Large deviation principle for the maximal eigenvalue of inhomogeneous Erdos-Renyi random graphs

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Large Deviation Principle for the Maximal Eigenvalue of Inhomogeneous Erdős-Rényi Random Graphs

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Abstract

We consider an inhomogeneous Erdős-Rényi random graph G_N with vertex set $[N] = \{1, \ldots, N\}$ for which the pair of vertices $i, j \in [N]$, $i \neq j$, is connected by an edge with probability $r(\frac{i}{N}, \frac{j}{N})$, independently of other pairs of vertices. Here, $r: [0, 1]^2 \to (0, 1)$ is a symmetric function that plays the role of a reference graphon. Let λ_N be the maximal eigenvalue of the adjacency matrix of G_N. It is known that λ_N/N satisfies a large deviation principle as $N \to \infty$. The associated rate function ψ_r is given by a variational formula that involves the rate function I_r of a large deviation principle on graphon space. We analyse this variational formula in order to identify the properties of ψ_r, specially when the reference graphon is of rank 1.

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1 Introduction and Main Results

1.1 Motivation

In the past 20 years, many properties have been derived about spectra of random matrices associated with random graphs, like the adjacency matrix and the Laplacian matrix [3,7,9,15,18,20–25,30,31]. The focus of the present paper is on inhomogeneous Erdős-Rényi random graphs, which are rooted in the theory of complex networks. We consider the dense regime, where the average degree of the vertices are proportional to the size of the graph, and analyse the rate function of the large deviation principle for the maximal eigenvalue of the adjacency matrix derived in [17]. In [10] the non-dense non-sparse regime was considered, where the degrees diverge but sublinearly in the size of the graph, and identified the scaling limit of the empirical spectral distribution of both the adjacency matrix and the Laplacian matrix. Recent results on the maximal eigenvalue in the sparse regime, where the degrees are stochastically bounded, can be found in [6].

Large deviations of Erdős-Rényi random graphs were studied in [12,13,27] with the help of the theory of graphons, in particular, subgraph densities and maximal eigenvalues. We refer to [12] for a comprehensive review of the literature. Large deviation theory for random matrices started in [4], with the study of large deviations of the empirical spectral distribution of β-ensembles with a quadratic potential. The rate was shown to be the square of the number of vertices, and the rate function was shown to be given by a non-commutative notion of entropy. The maximal eigenvalue for such ensembles was studied in [5]. Large deviations of the empirical spectral distribution of random matrices with non-Gaussian tails were derived in [8]. More recently, the maximal eigenvalue in that setting was studied in [1,2]. The adjacency matrix of an inhomogeneous Erdős-Rényi random graph does not fall in this regime, and hence different techniques are needed in the present paper.

As mentioned above, the large deviation principle for dense inhomogeneous Erdős-Rényi random graphs was recently proved in [17]. Subsequently, in [28] the conditions on the reference graphon were relaxed and a simplified form of the rate function was derived. The large deviation principle for the largest eigenvalue of the adjacency matrix follows through the contraction principle, because the spectral norm of the graphon operator is a "nice graph parameter" (see Definition 6.1 of [12]) on the space of graphons. The main goal of the present paper is to study the variational formula for the rate function that arises from this contraction.

It is not possible to explicitly solve the variational formula, which is why we study the rate function around three points, namely, its unique minimiser and its two boundary points. For the case where the reference graphon is rank 1, we identify the optimal perturbations around these three points, which reveal how the large deviation is realised by the random graph. Interestingly, it turns out that if the reference graphon is not constant, then the optimal graphon is not rank 1. We further derive some basic properties of the rate function. We do not know whether the rate function is convex or analytic.

Inhomogeneous random graphs are essential in the study of complex networks with community structure. A further discussion and a list open problems are provided in Sect. 1.5.

1.2 LDP for Inhomogeneous Erdos-Rényi Random Graphs

Let

$$\mathcal{W} = \left\{ h \colon [0,1]^2 \to [0,1] \colon h(x,y) = h(y,x) \ \forall x, y \in [0,1] \right\} \tag{1.1}$$

denote the set of graphons. Let $\mathcal{M}$ denote the set of Lebesgue measure-preserving bijective maps $\phi \colon [0,1] \mapsto [0,1]$. For two graphons $h_1, h_2 \in \mathcal{W}$, the cut-distance is defined by

$$d_\square(h_1, h_2) = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} \mathrm{d}x \, \mathrm{d}y \left[h_1(x,y) - h_2(x,y) \right] \right|, \tag{1.2}$$

and the cut-metric by

$$\delta_\square(h_1, h_2) = \inf_{\phi \in \mathcal{M}} d_\square(h_1, h_2^\phi), \tag{1.3}$$

where $h_2^\phi(x,y) = h_2(\phi(x), \phi(y))$. The cut-metric defines an equivalence relation $\sim$ on $\mathcal{W}$ by declaring $h_1 \sim h_2$ if and only if $\delta_\square(h_1, h_2) = 0$, and leads to the quotient space $\widetilde{\mathcal{W}} = \mathcal{W}/\sim$. For $h \in \mathcal{W}$, we write $\widetilde{h}$ to denote the equivalence class of h in $\widetilde{\mathcal{W}}$. The equivalence classes correspond to relabelings of the vertices of the graph. The pair $(\widetilde{\mathcal{W}}, \delta_\square)$ is a compact metric space [26].

Let $r \in \mathcal{W}$ be a reference graphon satisfying

$$\exists \eta > 0 \colon \quad \eta \leq r(x,y) \leq 1 - \eta \quad \forall (x,y) \in [0,1]^2. \tag{1.4}$$

Fix $N \in \mathbb{N}$ and consider the random graph G_N with vertex set $[N] = \{1, \ldots, N\}$ where the pair of vertices $i, j \in [N], i \neq j$, is connected by an edge with probability $r(\frac{i}{N}, \frac{j}{N})$, independently of other pairs of vertices. Write $\mathbb{P}_N$ to denote the law of G_N. Use the same symbol for the law on $\mathcal{W}$ induced by the map that associates with the graph G_N its graphon h^{G_N}, defined by

$$h^{G_N}(x,y) = \begin{cases} 1, \ if\, there\, is\, an\, edge\, between\, vertex\, \lceil Nx \rceil\, and\, vertex\, \lceil Ny \rceil, \\ 0, \ otherwise. \end{cases} \tag{1.5}$$

Write $\widetilde{\mathbb{P}}_N$ to denote the law of $\widetilde{h}^{G_N}$.

The following LDP is proved in [17] and is an extension of the celebrated LDP for homogeneous ERRG derived in [13] and further properties of the rate functions were derived in [27].

Theorem 1 *(LDP for inhomogeneous ERRG) Subject to (1.4), the sequence $(\widetilde{\mathbb{P}}_N)_{N\in\mathbb{N}}$ satisfies the large deviation principle on $(\widetilde{\mathcal{W}}, \delta_{\square})$ with rate $\binom{N}{2}$, i.e.,*

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{\binom{N}{2}} \log \widetilde{\mathbb{P}}_N(\mathcal{C}) &\leq - \inf_{\widetilde{h}\in\mathcal{C}} J_r(\widetilde{h}) \quad \forall\, \mathcal{C} \subset \widetilde{\mathcal{W}} \text{ closed}, \\
\liminf_{N\to\infty} \frac{1}{\binom{N}{2}} \log \widetilde{\mathbb{P}}_N(\mathcal{O}) &\geq - \inf_{\widetilde{h}\in\mathcal{O}} J_r(\widetilde{h}) \quad \forall\, \mathcal{O} \subset \widetilde{\mathcal{W}} \text{ open},
\end{aligned} \tag{1.6}$$

where the rate function $J_r \colon \widetilde{\mathcal{W}} \to \mathbb{R}$ is given by

$$J_r(\widetilde{h}) = \inf_{\phi\in\mathcal{M}} I_r(h^\phi), \tag{1.7}$$

where h is any representative of $\widetilde{h}$ and

$$I_r(h) = \int_{[0,1]^2} \mathrm{d}x\, \mathrm{d}y\, \mathcal{R}\big(h(x,y) \mid r(x,y)\big), \quad h \in \mathcal{W}, \tag{1.8}$$

with

$$\mathcal{R}(a \mid b) = a \log \tfrac{a}{b} + (1-a) \log \tfrac{1-a}{1-b} \tag{1.9}$$

the relative entropy of two Bernoulli distributions with success probabilities $a \in [0,1]$, $b \in (0,1)$ (with the convention $0 \log 0 = 0$).

It is clear that J_r is a good rate function, i.e., $J_r \not\equiv \infty$ and J_r has compact level sets. Note that (1.7) differs from the expression in [17], where the rate function is the lower semi-continuous envelope of $I_r(h)$. However, it was shown in [28] that, under the integrability conditions $\log r, \log(1-r) \in L^1([0,1]^2)$, the two rate functions are equivalent, since $J_r(\widetilde{h})$ is lower semi-continuous on $\widetilde{\mathcal{W}}$. Clearly, these integrability conditions are implied by (1.4).

1.3 Graphon Operators

With $h \in \mathcal{W}$ we associate a graphon operator acting on $L^2([0,1])$, defined as the linear integral operator

$$(T_h u)(x) = \int_{[0,1]} \mathrm{d}y\, h(x,y) u(y), \quad x \in [0,1], \tag{1.10}$$

with $u \in L^2([0,1])$. The operator norm of T_h is defined as

$$\|T_h\| = \sup_{\substack{u\in L^2([0,1]) \\ \|u\|_2=1}} \|T_h u\|_2, \tag{1.11}$$

where $\| \cdot \|_2$ denotes the L^2-norm. Given a graphon $h \in \mathcal{W}$, we have $\|T_h\| \leq \|h\|_2$. Hence, a graphon sequence converging in the L^2-norm also converges in the operator norm.

