac{s^{|\boldsymbol{v}|}}{4} \Big(\frac{1}{2^{(|\boldsymbol{w}|+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(|\boldsymbol{w}|+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} \Big).
$$

Writing $f(n, j, k) = \max(n - 2 - j - k, 0)$, we thus obtain that

$$
\begin{aligned}
&m_{\boldsymbol{p}} \times \lambda(\varphi > n) \\
&\leq \sum_{g \in \Sigma_G} \sum_{\boldsymbol{v} \in A_G} \sum_{\boldsymbol{b} \in \Sigma_B^*} \sum_{\substack{\boldsymbol{w} \in \Sigma^*: \\ |\boldsymbol{w}| \geq f(n,|\boldsymbol{v}|,|\boldsymbol{b}|)}} \frac{p_{\boldsymbol{v}\boldsymbol{b}g\boldsymbol{w}} s^{|\boldsymbol{v}|}}{4} \Big(\frac{1}{2^{(|\boldsymbol{w}|+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(|\boldsymbol{w}|+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} \Big) \\
&\leq \frac{1}{4} \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} \sum_{g \in \Sigma_G} \sum_{\boldsymbol{b} \in \Sigma_B^k} p_{\boldsymbol{b}} \sum_{l=f(n,j,k)}^{\infty} \Big(\frac{1}{2^{(l+1)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} - \frac{1}{2^{(l+2)\ell_{\boldsymbol{b}}^{-1} r_g^{-1}}} \Big) \qquad (3.25) \\
&\leq \frac{1}{4} \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{(f(n,j,k)+1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}}.
\end{aligned}
$$

This gives the result. □

The next lemma gives estimates for the last part of the expression on the right-hand side of the first inequality from Lemma 3.3.4 for an initial range of values of j. The number of values j for which we obtain an upper bound for the double sum grows logarithmically with n.

Lemma 3.3.5. *Let $\gamma \in (\gamma_1, 0)$, and for each $n \in \mathbb{N}$ define $j(n) = \lfloor \frac{(\gamma-1)\log n}{\log s} \rfloor$. Then there exist $C_1 > 0$ and $n_1 \in \mathbb{N}$ such that for all integers $n \geq n_1$ and $j = 0, 1, \ldots, j(n)$ we have*

$$\sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-j-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}} \leq C_1 \cdot n^{\gamma-1}. \tag{3.26}$$

Proof. We will split the sum over k in (3.26) into three pieces: $0 \leq k \leq k_1(n)$, $k_1(n) + 1 \leq k \leq k_2(n)$ and $k > k_2(n)$. To define $k_1(n)$, let $a = \frac{\gamma \log \ell_{\max}}{\log \theta} \in (0, 1)$. Then for each $n \in \mathbb{N}$ set $k_1(n) = \lfloor \frac{\log(n^a)}{\log \ell_{\max}} \rfloor = \lfloor \frac{\gamma \log n}{\log \theta} \rfloor$. The values a and $k_1(n)$ are such that $\theta^{k_1(n)+1} \leq n^\gamma$ and $\ell_{\max}^{k_1(n)} \leq n^a$. Since $j(n) + k_1(n) = O(\log n)$, we can find an $N_0 \in \mathbb{N}$ and a constant $K_1 > 0$ such that for all integers $n \geq N_0$ we have $n - 1 - j(n) - k_1(n) > 1$ and $(n - 1 - j(n) - k_1(n)) \cdot r_{\max}^{-1} \geq K_1 \cdot n$. Then, for all $n \geq N_0$, $0 \leq j \leq j(n)$ and $0 \leq k \leq k_1(n)$,

$$\max(n - 1 - j - k, 1) \geq \max(n - 1 - j(n) - k_1(n), 1) = n - 1 - j(n) - k_1(n)$$

and $\ell_{\boldsymbol{b}} \leq \ell_{\max}^{k_1(n)} \leq n^a$ for all $\boldsymbol{b} \in \Sigma_B^k$. Setting $p_B = \sum_{b \in \Sigma_B} p_b$ as before, this together gives

$$\begin{aligned} \sum_{k=0}^{k_1(n)} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-j-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}} &\leq \sum_{k=0}^{\infty} \frac{p_B^k}{2^{(n-1-j(n)-k_1(n))n^{-a} r_{\max}^{-1}}} \\ &\leq \frac{2^{-K_1 \cdot n^{1-a}}}{1 - p_B}. \end{aligned} \tag{3.27}$$

Secondly, for each $n \in \mathbb{N}$ set $k_2(n) = \lceil \frac{1}{2}(n - 1 - j(n)) \rceil$ and take an integer $N_1 \geq N_0$ and constant $K_2 > 0$ such that for all integers $n \geq N_1$ we have $k_2(n) \geq k_1(n) + 1$ and $\frac{1}{2}(n - 1 - j(n)) - 1 \geq K_2 \cdot n$. Noting that for each $d > 1$ the function f on $\mathbb{R}$ given by $f(x) = \frac{x}{d^x}$ has maximal value $\frac{1}{e \log d} < \frac{1}{\log d}$, we obtain for all integers $n \geq N_1$,

$$\begin{aligned} \sum_{k=k_1(n)+1}^{k_2(n)} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-j-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}} &= \sum_{k=k_1(n)+1}^{k_2(n)} \sum_{\boldsymbol{b} \in \Sigma_B^k} p_{\boldsymbol{b}} \ell_{\boldsymbol{b}} \frac{\ell_{\boldsymbol{b}}^{-1}}{(2^{(n-1-j-k) r_{\max}^{-1}})^{\ell_{\boldsymbol{b}}^{-1}}} \\ &\leq \sum_{k=k_1(n)+1}^{k_2(n)} \theta^k \frac{r_{\max}}{(n - 1 - j - k)\log 2} \\ &\leq \frac{\theta^{k_1(n)+1}}{1 - \theta} \cdot \frac{r_{\max}}{(n - 1 - j(n) - k_2(n))\log 2} \\ &\leq \frac{r_{\max}}{(1 - \theta) K_2 \log 2} \cdot n^{\gamma - 1}. \end{aligned} \tag{3.28}$$

Finally, for each $n \geq N_1$ we have

$$\sum_{k=k_2(n)+1}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-j-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}} \leq \sum_{k=k_2(n)+1}^{\infty} p_B^k = \frac{p_B^{k_2(n)+1}}{1 - p_B} \leq \frac{p_B^{K_2 \cdot n}}{1 - p_B}. \tag{3.29}$$

Combining (3.27), (3.28) and (3.29) yields

$$\sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-j-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\max}^{-1}}} \leq \frac{2^{-K_1 \cdot n^{1-a}} + p_B^{K_2 \cdot n}}{1 - p_B} + \frac{r_{\max}}{(1-\theta) K_2 \log 2} n^{\gamma - 1}.$$

Since the first term on the right-hand side decreases superpolynomially fast in n, this yields the existence of a constant $C_1 > 0$ and integer $n_1 \geq N_1$ for which the statement of the lemma holds. □

Lemma 3.3.6. *There exist $C_2 > 0$ and $n_2 \in \mathbb{N}$ such that for each integer $n \geq N_2$ we have*

$$\sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\min}^{-1}}} \geq C_2 \cdot n^{\gamma_2 - 1}.$$

Proof. Let $b \in \Sigma_B$ be such that $\gamma_2 = 1 + \frac{\log \pi_b}{\log \ell_b}$ with π_b as in (3.5). For each $k \in \mathbb{N}$ let

$$A_k = \{ \boldsymbol{b} = b_1 \cdots b_k \in \Sigma_B^k : \ell_{b_j} \geq \ell_b \text{ for each } j = 1, \dots, k \}.$$

Then $\sum_{\boldsymbol{b} \in A_k} p_{\boldsymbol{b}} = \pi_b^k$ and for each $\boldsymbol{b} \in A_k$ we have $\ell_{\boldsymbol{b}} \geq \ell_b^k$. This gives

