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Counting elliptic curves with prescribed level structures over number fields

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Abstract
Harron and Snowden (J. reine angew. Math. 729 (2017), 151–170) counted the number of elliptic curves over $\mathbb{Q}$ up to height $X$ with torsion group $G$ for each possible torsion group $G$ over $\mathbb{Q}$. In this paper, we generalize their result to all number fields and all level structures $G$ such that the corresponding modular curve $X_G$ is a weighted projective line $\mathbb{P}(w_0,w_1)$ and the morphism $X_G \rightarrow X(1)$ satisfies a certain condition. In particular, this includes all modular curves $X_1(m,n)$ with coarse moduli space of genus 0. We prove our results by defining a size function on $\mathbb{P}(w_0,w_1)$ following unpublished work of Deng (Preprint, https://arxiv.org/abs/math/9812082), and working out how to count the number of points on $\mathbb{P}(w_0,w_1)$ up to size $X$.

MSC (2020)
11G05 (primary), 11G18, 11G50, 14D23, 14G40 (secondary)

1 | INTRODUCTION

Let $E$ be an elliptic curve over a number field $K$. The Mordell–Weil theorem says that $E(K)$ is isomorphic to $\mathbb{Z}^r \times E(K)_{\text{tor}}$ for some $r \geq 0$, where $E(K)_{\text{tor}}$ is the (finite) torsion subgroup of $E(K)$. It is a natural question which groups appear as $E(K)_{\text{tor}}$, and moreover how often each one of these groups appears. Harron and Snowden [11] studied this question and answered it in the case $K = \mathbb{Q}$. The aim of this paper is to study the same problem, but to both allow $K$ to be any number field and to answer the more general question how often a prescribed $G$-level structure appears.
To make this question more precise, let $n$ be a positive integer, let $G$ be a subgroup of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$, and let $K$ be a number field. We say that an elliptic curve $E$ over $K$ admits a $G$-level structure if there exists a $(\mathbb{Z}/n\mathbb{Z})$-basis of $E[n]\bar{K}$ such that the Galois representation $\rho_{E,n}: \text{Gal}(\bar{K}/K) \to \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ defined by this basis has image contained in $G$. We write

$$\mathcal{E}_{G,K} = \{ \text{elliptic curves over } K \text{ admitting a } G\text{-level structure} \}/\cong.$$ 

We will define a size function $S_K$ from the set of isomorphism classes of elliptic curves over $K$ to $\mathbb{R}_{>0}$; see Definition 7.1. We define a function $N_{G,K}: \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}$ by

$$N_{G,K}(X) = \#\{ E \in \mathcal{E}_{G,K} \mid S_K(E)^{12} \leq X \}.$$ 

Let $X_G$ be the moduli stack of generalized elliptic curves with $G$-level structure. This is a one-dimensional proper smooth geometrically connected algebraic stack over the fixed field $K_G$ of the action of $G$ on $\mathbb{Q}(\zeta_n)$ given by $(g, \zeta_n) \mapsto \zeta_n^{\det g}$. We consider cases where $X_G$ is a weighted projective line $\mathbb{P}(w_0, w_1)$ over $K_G$. We can now state our main result (which is also stated in a slightly different form in Theorem 7.6).

**Theorem 1.1.** Let $n$ be a positive integer, and let $G$ be a subgroup of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$. Assume that the stack $X_G$ over $K_G$ is isomorphic to $\mathbb{P}(w)_{K_G}$, where $w = (w_0, w_1)$ is a pair of positive integers, and let $e(G)$ be the reduced degree of the canonical morphism $X_G \to X(1)$ (see Definition 4.2). Furthermore, assume $e(G) = 1$ or $w = (1, 1)$ holds. Then for every finite extension $K$ of $K_G$, we have

$$N_{G,K}(X) \asymp X^{1/d(G)} \text{ as } X \to \infty,$$

where

$$d(G) = \frac{12e(G)}{w_0 + w_1}.$$ 

As all modular curves $X_G = X_1(m,n)$ with coarse moduli space of genus 0 satisfy the assumptions of Theorem 1.1, our result generalizes [11, Theorem 1.2], where this statement was proved in the case where $K = \mathbb{Q}$ and where $G$ is one of the 15 groups corresponding to the torsion groups from Mazur’s theorem.

A recent result of Pizzo, Pomerance and Voight [16] is $N_{G,\mathbb{Q}}(X) \sim X^{1/2}$ for $G$ such that $X_G = X_0(3)$. Moreover, they determined the constant in front of the leading term of the function $N_{G,\mathbb{Q}}(X)$ as well as the first two lower order terms. This result falls outside of the reach of our results, as $X_0(3)$ is not a weighted projective line (cf. Remark 7.4).

Similarly, Pomerance and Schaefer [17] proved that $N_{G,\mathbb{Q}}(X) \sim X^{1/3}$ for $G$ such that $X_G = X_0(4)$, and determined the constants in front of the leading term and the first lower order term. Our result implies $N_{G,K} \asymp X^{1/3}$ for all number fields $K$; for $K = \mathbb{Q}$, this follows from the sharper results of [17].

Cullinan, Kenney and Voight [4, Theorem 1.3.3] proved a sharper version of Theorem 1.1 in the special case where $X_G$ is a projective line (that is, isomorphic to $\mathbb{P}^1 = \mathbb{P}(1,1)$) and $K = \mathbb{Q}$. More precisely, they give an asymptotic expression for $N_{G,\mathbb{Q}}(X)$ containing an effectively computable leading coefficient and an error term.
Boggess and Sankar [2] determined the growth rate of the number of elliptic curves over \( \mathbb{Q} \) with a cyclic \( n \)-isogeny for \( n \in \{2, 3, 4, 5, 6, 8, 9, 12, 16, 18\} \). Out of these, only the cases \( n = 2 \) and \( n = 4 \) (for which a more precise result was already proved in [11, 17]) correspond to weighted projective lines and are therefore generalized to number fields by Theorem 1.1.

Remark 1.2. The 12th power in the definition of \( N_{G,K}(X) \) is included for easier comparison with the height function in [11]; see Remark 7.2.

Remark 1.3. Our result gives a more conceptual interpretation of \( d(G) \); cf. [11, §1.2]. Namely, we show that \( d(G) \) can be expressed in terms of the pair of positive integers \( (w_0, w_1) \) for which \( X_G \) is isomorphic to the weighted projective line with weights \( (w_0, w_1) \), and \( e(G) \), an invariant (similar to the degree) of the morphism \( X_G \to X(1) \).

We also remark that our result shows how in certain cases one can count points in the image of a morphism of stacks, partially answering a question in [11, Remark 1.5].