The product of two graphons $h_1, h_2 \in \mathcal{W}$ is defined as

$$(h_1 h_2)(x, y) = \int_{[0,1]} \mathrm{d}z\, h_1(x, z) h_2(z, y), \qquad (x, y) \in [0, 1]^2, \qquad (1.12)$$

and the n-th power of a graphon $h \in \mathcal{W}$ as

$$h^n(x, y) = \int_{[0,1]^{n-1}} \mathrm{d}z_1 \cdots \mathrm{d}z_{n-1} h(x, z_1) \times \cdots \times h(z_{n-1}, y),$$
$$(x, y) \in [0, 1]^2, \ n \in \mathbb{N}. \qquad (1.13)$$

Definition 1 (Eigenvalues and eigenfunctions) $\mu \in \mathbb{R}$ is said to be an eigenvalue of the graphon operator T_h if there exists a non-zero function $u \in L^2([0, 1])$ such that

$$(T_h u)(x) = \mu u(x), \qquad x \in [0, 1]. \qquad (1.14)$$

The function u is said to be an eigenfunction of T_h associated with μ.

Proposition 1 *(Properties of the graphon operator) For any $h \in \mathcal{W}$:*
(i) The graphon operator T_h is self-adjoint, bounded and continuous.
(ii) The graphon operator T_h is diagonalisable and has countably many eigenvalues, all of which are real and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq 0$. Moreover, there exists a collection of eigenfunctions which form an orthonormal basis of $L^2([0, 1])$.
(iii) The maximal eigenvalue μ_1 of the graphon operator T_h is strictly positive and has an associated eigenfunction u_1 satisfying $u_1(x) > 0$ for all $x \in [0, 1]$. Moreover, $\mu_1 = \|T_h\|$, i.e., the maximal eigenvalue equals the operator norm.

Proof The claim is a special case of [29, Theorem 7.3] (when the compact Hermitian operators considered there are taken to be the graphon operators). See also [14, Theorem 19.2] and [19, Appendix A]. □

1.4 Main Theorems

Let λ_N be the maximal eigenvalue of the adjacency matrix A_N of G_N. Write $\mathbb{P}_N^*$ to denote the law of λ_N / N.

Theorem 2 *(LDP for the maximal eigenvalue) Subject to (1.4), the sequence $(\mathbb{P}_N^*)_{N \in \mathbb{N}}$ satisfies the large deviation principle on $\mathbb{R}$ with rate $\binom{N}{2}$ and with rate function*

$$\psi_r(\beta) = \inf_{\substack{\widetilde{h} \in \widetilde{\mathcal{W}} \\ \|T_{\widetilde{h}}\| = \beta}} J_r(\widetilde{h}) = \inf_{\substack{h \in \mathcal{W} \\ \|T_h\| = \beta}} I_r(h), \qquad \beta \in \mathbb{R}. \qquad (1.15)$$
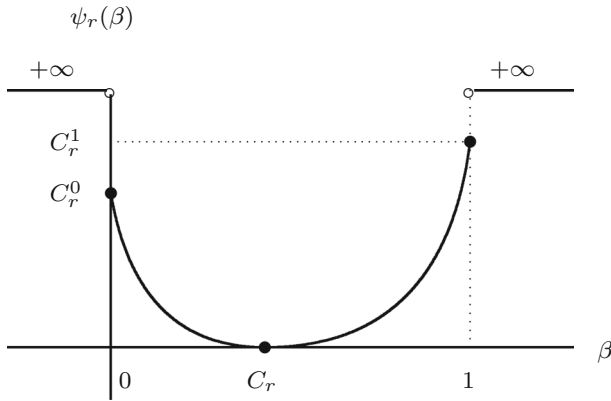
Fig. 1 Graph of $\beta \mapsto \psi_r(\beta)$

Proof Note that $\lambda_N / N = \|T_{h^{G_N}}\|$, where h is any representative of $\widetilde{h}$ (because $\|T_{\widetilde{h}}\| = \|T_{\widetilde{h\phi}}\|$ for all $\phi \in \mathcal{M}$). Also note that $\widetilde{h} \mapsto \|T_{\widetilde{h}}\|$ is a bounded and continuous function on $\widetilde{\mathcal{W}}$ [12, Exercises 6.1–6.2, Lemma 6.2]. Hence the claim follows from Theorem 1 via the contraction principle [16, Chapter 3]. □

Put

$$C_r = \|T_r\|. \tag{1.16}$$

When $\beta = C_r$, the graphon h that minimizes $I_r(h)$ such that $\|T_h\| = C_r$ is the reference graphon $h = r$ almost everywhere, for which $I_r(r) = 0$ and no large deviation occurs. When $\beta > C_r$, we are looking for graphons h with a larger operator norm. The large deviation cannot go above 1, which is represented by the constant graphon $h \equiv 1$, for which $I_r(1) = C_r^1$. Similarly, when $\beta < C_r$, we are looking for graphons h with a smaller operator norm. The large deviation cannot go below 0, which is represented by the constant graphon $h \equiv 0$, for which $I_r(0) = C_r^0$ (see Fig. 1).

Theorem 3 *(Properties of the rate function) Subject to (1.4):*
(i) ψ_r is continuous and unimodal on $[0, 1]$, with a unique zero at C_r.
(ii) ψ_r is strictly decreasing on $[0, C_r]$ and strictly increasing on $[C_r, 1]$.
(iii) For every $\beta \in [0, 1]$, the set of minimisers of the variational formula for $\psi_r(\beta)$ in (1.15) is non-empty and compact in $\widetilde{\mathcal{W}}$.

If the reference graphon r is of rank 1, i.e.,

$$r(x, y) = \nu(x)\, \nu(y), \quad (x, y) \in [0, 1]^2, \tag{1.17}$$

for some $\nu \colon [0, 1] \to [0, 1]$ that is bounded away from 0 and 1, then we are able to say more. Define

$$m_k = \int_{[0,1]} \nu^k, \qquad k \in \mathbb{N}. \tag{1.18}$$

Note that $C_r = m_2$. Abbreviate $B_r = \int_{[0,1]^2} r^3(1-r)$, and note that $B_r = m_3^2 - m_4^2$. Further abbreviate

$$N_r^1 = \int_{[0,1]^2} \frac{1-r}{r}, \qquad N_r^0 = \int_{[0,1]^2} \frac{r}{1-r}. \tag{1.19}$$

Recall that $\mathcal{M}$ is the set of Lebesgue measure-preserving bijective maps $\phi\colon [0,1] \to [0,1]$.

Theorem 4 *(Scaling of the rate function) Let ψ_r be the rate function in (1.15).*
(i) Subject to (1.4) and (1.17),

$$\psi_r(\beta) = [1 + o(1)] K_r (\beta - C_r)^2, \qquad \beta \to C_r, \tag{1.20}$$

with

$$K_r = \frac{(C_r)^2}{2 B_r} = \frac{m_2^2}{2(m_3^2 - m_4^2)}. \tag{1.21}$$

(ii) Subject to (1.4),

$$C_r^1 - \psi_r(\beta) = (1 - \beta) \left[\log\left(\frac{N_r^1}{1-\beta} \right) + 1 + o(1) \right], \qquad \beta \uparrow 1. \tag{1.22}$$

(iii) Subject to (1.4),

$$C_r^0 - \psi_r(\beta) = \beta \left[\log\left(\frac{N_r^0}{\beta} \right) + 1 + o(1) \right], \qquad \beta \downarrow 0. \tag{1.23}$$

Theorem 5 *(Scaling of the minimisers) Let $h_\beta \in \mathcal{W}$ be any minimiser of the second infimum in (1.15).*
(i) Subject to (1.4) and (1.17),

$$\lim_{\beta \to C_r} (\beta - C_r)^{-1} \| h_\beta - r - (\beta - C_r)\Delta \|_2 = 0, \tag{1.24}$$

with

$$\Delta(x, y) = \frac{C_r}{B_r} r(x, y)^2 [1 - r(x, y)], \qquad (x, y) \in [0, 1]^2. \tag{1.25}$$

(ii) Subject to (1.4),

$$\lim_{\beta \uparrow 1} (1 - \beta)^{-1} \| 1 - h_\beta - (1 - \beta)\Delta \|_2 = 0, \tag{1.26}$$

with

$$\Delta(x, y) = \frac{1}{N_r^1} \frac{1 - r(x, y)}{r(x, y)}, \qquad (x, y) \in [0, 1]^2. \tag{1.27}$$

(iii) Subject to (1.4),

$$\lim_{\beta \downarrow 0} \beta^{-1} \|h_\beta - \beta \Delta\|_2 = 0, \tag{1.28}$$

with

$$\Delta(x, y) = \frac{1}{N_r^0} \frac{r(x, y)}{1 - r(x, y)}, \qquad (x, y) \in [0, 1]^2. \tag{1.29}$$

1.5 Discussion and Outline

1. Theorem 3 confirms the picture of ψ_r drawn in Fig. 1. It remains open whether or not ψ_r is convex. We do not expect ψ_r to be analytic, because bifurcations may occur in the set of minimisers of ψ_r as β is varied.

2. Theorem 4 identifies the scaling of ψ_r near its zero and near its end points, provided r is of rank 1. Theorem 5 identifies the corresponding scaling of the minimiser h_β of ψ_r. Interestingly, the scaling corrections are not rank 1. It remains open to determine what happens near C_r when r is not of rank 1 (see the Appendix).

3. The inverse curvature $1/K_r$ equals the variance in the central limit theorem derived in [11]. This is in line with the standard folklore of large deviation theory.

4. It would be interesting to investigate to what extent the condition on the reference graphon in (1.4) can be weakened to some form of integrability condition. Especially for the upper bound in the LDP this is delicate, because the proof in [17] is based on block-graphon approximation (see [28]).

5. It would be interesting to generalise Theorems 4–5 to reference graphons of higher rank. This is possible in principle, but at the cost of more technicalities (see the Appendix).

Outline. The proof of Theorems 3–5 is given in Sect. 3 and relies on the variational formula in (1.15). Since the maximal eigenvalue is invariant under relabeling of the vertices, we can work directly with I_r in (1.8) without worrying about the equivalence classes. In Sect. 2 we derive an expansion for the operator norm of a graphon around any graphon of rank 1. This expansion will be needed in Sect. 3.