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in \Sigma_B^k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\min}^{-1}}} &\geq \sum_{k=0}^{\infty} \sum_{\boldsymbol{b} \in A_k} \frac{p_{\boldsymbol{b}}}{2^{\max(n-1-k,1)\ell_{\boldsymbol{b}}^{-1} r_{\min}^{-1}}} \\ &\geq \sum_{k=0}^{\infty} \frac{\pi_b^k}{2^{\max(n-1-k,1)\ell_b^{-k} r_{\min}^{-1}}}. \end{aligned} \tag{3.30}$$

For each $n \in \mathbb{N}$ we define $k_3(n) = \lceil \frac{(\gamma_2 - 1)\log n}{\log \pi_b} \rceil = \lceil \frac{\log n}{\log \ell_b} \rceil$. Then $\pi_b^{k_3(n)-1} \geq n^{\gamma_2 - 1}$ and $\ell_b^{-k_3(n)} \leq n^{-1}$. We take $N_2 \in \mathbb{N}$ and $K_3 > 0$ such that for each integer $n \geq N_2$ we have $n - 1 - k_3(n) \geq 1$ and $(n - 1 - k_3(n)) \cdot n^{-1} \cdot r_{\min}^{-1} \leq K_3$. Then we get

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\pi_b^k}{2^{\max(n-1-k,1)\ell_b^{-k} r_{\min}^{-1}}} &\geq \pi_b^{k_3(n)} \frac{1}{2^{(n-1-k_3(n))\ell_b^{-k_3(n)} r_{\min}^{-1}}} \\ &\geq \pi_b \cdot n^{\gamma_2 - 1} \frac{1}{2^{(n-1-k_3(n)) \cdot n^{-1} \cdot r_{\min}^{-1}}} \\ &\geq \pi_b \cdot 2^{-K_3} \cdot n^{\gamma_2 - 1} \end{aligned} \tag{3.31}$$

for each $n \geq N_2$. Combining (3.30) and (3.31) now yields the result with $C_2 = \pi_b \cdot 2^{-K_3}$. □

Proof of Proposition 3.3.3. First of all, note that

$$\int_Y \varphi \, dm_{\boldsymbol{p}} \times \lambda = \sum_{n=2}^{\infty} n \cdot m_{\boldsymbol{p}} \times \lambda(\varphi = n) \leq 2 \cdot \sum_{n=1}^{\infty} m_{\boldsymbol{p}} \times \lambda(\varphi > n).$$

Since for each $\gamma < 0$ we have $\sum_{n=1}^{\infty} n^{\gamma-1} < \infty$, (i) follows from (ii). For (ii), let $\gamma \in (\gamma_1, 0)$. It follows from Lemma 3.3.4 and Lemma 3.3.5 that for each integer $n \geq n_1$ we have

$$\begin{aligned} m_{\boldsymbol{p}} \times \lambda(\varphi > n) &\leq \frac{C_1}{4} \cdot n^{\gamma-1} \cdot \sum_{j=0}^{j(n)} s^j + \frac{1}{4} \sum_{j=j(n)+1}^{\infty} s^j \sum_{k=0}^{\infty} p_B^k \\ &\leq \frac{C_1}{4(1-s)} \cdot n^{\gamma-1} + \frac{1}{4(1-s)(1-p_B)} \cdot s^{j(n)+1}. \end{aligned}$$

By the definition of $j(n)$, we have $s^{j(n)+1} \leq n^{\gamma-1}$, which gives (ii).

Finally, it follows from Lemma 3.3.4 and Lemma 3.3.6 that for each integer $n \geq n_2$ we have

$$m_{\boldsymbol{p}} \times \lambda(\varphi > n) \geq \frac{\min\{p_g \,:\, g \in \Sigma_G\} \cdot C_2}{4} \cdot n^{\gamma_2 - 1}. \qquad \square$$

§3.3.3 Estimates on the induced map

Recall that $F^{\varphi}(\omega, x) = (\tau^{\varphi(\omega,x)}\omega, T_{\omega}^{\varphi(\omega,x)}(x))$. The second part of the next lemma shows in particular that F^{φ} projected on the second coordinate is expanding.

Lemma 3.3.7. *Let $(\omega, x) \in Y$.*

(i) For each $j = 1, \ldots, \varphi(\omega, x) - 1$ we have $DT_{\tau^j\omega}^{\varphi(\omega,x)-j}(T_{\omega}^j(x)) \geq \frac{1}{2}$.

(ii) $DT_{\omega}^{\varphi(\omega,x)}(x) \geq 2$.

Proof. Let $(\omega, x) \in Y$. Recall the definitions of $\kappa = \kappa(\omega, x)$ from (3.13), $l = l(\omega, x)$ from (3.14) and $m = m(\omega, x)$ from (3.16). Write[3]

$$\begin{aligned} \boldsymbol{u} &= \omega_2 \cdots \omega_m \in \Sigma^*, \\ \boldsymbol{d} &= \omega_{m+1} \cdots \omega_{\kappa-1} \in \Sigma_B^*, \\ g &= \omega_{\kappa} \in \Sigma_G. \end{aligned}$$

Then

$$T_{\omega}^{\varphi(\omega,x)}(x) = R^l \circ L_g \circ L_{\boldsymbol{d}} \circ L_{\boldsymbol{u}} \circ R(x).$$

We have $DL_b(y) \geq 1$ for all $y \in [0, x_b)$ and all $b \in \Sigma_B$. For $v \in \Sigma_G$ with $r_v > 1$ we obtained in Lemma 3.2.2(ii) that $x_v > L_v^{-1}(\frac{1}{2})$ and hence $DL_v(y) \geq 1$ for all $v \in \Sigma_G$ and $y \in [0, L_v^{-1}(\frac{1}{2}))$. It follows from the definition of m that, for each $j \in \{1, \ldots, m-1\}$,

$$DL_{u_j \cdots u_{m-1}}(L_{u_1 \cdots u_{j-1}} \circ R(x)) = \prod_{i=j}^{m-1} DL_{u_i}(L_{u_1 \cdots u_{i-1}} \circ R(x)) \geq 1. \tag{3.32}$$

[3] We use different letters here than for the partition elements $P_{\boldsymbol{u}\boldsymbol{v}\boldsymbol{b}g\boldsymbol{w}}$ from $\mathcal{P}$, since the subdivision here is different (and (ω, x)-dependent).

Let $\boldsymbol{q}, \boldsymbol{t} \in \Sigma_B^*$ be any two words such that $\boldsymbol{d} = \boldsymbol{q}\boldsymbol{t}$. Using Lemma 3.2.1 we find that for each $y \in [0, \frac{1}{2})$,

$$D(L_g \circ L_{\boldsymbol{t}})(y) = 2\ell_{\boldsymbol{t}} r_g (1-2y)^{\ell_{\boldsymbol{t}} r_g - 1}. \tag{3.33}$$

Furthermore, from (3.17) we see that

$$2^l \geq \frac{1}{4}(1 - 2L_{\boldsymbol{u}} \circ R(x))^{-\ell_{\boldsymbol{d}} r_g} \tag{3.34}$$

and applying Lemma 3.2.1 to $L_{\boldsymbol{q}}$ gives

$$(1 - 2L_{\boldsymbol{q}} \circ L_{\boldsymbol{u}} \circ R(x))^{\ell_{\boldsymbol{t}} r_g - 1} = (1 - 2L_{\boldsymbol{u}} \circ R(x))^{\ell_{\boldsymbol{d}} r_g - \ell_{\boldsymbol{q}}}. \tag{3.35}$$