2 WEIGHTED PROJECTIVE SPACES

Definition 2.1. Given an \( (n+1) \)-tuple \( w = (w_0, \ldots, w_n) \) of positive integers, the weighted projective space with weights \( w \) is the algebraic stack

\[
P(w) = \mathbb{G}_m \backslash \mathbb{A}^{n+1}_{\neq 0}
\]

over \( \mathbb{Z} \), where \( \mathbb{A}^{n+1}_{\neq 0} \) is the complement of the zero section in \( \mathbb{A}^{n+1} \) and \( \mathbb{G}_m \) acts on \( \mathbb{A}^{n+1}_{\neq 0} \) by

\[
(\lambda, (x_0, \ldots, x_n)) \mapsto (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n).
\]

It is known that \( P(w) \) is a proper smooth algebraic stack over \( \mathbb{Z} \), and in fact a complete smooth toric Deligne–Mumford stack in the sense of Fantechi, Mann and Nironi [10, Example 7.27]. For every ring \( R \), there is a groupoid of \( R \)-points of \( P(w) \). We will mostly be interested in the set of isomorphism classes of this groupoid, which we call the set of \( R \)-points of \( P(w) \) and denote by \( P(w)(R) \). Given a field \( L \), the set \( P(w)(L) \) can be described as

\[
P(w)(L) = L^\times \backslash (L^{n+1} \backslash \{0\}),
\]

where \( L^\times \) acts on \( L^{n+1} \backslash \{0\} \) by

\[
(\lambda, (x_0, \ldots, x_n)) \mapsto (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n).
\]

The image in \( P(w)(L) \) of an element \( x \in L^{n+1} \backslash \{0\} \) will be denoted by \([x]\).

Example 2.2. If \( w = (m) \) with \( m \) a positive integer, then \( P(m) \) is canonically isomorphic to the classifying stack of the group scheme \( \mu_m \). If \( L \) is a field, then we have

\[
P(m)(L) = (L^\times)^m \backslash L^\times.
\]
Let \( w \) be an \((n+1)\)-tuple as above, let \( K \) be a number field, and let \( \mathcal{O}_K \) be its ring of integers. On the set \( \mathcal{P}(w)(K) \), we define a size function similarly to Deng [7]; see Definition 3.4. We do not restrict to weighted projective spaces that are 'well-formed' in the sense of [7].

**Definition 3.1.** For \( x \in K^{n+1} \), the scaling ideal of \( x \), denoted by \( I_w(x) \), is the intersection of all fractional ideals \( a \) of \( \mathcal{O}_K \) satisfying \( x \in a^{w_0} \times \cdots \times a^{w_n} \). Similarly, for an \((n+1)\)-tuple \((b_0, \ldots, b_n)\) of fractional ideals of \( \mathcal{O}_K \), the scaling ideal of \((b_0, \ldots, b_n)\), denoted by \( I_w(b_0, \ldots, b_n) \), is the intersection of all fractional ideals \( a \) of \( \mathcal{O}_K \) satisfying \( b_i \subseteq a^{w_i} \) for all \( i \).

**Remark 3.2.** For all \( x \in K^{n+1} \setminus \{0\} \), the fractional ideal \( I_w(x) \) is nonzero and satisfies
\[
I_w(x)^{-1} = \{ a \in K \mid a^{w_i} x_i \in \mathcal{O}_K \text{ for } i = 0, \ldots, n \}.
\]

Similarly, for every \((n+1)\)-tuple \((b_0, \ldots, b_n)\) of fractional ideals of \( \mathcal{O}_K \), not all zero, the fractional ideal \( I_w(b_0, \ldots, b_n) \) is nonzero and satisfies
\[
I_w(b_0, \ldots, b_n)^{-1} = \{ a \in K \mid a^{w_i} b_i \subseteq \mathcal{O}_K \text{ for } i = 0, \ldots, n \}.
\]

**Definition 3.3.** Let \( \Omega_{K,\infty} \) denote the set of Archimedean places of \( K \), and for each \( v \in \Omega_{K,\infty} \), let \( | \cdot |_v \colon K \to \mathbb{R}_{\geq 0} \) be the corresponding normalized absolute value. The Archimedean size on \( K^{n+1} \setminus \{0\} \) is the function
\[
H_{w,\infty} : K^{n+1} \setminus \{0\} \to \mathbb{R}_{>0}
\]
\[
x \mapsto \prod_{v \in \Omega_{K,\infty}} \max_{0 \leq i \leq n} |x_i|_v^{1/w_i}.
\]

**Definition 3.4.** The size function on \( \mathcal{P}(w)(K) \) is the function
\[
S_{w,K} : \mathcal{P}(w)(K) \to \mathbb{R}_{>0}
\]
\[
[x] \mapsto N(I_w(x))^{-1} H_{w,\infty}(x).
\]

It is straightforward to check that \( S_{w,K}([x]) \) does not depend on the choice of the representative \( x \).

**Example 3.5.** If \( w = (m) \) with \( m \) a positive integer and \( x \in \mathbb{Z} \setminus \{0\} \) is \( m \)-th power free, we have
\[
S_{(m),\mathbb{Q}}([x]) = |x|^{1/m}.
\]

**Remark 3.6.** If \( L/K \) is an extension of number fields, we have
\[
S_{(1, \ldots, 1),L}(x) = S_{(1, \ldots, 1),K}(x)^{L:K},
\]
but for general weights \( w \) such a relation does not hold. For example, if \( w = (m) \) with \( m \geq 2 \) and \( x \in \mathbb{Z} \setminus \{0\} \) is \( m \)th power free, then

\[
S_{(m), \mathbb{Q}}([x]) = |x|^{1/m},
\]

but

\[
S_{(m), \mathbb{Q}(x^{1/m})}([x]) = S_{(m), \mathbb{Q}(x^{1/m})}(1) = 1.
\]

**Remark 3.7.** Definition 3.4 is a special case of the notion of height for rational points on algebraic stacks defined by Ellenberg, Satriano and Zureick-Brown [9]. Namely, as explained in [9, Section 3.3], we have

\[
\log S_{w, \mathbb{Q}}(x) = h_{\mathcal{L}}(x),
\]

where \( h_{\mathcal{L}} \) is the height function corresponding to the tautological line bundle \( \mathcal{L} \) on \( \mathbb{P}(w) \). The work of Ellenberg, Satriano and Zureick-Brown was recently used by Boggess and Sankar [2] to count elliptic curves over \( \mathbb{Q} \) with a rational \( n \)-isogeny for \( n \in \{2, 3, 4, 5, 6, 8, 9\} \), as mentioned in the introduction.

**Theorem 3.8.** Let \( n \) be a non-negative integer, let \( w = (w_0, \ldots, w_n) \) be an \((n+1)\)-tuple of positive integers, and let \( K \) be a number field. Let \( r_1, r_2, d_K, h_K, R_K, \mu_K \) and \( \zeta_K \) be the number of real places, number of nonreal complex places, discriminant, class number, regulator, number of roots of unity and Dedekind \( \zeta \)-function of \( K \), respectively. We write

\[
|w| = w_0 + w_1 + \cdots + w_n,
\]

\[
\mu^w_K = \frac{\mu_K}{\gcd(w_0, w_1, \ldots, w_n, \mu_K)}
\]

\[
C^w_K = \frac{h_K R_K}{\mu^w_K \zeta_K(|w|)} \left( \frac{2r_1(2\pi)^{r_2}}{\sqrt{|d_K|}} \right)^{n+1} |w|^{r_1 + r_2 - 1}.
\]

Then we have

\[
\# \{ x \in \mathbb{P}(w)(K) \mid S_{w, K}(x) \leq T \} \sim C^w_K T^{|w|} \quad \text{as } T \to \infty.
\]

**Proof.** This was proved by Deng [7, Theorem (A)] in the case where \( \mathbb{P}(w) \) is well-formed, that is, each \( n \) elements from \( w \) are coprime. However, the proof works in general with only minor changes: in the paragraph before [7, Proposition 4.2], the statement that the group of roots of unity acts effectively has to be replaced by the statement that all orbits of points with all coordinates nonzero contain \( \mu^w_K \) points, and the factor \( w \) (denoting the number of roots of unity) in [7, Proposition 4.2, Proposition 4.5, Corollary 4.6 and Theorem (A)] has to be replaced by \( \mu^w_K \). \( \square \)

**Remark 3.9.** Theorem 3.8 also follows from recent results of Darda [5] on counting rational points on weighted projective spaces with respect to more general height functions; see in particular [5, Remark 7.3.2.5].
In the remainder of this article, we will only consider weighted projective lines, that is, one-dimensional weighted projective spaces where the weight is given by a pair \((w_0, w_1)\).