2 Expansion Around Graphons of Rank 1

In order to prepare for the proof of Theorem 4, we show how we can expand the operator norm of a graphon around any graphon of rank 1.

Lemma 1 *(Rank 1 expansion) Consider a graphon $\bar{h} \in \mathcal{W}$ of rank 1 such that $\bar{h}(x, y) = \bar{v}(x)\bar{v}(y)$, $x, y \in [0, 1]$. For any $h \in \mathcal{W}$ such that $\|T_{h-\bar{h}}\| < \|T_h\|$, the operator norm $\mu = \|T_h\|$ is a solution of the equation*

$$\mu = \sum_{n \in \mathbb{N}_0} \frac{1}{\mu^n} \mathcal{F}_n(h, \bar{h}), \tag{2.1}$$

where

$$\mathcal{F}_n(h, \bar{h}) = \int_{[0,1]^2} dx\, dy\, \bar{v}(x)(h - \bar{h})^n(x, y)\bar{v}(y). \tag{2.2}$$

Proof By Proposition 1, we have

$$T_h u = \mu u, \tag{2.3}$$

where μ equals both the norm and the maximal eigenvalue of T_h, and u is the eigen-function of T_h associated with μ. Put $g = h - \bar{h}$ and we have $(\mu - T_g)u = T_{\bar{h}}u$. This gives

$$u = (\mu - T_g)^{-1}\bar{v}\langle \bar{v}, u\rangle \tag{2.4}$$

where we use that $\mu - T_g$ is invertible because $\|T_g\| = \|T_{h-\bar{h}}\| < \|T_h\|$. Hence, taking the inner product of u with $\bar{v}$ and observing that $\langle \bar{v}, u\rangle \neq 0$, we get

$$\langle \bar{v}, u\rangle = \langle \bar{v}, u\rangle\langle \bar{v}, (\mu - T_g)^{-1}\bar{v}\rangle \tag{2.5}$$

which gives

$$\mu = \langle \bar{v}, (1 - T_g/\mu)^{-1}\bar{v}\rangle. \tag{2.6}$$

We can expand the above to get

$$
\begin{aligned}
\mu &= \left\langle \bar{v}, \sum_{n\in\mathbb{N}_0} \left(\frac{T_g}{\mu}\right)^n \bar{v}\right\rangle \\
&= \sum_{n\in\mathbb{N}_0} \frac{1}{\mu^n} \int_{[0,1]^{n+1}} dx_0\, dx_1 \cdots dx_n\, \bar{v}(x_0)g(x_0, x_1) \times \cdots \times g(x_{n-1}, x_n)\bar{v}(x_n) \\
&= \sum_{n\in\mathbb{N}_0} \frac{1}{\mu^n} \mathcal{F}_n(h, \bar{h}),
\end{aligned}
\tag{2.7}
$$

and this completes the proof. □

Subject to (1.17), it follows from Lemma 1 with $h = \bar{h} = r$ that

$$C_r = \|T_r\| = m_2, \tag{2.8}$$

because only the term with $n = 0$ survives in the expansion.

Remark 1 (Higher rank) The expansion around reference graphons of rank 1 can be extended to finite rank. We provide the details in the Appendix. In this paper we focus on rank 1, for which Lemma 1 allows us to analyse the behaviour of $\psi_r(\beta)$ near the values $\beta = C_r$, $\beta = 1$ and $\beta = 0$. This corresponds to an expansion around the graphons $h = v \times v$, $h \equiv 1$ and $h \equiv 0$, which are all of rank 1.

3 Proofs of Main Theorems

Theorem 3 is proved in Sect. 3.1, Theorems 4–5 are proved in Sects. 3.2–3.4.

3.1 Continuity, Unimodality and Unique Minimisers

Proof We follow [12, Chapter 6]. Even though this monograph deals with constant reference graphons only, most arguments carry over to r satisfying (1.4).
(i), (iii) Define

$$\psi_r^+(\beta) = \inf_{\substack{h \in \mathcal{W} \\ \|T_h\| \geq \beta}} I_r(h), \quad \psi_r^-(\beta) = \inf_{\substack{h \in \mathcal{W} \\ \|T_h\| \leq \beta}} I_r(h), \qquad \beta \in \mathbb{R}. \tag{3.1}$$

Because $h \mapsto \|T_h\|$ is a nice graph parameter, in the sense of [12, Definition 6.1], it follows that $\beta \mapsto \psi_r^+(\beta)$ is non-decreasing and continuous, while $\beta \mapsto \psi_r^-(\beta)$ is non-increasing and continuous [12, Proposition 6.1]. (The proof requires the fact that $\|f_n - f\|_2 \to 0$ implies $I_r(f_n) \to I_r(f)$ and that $I_r(f)$ is lower semi-continuous on $\mathcal{W}$.)

The variational formulas in (3.1) achieve minimisers. In fact, the sets of minimiser are non-empty compact subsets of $\widetilde{\mathcal{W}}$ [12, Theorem 6.2]. In addition, all minimisers h of $\phi_r^+(h)$ satisfy $h \geq r$ almost everywhere, while all minimisers h of ϕ_r^- satisfy $h \leq r$ almost everywhere [12, Lemma 6.3]. Moreover, because

$$\begin{aligned} h_1 \geq h_2 \geq r &\implies \|T_{h_1}\| \geq \|T_{h_2}\|, \ I_r(h_1) \geq I_r(h_2), \\ h_1 \leq h_2 \leq r &\implies \|T_{h_1}\| \leq \|T_{h_2}\|, \ I_r(h_1) \leq I_r(h_2), \end{aligned} \tag{3.2}$$

(use that $a \mapsto \mathcal{R}(a \mid b)$ is unimodal on $[0, 1]$ with unique zero at b), it follows that both variational formulas achieve minimisers with norm equal to β, and so

$$\psi_r(\beta) = \begin{cases} \psi_r^+(\beta), \ \beta \geq C_r, \\ \psi_r^-(\beta), \ \beta \leq C_r. \end{cases} \tag{3.3}$$

Hence, ψ_r is continuous and unimodal on $[0, 1]$. Since $I_r(h) = 0$ if and only if $h = r$ almost everywhere, it is immediate that C_r is the unique zero of ψ_r.
(ii) The proof is by contradiction. Suppose that $\beta \mapsto \psi_r^+(\beta)$ is not strictly increasing on $[C_r, 1]$. Then there exist $\beta_1, \beta_2 \in [C_r, 1]$ with $\beta_1 < \beta_2$ such that ψ_r^+ is constant on $[\beta_1, \beta_2]$. Consequently, there exist minimisers $h_{\beta_1}^{\phi_1}, h_{\beta_2}^{\phi_2}$ with $\phi_1, \phi_2 \in \mathcal{M}$ satisfying $r \leq h_{\beta_1}^{\phi_1} \leq h_{\beta_2}^{\phi_2}$ such that

$$I_r(h_{\beta_1}^{\phi_1}) = I_r(h_{\beta_2}^{\phi_2}), \qquad \|T_{h_{\beta_1}^{\phi_1}}\| = \beta_1 < \beta_2 = \|T_{h_{\beta_2}^{\phi_2}}\|. \tag{3.4}$$

However, since $a \mapsto \mathcal{R}(a \mid b)$ is strictly increasing on $[b, 1]$ (recall (1.8)), it follows that $h_{\beta_1}^{\phi_1} = h_{\beta_2}^{\phi_2}$ almost everywhere. This in turn implies that $\|T_{h_{\beta_1}^{\phi_1}}\| = \|T_{h_{\beta_2}^{\phi_2}}\|$, which

is a contradiction. A similar argument shows that $\beta \mapsto \psi_r^-(\beta)$ cannot have a flat piece on $[0, C_r]$. □

3.2 Perturbation Around the Minimum

In this section we study $\psi_r(\beta)$ around the point C_r when the reference graphon is of rank 1. Along the way we prove Theorem 4(i) and Theorem 5(i). The proof is split into 4 steps.

> **Step 1.** In Lemma 2 we show that if we take a perturbation around C_r and consider the constraint $\|T_{r+\Delta_\epsilon}\| = C_r + \epsilon$ for some perturbation Δ_ϵ of the reference graphon r, then the constraint can be reduced to a much simpler expression with the help of Lemma 1.
> **Step 2.** In Lemma 3 we show that the cost function $I_r(r + \Delta_\epsilon)$ is at least $2\epsilon^2$ for any perturbation around r. In fact, we show that the optimal perturbation has L^2-norm of order ϵ. From the proof of this fact we deduce that if $\Delta_\epsilon = \epsilon \Delta$ for some $\Delta \in L^2([0, 1]^2)$, then we get the best possible perturbation.
> **Step 3.** The proof proceeds by looking at block graphons. Since block graphons are constant on blocks, the geometry of the perturbation no longer plays a role. In Lemma 4 we compute the contribution of a single block. In Lemma 5 we show that the cost function is well approximated by block graphons. In Lemma 6 we show that the blocks contribute uniformly to the cost. In Lemma 7 we compute the cost function for perturbations of the form $\epsilon \Delta$. These computations can also be used for block graphons, which allows us to show that unbalanced perturbations do not contribute optimally to the cost function. This implies that we can restrict to perturbations of the form $\Delta_\epsilon = \epsilon \Delta$, with Δ not depending on ϵ.
> **Step 4.** Via the proof of Lemma 7, we identify the optimal Δ and hence the scaling of the variational formula around C_r.