Combining (3.33), (3.34) and (3.35) yields

$$\begin{aligned} & D(R^l \circ L_g \circ L_{\boldsymbol{t}})(L_{\boldsymbol{q}} \circ L_{\boldsymbol{u}} \circ R(x)) \\ &= 2^l \cdot 2\ell_{\boldsymbol{t}} r_g (1 - 2L_{\boldsymbol{q}} \circ L_{\boldsymbol{u}} \circ R(x))^{\ell_{\boldsymbol{t}} r_g - 1} \\ &\geq \frac{1}{4}(1 - 2L_{\boldsymbol{u}} \circ R(x))^{-\ell_{\boldsymbol{d}} r_g} \cdot 2\ell_{\boldsymbol{t}} r_g (1 - 2L_{\boldsymbol{u}} \circ R(x))^{\ell_{\boldsymbol{d}} r_g - \ell_{\boldsymbol{q}}} \\ &= \frac{1}{2}\ell_{\boldsymbol{t}} r_g (1 - 2L_{\boldsymbol{u}} \circ R(x))^{-\ell_{\boldsymbol{q}}}, \end{aligned} \tag{3.36}$$

which we can lower bound by $\frac{1}{2}$. To prove (i), for any $j \in \{1, \ldots, m-1\}$ taking $\boldsymbol{q} = \epsilon$ (which means $\ell_{\boldsymbol{q}} = 1$) and $\boldsymbol{t} = \boldsymbol{d}$ we obtain using (3.32) and (3.36) that

$$DT_{\tau^j \omega}^{\varphi(\omega,x)-j}(T_\omega^j x) = D(R^l \circ L_g \circ L_{\boldsymbol{d}})(L_{\boldsymbol{u}} \circ R(x)) \cdot DL_{u_j \cdots u_{m-1}}(L_{u_1 \cdots u_{j-1}} \circ R(x)) \geq \frac{1}{2}.$$

For $j \in \{m, \ldots, \kappa - 1\}$ we take $\boldsymbol{q} = \omega_{m+1} \cdots \omega_j$ and $\boldsymbol{t} = \omega_{j+1} \cdots \omega_{\kappa-1}$ (so $\boldsymbol{q} = \epsilon$ in case $j = m$ and $\boldsymbol{t} = \epsilon$ in case $j = \kappa - 1$) and get

$$DT_{\tau^j \omega}^{\varphi(\omega,x)-j}(T_\omega^j(x)) = D(R^l \circ L_g \circ L_{\boldsymbol{t}})(L_{\boldsymbol{q}} \circ L_{\boldsymbol{u}} \circ R(x)) \geq \frac{1}{2}$$

by (3.36). Finally, if $j \in \{\kappa, \ldots, \varphi(\omega, x) - 1\}$, then

$$DT_{\tau^j \omega}^{\varphi(\omega,x)-j}(T_\omega^j(x)) = 2^{\kappa + l - j} \geq \frac{1}{2}.$$

This proves (i). For (ii), we write

$$DT_\omega^{\varphi(\omega,x)}(x) = D(R^l \circ L_g \circ L_{\boldsymbol{d}})(L_{\boldsymbol{u}} \circ R(x)) \cdot DL_{\boldsymbol{u}}(R(x)) \cdot DR(x).$$

We have $DR(x) = 2$ and by (3.32) with $j = 1$ we get $DL_{\boldsymbol{u}}(R(x)) \geq 1$. What is left is to estimate the first factor. From (3.36) with $\boldsymbol{q} = \epsilon$ and $\boldsymbol{t} = \boldsymbol{d}$ we see that

$$D(R^l \circ L_g \circ L_{\boldsymbol{d}})(L_{\boldsymbol{u}} \circ R(x)) \geq \frac{1}{2}\ell_{\boldsymbol{d}} r_g (1 - 2L_{\boldsymbol{u}} \circ R(x))^{-1}.$$

Note that by the definition of m we have $L_{\boldsymbol{u}} \circ R(x) \in (L_g^{-1}(\frac{1}{2}), \frac{1}{2})$ if $m = \kappa - 1$, so if $\boldsymbol{d} = \epsilon$, and $L_{\boldsymbol{u}} \circ R(x) \in [x_{d_1}, \frac{1}{2})$ if $m < \kappa - 1$. In case $m = \kappa - 1$ we obtain that

$$\ell_{\boldsymbol{d}} r_g (1 - 2L_{\boldsymbol{u}} \circ R(x))^{-1} \geq r_g \Big(1 - 2L_g^{-1}\Big(\frac{1}{2}\Big)\Big)^{-1} = r_g \cdot 2^{1/r_g} \geq 2,$$

where we used the expression for $L_g^{-1}(\frac{1}{2})$ from (3.8) and the fact that $x \cdot 2^{1/x} \geq 2$ for all $x \geq 1$. In case $m < \kappa - 1$, we have

$$\ell_{\boldsymbol{d}} r_g (1 - 2L_{\boldsymbol{u}} \circ R(x))^{-1} \geq \ell_{d_1}(1 - 2x_{d_1})^{-1} = \ell_{d_1}^{1+(\ell_{d_1}-1)^{-1}} \geq 2,$$

where we used (3.9) and the fact that $x^{1+(x-1)^{-1}} > 2$ for all $x > 1$. Hence, in all cases

$$DT_\omega^{\varphi(\omega,x)}(x) \geq \frac{1}{2} \cdot 2 \cdot 1 \cdot 2 = 2.$$

□

Recall the first return time partition $\mathcal{P}$ from (3.20). For $P = P_{\boldsymbol{usgw}} \in \mathcal{P}$ set

$$\pi_2(P) := (R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{s}} \circ R)^{-1}\Big(\frac{1}{2}, \frac{3}{4}\Big)$$

and write S_P for the restriction of the map $T_\omega^{\varphi(\omega,x)}$ to $\pi_2(P)$, so

$$S_P := T_\omega^{\varphi(\omega,x)}|_{\pi_2(P)} = R^{|\boldsymbol{w}|} \circ L_g \circ L_{\boldsymbol{s}} \circ R|_{\pi_2(P)}.$$

We give two lemmas on the maps S_P, that will be useful when verifying (t6) for the Young tower in the next section.

Lemma 3.3.8. *There exists a constant $C_3 > 0$ such that for each $P \in \mathcal{P}$ and all $(\omega, x), (\omega', y) \in P$ we have*

$$\Big|\frac{J_{m_{\boldsymbol{p}} \times \lambda} F^\varphi(\omega, x)}{J_{m_{\boldsymbol{p}} \times \lambda} F^\varphi(\omega', y)} - 1\Big| \leq C_3 \cdot \big|S_P(x) - S_P(y)\big|.$$

Proof. For each $P \in \mathcal{P}$ and all $(\omega, x), (\omega', y) \in P$ we have $\varphi(\omega, x) = \varphi(\omega', y) = \varphi_P$ and $\omega_j = \omega'_j$ for all $1 \leq j \leq \varphi_P$. Hence, for each measurable set $A \subseteq P$ we have

$$m_{\boldsymbol{p}} \times \lambda(F^\varphi(A)) = \int_A \Big(\prod_{j=1}^{\varphi(\omega,x)} p_{\omega_j}^{-1}\Big) DT_\omega^{\varphi(\omega,x)}(x)\, dm_{\boldsymbol{p}} \times \lambda(\omega, x).$$

By Proposition 1.2.19 we obtain

$$J_{m_{\boldsymbol{p}} \times \lambda} F^\varphi(\omega, x) = \Big(\prod_{j=1}^{\varphi(\omega,x)} p_{\omega_j}^{-1}\Big) DT_\omega^{\varphi(\omega,x)}(x),$$

which, for each $P \in \mathcal{P}$ and all $(\omega, x), (\omega', y) \in P$, gives

$$\Big|\frac{J_{m_{\boldsymbol{p}} \times \lambda} F^\varphi(\omega, x)}{J_{m_{\boldsymbol{p}} \times \lambda} F^\varphi(\omega', y)} - 1\Big| = \Big|\frac{DT_\omega^{\varphi(\omega,x)}(x)}{DT_{\omega'}^{\varphi(\omega',y)}(y)} - 1\Big| = \Big|\frac{DS_P(x)}{DS_P(y)} - 1\Big|.$$

Let $c > 0$. As compositions of good and bad maps each S_P has non-positive Schwarzian derivative and by Lemma 3.3.7(ii) each S_P satisfies $DS_P \geq 2$. For this reason for each $P \in \mathcal{P}$ we can extend the domain $\pi_2(P)$ of S_P on both sides to an interval $I_P \supseteq \pi_2(P)$ such that there exists an extension $\tilde{S}_P : I_P \to \mathbb{R}$ of S_P, i.e. $\tilde{S}_P|_{\pi_2(P)} =$

$S_P|_{\pi_2(P)}$, that has non-positive Schwarzian derivative and for which both components of $\tilde{S}_P(I_P)\backslash(\frac{1}{2},\frac{3}{4})$ have length at least $\frac{c}{4}$. Applying for each $P \in \mathcal{P}$ the Koebe Principle (1.15) with $I = I_P$ and $J = \pi_2(P)$ then gives a constant $C_3 > 0$ that depends only on c such that for each $P \in \mathcal{P}$ and each $x, y \in \pi_2(P)$ we have

$$\Big|\frac{DS_P(x)}{DS_P(y)} - 1\Big| \le C_3 \cdot |S_P(x) - S_P(y)|.$$

This gives the lemma. □

Recall the definition of the separation time from (3.10). We have the following lemma.