## 4  MORPHISMS BETWEEN WEIGHTED PROJECTIVE LINES

Let \(u = (u_0, u_1)\), \(w = (w_0, w_1)\) be two pairs of positive integers. In this section, we classify the morphisms of stacks from \(\mathbb{P}(w)\) to \(\mathbb{P}(u)\) over a field. These morphisms form a groupoid, but for simplicity we will only be interested in the set of isomorphism classes of this groupoid, or in other words the set of morphisms from \(\mathbb{P}(w)\) to \(\mathbb{P}(u)\). We also prove some facts about such morphisms generalizing the corresponding facts about morphisms \(\mathbb{P}^1 \to \mathbb{P}^1\).

**Lemma 4.1.** Let \(K\) be a field, and let \(u = (u_0, u_1)\), \(w = (w_0, w_1)\) be two pairs of positive integers. We consider \(R = K[x_0, x_1]\) as a graded \(K\)-algebra where \(x_0\) and \(x_1\) are homogeneous of degrees \(w_0\) and \(w_1\), respectively. Let \(P_{u,w}(K)\) be the set of pairs \((f_0, f_1) \in R \times R\) with the following properties.

1. There exists \(e = e(f_0, f_1) \in \mathbb{Z}_{\geq 0}\) for which \(f_0\) and \(f_1\) are homogeneous of degrees \(e u_0\) and \(e u_1\), respectively.
2. The homogeneous ideal \(\sqrt{(f_0, f_1)} \subseteq R\) contains \((x_0, x_1)\).

Let \(K^\times\) act on \(P_{u,w}(K)\) by \(c(f_0, f_1) = (c^{u_0} f_0, c^{u_1} f_1)\). Then there is a natural bijection from \(K^\times \backslash P_{u,w}(K)\) to the set of morphisms \(\mathbb{P}(w)_K \to \mathbb{P}(u)_K\) sending the class of \((f_0, f_1) \in P_{u,w}(K)\) to the morphism induced by the \(K\)-algebra homomorphism

\[
K[y_0, y_1] \longrightarrow K[x_0, x_1]
\]

\[
y_0 \longmapsto f_0
\]

\[
y_1 \longmapsto f_1.
\]

**Proof.** We apply Lemma A.2 to the following data over \(K\):

- \(X = \mathbb{A}^2_{\neq 0}\) with coordinates \(x = (x_0, x_1)\),
- \(Y = \mathbb{A}^2_{\neq 0}\) with coordinates \(y = (y_0, y_1)\),
- \(G = \mathbb{G}_m\) with coordinate \(g\),
- \(H = \mathbb{G}_m\) with coordinate \(h\),
- \(a : G \times X \to X\) is the weight \(w\) action, given on points by \(a(g,x) = (g^{u_0} x_0, g^{u_1} x_1)\),
- \(b : H \times Y \to Y\) is the weight \(u\) action, given on points by \(b(h,y) = (h^{u_0} y_0, h^{u_1} y_1)\).

(Note that the lemma applies because the Picard group of \(X\) is trivial.)

We first determine the morphisms \(h : G \times X \to H\) satisfying the ‘cocycle condition’ (A.1) of Lemma A.2. A morphism \(h : G \times X \to H\) is given by a monomial of the form \(h(g,x) = \lambda g^e\) with \(\lambda \in K^\times\) and \(e \in \mathbb{Z}\), and \(h\) satisfies (A.1) if and only if \(\lambda = 1\), i.e. \(h\) is of the form \(h(g,x) = g^e\).

Given \(h\) as above, we now determine the morphisms \(f : X \to Y\) such that the pair \((f, h)\) satisfies condition (A.2) of Lemma A.2. Every such \(f\) is given by a pair \((f_0, f_1) \in R \times R\), and \((f_0, f_1)\) determines a morphism \(X \to Y\) if and only if \(\sqrt{(f_0, f_1)}\) contains \((x_0, x_1)\). It is straightforward to check that condition (A.2) translates to the condition that \(f_j\) is homogeneous of degree \(e u_j\) for \(j = 0, 1\). In particular, morphisms \(f : X \to Y\) such that \((f, h)\) defines a morphism \([G\backslash X] \to [H\backslash Y]\) only exist if \(e > 0\); moreover, \(e\) and therefore \(h\) are uniquely determined by \(f\).
Finally, the group $H(X)$ is canonically isomorphic to $K^X$, and if $(f, h)$ is a pair as above where $f$ is defined by $(f_0, f_1)$, and $c \in H(X)$, then we have $c(f, h) = (f', h)$ where $f'$ is defined by $(c^{u_0}f_0, c^{u_1}f_1)$. The lemma therefore follows from Lemma A.2.

**Definition 4.2.** Let $K$ be a field, let $u, w$ be two pairs of positive integers, and let $\phi : \mathbb{P}(w)_K \to \mathbb{P}(u)_K$ be a morphism. The reduced degree of $\phi$, denoted by $\deg_{\text{red}} \phi$, is the unique integer $e \geq 0$ satisfying Lemma 4.1(i) for some (hence every) pair $(f_0, f_1)$ giving rise to $\phi$ via the bijection of Lemma 4.1.

**Remark 4.3.** A morphism $\phi : \mathbb{P}(u)_K \to \mathbb{P}(w)_K$ is representable (by which we mean representable in algebraic spaces) if and only if $\phi$ is faithful as a functor [18, tag 04Y5]. Moreover, it suffices to check this condition on geometric fibres [3, Corollary 2.2.7]. From this one can deduce that $\phi$ is representable if and only if its reduced degree $e = \deg_{\text{red}} \phi$ satisfies

$$\gcd(u_0, e) = \gcd(w_1, e) = 1.$$  

**Lemma 4.4.** In the setting of Lemma 4.1, let $(f_0, f_1) \in P_{u, w}(K)$ and assume $e(f_0, f_1) > 0$. Then $R$ is finite over its graded subalgebra $S = K[f_0, f_1]$.

**Proof.** We write $R_+ = Rx_0 + Rx_1, S_+ = Sf_0 + Sf_1$ and $I = Rf_0 + Rf_1 = RS_+$. By condition (ii) of Lemma 4.1 and the fact that $f_0$ and $f_1$ are nonconstant, we have $\sqrt{I} = R_+$. Hence for $m$ sufficiently large, we have $R_m \subseteq I$, so the graded $K$-algebra $R/I$ is a quotient of $R/R_m$ and is therefore finite-dimensional over $K$. Choose homogeneous elements $g_1, \ldots, g_r \in R$ such that their images in $R/I$ are a $K$-basis of $R/I$. In particular, the $g_i$ generate $R/I = R/RS_+$ over $S$, so we have

$$R = RS_+ + Sg_1 + \cdots + Sg_r.$$  

Hence the $\mathbb{Z}_{\geq 0}$-graded $S$-module $M = R/(Sg_1 + \cdots + Sg_r)$ satisfies $S_+M = M$. It follows from a variant of Nakayama’s lemma (see, for example, Eisenbud [8, Exercise 4.6]) that $M = 0$ and hence $R = Sg_1 + \cdots + Sg_r$. □

**Lemma 4.5.** Let $K$ be a field, let $u, w$ be two pairs of positive integers, and let $\phi : \mathbb{P}(w)_K \to \mathbb{P}(u)_K$ be a nonconstant representable morphism. Then $\phi$ is finite.