Note that when $\beta = C_r$, the infimum in (1.15) is attained at $h = r$ and $\psi_r(C_r) = 0$. Take $\beta = C_r + \epsilon$ with $\epsilon > 0$ small, and assume that the infimum is attained by a graphon of the form $h = r + \Delta_\epsilon$, where $\Delta_\epsilon \colon [0, 1]^2 \to \mathbb{R}$ represents a perturbation of the graphon r. Note that $r + \Delta_\epsilon \in \mathcal{W}$, and so we are dealing with a perturbation Δ_ϵ that is symmetric and bounded. We compare

$$\psi_r(C_r + \epsilon) = \inf_{\substack{\Delta_\epsilon \colon [0,1]^2 \to \mathbb{R} \\ r+\Delta_\epsilon \in \mathcal{W} \\ \|T_{r+\Delta_\epsilon}\| = C_r + \epsilon}} I_r(r + \Delta_\epsilon) \tag{3.5}$$

with $\psi_r(C_r) = 0$ by computing the difference

$$\delta_r(\epsilon) = \psi_r(C_r + \epsilon) - \psi_r(C_r) = \psi_r(C_r + \epsilon) \tag{3.6}$$

and studying its behaviour as $\epsilon \to 0$. To study this we use the following lemma. The next lemma shows that in the rank 1 case we can reduce the constraint $\|T_{r+\Delta_\epsilon}\| = C_r + \epsilon$ to something simpler.

Lemma 2 *Let $r(x, y) = v(x)v(y)$ be the reference graphon. Let $\Delta_\epsilon \in L^2([0, 1]^2)$ be such that $r + \Delta_\epsilon \in \mathcal{W}$ and $\|\Delta_\epsilon\|_2 \to 0$ as $\epsilon \to 0$. Then the constraint $\|T_{r+\Delta_\epsilon}\| = C_r + \epsilon$ is equivalent to*

$$\int_{[0,1]^2} r\Delta_\epsilon = \epsilon C_r + \epsilon^2 + O\left(\|\Delta_\epsilon\|_2^2\right). \tag{3.7}$$

In particular, if $\Delta_\epsilon = \epsilon\Delta$ for some $\Delta \in L^2([0, 1]^2)$, then (3.7) reads as

$$\int_{[0,1]^2} r\Delta = C_r\left[1 + o(1)\right]. \tag{3.8}$$

Proof Since $r(x, y) = v(x)v(y)$, $x, y \in [0, 1]$, we can use Lemma 1 to control the norm of $T_h = T_{r+\Delta_\epsilon}$. Pick $\bar{h} = r$ and $h = r + \Delta_\epsilon$ in (2.1) such that $\|\Delta_\epsilon\|_2 \to 0$ as $\epsilon \to 0$. Note that $\|T_{\Delta_\epsilon}\| \leq \|\Delta_\epsilon\|_2 < C_r$ for ϵ small enough. Hence, writing out the expansion for the norm, we get

$$\|T_{r+\Delta_\epsilon}\| = C_r + \sum_{n \in \mathbb{N}} \frac{1}{\|T_{r+\Delta_\epsilon}\|^n} \mathcal{F}_n(r + \Delta_\epsilon, r). \tag{3.9}$$

Since $\|T_{r+\Delta_\epsilon}\| = C_r + \epsilon$, we have

$$C_r + \epsilon = C_r + \frac{\langle v, \Delta_\epsilon v \rangle}{C_r + \epsilon} + \sum_{n \in \mathbb{N}\setminus\{1\}} \frac{1}{(C_r + \epsilon)^n} \langle v, \Delta_\epsilon^n v \rangle \tag{3.10}$$

with $\langle v, \Delta_\epsilon v \rangle = \int_{[0,1]^2} r\Delta_\epsilon$. So

$$\epsilon(C_r + \epsilon) = \int_{[0,1]^2} r\Delta_\epsilon + \sum_{n \in \mathbb{N}\setminus\{1\}} \frac{1}{(C_r + \epsilon)^{n-1}} \langle v, \Delta_\epsilon^n v \rangle. \tag{3.11}$$

Since v is bounded, using the generalised Hölder's inequality [27, Theorem 3.1] we get

$$|\langle v, \Delta_\epsilon^n v \rangle| \leq \|\Delta_\epsilon\|_2^n. \tag{3.12}$$

Since $\|\Delta_\epsilon\|_2 \to 0$ as $\epsilon \to 0$, we can choose ϵ small enough such that $\|\Delta_\epsilon\|_2 < \frac{1}{2}(C_r + \epsilon)$, which gives

$$\sum_{n \in \mathbb{N}\setminus\{1\}} \frac{1}{(C_r + \epsilon)^{n-1}} \langle v, \Delta_\epsilon^n v \rangle = O\left(\|\Delta_\epsilon\|_2^2\right). \tag{3.13}$$

The constraint $\|T_{r+\Delta_\epsilon}\| = C_r + \epsilon$ therefore reads

$$\int_{[0,1]^2} r\Delta_\epsilon = \epsilon C_r + \epsilon^2 + \mathrm{O}\left(\|\Delta_\epsilon\|_2^2\right). \tag{3.14}$$

$\square$

3.2.1 Small Perturbation on a Given Region

In what follows we use the standard notation $o(\cdot)$, $O(\cdot)$, $\asymp$ to describe the asymptotic behaviour in the limit as $\epsilon \to 0$.

We are interested in finding the asymptotic behaviour of (3.5) as $\epsilon \to 0$. In the next lemma we show that the cost is bounded below by $2\epsilon^2$ for any perturbation Δ_ϵ, and that if Δ_ϵ is of the form $\epsilon\Delta$ for some $\Delta \in L^2([0,1]^2)$, then the cost is of order ϵ^2.

Lemma 3 *(Order of minimal cost) Let $\Delta_\epsilon : [0,1]^2 \to \mathbb{R}$ be such that $r + \Delta_\epsilon \in \mathcal{W}$ and $\|T_{r+\Delta_\epsilon}\| = C_r + \epsilon$. Then*

$$I_r(r + \Delta_\epsilon) \geq 2\epsilon^2. \tag{3.15}$$

Moreover, if $\Delta_\epsilon = \epsilon\Delta$, then

$$I_r(r + \epsilon\Delta) = [1 + o(1)]\, 2\epsilon^2 \int_{[0,1]^2} \frac{\Delta^2}{4r(1-r)}, \qquad \epsilon \to 0. \tag{3.16}$$

Proof Fix $b \in [0,1]$ and abbreviate (recall (1.9))

$$\chi(a) = \mathcal{R}(a \mid b) = a \log \frac{a}{b} + (1-a) \log \frac{1-a}{1-b}, \qquad a \in [0,1]. \tag{3.17}$$

Note that

$$\chi(b) = \chi'(b) = 0, \quad \chi''(a) \geq 4, \quad a \in [0,1]. \tag{3.18}$$

Consequently,

$$\chi(a) \geq 2(a-b)^2, \qquad a \in [0,1], \tag{3.19}$$

and hence

$$I_r(r + \Delta_\epsilon) \geq 2 \int_{[0,1]^2} \Delta_\epsilon^2 = 2\|\Delta_\epsilon\|_2^2. \tag{3.20}$$

Next observe that

$$C_r + \epsilon = \|T_{r+\Delta_\epsilon}\| = \|T_r + T_{\Delta_\epsilon}\| \leq \|T_r\| + \|T_{\Delta_\epsilon}\| \leq C_r + \|\Delta_\epsilon\|_2, \tag{3.21}$$

which gives $\|\Delta_\epsilon\|_2 \geq \epsilon$. Inserting this lower bound into (3.20), we get (3.15). To get (3.16), we need a higher-order expansion of χ, namely, $\chi(x) = \frac{1}{2}\chi''(b)(x - b)^2 + O((x - b)^3)$, $x \to b$. Since r is bounded away from 0 and 1, and the constraint $r + \Delta_\epsilon \in \mathcal{W}$ implies that $\Delta_\epsilon(x, y) \in [-1, 1]$, we see that the third-order term is smaller than the second-order term when $\Delta_\epsilon = \epsilon\Delta$. Hence (3.16) follows.   □

Remark 2 (Order) In the above proof we found that $\|\Delta_\epsilon\|_2 \geq \epsilon$. Moreover, the optimal Δ_ϵ for the minimisation problem in (3.5) must satisfy $\|\Delta_\epsilon\|_2 = O(\epsilon)$ because

$$\inf_{\substack{\Delta_\epsilon : [0,1]^2 \to \mathbb{R} \\ r + \Delta_\epsilon \in \mathcal{W} \\ \|T_{r+\Delta_\epsilon}\| = C_r + \epsilon}} I_r(r + \Delta_\epsilon) \leq \inf_{\substack{\Delta \in L^2([0,1]^2) \\ r + \epsilon\Delta \in \mathcal{W} \\ \|T_{r+\epsilon\Delta}\| = C_r + \epsilon}} I_r(r + \epsilon\Delta), \tag{3.22}$$

where in the right-hand side we restrict to the special class $\Delta_\epsilon = \epsilon\Delta$. From (3.16) it follows that the right-hand side is $O(\epsilon^2)$. Now, if there is an optimal Δ_ϵ^* that minimises the cost function in the left-hand side of (3.22), then by the lower bound in (3.20) it follows that $\|\Delta_\epsilon^*\|_2 = O(\epsilon)$.

Lemma 4 (*Cost of small perturbations*) *Let* $B \subseteq [0, 1]^2$ *be a measurable region with area* $|B|$. *Suppose that* $\Delta_\epsilon = \epsilon^\alpha\Delta$ *on* B, *with* $\epsilon > 0$, $\alpha > 0$ *and* $\Delta : [0, 1]^2 \to \mathbb{R}$. *Then the contribution of* B *to the cost* $I_r(h)$ *is*

$$\int_B dx\, dy\, \mathcal{R}(h(x, y) \mid r(x, y)) = [1 + o(1)]\,\epsilon^{2\alpha} \int_B \frac{\Delta^2}{2r(1 - r)}, \quad \epsilon \to 0. \tag{3.23}$$

If the integral diverges, then the contribution decays slower than $\epsilon^{2\alpha}$.