Lemma 3.3.9. *Let* $(\omega, x), (\omega', y) \in Y$*. Then*

$$|x - y| \le 2^{-s((\omega,x),(\omega',y))}. \tag{3.37}$$

Furthermore, if $(\omega, x), (\omega', y) \in P$ *for some* $P \in \mathcal{P}$*, then*

$$\big|S_P(x) - S_P(y)\big| \le 2^{-s(F^\varphi(\omega,x),F^\varphi(\omega',y))}. \tag{3.38}$$

Proof. Write $n = s((\omega, x), (\omega', y))$ and, for each $k \in \{0, 1, \dots, n-1\}$, let $P^{(k)} \in \mathcal{P}$ be such that $(F^\varphi)^k(\omega, x), (F^\varphi)^k(\omega', y) \in P^{(k)}$. Then for each $k \in \{0, 1, \dots, n-1\}$ the points $(S_{P^{(k-1)}} \circ \cdots \circ S_{P^{(0)}})(x)$ and $(S_{P^{(k-1)}} \circ \cdots \circ S_{P^{(0)}})(y)$ lie in the domain $\pi_2(P^{(k)})$ of $S_{P^{(k)}}$, so it follows from Lemma 3.3.7(ii) together with the Mean Value Theorem that

$$\frac{|S_{P^{(k)}} \circ \cdots \circ S_{P^{(0)}}(x) - S_{P^{(k)}} \circ \cdots \circ S_{P^{(0)}}(y)|}{|S_{P^{(k-1)}} \circ \cdots \circ S_{P^{(0)}}(x) - S_{P^{(k-1)}} \circ \cdots \circ S_{P^{(0)}}(y)|} \ge \inf DS_{P^{(k)}} \ge 2.$$

We conclude that

$$\begin{aligned} |x - y| &\le 2^{-1}|S_{P^{(0)}}(x) - S_{P^{(0)}}(y)| \\ &\le \cdots \le 2^{-n}|S_{P^{(n-1)}} \circ \cdots \circ S_{P^{(0)}}(x) - S_{P^{(n-1)}} \circ \cdots \circ S_{P^{(0)}}(y)| \le 2^{-n}, \end{aligned}$$

which gives the first part of the lemma. For the second part, note that if $(\omega, x), (\omega', y) \in P$ for some $P \in \mathcal{P}$, then (3.38) follows by applying (3.37) to the points $F^\varphi(\omega, x) = (\tau^{\varphi_P}\omega, S_P(x))$ and $F^\varphi(\omega', y) = (\tau^{\varphi_P}\omega', S_P(y))$. □

§3.4 A Young tower for the random map

§3.4.1 The acs probability measure

We are now in the position to construct a Young tower for the skew product F according to the setup from [Y99, Section 1.1] that we outlined in Subsection 3.2.2.

As the base for the Young tower we take the set Y. The Young tower Δ, the l^{th} levels of the tower Δ_l and the tower map $G : \Delta \to \Delta$ are defined in Subsection 3.2.2, as well as the reference measure m and the partition η on Δ. Following the general

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setup in [Y99], inducing the map G on $\Delta_0 = Y \times \{0\}$ yields a transformation G^φ on Δ_0 given by $G^\varphi(z,0) = G^{\varphi(z)}(z,0)$. Recall that we identify G^φ with F^φ by identifying Δ_0 with Y and using the correspondence $G^\varphi(z,0) = (F^\varphi(z),0)$. We check that the conditions (t1)–(t6) from Section 3.2.2 hold for this construction.

Proposition 3.4.1. *The conditions (t1)–(t6) hold for the map G on the Young tower Δ defined above.*

Proof. From the first return time partition $\mathcal{P}$ it is clear that (t1), (t2), (t3) and (t5) hold.

For (t4) it is enough to show that the collection

$$\bigvee_{n\geq 0} G^{-n}\eta = \{E_0 \cap G^{-1}E_1 \cap \cdots \cap G^{-n}E_n \,:\, E_i \in \eta,\, 1 \leq i \leq n,\, n \geq 0\}$$

separates points. To show this, let $(z_1,l_1),(z_2,l_2) \in \Delta$ be two points. If $z_1 = z_2$ and $l_1 \neq l_2$ and $P \in \mathcal{P}$ is such that $z_1 \in P$, then $\Delta_{l_1,P}, \Delta_{l_2,P} \in \eta$ are sets that separate (z_1,l_1) and (z_2,l_2). Assume that $z_1 \neq z_2$. Lemma 3.3.7(ii) implies that the map F^φ is expanding on Y, so there exist an $N \geq 0$ and two disjoint sets $A, E \in \bigvee_{n=0}^{N}(F^\varphi)^{-n}\mathcal{P}$ such that $z_1 \in A$ and $z_2 \in E$. On these sets the first N first return times to Y are constant, meaning that if $K > 0$ is such that $G^K(z_1,0) = ((F^\varphi)^N(z_1),0)$, then $A \times \{l_1\} \in \bigvee_{n=0}^{K+l_1} G^{-n}\eta$ and if $L > 0$ such that $G^L(z_2,0) = ((F^\varphi)^N(z_2),0)$, then $E \times \{l_2\} \in \bigvee_{n=0}^{L+l_2} G^{-n}\eta$. Note that $(z_1,l_1) \in A \times \{l_1\}$ and $(z_2,l_2) \in E \times \{l_2\}$ and $(A \times \{l_1\}) \cap (E \times \{l_2\}) = \emptyset$. Hence, (t4) holds.

Finally, from Lemma 3.3.8 and (3.38) we obtain that

$$\left| \frac{J_{m_p \times \lambda} F^\varphi(z_1)}{J_{m_p \times \lambda} F^\varphi(z_2)} - 1 \right| \leq C_3 \cdot 2^{-s(F^\varphi(z_1),F^\varphi(z_2))} \tag{3.39}$$

for each $P \in \mathcal{P}$ and all $z_1, z_2 \in P$. This gives (t6) with $\beta = \frac{1}{2}$ and the proposition follows. ☐

Now Proposition 3.3.3(i) and Theorem 3.2.3 imply the existence of a probability measure ν on $(\Delta, \mathcal{B})$ that is G-invariant, exact and absolutely continuous with respect to m with a density that is bounded (because it is in $\mathcal{C}_\beta$) and bounded away from zero and that satisfies (3.12). We use this to construct the invariant measure for F that is promised in Theorem 3.1.2. Define

$$\pi : \Delta \to \Sigma^{\mathbb{N}} \times [0,1],\ (z,l) \mapsto F^l(z).$$

Then

$$\begin{aligned} \pi(G(z,l)) &= \pi(z,l+1) = F^{l+1}(z) = F(\pi(z,l)), \qquad && l < \varphi(z) - 1, \\ \pi(G(z,l)) &= \pi(F^\varphi(z),0) = F^\varphi(z) = F(\pi(z,l)), \qquad && l = \varphi(z) - 1. \end{aligned}$$

So $\pi \circ G = F \circ \pi$. Let $\rho = \nu \circ \pi^{-1}$ be the pushforward measure of ν under π.