**Proof.** Since $\mathbb{P}(w)_K$ and $\mathbb{P}(u)_K$ are Deligne–Mumford stacks and $\phi$ is representable, we may choose a Cartesian diagram

$$\begin{array}{ccc}
T & \phi' & S \\
\downarrow & & \downarrow \\
\mathbb{P}(w)_K & \phi & \mathbb{P}(u)_K
\end{array}$$

where $S$ and $T$ are algebraic spaces and the vertical maps are étale coverings. Then $\phi'$ is proper because $\phi$ is proper, and is locally quasi-finite because $\phi'$ has relative dimension 0 [18, tag 04NV].
In particular, $\phi'$ is representable in schemes [18, tag 0418] and is finite [18, tag 0A4X]. It follows that $\phi$ is finite.

Remark 4.6. Alternatively, Lemma 4.5 may be proved using Lemma 4.4.

Corollary 4.7. With the notation of Lemma 4.5, let $V \subseteq \mathbb{P}(w)_K$ be a dense open substack. Then $\mathbb{P}(w)_K$ is the integral closure of $\mathbb{P}(u)_K$ in $V$.

Proof. By Lemma 4.5, the morphism $\phi$ is finite and in particular integral. Furthermore, $\mathbb{P}(w)_K$ is normal because $K[x_0, x_1]$ is integrally closed. This proves the claim.

5 SOME RESULTS ON SCALING IDEALS

Let $K$ be a number field. We prove two elementary results about scaling ideals.

Lemma 5.1. Let $w = (w_0, w_1)$ be a pair of positive integers. We consider $K[x_0, x_1]$ as a graded $K$-algebra by assigning weight $w_i$ to $x_i$. Let $f \in K[x_0, x_1]$ be homogeneous of degree $d$. Let $\mathfrak{a}(f)$ be the fractional ideal generated by the coefficients of $f$. Then for all $z \in K^2$, we have

$$f(z) \in \mathfrak{a}(f) I_w(z)^d.$$  

Proof. We abbreviate

$$m = I_w(z),$$

so we have $z_0 \in m^{w_0}$ and $z_1 \in m^{w_1}$. We write

$$f = \sum_{k_0, k_1} a_{k_0, k_1} x_0^{k_0} x_1^{k_1}$$

where the sum ranges over all pairs $(k_0, k_1)$ of nonnegative integers such that $k_0 w_0 + k_1 w_1 = d$, and $a_{k_0, k_1} \in K$. We now compute

$$f(z_0, z_1) = \sum_{k_0, k_1} a_{k_0, k_1} z_0^{k_0} z_1^{k_1}$$

$$\in \sum_{k_0, k_1} a_{k_0, k_1} (m^{w_0})^{k_0} (m^{w_1})^{k_1}$$

$$= \sum_{k_0, k_1} a_{k_0, k_1} m^d$$

$$= \mathfrak{a}(f) m^d,$$

which proves the claim.
Lemma 5.2. Let \( z \in K \), and let
\[
h = x^d + c_1 x^{d-1} + \cdots + c_d \in K[x]
\]
be a monic polynomial such that \( h(z) = 0 \). Suppose \( b_1, \ldots, b_d \) are fractional ideals of \( K \) such that \( c_i \in b_i \) for all \( i \). Then we have
\[
z \in I_{(1, \ldots, d)} (b_1, \ldots, b_d).
\]

Proof. If all the \( b_i \) are zero, then \( z \) vanishes and the claim is trivial. Now assume not all of the \( b_i \) are zero. We write
\[
a = I_{(1, \ldots, d)} (b_1, \ldots, b_d)^{-1} = \{ a \in K \mid a b_1, a^2 b_2, \ldots, a^d b_d \subseteq \mathcal{O}_K \}.
\]
Then for all \( a \in a \) we have
\[
0 = a^d h(z) = (az)^d + (a c_1)(az)^{d-1} + \cdots + (a^d c_d).
\]
By assumption, each \( a^i c_i \) lies in \( a^i b_i \) and hence in \( \mathcal{O}_K \). This shows that \( az \) is integral over \( \mathcal{O}_K \). Thus we have \( az \subseteq \mathcal{O}_K \) and hence \( z \in a^{-1} \).

6 | BEHAVIOUR OF SIZE FUNCTIONS UNDER MORPHISMS

Let \( K \) be a number field. Let \( w = (w_0, w_1) \) and \( u = (u_0, u_1) \) be two pairs of positive integers, and let \( \phi : \mathbb{P}(w)_K \rightarrow \mathbb{P}(u)_K \) be a nonconstant morphism. Our goal in this section will be to study how the size of a point in \( \mathbb{P}(w)(K) \) relates to the size of its image under \( \phi \).

By Lemma 4.1, the morphism \( \phi \) is defined by a pair of nonconstant homogeneous polynomials \( f_0, f_1 \in K[x_0, x_1] \) of degrees \( eu_0 \) and \( eu_1 \), respectively, where \( e \) is the reduced degree of \( \phi \). For \( i \in \{0, 1\} \), let \( a_i \) be the fractional ideal generated by the coefficients of \( f_i \).

Lemma 6.1. For all \( z \in K^2 \), we have
\[
I_u(f(z)) \subseteq I_u(a_0, a_1) I_u(z)^e.
\]

Proof. We abbreviate
\[
m = I_u(z).
\]
Since \( f_i \) is homogeneous of degree \( eu_i \), Lemma 5.1 gives
\[
f_i(z) \in a_i m^{eu_i}.
\]
It follows that
\[
I_u(f(z)) \subseteq I_u(a_0 m^{eu_0}, a_1 m^{eu_1}) = I_u(a_0, a_1) m^e,
\]
which proves the claim.
For \( i \in \{0,1\} \), we write the rational number \( w_i/e \) in reduced form as

\[
\frac{w_i}{e} = \frac{\nu_i}{\delta_i}
\]

with \( \nu_i, \delta_i \) coprime positive integers.