Proof The proof is similar to that of Lemma 3.   □

3.2.2 Approximation by Block Graphons

We next introduce block graphons, which will be useful for our perturbation analysis. It follows from Remark 2 that optimal perturbations Δ_ϵ^* must satisfy $\|\Delta_\epsilon^*\|_2 \asymp \epsilon$. We know that if $\Delta_\epsilon = \epsilon\Delta$, then the order of the cost is ϵ^2 and $\|\Delta_\epsilon\|_2 \asymp \epsilon$. We argue through block graphon approximation that perturbations of the form $\epsilon\Delta$ are optimal. The usefulness of block graphons is that they are constant on blocks, so that no optimisation is needed over the shape of Δ_ϵ inside blocks. We first show in Lemma 5 that the cost function is well approximated by block graphons. Afterwards we apply Lemma 4 to find the cost function for block graphon perturbations. The restriction to block graphons simplifies the optimisation problem, i.e., it reduces the optimisation in (3.5) to a finite-dimensional problem.

Definition 2 (Block graphons) Let $\mathcal{W}_N \subset \mathcal{W}$ be the space of graphons with N blocks having a constant value on each of the blocks, i.e., $f \in \mathcal{W}_N$ is of the form

$$f(x, y) = \begin{cases} f_{i,j}, & (x, y) \in B_i \times B_j, \\ 0, & otherwise, \end{cases} \tag{3.24}$$

where $B_i = [\frac{i-1}{N}, \frac{i}{N})$, $1 \leq i \leq N - 1$ and $B_N = [\frac{N-1}{N}, 1]$ and $f_{i,j} \in [0, 1]$. Write $B_{i,j} = B_i \times B_j$. With each $f \in \mathcal{W}$ associate the block graphon $f_N \in \mathcal{W}_N$ given by

$$f_N(x, y) = N^2 \int_{B_{i,j}} dx' \, dy' \, f(x', y') = \bar{f}_{N,ij}, \quad (x, y) \in B_{i,j}. \tag{3.25}$$

Observe that if f_N is the block graphon associated with a graphon f, then

$$\|T_{f_N} - T_f\| = \|T_{f_N - f}\| \leq \|f_N - f\|_2. \tag{3.26}$$

We know from [12, Proposition 2.6] that $\|f_N - f\|_2 \to 0$, and hence $\lim_{N \to \infty} \|T_{f_N}\| = \|T_f\|$ for any $f \in \mathcal{W}$ and its associated sequence of block graphons $(f_N)_{N \in \mathbb{N}}$. The following lemma shows that the cost function associated with the graphons r and f is well approximated by the cost function associated with the block graphons r_N and f_N.

Lemma 5 *(Convergence of the cost function)* $\lim_{N \to \infty} I_{r_N}(f_N) = I_r(f)$ *for any* $f \in \mathcal{W}$.

Proof Since $f \in L^2([0, 1]^2)$, f_N is bounded. The assumption in (1.4) implies that $\eta \leq r_N \leq 1 - \eta$ for all $N \in \mathbb{N}$. We know from [17, Lemma 2.3] that there exists a constant $c > 0$ independent of f such that

$$|I_{r_N}(f) - I_r(f)| \leq c \, \|r_N - r\|_1 \leq c \, \|r_N - r\|_2. \tag{3.27}$$

Hence

$$\begin{aligned}
|I_{r_N}(f_N) - I_r(f)| &\leq |I_{r_N}(f_N) - I_r(f_N)| + |I_r(f_N) - I_r(f)| \\
&\leq c \, \|r_N - r\|_2 + |I_r(f_N) - I_r(f)|.
\end{aligned} \tag{3.28}$$

But we know from [12, Proposition 2.6] that $\lim_{N \to \infty} \|r_N - r\|_2 = 0$ and $\lim_{N \to \infty} \|f_N - f\|_2 = 0$, while we know from [28, Lemma 3.4] that I_r is continuous in the L^2-topology on $\mathcal{W}$. □

3.2.3 Block Graphon Perturbations

In what follows we fix $N \in \mathbb{N}$, analyse different types of perturbation and identify which one is optimal. For each $N \in \mathbb{N}$, we associate with the perturbed graphon $h = r + \Delta_\epsilon$ the block graphon $h_N \in \mathcal{W}_N$ given by (recall that $B_{i,j} = B_i \times B_j$)

$$\overline{h}_{N,ij}(x, y) = \overline{r}_{N,ij}(x, y) + \overline{\Delta_\epsilon}_{N,ij}(x, y), \quad (x, y) \in B_{i,j}, \tag{3.29}$$

where

$$\overline{r}_{N,ij} = N^2 \int_{B_{i,j}} dx' \, dy' \, r(x', y'), \qquad \overline{\Delta_\epsilon}_{N,ij} = N^2 \int_{B_{i,j}} dx' \, dy' \, \Delta_\epsilon(x', y'). \tag{3.30}$$

Observe that optimal perturbations must have $\|\Delta_\epsilon\|_2 \asymp \epsilon$ (see Remark 2), and hence the constraint in (3.14) becomes

$$\sum_{i,j=1}^{N} \int_{B_{i,j}} r\Delta_\epsilon = \sum_{i,j=1}^{N} \frac{1}{N^2} \overline{r\Delta_\epsilon}_{N,ij} = [1 + o(1)]\,C_r\epsilon, \qquad \epsilon \to 0. \qquad (3.31)$$

The block constraint in (3.31) implies that the sum over each block must be of order ϵ. We therefore must have that

$$\overline{r\Delta_\epsilon}_{N,ij} = O(\epsilon), \qquad \epsilon \to 0 \qquad \forall\,(i,j), \qquad (3.32)$$

which means that

$$\overline{\Delta_\epsilon}_{N,ij} = O(\epsilon), \qquad \epsilon \to 0 \qquad \forall\,(i,j), \qquad (3.33)$$

since (1.4) implies that $\overline{r\Delta_\epsilon}_{N,ij} \asymp \overline{\Delta_\epsilon}_{N,ij}$. There are two possible cases:

(I) All blocks contribute to the constraint with a term of order ϵ (balanced perturbation).
(II) Some blocks contribute to the constraint with a term of order ϵ and some with $o(\epsilon)$ (unbalanced perturbation).

Perturbations of type (I) consist of a small perturbation on each block, i.e., $\overline{\Delta_\epsilon}_{N,ij} \asymp \epsilon$ for each block $B_{i,j}$. By Lemma 4, this contributes a term of order ϵ^2 to the total cost. Since all blocks have the same type of perturbation, they all contribute in the same way, and so we get $I_{r_N}(h_N) \asymp \epsilon^2$.

The following lemma shows that perturbations of type (II) are worse than perturbations of type (I).

Lemma 6 *(Unbalanced perturbations) Perturbations of order ϵ that are not balanced, i.e., that do not cover the entire unit square $[0,1]^2$, are worse than the balanced perturbation in Lemma 7.*

We defer the proof of the above lemma to the next section, since it uses similar computations as in the proof of Lemma 7. Let $1 \le k \le N^2 - 1$ be the number of blocks that contribute a term of order $o(\epsilon)$ to the constraint, i.e., $\overline{\Delta_\epsilon}_{N,ij} = o(\epsilon)$. By Lemma 4, such blocks contribute order $o(\epsilon^2)$ to the total cost. The remaining blocks must fall in the class of blocks of type (I), with a perturbation of order ϵ on each of them. Lemma 6 shows that the cost function attains its minimum when the perturbation of order ϵ is uniform on $[0,1]^2$. This shows that without loss of generality we may assume that the optimal perturbation Δ_ϵ^* is of the form $\epsilon\Delta$ for some $\Delta \in L^2([0,1]^2)$.

3.2.4 Optimal Perturbation

Since we have shown that the optimal perturbations are of the form $\epsilon\Delta$, we can now solve (3.5). We know that the order of the cost is ϵ^2. Below we identify the proportionality constant. In the course of the proof we also find the optimal Δ.

Lemma 7 *(Balanced perturbations) Suppose that $\Delta_\epsilon = \epsilon \Delta$ with $\Delta: [0,1]^2 \to \mathbb{R}$.
Let $\mathcal{M}$ be the set of Lebesgue measure-preserving bijective maps. Then*

$$\delta_r(\epsilon) = [1 + \mathrm{o}(1)] K_r \epsilon^2, \qquad \epsilon \to 0, \tag{3.34}$$

with

$$K_r = \frac{1}{2}(C_r)^2 \inf_{\phi \in \mathcal{M}} \frac{D_r^\phi}{(B_r^\phi)^2}, \tag{3.35}$$

where $B_r^\phi = \int_{[0,1]^2} r^\phi r^2(1-r)$ and $D_r^\phi = \int_{[0,1]^2} (r^\phi)^2 r(1-r)$.

Proof The constraint in (3.14) becomes

$$\int_{[0,1]^2} r\Delta = [1 + \mathrm{o}(1)] C_r, \qquad \epsilon \to 0, \tag{3.36}$$

and we get

$$
\begin{aligned}
\delta_r(\epsilon) &= \inf_{\substack{\Delta: [0,1]^2 \to \mathbb{R} \\ r+\epsilon\Delta \in \mathcal{W} \\ \int_{[0,1]^2} r\Delta = [1+\mathrm{o}(1)] C_r}} I_r(r + \epsilon\Delta) \\
&= \inf_{\substack{\Delta: [0,1]^2 \to \mathbb{R} \\ r+\epsilon\Delta \in \mathcal{W} \\ \int_{[0,1]^2} r\Delta = [1+\mathrm{o}(1)] C_r}} \int_{[0,1]^2} \mathrm{d}x\, \mathrm{d}y\, \mathcal{R}\big((r+\epsilon\Delta)(x,y) \mid r(x,y)\big).
\end{aligned}
\tag{3.37}
$$

By Lemma 4 (with $\alpha = 1$), we have

$$\delta_r(\epsilon) = [1 + \mathrm{o}(1)] K_r \epsilon^2, \qquad \epsilon \to 0, \tag{3.38}$$

with

$$K_r = \inf_{\substack{\Delta: [0,1]^2 \to \mathbb{R} \\ \int_{[0,1]^2} r\Delta = C_r}} \int_{[0,1]^2} \frac{\Delta^2}{2r(1-r)}. \tag{3.39}$$

The prefactor $1 + \mathrm{o}(1)$ in (3.38) arises after we scale Δ by $1 + \mathrm{o}(1)$ in order to force $\int_{[0,1]^2} r\Delta = C_r$. Note that the optimisation problem in (3.39) no longer depends on ϵ.