Lemma 3.4.2. *The probability measure ρ satisfies the following properties.*

(i) F is measure preserving and mixing with respect to ρ.

(ii) ρ is absolutely continuous with respect to $m_{\boldsymbol{p}} \times \lambda$.

(iii) We have

$$\rho(A \cap Y) = \nu((A \cap Y) \times \{0\}), \qquad A \in \mathcal{F}.$$

Proof. Part (i) immediately follows from the properties of the measure ν and the fact that $\pi \circ G = F \circ \pi$. For (ii), let $A \in \mathcal{F}$ be such that $m_{\boldsymbol{p}} \times \lambda(A) = 0$. Using that F is non-singular with respect to $m_{\boldsymbol{p}} \times \lambda$, we obtain that

$$m(\pi^{-1}(A)) = m\Big(\Delta \cap \Big(\bigcup_{l \geq 0} F^{-l}(A) \times \{l\}\Big)\Big) \leq \sum_{l \geq 0} m_{\boldsymbol{p}} \times \lambda(F^{-l}(A)) = 0.$$

Since ν is absolutely continuous with respect to m, it follows that $\rho(A) = \nu(\pi^{-1}A) = 0$. For (iii) let $A \in \mathcal{F}$. We have

$$\pi^{-1}(A \cap Y) = \bigcup_{P \in \mathcal{P}} \bigcup_{l=0}^{\varphi_P - 1} (F^{-l}(A \cap Y) \cap P) \times \{l\}.$$

By definition of φ_P, we have $F^l(z) \notin Y$ for each $z \in P$ and each $l \in \{1, \ldots, \varphi_P - 1\}$. Therefore

$$\pi^{-1}(A \cap Y) = \bigcup_{P \in \mathcal{P}} (A \cap P) \times \{0\} = (A \cap Y) \times \{0\}.$$

□

Combining Lemma 3.4.2 with Lemma 1.4.1 yields that there exists a probability measure μ that is absolutely continuous with respect to λ and such that $\rho = m_{\boldsymbol{p}} \times \mu$. In other words, μ is an acs measure for $(\mathcal{T}, \boldsymbol{p})$. We will now prove Theorem 3.1.2, which shows that μ is in fact the only acs measure for $(\mathcal{T}, \boldsymbol{p})$.

Proof of Theorem 3.1.2. It follows from Lemma 3.4.2(i) that F is mixing with respect to $m_{\boldsymbol{p}} \times \mu$. Hence, to obtain that μ is the only acs probability measure for $(\mathcal{T}, \boldsymbol{p})$, it suffices according to Theorem 1.2.6 to show that $\frac{d\mu}{d\lambda} > 0$ holds λ-a.e. Theorem 3.2.3 asserts that there is a constant $C_4 \geq 1$ such that

$$\frac{1}{C_4} \leq \frac{d\nu}{dm} \leq C_4. \tag{3.40}$$

Let $B \subseteq (\frac{1}{2}, \frac{3}{4})$ be a Borel set. Lemma 3.4.2(iii) and (3.40) imply that

$$\mu(B) = \nu(((\tilde{\Omega} \times B) \cap Y) \times \{0\}) \geq C_4^{-1} \cdot m(((\tilde{\Omega} \times B) \cap Y) \times \{0\}) = C_4^{-1} \cdot \lambda(B).$$

Since B was arbitrary, we have $\frac{d\mu}{d\lambda}(x) \geq C_4^{-1}$ for λ-a.e. $x \in (\frac{1}{2}, \frac{3}{4})$. Recall that the density $\frac{d\mu}{d\lambda}$ is a fixed point of the Perron-Frobenius operator $\mathcal{P}_{\mathcal{T},\boldsymbol{p}}$ being of the form as in (1.20). Fix some $g \in \Sigma_G$. The map $T_g^2|_{(\frac{1}{2}, \frac{3}{4})} : (\frac{1}{2}, \frac{3}{4}) \to (0,1)$ is a measurable bijection with measurable inverse. For each $x \in (0,1)$ let y_x be the unique element in $(\frac{1}{2}, \frac{3}{4})$ that satisfies $x = T_g^2(y_x)$. Furthermore, note that $\sup_{y \in (\frac{1}{2}, \frac{3}{4})} DT_g^2(y) \leq 2DT_g(0)$. We conclude that for λ-a.e. $x \in (0,1)$

$$\frac{d\mu}{d\lambda}(x) = \mathcal{P}_{\mathcal{T},\boldsymbol{p}}^2 \frac{d\mu}{d\lambda}(x) \geq p_g^2 \frac{\frac{d\mu}{d\lambda}(y_x)}{DT_g^2(y_x)} \geq \frac{p_g^2 C_4^{-1}}{2DT_g(0)} > 0.$$

This gives that $\frac{d\mu}{d\lambda}$ is bounded away from zero and therefore that μ is the unique acs measure. □

Remark 3.4.3. Besides Theorems 3.1.1 and 3.1.2 it can also be shown that all the results from Theorem 2.1.2 carry over. Namely, by following the same steps as in Section 2.2 it can be shown that $(\mathcal{T}, \boldsymbol{p})$ admits, independent of the value of θ, a unique (up to scalar multiplication) acs measure that is σ-finite and ergodic and for which the density is bounded away from zero, is locally Lipschitz on $(0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ and is not in L^q for any $q > 1$. This measure is infinite if $\theta \geq 1$ and coincides with μ if $\theta < 1$. Similar as in Chapter 2, if $\theta \geq 1$, it follows from applying Aaronson's Ergodic Theorem [A97, Theorem 2.4.2] to this infinite measure that no physical measure (see footnote 1 on page 39) for F exists.

§3.4.2 Decay of correlations

Recall that for $\alpha \in (0,1)$ we have set $\mathcal{H}_\alpha$ for the set of α-Hölder continuous functions on $\Sigma^{\mathbb{N}} \times [0,1]$ with metric d as in (3.3). Also recall the definition of the function spaces $\mathcal{C}_\delta$ on Δ from (3.11).

Lemma 3.4.4. *Let $\alpha \in (0,1)$ and $h \in \mathcal{H}_\alpha$. Then $h \circ \pi \in \mathcal{C}_{1/2^\alpha}$.*

Proof. Since $h \in \mathcal{H}_\alpha$, there exists a constant $C_5 > 0$ such that

$$|h(z_1) - h(z_2)| \leq C_5 \cdot d(z_1, z_2)^\alpha \quad \text{for all } z_1, z_2 \in \Sigma^{\mathbb{N}} \times [0,1]. \tag{3.41}$$

From this it is easy to see that $\|h\|_\infty < \infty$. Let $\upsilon_1 = (z_1, l_1), \upsilon_2 = (z_2, l_2) \in \Delta$. If $l_1 \neq l_2$ or if z_1 and z_2 lie in different elements of $\mathcal{P}$, then $s(\upsilon_1, \upsilon_2) = 0$ and

$$|h \circ \pi(\upsilon_1) - h \circ \pi(\upsilon_2)| = |h(F^{l_1}(z_1)) - h(F^{l_2}(z_2))| \leq 2\|h\|_\infty = 2\|h\|_\infty \cdot 2^{-\alpha s(\upsilon_1, \upsilon_2)}. \tag{3.42}$$

Hence, to prove that $h \circ \pi \in \mathcal{C}_{1/2^\alpha}$, it remains to consider the case that $z_1, z_2 \in P$ for some $P \in \mathcal{P}$ and $l_1 = l_2 = l \in \{0, \dots, \varphi_P - 1\}$. Write $z_1 = (\omega, x)$ and $z_2 = (\omega', y)$. Note that $\omega_j = \omega'_j$ for each $j \in \{1, 2, \dots, \varphi_P\}$. Hence,

$$2^{-\min\{i \in \mathbb{N} : \omega_{l+i} \neq \omega'_{l+i}\}} \leq 2^{-\min\{i \in \mathbb{N} : \omega_{\varphi_P - 1 + i} \neq \omega'_{\varphi_P - 1 + i}\}} \leq 2^{-s(z_1, z_2)}.$$