By Lemma 4.4, there are integers \( d_i > 0 \) and polynomials \( g_{i,j} \in K[y_0, y_1] \) (for \( i = 0,1 \) and \( j = 1,\ldots,d_i \)) satisfying

\[
x_i^{d_i} + g_{i,1}(f_0, f_1)x_i^{d_i-1} + \cdots + g_{i,d_i}(f_0, f_1) = 0 \quad \text{in } K[x_0, x_1]. \tag{6.1}
\]

After taking homogeneous components of degree \( d_i w_i \), we may and do assume that each \( g_{i,j}(f_0, f_1) \) is homogeneous of degree \( jw_i \). After dividing by a power of \( x_i \) if necessary, we may and do also assume \( g_{i,d_i} \neq 0 \). We write

\[
g_{i,j} = \sum_{k_0, k_1 \geq 0; e(k_0u_0 + k_1u_1) = jw_i} \gamma_{i,j,(k_0,k_1)} y_0^{k_0} y_1^{k_1} \quad \text{with } \gamma_{i,j,(k_0,k_1)} \in K.
\]

In particular, if \( g_{i,j} \neq 0 \), then \( e \) divides \( jw_i \), so \( j \) is a multiple of the denominator of \( w_i/e \); in other words, there is a positive integer \( l \) with \( j = lw_i \). Since we have ensured that \( g_{i,d_i} \) is nonzero, we obtain in particular a positive integer \( m_i \) with

\[
d_i = m_i \delta_i,
\]

and all \( j \) for which \( g_{i,j} \) does not vanish are of the form \( j = lw_i \) with \( 1 \leq l \leq m_i \). We can therefore rewrite (6.1) as

\[
x_i^{m_i \delta_i} + \sum_{l=1}^{m_i} g_{i,l \delta_i}(f_0, f_1)x_i^{(m_i-l)\delta_i} = 0 \quad \text{in } K[x_0, x_1] \tag{6.2}
\]

and note that

\[
g_{i,l \delta_i} = \sum_{k_0, k_1 \geq 0; k_0u_0 + k_1u_1 = lw_i} \gamma_{i,l \delta_i,(k_0,k_1)} y_0^{k_0} y_1^{k_1}.
\]

For \( i \in \{0,1\} \) and \( 1 \leq l \leq m_i \), we write \( c_{i,l} \) for the fractional ideal generated by the coefficients of \( g_{i,l \delta_i} \), that is,

\[
c_{i,l} = (\gamma_{i,l \delta_i,(k_0,k_1)} \mid k_0, k_1 \geq 0, k_0u_0 + k_1u_1 = lw_i).
\]

For \( i \in \{0,1\} \), we write

\[
d_i = I_{(1,\ldots,m_i)}(c_{i,1}, \ldots, c_{i,m_i}).
\]
Lemma 6.2. For all \( z \in K^2 \) and \( i \in \{0, 1\} \), we have
\[
z^\delta_i \in \mathfrak{d}_i I_u(f(z))^{\nu_i}.
\]

Proof. For \( i = 0, 1 \) and \( l = 0, \ldots, m_i \), we write
\[
c_{i,l} = g_{i,l} \delta_i(f(z)) \in K.
\]
Substituting \((x_0, x_1) = (z_0, z_1)\) in (6.2), we obtain
\[
(z^\delta_i)^{m_i} + \sum_{l=1}^{m_i} c_{i,l} (z^\delta_i)^{m_i-l} = 0 \quad \text{for } i = 0, 1.
\]
We abbreviate
\[
m = I_u(f(z)).
\]
Since \( g_{i,l} \delta_j \) is homogeneous of degree \( l\nu_i \), Lemma 5.1 gives
\[
c_{i,l} \in \mathfrak{c}_{i,l} m^{l\nu_i}.
\]
Applying Lemma 5.2, we obtain
\[
z^\delta_i \in I_{(1, \ldots, m_i)}(c_{i,1} m^{\nu_1}, \ldots, c_{i,m_i} m^{m_i\nu_i}) \quad \text{for } i = 0, 1.
\]
This last ideal equals \( I_{(1, \ldots, m_i)}(c_{i,1}, \ldots, c_{i,m_i}) m^{\nu_i} = \mathfrak{d}_i m^{\nu_i} \). \( \square \)

Corollary 6.3. For all \((z_0, z_1) \in K^2 \) and \( i \in \{0, 1\} \), we have
\[
I_{(\nu_0, \nu_1)}(z^\delta_0, z^\delta_1) \subseteq I_{(\nu_0, \nu_1)}(\mathfrak{d}_0, \mathfrak{d}_1) I_u(f(z)).
\]

Theorem 6.4. Let \( K \) be a number field, let \( u, w \) be two pairs of positive integers, and let \( \phi : \mathbb{P}(w)_K \to \mathbb{P}(u)_K \) be a nonconstant morphism. Let \( e \) be the reduced degree of \( \phi \) (see Definition 4.2), and for \( i = 0, 1 \) write \( w_i/e = \nu_i/\delta_i \) with \( \nu_i, \delta_i \) coprime positive integers. Then for all \( z \in \mathbb{P}(w)(K) \), we have
\[
S_u(\phi(z)) \ll S_w(z)^e
\]
and
\[
S_u(\phi(z)) \gg S_{(\nu_0, \nu_1)}(z^\delta_0, z^\delta_1),
\]
where the implied constants depend only on \( K, u, w \) and \( \phi \).

Proof. Lemma 4.1 gives us homogeneous polynomials \( f_0, f_1 \in K[x_0, x_1] \) such that \( \phi \) is defined by \((f_0, f_1)\). For every Archimedean place \( v \) of \( K \), the set \( \mathbb{P}(w)(K_v) \) of points of \( \mathbb{P}(w) \) over the
completion $K_v$ of $K$ at $v$ is in a natural way a compact topological space. We consider the function

$$q_v : \mathbb{P}(w)(K_v) \rightarrow \mathbb{R}_{>0}$$

$$z \mapsto \max_{0 \leq i \leq 1} |f_i(z)|_v^{1/u_i} \max_{0 \leq i \leq 1} |z_i|^{e/w_i}. $$

Using the definitions of the size functions and the $q_v$, we compute

$$S_u(\phi(z)) = \frac{N(I_u(f(z)))^{-1} H_{u,\infty}(f(z))}{N(I_w(z))^{-e} H_{w,\infty}(z)^e} = N(I_w(z)^e I_u(f(z))^{-1}) \prod_{v \in \Omega_{K,\infty}} q_v(z)$$

and

$$S_u(\phi(z)) = \frac{N(I_u(f(z)))^{-1} H_{u,\infty}(f(z))}{N(I_{(\nu_0,\nu_1)}(z_0^\delta, z_1^\delta))^{-1} H_{(\nu_0,\nu_1),\infty}(z_0^\delta, z_1^\delta)} = N(I_{(\nu_0,\nu_1)}(z_0^\delta, z_1^\delta) I_u(f(z))^{-1}) \prod_{v \in \Omega_{K,\infty}} q_v(z).$$

Let $a_i, b_i$ ($i = 0, 1$) be the fractional ideals defined earlier. By Lemma 6.1, we have

$$I_w(z)^e I_u(f(z))^{-1} \supseteq I_u(a_0, a_1)^{-1},$$

and hence

$$N(I_w(z)^e I_u(f(z))^{-1}) \leq N(I_u(a_0, a_1))^{-1}.$$  

By Corollary 6.3, we have

$$I_{(\nu_0,\nu_1)}(z_0^\delta, z_1^\delta) I_u(f(z))^{-1} \subseteq I_{(\nu_0,\nu_1)}(b_0, b_1),$$

and hence

$$N(I_{(\nu_0,\nu_1)}(z_0^\delta, z_1^\delta) I_u(f(z))^{-1}) \geq N(I_{(\nu_0,\nu_1)}(b_0, b_1)).$$

Finally, for each $v \in \Omega_{K,\infty}$, the function $q_v : \mathbb{P}(w)(K_v) \rightarrow \mathbb{R}_{>0}$ is bounded by compactness. From this the theorem follows. 

\[\square\]

**Corollary 6.5.** In the setting of Theorem 6.4, suppose $e = 1$ or $w = (1,1)$ holds. Then for all $z \in \mathbb{P}(w)(K)$, we have

$$S_u(\phi(z)) \asymp S_w(z)^e,$$

where the implied constants depend only on $K, u, w$ and $\phi$. 