We can apply the method of Lagrange multipliers to solve this constrained optimisation problem. To that end we define the Lagrangian

$$\mathcal{L}_{A_r}(\Delta) = \int_{[0,1]^2} \frac{\Delta^2}{2r(1-r)} + A_r \int_{[0,1]^2} r\Delta, \tag{3.40}$$

where A_r is a Langrange multiplier. Since $\int_{[0,1]^2} r = \int_{[0,1]^2} r^\phi$ for any Lebesgue measure-preserving bijective map $\phi \in \mathcal{M}$, we get that the minimiser (in the space of functions from $[0,1]^2 \to \mathbb{R}$) is of the form

$$\Delta^\phi = -A_r \, r^\phi r(1-r), \quad \phi \in \mathcal{M}. \tag{3.41}$$

We pick A_r such that the constraint is satisfied, i.e.,

$$-A_r B_r^\phi = [1 + o(1)] \, C_r \tag{3.42}$$

with

$$B_r^\phi = \int_{[0,1]^2} r^\phi r^2 (1-r). \tag{3.43}$$

We get

$$\Delta^\phi = \frac{C_r}{B_r^\phi} r^\phi r(1-r), \quad \phi \in \mathcal{M}, \tag{3.44}$$

and

$$K_r = \inf_{\phi \in \mathcal{M}} \int_{[0,1]^2} \frac{(\Delta^\phi)^2}{2r(1-r)} = \frac{1}{2}(C_r)^2 \inf_{\phi \in \mathcal{M}} \frac{D_r^\phi}{(B_r^\phi)^2} \tag{3.45}$$

with

$$D_r^\phi = \int_{[0,1]^2} (r^\phi)^2 r(1-r). \tag{3.46}$$

$\square$

We next show that the infimum in (3.45) is uniquely attained when ϕ is the identity. For this we will show that $D_r^\phi/(B_r^\phi)^2 \geq 1/B_r$ with equality if and only if $\phi = \mathrm{Id}$.

Indeed, write

$$
B_r D_r^\phi - (B_r^\phi)^2
$$
$$
= \int_{[0,1]^2} dx\, dy\, r(x,y)[1 - r(x,y)] \int_{[0,1]^2} d\bar{x}\, d\bar{y}\, r(\bar{x},\bar{y})[1 - r(\bar{x},\bar{y})]
$$
$$
\times \left\{ r(x,y)^2\, r^\phi(\bar{x},\bar{y})^2 - r(x,y)r^\phi(x,y)\, r(\bar{x},\bar{y})r^\phi(\bar{x},\bar{y}) \right\}
$$
$$
= \int_{[0,1]^2} dx\, dy\, r(x,y)[1 - r(x,y)] \int_{[0,1]^2} d\bar{x}\, d\bar{y}\, r(\bar{x},\bar{y})[1 - r(\bar{x},\bar{y})] \quad (3.47)
$$
$$
\times \tfrac{1}{2} \left\{ r(x,y)^2\, r^\phi(\bar{x},\bar{y})^2 + r^\phi(x,y)^2\, r(\bar{x},\bar{y})^2 - 2r(x,y)r^\phi(x,y)\, r(\bar{x},\bar{y})r^\phi(\bar{x},\bar{y}) \right\}
$$
$$
= \int_{[0,1]^2} dx\, dy\, r(x,y)[1 - r(x,y)] \int_{[0,1]^2} d\bar{x}\, d\bar{y}\, r(\bar{x},\bar{y})[1 - r(\bar{x},\bar{y})]
$$
$$
\times \tfrac{1}{2} \left(r(x,y)r^\phi(\bar{x},\bar{y}) - r^\phi(x,y)r(\bar{x},\bar{y}) \right)^2,
$$

where the second equality uses the symmetry between the integrals. Hence $B_r D_r^\phi - (B_r^\phi)^2 \geq 0$, with equality if and only if $r(x,y)/r^\phi(x,y) = C$ for almost every $(x,y) \in [0,1]^2$. Clearly, this can hold only for $C = 1$, which amounts to $\phi = \mathrm{Id}$.

We conclude that the infimum in (3.45) equals $1/B_r$, and so we find that

$$
K_r = \frac{(C_r)^2}{2B_r}. \quad (3.48)
$$

Finally, note that $C_r = m_2$ by (2.8), and that $B_r = m_3^2 - m_4^2$ by (1.18). This settles the expression for K_r in (1.21).

Proof (Proof of Lemma 6) The argument of the variational formula can be reduced to an integral that considers only those regions that contribute order ϵ^2, which constitute a subset $A \subset [0,1]^2$. Applying the method of Lagrange multipliers as in Lemma 7, we obtain that the solution is given by

$$
\delta_r(\epsilon) = [1 + o(1)]\, K_r' \epsilon^2, \qquad \epsilon \to 0, \quad (3.49)
$$

with $K_r' > K_r$. The strict inequality comes from the fact that the optimal balanced perturbation Δ^{Id} found in (3.41) is non-zero everywhere. □

In conclusion, we have shown that a balanced perturbation is optimal and we have identified in (3.44) the form of the optimal balanced perturbation. Lemma 7 settles the claim in Theorem 4(i), while (3.44) settles the claim in Theorem 5(i).

3.3 Perturbation Near the Right End

Take $\beta = 1 - \epsilon$ and consider a graphon of the form $h = 1 - \Delta_\epsilon$, where $\Delta_\epsilon \colon [0,1]^2 \to [0,\infty)$ represents a symmetric and bounded perturbation of the constant graphon

$h \equiv 1$. We compare

$$\psi_r(1-\epsilon) = \inf_{\substack{\Delta_\epsilon:\ [0,1]^2 \to [0,\infty) \\ 1-\Delta_\epsilon \in \mathcal{W} \\ \|T_{1-\Delta_\epsilon}\|=1-\epsilon}} I_r(1-\Delta_\epsilon) \tag{3.50}$$

with

$$C_r^1 = I_r(1) \tag{3.51}$$

by computing the difference

$$\delta_r(\epsilon) = \psi_r(1) - \psi_r(1-\epsilon) \tag{3.52}$$

and studying its behaviour as $\epsilon \downarrow 0$. Since $I_r(1)$ is a constant, we can write

$$\delta_r(\epsilon) = \sup_{\substack{\Delta_\epsilon:\ [0,1]^2 \to [0,\infty) \\ 1-\Delta_\epsilon \in \mathcal{W} \\ \|T_{1-\Delta_\epsilon}\|=1-\epsilon}} [I_r(1) - I_r(1-\Delta_\epsilon)]. \tag{3.53}$$

We again use the expansion in Lemma 1. Pick $\bar{h} = 1$ and $h = 1 - \Delta_\epsilon$ in (2.1), to get

$$\|T_{1-\Delta_\epsilon}\| = 1 + \sum_{n \in \mathbb{N}} \frac{1}{\|T_{1-\Delta_\epsilon}\|^n} \mathcal{F}_n(1 - \Delta_\epsilon, 1). \tag{3.54}$$

Since $\|T_{1-\Delta_\epsilon}\| = 1 - \epsilon$, this gives

$$1 - \epsilon = 1 + \frac{\langle 1, (-\Delta_\epsilon)1 \rangle}{1-\epsilon} + \frac{\langle 1, (-\Delta_\epsilon)^2 1 \rangle}{(1-\epsilon)^2} + \sum_{n \in \mathbb{N}\setminus\{1,2\}} \frac{\langle 1, (-\Delta_\epsilon)^n 1 \rangle}{(1-\epsilon)^n}. \tag{3.55}$$

For $\epsilon \downarrow 0$ we have $\|\Delta_\epsilon\|_2 \downarrow 0$ and $|\langle 1, (-\Delta_\epsilon)^n 1 \rangle| = O(\|\Delta_\epsilon\|_2^n)$. Therefore

$$\epsilon(1-\epsilon) = \int_{[0,1]^2} \Delta_\epsilon - \frac{\langle 1, \Delta_\epsilon^2 1 \rangle}{(1-\epsilon)} + O\left(\|\Delta_\epsilon\|_2^3\right). \tag{3.56}$$

The restriction $1 - \Delta_\epsilon \in \mathcal{W}$ implies that $\Delta_\epsilon \in [0,1]$. Hence $\|\Delta_\epsilon\|_2^2 \leq \|\Delta_\epsilon\|_1$. Moreover,

$$1 - \epsilon = \|T_{1-\Delta_\epsilon}\| \leq \|1 - \Delta_\epsilon\|_2 \leq \sqrt{\|1 - \Delta_\epsilon\|_1}. \tag{3.57}$$

Since $\|1 - \Delta_\epsilon\|_1 = 1 - \|\Delta_\epsilon\|_1$, we have

$$\|\Delta_\epsilon\|_1 \leq 1 - (1-\epsilon)^2 = \epsilon(2-\epsilon). \tag{3.58}$$

Since $\|\Delta_\epsilon\|_2^3 = O(\epsilon^{3/2})$, (3.56) reads

$$\frac{1}{1-\epsilon} \int_{[0,1]^3} dx\, dy\, dz\, \Delta_\epsilon(x,y)[1 - \Delta_\epsilon(y,z)] - \frac{\epsilon}{1-\epsilon} \|\Delta_\epsilon\|_1 = \epsilon(1-\epsilon) + O(\epsilon^{3/2}), \tag{3.59}$$

which, because $\|\Delta_\epsilon\|_1 = O(\epsilon)$, further reduces to

$$\int_{[0,1]^3} dx\, dy\, dz\, \Delta_\epsilon(x,y)[1 - \Delta_\epsilon(y,z)] = \epsilon\,[1 + O(\epsilon^{1/2})]. \tag{3.60}$$

Note that when $\Delta_\epsilon = \epsilon\Delta$, the constraint reads

$$\int_{[0,1]^2} \Delta = 1 + O(\epsilon^{1/2}), \qquad \epsilon \downarrow 0. \tag{3.61}$$

The following lemma gives an upper bound for $I_r(1) - I_r(1 - \Delta_\epsilon)$.