Furthermore, it follows from the Mean Value Theorem together with Lemma 3.3.7(i) that

$$\frac{|T_\omega^{\varphi_P}(x) - T_\omega^{\varphi_P}(y)|}{|T_\omega^l(x) - T_\omega^l(y)|} = \frac{|T_{\tau^l \omega}^{\varphi_P - l}(T_\omega^l(x)) - T_{\tau^l \omega}^{\varphi_P - l}(T_\omega^l(y))|}{|T_\omega^l(x) - T_\omega^l(y)|} \geq \frac{1}{2}.$$

Combining this with (3.38) yields that

$$|T_\omega^l(x) - T_\omega^l(y)| \leq 2 \cdot 2^{-s(F^\varphi(z_1), F^\varphi(z_2))} = 4 \cdot 2^{-s(z_1, z_2)}$$

and hence by (3.41),

$$|h(F^l(z_1)) - h(F^l(z_2))| \leq C_5 (2^{-s(z_1, z_2)} + 4 \cdot 2^{-s(z_1, z_2)})^\alpha = 5^\alpha C_5 \cdot 2^{-\alpha s(z_1, z_2)}.$$

Together with (3.42) this gives the result. □

We now have all the ingredients to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. To prove the theorem, we would like to use Theorem 3.2.3(iv), which requires us to bound $m(\hat{\varphi} > n)$, where

$$\hat{\varphi} : \Delta \to \mathbb{N}_0,\ v \mapsto \inf\{n \geq 0 \,:\, G^n(v) \in \Delta_0\}.$$

Since

$$\{\hat{\varphi} = 0\} = \Delta_0 = \bigcup_{P \in \mathcal{P} : \varphi_P > 0} \Delta_{0,P},$$
$$\{\hat{\varphi} = n\} = \bigcup_{P \in \mathcal{P} : \varphi_P > n} \Delta_{\varphi_P - n, P}, \qquad n \geq 1,$$

we have for each $n \geq 0$ that

$$m(\hat{\varphi} = n) = \sum_{P \in \mathcal{P} : \varphi_P > n} m_{\boldsymbol{p}} \times \lambda(P) = m_{\boldsymbol{p}} \times \lambda(\varphi > n).$$

It follows from Proposition 3.3.3(ii) that for each $\gamma \in (\gamma_1, 0)$ there is an $M > 0$ and an $N \geq 1$ such that for each $n \geq N$,

$$m_{\boldsymbol{p}} \times \lambda(\varphi > n) \leq M \cdot n^{\gamma - 1}.$$

Thus, for all $n \geq N$,

$$m(\hat{\varphi} > n) = \sum_{k > n} m(\hat{\varphi} = k) \leq M \sum_{k \geq n} k^{\gamma - 1} \leq M \cdot n^{\gamma - 1} + M \int_n^\infty x^{\gamma - 1} dx. \tag{3.43}$$

So, $m(\hat{\varphi} > n) = O(n^\gamma)$. Combining Proposition 3.3.3(i), Proposition 3.4.1 and Theorem 3.2.3(iv) now gives that for each $\gamma \in (\gamma_1, 0)$, $\hat{f} \in L^\infty(\Delta, \nu)$, $\delta \in (0,1)$ and $\hat{h} \in \mathcal{C}_\delta$,

$$\Big| \int_\Delta \hat{f} \circ G^n \cdot \hat{h} \, d\nu - \int_\Delta \hat{f} \, d\nu \int_\Delta \hat{h} \, d\nu \Big| = O(n^\gamma). \tag{3.44}$$

Now, let $\gamma \in (\gamma_1, 0)$, $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ and $h \in \mathcal{H}$. Using that $m_{\boldsymbol{p}} \times \mu = \nu \circ \pi^{-1}$ and $\pi \circ G = F \circ \pi$, it then follows that

$$|\mathrm{Cor}_n(f,h)| = \Big| \int_\Delta (f \circ \pi) \circ G^n \cdot (h \circ \pi) \, d\nu - \int_\Delta f \circ \pi \, d\nu \int_\Delta h \circ \pi \, d\nu \Big|.$$

Since $h \in \mathcal{H}$, it holds that $h \in \mathcal{H}_\alpha$ for some $\alpha \in (0,1)$, so $h \circ \pi \in \mathcal{C}_{1/2^\alpha}$ by Lemma 3.4.4. Since also $f \circ \pi \in L^\infty(\Delta, \nu)$, we obtain the result from (3.44) with $\hat{f} = f \circ \pi$ and $\hat{h} = h \circ \pi$. □

In order to prove Theorem 3.1.4, we need the following lemma.

Lemma 3.4.5. *There exists $C_6 > 0$ such that for each $P \in \mathcal{P}$ and $z_1, z_2 \in P$,*

$$\Big| \log \frac{J_{m_{\boldsymbol{p}} \times \mu} F^\varphi(z_1)}{J_{m_{\boldsymbol{p}} \times \mu} F^\varphi(z_2)} \Big| \leq C_6 \cdot 2^{-s(z_1, z_2)}.$$

Proof. From the definition of the Jacobian we see that $J_m G^\varphi|_{\Delta_0} = J_{m_{\boldsymbol{p}}\times\lambda} F^\varphi$ with the identification of Δ_0 and Y. Lemma 3.4.2(iii) and Lemma 1.2.20(a) give us that for each $P \in \mathcal{P}$ and each measurable set $A \subseteq P$,

$$\begin{aligned} m_{\boldsymbol{p}} \times \mu(F^\varphi(A)) &= \nu(G^\varphi(A \times \{0\})) \\ &= \int_{A\times\{0\}} \Big(\frac{d\nu}{dm} \circ G^\varphi\Big) J_m G^\varphi \, dm \\ &= \int_A \frac{d\nu}{dm}(F^\varphi(z), 0) \cdot J_{m_{\boldsymbol{p}}\times\lambda} F^\varphi(z) \cdot \frac{dm}{d\nu}(z,0)\, dm_{\boldsymbol{p}} \times \mu(z). \end{aligned}$$

This gives

$$J_{m_{\boldsymbol{p}}\times\mu} F^\varphi(z) = \frac{d\nu}{dm}(F^\varphi(z), 0) \cdot J_{m_{\boldsymbol{p}}\times\lambda} F^\varphi(z) \cdot \frac{dm}{d\nu}(z,0), \qquad z \in Y,$$

and thus, for each $z_1, z_2 \in Y$,

$$\Big|\log \frac{J_{m_{\boldsymbol{p}}\times\mu} F^\varphi(z_1)}{J_{m_{\boldsymbol{p}}\times\mu} F^\varphi(z_2)}\Big| \le \Big|\log \frac{\frac{d\nu}{dm}(F^\varphi(z_1),0)}{\frac{d\nu}{dm}(F^\varphi(z_2),0)}\Big| + \Big|\log \frac{J_{m_{\boldsymbol{p}}\times\lambda} F^\varphi(z_1)}{J_{m_{\boldsymbol{p}}\times\lambda} F^\varphi(z_2)}\Big| + \Big|\log \frac{\frac{d\nu}{dm}(z_2,0)}{\frac{d\nu}{dm}(z_1,0)}\Big|.$$

Combining Proposition 3.3.3(i), Proposition 3.4.1 and Theorem 3.2.3(ii) gives the existence of a constant $C^+ > 0$ such that, for each $\Delta_{l,P} \in \eta$ and $\upsilon_1, \upsilon_2 \in \Delta_{l,P}$,

$$\Big|\frac{\frac{d\nu}{dm}(\upsilon_1)}{\frac{d\nu}{dm}(\upsilon_2)} - 1\Big| \le C^+ \cdot 2^{-s(\upsilon_1,\upsilon_2)}. \tag{3.45}$$