Proof. First suppose \( e = 1 \). Then we have \( \delta_i = 1 \) and \( \nu_i = w_i \) for \( i \in \{0, 1\} \), and hence

\[
S_{\nu_0, \nu_1}(z_0^{\delta_0}, z_1^{\delta_1}) = S_w(z) = S_w(z)^e.
\]

Next suppose \( w = (1, 1) \). Then we have \( \delta_i = e \) and \( \nu_i = 1 \) for \( i \in \{0, 1\} \), and hence

\[
S_{(\nu_0, \nu_1)}(z_0^{\delta_0}, z_1^{\delta_1}) = S_{(1, 1)}(z_0^e, z_1^e) = S_{(1, 1)}(z_0, z_1)^e = S_w(z)^e.
\]

In both cases, Theorem 6.4 gives the result. \( \square \)

Remark 6.6. The condition ‘\( e = 1 \) or \( w = (1, 1) \)’ in Corollary 6.5 is reminiscent of the condition ‘\( n = 1 \) or \( m = 1 \)’ in [11, Proposition 2.1].

Remark 6.7. By Remark 4.3, the assumption \( e = 1 \) or \( w = (1, 1) \) implies that every morphism satisfying the conditions of Corollary 6.5 is representable. However, the conclusion of Corollary 6.5 no longer holds when ‘\( e = 1 \) or \( w = (1, 1) \)’ is weakened to ‘\( \phi \) is representable’. For example, take \( u = (1, 3) \) and \( w = (1, 3) \), and consider the morphism

\[
\phi : \mathbb{P}(1, 3) \rightarrow \mathbb{P}(1, 3)
\]

\[
(x_0, x_1) \mapsto (x_0^2, x_1^2),
\]

which has \( e = 2 \) and is therefore representable. For all primes \( p \), taking \( x = (p, p^2) \in \mathbb{P}(1, 3)(\mathbb{Q}) \), we get

\[
S_u(x) = S_{(1, 3)}(p, p^2) = p,
\]

\[
S_u(\phi(x)) = S_{(1, 3)}(p^2, p^4) = S_{(1, 3)}(p, p) = p.
\]

On the other hand, for all primes \( p \), taking \( x = (1, p) \in \mathbb{P}(1, 3)(\mathbb{Q}) \), we get

\[
S_u(x) = S_{(1, 3)}(1, p) = p^{1/3},
\]

\[
S_u(\phi(x)) = S_{(1, 3)}(1, p^2) = p^{2/3}.
\]

This shows that the ratio between \( S_u(\phi(x)) \) and any fixed power of \( S_u(x) \) is unbounded as \( x \) varies.

7 | POINTS OF BOUNDED SIZE ON MODULAR CURVES

Let \( Y(1) \) be the moduli stack over \( \mathbb{Q} \) of elliptic curves. There is an open immersion

\[
i : Y(1) \hookrightarrow \mathbb{P}(4, 6)_{\mathbb{Q}}
\]

defined as follows: given an elliptic curve \( E \) over a \( \mathbb{Q} \)-scheme \( S \), then Zariski locally on \( S \) we can choose a nonzero differential \( \omega \) and define

\[
i(E) = (c_4(E, \omega), c_6(E, \omega)),
\]
where \( c_4 \) and \( c_6 \) are defined in the usual way. A different choice of \( \omega \) gives the same point of \( \mathbb{P}(4,6)_{\mathbb{Q}} \), so \( \iota \) is well defined.

**Definition 7.1.** Let \( K \) be a number field. Using the morphism \( \iota \), we define the size function

\[
S_K : Y(1)(K) \rightarrow \mathbb{R}_{>0}
\]

as the composition

\[
Y(1)(K) \xrightarrow{\iota(K)} \mathbb{P}(4,6)(K) \xrightarrow{S_{(4,6),K}} \mathbb{R}_{>0}.
\]

**Remark 7.2.** If \( E \) is given in short Weierstrass form as

\[
E : y^2 = x^3 + ax + b,
\]

then we have

\[
\iota(E) = (-48a, -864b)
\]

and hence

\[
S_K(E) = S_{(4,6),K}(-48a, -864b) \asymp \max\{|a|^{1/4}, |b|^{1/6}\}.
\]

This shows that if \( E \) is an elliptic curve over \( \mathbb{Q} \), then the ratio between \( S_{\mathbb{Q}}(E)^{12} \) and the height of \( E \)

\[
h(E) = \max\{|a|^3, |b|^2\},
\]

as defined in [11], is bounded from above and below by a constant.

Now let \( n \) be a positive integer, and let \( G \) be a subgroup of \( \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \). Let \( K_G \) be the subfield of the cyclotomic field \( \mathbb{Q}(\zeta_n) \) fixed by \( G \), where \( G \) acts on \( \mathbb{Q}(\zeta_n) \) by \( (g, \zeta_n) \mapsto \zeta_n^{\det g} \). Let \( Y_G \) be the moduli stack of elliptic curves with \( G \)-level structure, viewed as an algebraic stack over \( K_G \). There is a canonical morphism of stacks

\[
\pi_G : Y_G \rightarrow Y(1)_{K_G}.
\]

Let \( K \) be a finite extension of \( K_G \). We define

\[
\mathcal{E}_{G,K} = \{ \text{elliptic curves admitting a } G\text{-level structure over } K \}/\cong
\]

and

\[
N_{G,K}(X) = \# \{ E \in \mathcal{E}_{G,K} \mid S_K(E)^{12} \leq X \}.
\]
**Lemma 7.3.** Let \( n \) be a positive integer, let \( G \) be a subgroup of \( \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \), and let \( w \) be a pair of positive integers. The following are equivalent.

(i) There is a commutative diagram

\[
\begin{array}{ccc}
Y_G & \xrightarrow{\iota_G} & \mathbb{P}(w)_{K_G} \\
\downarrow{\kappa_G} & & \downarrow{\phi} \\
Y(1)_{K_G} & \xrightarrow{i} & \mathbb{P}(4,6)_{K_G}
\end{array}
\]

of algebraic stacks over \( K_G \), where \( \iota_G \) is an open immersion and \( \phi \) is representable.

(ii) The integral closure of \( X(1) = \mathbb{P}(4,6) \) in the function field of \( Y_G \) is isomorphic to \( \mathbb{P}(w) \).

(iii) The moduli space of generalized elliptic curves with \( G \)-level structure is isomorphic to \( \mathbb{P}(w) \).

**Proof.** The equivalence of (ii) and (iii) follows from the fact that the integral closure from (ii) is canonically isomorphic to the moduli space of generalized elliptic curves with \( G \)-level structure [6, IV, Théorème 6.7(ii)].

The implication (ii) \( \implies \) (i) follows from the fact that the integral closure of \( X(1) \) in the function field of \( Y_G \) fits in a commutative diagram as above.