Lemma 8 *(Order of minimal cost) Let $\Delta_\epsilon : [0,1]^2 \to [0,1]$ be such that $1 - \Delta_\epsilon \in \mathcal{W}$ and $\|T_{1-\Delta_\epsilon}\| = 1 - \epsilon$. Then, for ϵ small enough,*

$$I_r(1) - I_r(1 - \Delta_\epsilon) \leq \|\Delta_\epsilon\|_1 \log \frac{1}{\|\Delta_\epsilon\|_1} + O(\|\Delta_\epsilon\|_1). \tag{3.62}$$

Moreover, $\delta_r(\epsilon) \leq \epsilon \log \frac{1}{\epsilon} + O(\epsilon)$.

Proof Abbreviate (recall (1.9))

$$\chi(a) = \mathcal{R}(a \mid r) = a \log \frac{a}{r} + (1-a) \log \frac{1-a}{1-r}, \quad a \in [0,1]. \tag{3.63}$$

Then

$$\begin{aligned} &\chi(1) - \chi(1 - \Delta_\epsilon(x,y)) \\ &= \Delta_\epsilon(x,y) \log \left(\frac{1 - \Delta_\epsilon(x,y)}{\Delta_\epsilon(x,y)} \frac{1 - r(x,y)}{r(x,y)} \right) - \log(1 - \Delta_\epsilon(x,y)), \end{aligned} \tag{3.64}$$

and so

$$I_r(1) - I_r(1 - \Delta_\epsilon) = \int_{[0,1]^2} \left[\Delta_\epsilon \log \left(\frac{1 - \Delta_\epsilon}{\Delta_\epsilon} \frac{1-r}{r} \right) - \log(1 - \Delta_\epsilon) \right]. \tag{3.65}$$

Let μ_ϵ be the probability measure on $[0,1]^2$ whose density with respect to the Lebesgue measure is $Z_\epsilon^{-1}(1 - \Delta_\epsilon(x,y))$, where $Z_\epsilon = \int_{[0,1]^2}(1 - \Delta_\epsilon) = 1 - O(\epsilon)$. Since $u \mapsto \bar{s}(u) = u \log(1/u)$ is strictly concave, by Jensen's inequality we have

$$\int_{[0,1]^2} \Delta_\epsilon \log\left(\frac{1-\Delta_\epsilon}{\Delta_\epsilon}\right) = Z_\epsilon \int_{[0,1]^2} \mu_\epsilon \bar{s}\left(\frac{\Delta_\epsilon}{1-\Delta_\epsilon}\right) \leq Z_\epsilon \bar{s}\left(Z_\epsilon^{-1}\|\Delta_\epsilon\|_1\right)$$

$$= \|\Delta_\epsilon\|_1 \log\left(\frac{Z_\epsilon}{\|\Delta_\epsilon\|_1}\right). \tag{3.66}$$

Moreover,

$$\int_{[0,1]^2} \Delta_\epsilon \log\left(\frac{1-r}{r}\right) = O(\|\Delta_\epsilon\|_1), \qquad -\int_{[0,1]^2} \log(1-\Delta_\epsilon) = O(\|\Delta_\epsilon\|_1) \tag{3.67}$$

Hence

$$I_r(1) - I_r(1-\Delta_\epsilon) \leq \|\Delta_\epsilon\|_1 \log\frac{1}{\|\Delta_\epsilon\|_1} + O(\|\Delta_\epsilon\|_1), \qquad \epsilon \downarrow 0, \tag{3.68}$$

and since $\|\Delta_\epsilon\|_1 = O(\epsilon)$ also $\delta_r(\epsilon) \leq \epsilon \log\frac{1}{\epsilon} + O(\epsilon)$. $\qquad \square$

The following is the analogue of Lemma 4 for perturbations near the right end.

Lemma 9 *(Cost of small perturbations) Let $B \subseteq [0,1]^2$ be a measurable region of area $|B|$. Suppose that $\Delta_\epsilon = \epsilon^\alpha \Delta$ on B with $\epsilon > 0$, $\alpha > 0$ and $\Delta\colon [0,1]^2 \to [0,\infty)$. Then the contribution of B to the cost $I_r(h)$ is*

$$\int_B dx\, dy \left[\mathcal{R}(1 \mid r(x,y)) - \mathcal{R}(1 - \epsilon^\alpha \Delta(x,y) \mid r(x,y))\right]$$

$$= [1 + o(1)] \int_B \epsilon^\alpha \Delta \log\left(\frac{1-r}{\epsilon^\alpha \Delta r}\right), \qquad \epsilon \downarrow 0. \tag{3.69}$$

Proof The proof is the same as that of Lemma 8, with the observation that

$$\mathcal{R}(1 \mid r) - \mathcal{R}(1 - \epsilon^\alpha \Delta \mid r) = [1 + o(1)] \epsilon^\alpha \Delta \log\left(\frac{1-r}{\epsilon^\alpha \Delta r}\right), \qquad \epsilon \downarrow 0. \tag{3.70}$$

$\qquad \square$

Following the argument in Sect. 3.2, we can approximate the cost function by using block graphons. We see

$$\sum_{i,j=1}^N \int_{B_{i,j}} dx\, dy\, \Delta_{\epsilon,N}(x,y) = \sum_{i,j=1}^N \frac{1}{N^2} \overline{\Delta_{\epsilon N,ij}} = \epsilon[1 + o(1)], \qquad \epsilon \downarrow 0. \tag{3.71}$$

The block constraint in (3.71) implies that the sum over each block must be of order ϵ. Hence

$$\overline{\Delta_{\epsilon N,ij}} = O(\epsilon), \qquad \epsilon \downarrow 0 \quad \forall (i,j). \tag{3.72}$$

There are two cases to distinguish: all blocks contribute to the constraint with a term of order ϵ (balanced perturbation), or some of the blocks contribute to the constraint with a term of order ϵ and some with $o(\epsilon)$. Analogously to the analysis in Sect. 3.2, by using Lemma 9 we can compute the total cost that different types of block perturbations produce. This again shows that the optimal perturbations are the balanced perturbations, consisting of perturbations of order ϵ on every block. As $N \to \infty$, a sequence of such perturbations converges to a perturbation $\Delta_\epsilon = \epsilon\Delta$ with $\Delta \colon [0, 1]^2 \to [0, \infty)$, which we analyse next.

Lemma 10 (*Balanced perturbations*) *Suppose that* $\Delta_\epsilon = \epsilon\Delta$ *with* $\Delta \colon [0, 1]^2 \to [0, \infty)$. *Then*

$$\delta_r(\epsilon) = [1 + O(\epsilon^{1/2})] \left\{ \epsilon + \epsilon \log \left(\frac{N_r^1}{\epsilon} \right) \right\} + O(\epsilon^2), \qquad \epsilon \downarrow 0. \tag{3.73}$$

Proof By (3.61) and (3.65),

$$
\begin{aligned}
\delta_r(\epsilon) &= \sup_{\substack{\Delta \colon [0,1]^2 \to [0,\infty) \\ 1-\epsilon\Delta \in \mathcal{W} \\ \int_{[0,1]^2} \Delta = 1 + O(\epsilon^{1/2})}} \left[I_r(1) - I_r(1 - \epsilon\Delta) \right] \\
&= \sup_{\substack{\Delta \colon [0,1]^2 \to [0,\infty) \\ 1-\epsilon\Delta \in \mathcal{W} \\ \int_{[0,1]^2} \Delta = 1 + O(\epsilon^{1/2})}} \int_{[0,1]^2} \left[\epsilon\Delta \log \left(\frac{1-\epsilon\Delta}{\epsilon\Delta} \frac{1-r}{r} \right) - \log(1-\epsilon\Delta) \right].
\end{aligned}
\tag{3.74}
$$

The integral in (3.74) equals

$$
\begin{aligned}
\int_{[0,1]^2} &\left[\epsilon\Delta \log \left(\frac{1-r}{\epsilon\Delta r} \right) - (1-\epsilon\Delta)\log(1-\epsilon\Delta) \right] \\
&= \int_{[0,1]^2} \epsilon\Delta \log \left(\frac{1-r}{\epsilon\Delta r} \right) + \epsilon \int_{[0,1]^2} \Delta + O(\epsilon^2).
\end{aligned}
\tag{3.75}
$$

Hence

$$\delta_r(\epsilon) = [1 + O(\epsilon^{1/2})] \left\{ \epsilon + \sup_{\substack{\Delta \colon [0,1]^2 \to [0,\infty) \\ \int_{[0,1]^2} \Delta = 1}} \int_{[0,1]^2} \epsilon\Delta \log \left(\frac{1-r}{\epsilon\Delta r} \right) \right\} + O(\epsilon^2),
\tag{3.76}$$

where the prefactor arises after we scale Δ by $1+O(\epsilon^{1/2})$ in order to force $\int_{[0,1]^2} \Delta = 1$. Note that the constraint under the supremum no longer depends on ϵ.