Using that $|\log x| \le \max\{|x-1|, |x^{-1}-1|\}$ for all $x > 0$, we obtain from (3.39) and (3.45) that

$$\Big|\log \frac{J_{m_{\boldsymbol{p}}\times\mu} F^\varphi(z_1)}{J_{m_{\boldsymbol{p}}\times\mu} F^\varphi(z_2)}\Big| \le C^+ \cdot 2^{-s(F^\varphi(z_1),F^\varphi(z_2))} + C_3 \cdot 2^{-s(F^\varphi(z_1),F^\varphi(z_2))} + C^+ \cdot 2^{-s(z_1,z_2)}$$

for all $z_1, z_2 \in P$, $P \in \mathcal{P}$. The lemma thus holds with $C_6 = 3C^+ + 2C_3$. □

Proof of Theorem 3.1.4. Let $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ and $h \in \mathcal{H}$ be such that both f and h are identically zero on $\Sigma^{\mathbb{N}} \times \big([0,\frac{1}{2}] \cup [\frac{3}{4},1]\big)$ and such that $\int f\, dm_{\boldsymbol{p}} \times \mu \cdot \int h \, dm_{\boldsymbol{p}} \times \mu > 0$. Let $\gamma \in (\gamma_1, \min\{\gamma_2+1, -1\})$ if $\gamma_1 < -1$ and $\gamma \in (\gamma_1, \frac{\gamma_2}{2})$ if $-1 \le \gamma_1 < 0$. This is possible by assumption. Our strategy is to apply Theorem 3.2.4 with Y as before. For this, we verify (g1), (g2) and (g3).[4]

For (g3), $h \in \mathcal{H}$ implies that $h \in \mathcal{H}_\alpha$ for some $\alpha \in (0,1)$ and thus $h \circ \pi \in \mathcal{C}_{1/2^\alpha}$ by Lemma 3.4.4. In particular this yields (g3) with $\delta = 2^{-\alpha}$. For (g2), Lemma 3.4.2(iii) and (3.40) give

$$m_{\boldsymbol{p}} \times \mu(\varphi > n) = \int_{\{\varphi>n\}\times\{0\}} \frac{d\nu}{dm}\, dm \le C_4 \cdot m_{\boldsymbol{p}} \times \lambda(\varphi > n).$$

[4] More precisely, we apply Theorem 3.2.4 to versions of f and h that are also zero on $\big(\Sigma^{\mathbb{N}} \times (\frac{1}{2}, \frac{3}{4})\big) \backslash Y$.

Together with Proposition 3.3.3(ii) this implies that

$$m_{\boldsymbol{p}} \times \mu(\varphi > n) = O(n^{\gamma-1}).$$

Finally, (g1) with $\delta = 2^{-\alpha}$ follows from Lemma 3.4.5 by setting $C^* = C_6$ and noting that $\frac{1}{2^\alpha} > \frac{1}{2}$. Hence, we satisfy all the conditions of Theorem 3.2.4 with $\delta = 2^{-\alpha}$ and $\zeta = 1-\gamma$. Note that

$$K_{1-\gamma}(n) = \begin{cases} n^{\gamma-1}, & \text{if } 1-\gamma > 2, \\ \frac{\log n}{n^2}, & \text{if } 1-\gamma = 2, \\ n^{2\gamma}, & \text{if } 1-\gamma \in (1,2). \end{cases}$$

If $\gamma_1 < -1$, then $1-\gamma \in (\max\{-\gamma_2, 2\}, 1-\gamma_1) \subseteq (2,\infty)$ and if $-1 \leq \gamma_1 < 0$, then $1-\gamma \in (1-\frac{\gamma_2}{2}, 1-\gamma_1) \subseteq (1,2)$. We can thus conclude from Theorem 3.2.4 that

$$\left|\mathrm{Cor}_n(f,h) - \Big(\sum_{k>n}^{\infty} m_{\boldsymbol{p}} \times \mu(\varphi > k)\Big) \int f \, dm_{\boldsymbol{p}} \times \mu \int h \, dm_{\boldsymbol{p}} \times \mu\right| = O(n^{\xi}),$$

where $\xi = \gamma - 1$ if $\gamma_1 < -1$ and $\xi = 2\gamma$ if $-1 \leq \gamma_1 < 0$. As above it follows from Proposition 3.3.3(iii) combined with Lemma 3.4.2(iii) and (3.40) that $m_{\boldsymbol{p}} \times \mu(\varphi > n) = \Omega(n^{\gamma_2 - 1})$ and thus

$$\sum_{k>n}^{\infty} m_{\boldsymbol{p}} \times \mu(\varphi > k) = \Omega(n^{\gamma_2}).$$

The result now follows from observing that $\gamma_2 > \xi$. □

We provide some examples of combinations of parameters for which the conditions of Theorem 3.1.4 hold. As before set $\ell_{\min} = \min\{\ell_b : b \in \Sigma_B\}$ and $p_B = \sum_{j\in\Sigma_B} p_j$ and set $\pi_B = \sum_{j\in\Sigma_B : \ell_j = \ell_{\max}} p_j$. Examples that satisfy the conditions of Theorem 3.1.4 include the following.

- If Σ_B consists of one element, then $\gamma_1 = \gamma_2$.
- If $p_B^{-1/3} < \ell_{\min} \leq \ell_{\max} < p_B^{-1/2}$, or equivalently $\ell_{\min} > \ell_{\max}^{2/3}$ and $p_B \in (\ell_{\min}^{-3}, \ell_{\max}^{-2})$, then $\theta \leq p_B \cdot \ell_{\max} < \ell_{\max}^{-1}$, so $\theta < 1$ and $\gamma_1 < -1$, and

$$\gamma_2 \geq 1 + \frac{\log p_B}{\log \ell_{\min}} > -2 \geq \gamma_1 - 1.$$

- If $\pi_B > p_B^{4/3}$ (or equivalently $\pi_B^{-1/2} < p_B^{-2}\pi_B$) and $\ell_{\max} \in [\pi_B^{-1/2}, p_B^{-2}\pi_B)$, then $\theta \leq p_B \cdot \ell_{\max} \leq p_B^{-1}\pi_B < 1$ and $\theta \geq \pi_B \cdot \ell_{\max} \geq \ell_{\max}^{-1}$, i.e. $\gamma_1 \geq -1$, and

$$\gamma_2 \geq 1 + \frac{\log \pi_B}{\log \ell_{\max}} > \frac{2\log(p_B \cdot \ell_{\max})}{\log \ell_{\max}} \geq 2\gamma_1.$$

§3.5 Further results and final remarks

We can obtain more information from the results from [G04]. First of all, using the last part of [G04, Theorem 6.3] the upper bound in Theorem 3.1.3 can be improved for a specific class of test functions.

Theorem 3.5.1. *Assume that $\theta < 1$. Let $f \in L^\infty(\Sigma^{\mathbb{N}} \times [0,1], m_{\boldsymbol{p}} \times \mu)$ and $h \in \mathcal{H}$ be such that both f and h are identically zero on $\Sigma^{\mathbb{N}} \times \big([0,\frac{1}{2}] \cup [\frac{3}{4},1]\big)$ and $\int h \, dm_{\boldsymbol{p}} \times \mu = 0$. Let $\gamma \in (\gamma_1, 0)$. Then*

$$|\mathrm{Cor}_n(f,h)| = O(n^{\gamma-1}).$$

Proof. The statement follows by applying the last part of [G04, Theorem 6.3]. For this, (g1), (g2) and (g3) need to be verified. This is done before in the proof of Theorem 3.1.4. □

In [G04, Theorem 6.13] a Central Limit Theorem is derived for a specific class of functions in $\mathcal{H}$ with zero integral. This result immediately carries over to our setting and is given in the next theorem.