The implication (i) \( \implies \) (ii) follows from Corollary 4.7 applied to \( V = \iota_G(Y_G) \). \( \square \)

**Remark 7.4.** If \( G \) is a group satisfying the equivalent conditions of Lemma 7.3, then the coarse moduli space of \( X_G \) is isomorphic to \( \mathbb{P}^1 \). The converse does not hold. For example, taking \( G \) to be the group of upper-triangular matrices in \( \text{GL}_2(\mathbb{Z}/3\mathbb{Z}) \) gives the modular curve \( X_G = X_0(3) \). The coarse moduli space of \( X_0(3) \) is isomorphic to \( \mathbb{P}^1 \), but \( X_0(3) \) itself is not a weighted projective line. One way to see this is to note that the Picard group of a weighted projective line is infinite cyclic, generated by the class of the tautological line bundle [10, Example 7.27], but considering dimensions of spaces of global sections shows that the line bundle of modular forms on \( X_0(3) \) cannot be identified with any power of the tautological bundle on a weighted projective line.

**Remark 7.5.** The equivalent conditions of Lemma 7.3 hold if the graded \( K_G \)-algebra of modular forms for \( G \) is generated by two homogeneous elements. Over \( \mathbb{C} \), the groups for which this happens were classified by Bannai, Koike, Munemasa and Sekiguchi [1].

**Theorem 7.6.** Let \( n \) be a positive integer, and let \( G \) be a subgroup of \( \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \). Let \( K_G \) be the fixed field of the action of \( G \) on \( \mathbb{Q}(\zeta_n) \) given by \((g, \zeta_n) \mapsto \zeta_n^{\det g}\). Assume that \( G \) satisfies the equivalent conditions of Lemma 7.3 for some \((w_0, w_1)\), and let \( e(G) \) be the reduced degree of the canonical morphism \( X_G \to X(1) \) (see Definition 4.2). Furthermore, assume \( e(G) = 1 \) or \( w = (1,1) \) holds. Then for every finite extension \( K \) of \( K_G \), we have

\[
N_{G,K}(X) \approx X^{1/d(G)} \quad \text{as } X \to \infty,
\]

where

\[
d(G) = \frac{12e(G)}{w_0 + w_1}.
\]
Proof. Using the commutative diagram of Lemma 7.3 and noting that for counting purposes we may ignore the cusps (cf. [7, Remark 6.2]), we obtain

\[ N_{G,K}(X) = \#\{z \in \mathbb{P}(w)(K) \mid S_{(4,6)}(\phi(z))^{12} \leq X\}. \]

By Corollary 6.5 with \( u = (4, 6) \), the quotient \( S_{(4,6)}(\phi(z))/S_w(z)^e \) is bounded. This implies

\[ N_{G,K}(X) \leq \#\{z \in \mathbb{P}(w)(K) \mid S_w(z) \leq X^{1/(12e(G))}\}. \]

Applying Theorem 3.8, we obtain

\[ N_{G,K}(X) \leq X^{(w_0 + w_1)/(12e(G))}. \]

This proves the claim. \( \square \)

8 | EXAMPLES

The groups corresponding to the 15 torsion groups from Mazur’s theorem satisfy the conditions of Lemma 7.3. In Table 1, we list these groups and a few more satisfying these conditions.

For positive integers \( m \mid n \), we write

\[ G(m,n) = \left\{ g \in \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \mid g = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \text{ and } g \equiv \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \pmod{m} \right\}. \]

We also put

\[ G_1(n) = G(1, n) \]

and

\[ G_0(n) = \left\{ g \in \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \mid g = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \right\}. \]

For each group \( G \), we give its inverse image \( \Gamma \) under the canonical group homomorphism \( \text{SL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \), the index of \( \Gamma \) in \( \text{SL}_2(\mathbb{Z}) \), the weights of the corresponding weighted projective line, and the values \( e(G) \) and \( d(G) \). The first 12 groups can also be found in [13, Examples 2.1 and Example 2.5], and the 12 groups with \( e(G) = 1 \) can also be found in [1, Table 1]. By construction, for all groups \( G \) in the table, the determinant \( G \to (\mathbb{Z}/n\mathbb{Z})^\times \) is surjective, hence the index \( [\text{GL}_2(\mathbb{Z}/n\mathbb{Z}) : G] \) equals \( [\text{SL}_2(\mathbb{Z}) : \Gamma] \), and \( K_G \) equals \( \mathbb{Q} \). Furthermore, we note that the numbers \( e(G) \) and \( d(G) \) can be expressed as

\[ e(G) = \frac{w_0w_1}{24}[\text{SL}_2(\mathbb{Z}) : \Gamma], \]

\[ d(G) = \frac{w_0w_1}{2(w_0 + w_1)}[\text{SL}_2(\mathbb{Z}) : \Gamma]. \]
The first 15 groups are those appearing in Mazur’s theorem.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\Gamma$</th>
<th>$[\text{SL}_2(Z) : \Gamma]$</th>
<th>$(w_0, w_1)$</th>
<th>$e(G)$</th>
<th>$d(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1(1)$</td>
<td>$\Gamma_1(1) = \text{SL}_2(Z)$</td>
<td>1</td>
<td>(4,6)</td>
<td>1</td>
<td>6/5</td>
</tr>
<tr>
<td>$G_1(2)$</td>
<td>$\Gamma_1(2) = \Gamma_0(2)$</td>
<td>3</td>
<td>(2,4)</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$G_1(3)$</td>
<td>$\Gamma_1(3)$</td>
<td>8</td>
<td>(1,3)</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$G_1(4)$</td>
<td>$\Gamma_1(4)$</td>
<td>12</td>
<td>(1,2)</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$G_1(5)$</td>
<td>$\Gamma_1(5)$</td>
<td>24</td>
<td>(1,1)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$G_1(6)$</td>
<td>$\Gamma_1(6)$</td>
<td>24</td>
<td>(1,1)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$G_1(7)$</td>
<td>$\Gamma_1(7)$</td>
<td>48</td>
<td>(1,1)</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$G_1(8)$</td>
<td>$\Gamma_1(8)$</td>
<td>48</td>
<td>(1,1)</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$G_1(9)$</td>
<td>$\Gamma_1(9)$</td>
<td>72</td>
<td>(1,1)</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>$G_1(10)$</td>
<td>$\Gamma_1(10)$</td>
<td>72</td>
<td>(1,1)</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>$G_1(12)$</td>
<td>$\Gamma_1(12)$</td>
<td>96</td>
<td>(1,1)</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>$G_2(2)$</td>
<td>$\Gamma_2(2)$</td>
<td>6</td>
<td>(2,2)</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$G_2(4)$</td>
<td>$\Gamma_2(4)$</td>
<td>24</td>
<td>(1,1)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$G_2(6)$</td>
<td>$\Gamma_2(6)$</td>
<td>48</td>
<td>(1,1)</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$G_2(8)$</td>
<td>$\Gamma_2(8)$</td>
<td>96</td>
<td>(1,1)</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>$G_3(4)$</td>
<td>$\Gamma_3(4)$</td>
<td>6</td>
<td>(2,2)</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$G_4(4)$</td>
<td>$\Gamma_4(4)$</td>
<td>48</td>
<td>(1,1)</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$G_0(8) \cap G_1(4)$</td>
<td>$\Gamma_0(8) \cap \Gamma_1(4)$</td>
<td>24</td>
<td>(1,1)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$G_3(3)$</td>
<td>$\Gamma_3(3)$</td>
<td>24</td>
<td>(1,1)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$G_3(6)$</td>
<td>$\Gamma_3(6)$</td>
<td>72</td>
<td>(1,1)</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>$G_3(9) \cap G_1(3)$</td>
<td>$\Gamma_3(9) \cap \Gamma_1(3)$</td>
<td>24</td>
<td>(1,1)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$G_5(5)$</td>
<td>$\Gamma_5(5)$</td>
<td>120</td>
<td>(1,1)</td>
<td>5</td>
<td>30</td>
</tr>
</tbody>
</table>

9 | FUTURE WORK

In work of Manterola Ayala and the first author (see [12]), results are proved that make it possible to count points of a moduli stack of the form $\mathbb{P}(w)$ directly with respect to the pull-back of the size function from $X(1)$, rather than first relating this pull-back to the standard size function on $\mathbb{P}(w)$. This approach requires extending the work of Deng [7], but is conceptually simpler than the approach we have taken here. A similar result has been proved independently by Phillips [15, Theorem 1.2.2].