We can solve the optimisation problem by applying the method of Lagrange multipliers. To that end we define the Lagrangian

$$\mathcal{L}_{A_r}(\Delta) = \int_{[0,1]^2} \epsilon \Delta \log\left(\frac{1-r}{\epsilon \Delta r}\right) + A_r \int_{[0,1]^2} \Delta, \qquad (3.77)$$

where A_r is a Langrange multiplier. Since $\int_{[0,1]^2} \log\frac{1-r}{r} = \int_{[0,1]^2} \log\frac{1-r^\phi}{r^\phi}$ for any Lebesgue measure-preserving bijective map $\phi \in \mathcal{M}$, we get that the minimiser (in the space of functions from $[0,1]^2 \to \mathbb{R}$) is of the form

$$\Delta^\phi = e^{-\frac{\epsilon - A_r}{\epsilon}} \frac{1}{\epsilon} \frac{1 - r^\phi}{r^\phi}, \quad \phi \in \mathcal{M}. \qquad (3.78)$$

We pick A_r such that the constraint $\int_{[0,1]^2} \Delta = 1$ is satisfied. This gives

$$\Delta^\phi = \frac{1}{N_r^1} \frac{1 - r^\phi}{r^\phi}, \quad \phi \in \mathcal{M}, \qquad (3.79)$$

with $N_r^1 = \int_{[0,1]^2} \frac{1-r}{r}$. Hence the supremum in (3.76) becomes

$$\sup_{\phi \in \mathcal{M}} \int_{[0,1]^2} \epsilon \Delta^\phi \log\left(\frac{1-r}{\epsilon \Delta^\phi r}\right). \qquad (3.80)$$

We have

$$\int_{[0,1]^2} \epsilon \Delta^\phi \log\left(\frac{1-r}{\epsilon \Delta^\phi r}\right) = \epsilon \log\left(\frac{N_r^1}{\epsilon}\right) - \epsilon \int_{[0,1]^2} \Delta^\phi \log\left(\frac{\Delta^\phi}{\Delta}\right), \quad (3.81)$$

where we use that $\int_{[0,1]^2} \Delta^\phi = 1$. Since the function $u \mapsto s(u) = u \log u$ is strictly convex on $[0, \infty)$, Jensen's inequality gives

$$\int_{[0,1]^2} \Delta^\phi \log\left(\frac{\Delta^\phi}{\Delta}\right) = \int_{[0,1]^2} \Delta \, s\left(\frac{\Delta^\phi}{\Delta}\right) \geq s\left(\int_{[0,1]^2} \Delta \frac{\Delta^\phi}{\Delta}\right)$$

$$= s\left(\int_{[0,1]^2} \Delta^\phi\right) = s(1) = 0, \qquad (3.82)$$

where we use that $\int_{[0,1]^2} \Delta = 1$. Equality holds if and only if $\Delta = \Delta^\phi$ almost everywhere on $[0,1]^2$, which amounts to $\phi = \mathrm{Id}$. Hence the supremum in (3.80) is uniquely attained at $\phi = \mathrm{Id}$ and equals

$$\int_{[0,1]^2} \epsilon \frac{1}{N_r^1} \frac{(1-r)}{r} \log\left(\frac{N_r^1}{\epsilon}\right) = \epsilon \log\left(\frac{N_r^1}{\epsilon}\right). \qquad (3.83)$$

Consequently, (3.76) gives (3.73).   □

Lemma 10 settles the claim in Theorem 4(ii). Since we have shown that a balanced perturbation is optimal, (3.79) settles the claim in Theorem 5(ii).

3.4 Perturbation Near the Left End

Take $\beta = \epsilon$ and consider a graphon of the form $h = \Delta_\epsilon$, where $\Delta_\epsilon : [0, 1]^2 \to [0, \infty)$ represents a symmetric and bounded perturbation of the constant graphon $h \equiv 0$. We compare

$$\psi_r(\epsilon) = \inf_{\substack{\Delta_\epsilon : \ [0,1]^2 \to [0,\infty) \\ \Delta_\epsilon \in \mathcal{W} \\ \|T_{\Delta_\epsilon}\| = \epsilon}} I_r(\Delta_\epsilon) \tag{3.84}$$

with

$$\psi_r(0) = I_r(0) \tag{3.85}$$

by computing the difference

$$\delta_r(\epsilon) = \psi_r(\epsilon) - \psi_r(0) \tag{3.86}$$

and studying its behaviour as $\epsilon \to 0$.

We claim that analysing (3.86) is equivalent to analysing

$$\delta_{\hat{r}}(\epsilon) = \phi_{\hat{r}}(1) - \phi_{\hat{r}}(1 - \epsilon), \tag{3.87}$$

where $\hat{r}$ is the reflection of r defined as

$$\hat{r}(x, y) = 1 - r(x, y), \qquad x, y \in [0, 1]. \tag{3.88}$$

Indeed,

$$I_r(0) = \int_{[0,1]^2} \mathcal{R}(0 \mid r) = \int_{[0,1]^2} \log\left(\frac{1}{1-r}\right) = \int_{[0,1]^2} \mathcal{R}(1 \mid \hat{r}) = I_{\hat{r}}(1) \tag{3.89}$$

and

$$I_r(\Delta_\epsilon) = \int_{[0,1]^2} \mathcal{R}(\Delta_\epsilon \mid r) = \int_{[0,1]^2} \left[\Delta_\epsilon \log\left(\frac{\Delta_\epsilon}{r}\right) + (1 - \Delta_\epsilon) \log\left(\frac{1 - \Delta_\epsilon}{1 - r}\right)\right]$$

$$= \int_{[0,1]^2} \mathcal{R}(1 - \Delta_\epsilon \mid \hat{r}) = I_{\hat{r}}(1 - \Delta_\epsilon). \tag{3.90}$$

We can therefore use the results in Sect. 3.3. From Lemma 10 we know that

$$\delta_{\hat{r}}(\epsilon) = [1 + \mathrm{O}(\epsilon^{1/2})] \left\{\epsilon + \epsilon \log\left(\frac{N_{\hat{r}}^1}{\epsilon}\right)\right\} + \mathrm{O}(\epsilon^2), \qquad \epsilon \downarrow 0, \tag{3.91}$$

and hence we obtain

$$
\delta_r(\epsilon) = [1 + \mathrm{O}(\epsilon^{1/2})] \left\{ \epsilon + \epsilon \log\left(\frac{N_r^0}{\epsilon}\right) \right\} + \mathrm{O}(\epsilon^2), \qquad \epsilon \downarrow 0. \tag{3.92}
$$

Consequently, the optimal perturbation is given by the balanced perturbation $\Delta_\epsilon = \epsilon \Delta$ with

$$
\Delta = \frac{1}{N_r^0} \frac{r}{1-r}, \tag{3.93}
$$

with $N_r^0 = \int_{[0,1]^2} \frac{r}{1-r}$.

The scaling in (3.92) settles the claim in Theorem 4(iii). Since we have shown that a balanced perturbation is optimal, (3.93) settles the claim in Theorem 5(iii).

A Appendix

Lemma 11 *(Finite-rank expansion) Consider a graphon $\bar{h} \in \mathcal{W}$ such that*

$$
\bar{h}(x, y) = \sum_{i=1}^{k} \theta_i \bar{v}_i(x) \bar{v}_i(y), \qquad x, y \in [0, 1], \tag{A.1}
$$

for some $k \in \mathbb{N}$, where $\theta_1 > \theta_2 \geq \ldots \geq \theta_k \geq 0$ and $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is an orthonormal set in $L^2([0, 1])$. Then there exists an $\varepsilon > 0$ such that, for any $h \in \mathcal{W}$ satisfying $\|T_{h-\bar{h}}\| < \min(\varepsilon, \|T_h\|)$, the operator norm $\|T_h\|$ solves the equation

$$
\|T_h\| = \lambda_k \left(\sum_{n \in \mathbb{N}_0} \|T_h\|^{-n} \mathcal{F}_n(h, \bar{h}) \right), \tag{A.2}
$$

where $\lambda_k(M)$ is the largest eigenvalue of a $k \times k$ Hermitian matrix M, and $\mathcal{F}_n(h, \bar{h})$ is a $k \times k$ matrix whose (i, j)-th entry is

$$
\sqrt{\theta_i \theta_j} \int_{[0,1]^2} \mathrm{d}x\, \mathrm{d}y\, \bar{v}_i(x)(h - \bar{h})^n(x, y)\bar{v}_j(y) \tag{A.3}
$$

for $1 \leq i, j \leq k$ and $n \in \mathbb{N}_0$.

Proof Put $\mu = \|T_h\|$, and let u be the eigenfunction of h corresponding to μ, i.e.,

$$
T_h u = \mu u. \tag{A.4}
$$

Put $g = h - \bar{h}$ and rewrite the above as

$$(\mu - T_g)u = T_{\bar{h}}u. \tag{A.5}$$

The assumption $\|T_{h-\bar{h}}\| < \|T_h\|$ implies that $\mu - T_g$ is invertible, which allows us to write

$$u = (\mu - T_g)^{-1} T_{\bar{h}} u = \sum_{j=1}^{k} \theta_j \langle \bar{v}_j, u \rangle (\mu - T_g)^{-1} \bar{v}_j. \tag{A.6}$$

For fixed $1 \le i \le k$, it follows that

$$\langle \bar{v}_i, u \rangle = \sum_{j=1}^{k} \theta_j \langle \bar{v}_j, u \rangle \langle \bar{v}_i, (\mu - T_g)^{-1} \bar{v}_j \rangle. \tag{A.7}$$

Multiplying both sides by $\mu \sqrt{\theta_i}$, we get

$$Mv = \mu v, \tag{A.8}$$

where $M = (M_{ij})_{1 \le i, j \le k}$ is the $k \times k$ real symmetric matrix with elements

$$M_{ij} = \sqrt{\theta_i \theta_j} \left\langle \bar{v}_i, \left(1 - \frac{T_g}{\mu}\right)^{-1} \bar{v}_j \right\rangle, \qquad 1 \le i, j \le k, \tag{A.9}$$

and

$$v = \left[\sqrt{\theta_1} \langle \bar{v}_1, u \rangle, \dots, \sqrt{\theta_k} \langle \bar{v}_k, u \rangle \right]'. \tag{A.10}$$

The first entry of v is non-zero for ε small with $\|T_g\| < \varepsilon$. Thus, (A.8) means that μ is an eigenvalue of M. By studying the diagonal entries of M, we can shown with the help of the Gershgorin circle theorem that, for small $\|T_g\|$,

$$\mu = \lambda_k(M). \tag{A.11}$$

With the help of the observation

$$M_{ij} = \sqrt{\theta_i \theta_j} \sum_{n \in \mathbb{N}_0} \mu^{-n} \langle \bar{v}_i, g^n \bar{v}_j \rangle, \qquad 1 \le i, j \le k, \tag{A.12}$$

i.e.,

$$M = \sum_{n \in \mathbb{N}_0} \mu^{-n} \mathcal{F}_n(h, \bar{h}), \tag{A.13}$$

this completes the proof.   □

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