Theorem 3.5.2 (cf. Theorem 6.13 in [G04]). *Assume that $\theta < 1$. Let $h \in \mathcal{H}$ be identically zero on $\Sigma^{\mathbb{N}} \times \big([0,\frac{1}{2}] \cup [\frac{3}{4},1]\big)$ and with $\int h \, dm_{\boldsymbol{p}} \times \mu = 0$. Then the sequence $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h \circ F^k$ converges in distribution with respect to $m_{\boldsymbol{p}} \times \mu$ to a normally distributed random variable with zero mean and finite variance σ^2 given by*

$$\sigma^2 = -\int h^2 \, dm_{\boldsymbol{p}} \times \mu + 2 \sum_{n=0}^{\infty} \int h \cdot h \circ F^n \, dm_{\boldsymbol{p}} \times \mu.$$

Furthermore, we have $\sigma = 0$ if and only if there exists a measurable function ψ on $\Sigma^{\mathbb{N}} \times [0,1]$ such that $h \circ F = \psi \circ F - \psi$. Such a function ψ then satisfies $\sup_{z_1,z_2 \in Y} \frac{|\psi(z_1) - \psi(z_2)|}{(2^{-\alpha})^{s(z_1,z_2)}} < \infty$ and $\psi(F^j(z)) = \psi(z)$ for each $z \in Y$ and each $j = 0, 1, \ldots, \varphi(z) - 1$.

Using [Y99, Theorem 4] we can also derive a Central Limit Theorem, this time for a more general class of functions in $\mathcal{H}$ with zero integral but under the more restrictive assumption that $\theta < \ell_{\max}^{-1}$.

Theorem 3.5.3. *Assume that $\theta < \ell_{\max}^{-1}$. Let $h \in \mathcal{H}$ be such that $\int h \, dm_{\boldsymbol{p}} \times \mu = 0$. Then the sequence $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h \circ F^k$ converges in distribution with respect to $m_{\boldsymbol{p}} \times \mu$ to a normally distributed random variable with zero mean and finite variance σ^2. Furthermore, we have $\sigma = 0$ if and only if there exists a measurable function ψ on Δ such that $h \circ \pi \circ G = \psi \circ G - \psi$.*

Proof. The result from [Y99, Theorem 4] gives a statement for G on Δ. We have already seen that $h \in \mathcal{H}$ implies $h \circ \pi \in \mathcal{C}_{1/2^\alpha}$ for some $\alpha \in (0,1)$. The assumption that $\theta < \ell_{\max}^{-1}$ implies $\gamma_1 < -1$. Take $\gamma \in (\gamma_1, -1)$. We saw in (3.43) that $m(\hat{\varphi} > n) = O(n^\gamma)$. It then follows from [Y99, Theorem 4] that $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h \circ \pi \circ G^k$ converges in distribution with respect to ν to a normally distributed random variable with zero mean and finite variance σ^2, with $\sigma > 0$ if and only if $h \circ \pi \circ G \neq \psi \circ G - \psi$ for any measurable function ψ on Δ. Since $m_{\boldsymbol{p}} \times \mu = \nu \circ \pi^{-1}$ and $F \circ \pi = \pi \circ G$, we get for any $u \in \mathbb{R}$ that

$$m_{\boldsymbol{p}} \times \mu\Big(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h \circ F^k \leq u\Big) = \nu\Big(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (h \circ \pi) \circ G^k \leq u\Big).$$

The result now follows. □

Under an additional assumption on $r_{\min} = \min\{r_g : g \in \Sigma_G\}$ and $\ell_{\min} = \{\ell_b : b \in \Sigma_B\}$ we can weaken the assumption that the test functions in Theorem 3.1.4, Theorem 3.5.1 and Theorem 3.5.2 should be identically zero on $\Sigma^{\mathbb{N}} \times \big([0, \frac{1}{2}] \cup [\frac{3}{4}, 1]\big)$. Namely, if for an integer $l \geq 2$ we have

$$2^{-l} \cdot \min\Big\{r_{\min} \cdot 2^{1/r_{\min}}, \ell_{\min}^{1+1/(\ell_{\min}-1)}\Big\} \geq 1,$$

then it suffices to assume that these test functions are identically zero on $\Sigma^{\mathbb{N}} \times \big([0, \frac{1}{2}] \cup [1 - \frac{1}{2^{l+1}}, 1]\big)$. Indeed, in this case Lemma 3.3.7 and also Proposition 3.3.3 still carry over if we induce the random map on $(\frac{1}{2}, 1 - \frac{1}{2^{l+1}})$ instead, and the result then follows by applying [G04, Theorem 6.3 and Theorem 6.13] to this induced system in the same way as has been done in the proofs of Theorem 3.1.4, Theorem 3.5.1 and Theorem 3.5.2. The step in Lemma 3.4.5 where (3.45) is applied will then be replaced by applying

$$\left|\frac{\frac{d\mu}{d\lambda}(x)}{\frac{d\mu}{d\lambda}(y)} - 1\right| \leq C \cdot |x - y|, \qquad \forall x, y \in \Big[\frac{1}{2}, 1 - \frac{1}{2^{l+1}}\Big], \qquad \text{for some } C > 0,$$

which can be shown using that $\frac{d\mu}{d\lambda}$ is bounded away from zero and is locally Lipschitz on $[\frac{1}{2}, 1)$. As remarked in Remark 3.4.3, the latter can be shown by following the same steps as in Section 2.2. It in particular shows that $\frac{d\mu}{d\lambda}$ is bounded on $[\frac{1}{2}, 1 - \frac{1}{2^{l+1}}]$, which replaces the step in the proof of Theorem 3.1.4 where Lemma 3.4.2(iii) and (3.40) are applied.

We can extend the results in this chapter to the following more general classes of good and bad maps. Fix a $c \in (0, 1)$, and let the class of good maps $\mathfrak{G}$ consist of maps $T_g : [0, 1] \to [0, 1]$ given by

$$T_g(x) = \begin{cases} 1 - c^{-r_g}(c - x)^{r_g}, & \text{if} \quad x \in [0, c), \\ \frac{x-c}{1-c}, & \text{if} \quad x \in [c, 1], \end{cases}$$

where $r_g \geq 1$, and the class of bad maps $\mathfrak{B}$ consist of maps $T_b : [0, 1] \to [0, 1]$ given by

$$T_b(x) = \begin{cases} c - c^{-\ell_b+1}(c - x)^{\ell_b}, & \text{if} \quad x \in [0, c), \\ \frac{x-c}{1-c}, & \text{if} \quad x \in [c, 1], \end{cases}$$

where $\ell_b > 1$. Again, one easily computes that each map from $\mathfrak{G} \cup \mathfrak{B}$ has non-positive Schwarzian derivative when restricted to $[0, \frac{1}{2})$ or $[\frac{1}{2}, 1]$. For these collections of maps Lemma 3.2.2 carries over, replacing $\frac{1}{2}$ with c and $\frac{1}{4}$ with c^2, under the additional assumptions that $c < r_g \cdot (1-c)^{1-r_g^{-1}}$ holds for all $g \in \Sigma_G$, and that $1-c > \ell_b^{\ell_b/(1-\ell_b)}$ holds for all $b \in \Sigma_B$. Furthermore, Lemma 3.3.7 carries over with lower bounds $\frac{(1-c)^2}{c}$ and $\frac{1}{1-c}$ in (i) and (ii) instead of $\frac{1}{2}$ and 2, respectively, under the additional assumption that

$$\frac{(1-c)^2}{c} \cdot \min\left\{ \min_{b \in \Sigma_B} \{r_b \cdot (1-c)^{-1/r_b}\}, \ell_{\min}^{1+1/(\ell_{\min}-1)} \right\} \geq 1.$$

By equipping $\Sigma^{\mathbb{N}}$ with the metric $d_{\Sigma^{\mathbb{N}}}(\omega, \omega') = (1-c)^{\min\{i \in \mathbb{N} : \omega_i \neq \omega'_i\}}$, it can be shown that under these additional conditions all the results formulated in Sections 3.1 and 3.5 carry over and are proven in the same way.

Finally, polynomial decay of correlations is expected to hold for a more general class of good and bad maps for which random compositions show critical intermittency, but the proofs may become more cumbersome. Our assumption that all maps are identical on the interval $[\frac{1}{2}, 1]$ made it easier to find a suitable inducing domain, but does not seem necessary. Furthermore, the linearity of this right branch and the explicit forms of the left branches of the good and bad maps made the series in (3.24) and (3.25) telescopic. A first step to generalise our results to a more general class might be to require this explicit form of the left branch only close to c, though any generalisations will inevitably make the calculations more complicated.