Phillips has also obtained a result similar to Theorem 7.6 for moduli stacks of elliptic curves that are of the form [15, Theorem 5.1.4]. An example of such a moduli stack is $X_0(6)$, so this result enables one to count elliptic curves with a 6-isogeny over any number field.

APPENDIX A: MORPHISMS BETWEEN QUOTIENT STACKS

In this appendix, we assume some knowledge of stacks. We place ourselves in the following situation. Let $S$ be a scheme, let $G$ and $H$ be two group schemes over $S$, and let $m_G : G \times_S G \to G$ and $m_H : H \times_S H \to H$ be the group operations. Let $X$ and $Y$ be two $S$-schemes, let $a : G \times_S X \to X$ be a left action of $G$ on $X$, and let $b : H \times_S Y \to Y$ be a left action of $H$ on $Y$. Let $p_2 : G \times_S X \to X$
be the second projection, and let \( p_{2,3} : G \times_S G \times_S X \to G \times_S X \) be the projection onto the second and third factors.

We consider the quotient stacks \([G \backslash X]\) and \([H \backslash Y]\) over (the \(fpf\) site of) \( S\), writing quotients on the left because \( a \) and \( b \) are left actions. Below we give an explicit description of the groupoid of morphisms \([G \backslash X] \to [H \backslash Y]\) of stacks over \( S\). For this, we will use the following description of morphisms from the quotient stack \([G \backslash X]\) to another stack \( \mathcal{Y} \) given by Noohi [14, Proposition 3.19]; see also [18, tag 044U] for part of this statement.

**Lemma A.1.** Let \( \mathcal{Y} \) be a stack in groupoids over \( S\), and let \( C([G \backslash X], \mathcal{Y}) \) be the following groupoid. The objects are the pairs \((f, h)\) where \( f : X \to \mathcal{Y} \) is a morphism of stacks and \( h \) is a descent datum for \( f \), that is, an isomorphism \( h : f \circ p_2 \sim \to f \circ a \) of functors \( G \times_S X \to \mathcal{Y} \) satisfying

\[
(m_G \times \text{id}_X)^* h = (\text{id}_G \times a)^* h \circ p_{2,3}^* h.
\]

The morphisms from \((f, h)\) to \((f', h')\) are the isomorphisms \( c : f \sim \to f' \) of functors \( X \to \mathcal{Y} \) satisfying

\[
a^* c \circ h = h' \circ p_2^* c.
\]

Then the groupoid of morphisms \([G \backslash X] \to \mathcal{Y}\) is canonically equivalent to \( C([G \backslash X], \mathcal{Y})\).

To state the next lemma, we recall the following. Given a left action of a group \( \Gamma \) on a set \( Z\), the quotient groupoid \( \Gamma \backslash \backslash Z \) is the following groupoid: the set of objects is \( Z\), the morphisms \( z \to z'\) are the elements \( \gamma \in \Gamma \) with \( \gamma z = z'\), and composition of morphisms is the group operation in \( \Gamma\). The set of isomorphism classes of \( \Gamma \backslash \backslash Z \) is just the quotient set \( \Gamma \backslash Z\).

**Lemma A.2.** In the above situation, assume in addition that all \( H \)-torsors on \( X\) are trivial. Let \( Z \) be the set of pairs \((f : X \to Y, h : G \times_S X \to H)\) of morphisms of \( S\)-schemes such that for all \( S\)-schemes \( T\), all \( x \in X(T)\) and all \( g, g' \in G(T)\) we have

\[
h(g', g, x) = h(g', gx)h(g, x)
\]

and

\[
f(a(g, x)) = b(h(g, x), f(x)).
\]

Let the group \( H(X) \) act on \( Z\) by

\[
(c, (f, h)) \mapsto (f', h'),
\]

where \( f' \) and \( h' \) are defined on points as follows: for all \( S\)-schemes \( T\), all \( x \in X(T)\) and all \( g \in G(T)\), we have

\[
f'(x) = b(c(x), f(x))
\]
and
\[ h'(g, x) = c(a(g, x))h(g, x)c(x)^{-1}. \]

Then the groupoid of morphisms \([G \backslash X] \to [H \backslash Y]\) is canonically equivalent to the quotient groupoid \(H(X) \backslash \backslash Z\). In particular, there is a canonical bijection between the set of isomorphism classes of such morphisms and the quotient set \(H(X) \backslash Z\).

**Proof.** We apply Lemma A.1 with \(Y = [H \backslash Y]\). Because all \(H\)-torsors on \(X\) are trivial, the groupoid of morphisms \(X \to [H \backslash Y]\) is canonically equivalent to the groupoid \(D(X, [H \backslash Y])\) defined as follows: the objects of \(D(X, [H \backslash Y])\) are the morphisms \(f : X \to Y\) of schemes, and the isomorphisms \(f \sim f'\) in \(D(X, [H \backslash Y])\) are the elements \(c \in H(X)\) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & Y \\
\downarrow{(c, f)} & & \\
H \times_S Y
\end{array}
\]

is commutative. Similarly, the isomorphisms \(f \circ p_2 \sim f \circ \alpha\) in the groupoid of morphisms \(G \times_S X \to [H \backslash Y]\) correspond to the elements \(h \in H(G \times_S X)\) such that the diagram

\[
\begin{array}{ccc}
G \times_S X & \xrightarrow{f \circ \alpha} & X \\
\downarrow{(h, f \circ p_2)} & & \\
H \times_S Y
\end{array}
\]

is commutative. Furthermore, such an \(h\) is a descent datum for \(f\) if and only if the diagram

\[
\begin{array}{ccc}
G \times_S G \times_S X & \xrightarrow{m_0 \times 0\times d_X} & G \times_S X \\
\downarrow{(id_X \times 0 \times p_2)} & & \\
(G \times_S X) \times_S (G \times_S X) & \xrightarrow{h \circ h} & H \times_S H
\end{array}
\]

is commutative. On \(T\)-valued points, the commutativity of the last two diagrams comes down to (A.2) and (A.1), respectively, so the objects of \(C([G \backslash X], [H \backslash Y])\) correspond to the elements of \(Z\). The isomorphisms \((f, h) \sim (f', h')\) in \(C([G \backslash X], [H \backslash Y])\) correspond to the elements \(c \in H(X)\) as above such that in addition the diagram

\[
\begin{array}{ccc}
G \times_S X & \xrightarrow{(c \circ a, h)} & H \times_S H \\
\downarrow{(0 \circ f \circ p_2)} & & \\
H \times_S H & \xrightarrow{m_0} & H
\end{array}
\]

is commutative. Equivalently, these isomorphisms correspond to the elements \(c \in H(X)\) sending \((f, h)\) to \((f', h')\) under the given action of \(H(X)\) on \(Z\).
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