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Explicit computation of the height of a Gross-Schoen Cycle

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Citation

Wang, R. (2022, October 18). *Explicit computation of the height of a Gross-Schoen Cycle*. Retrieved from <https://hdl.handle.net/1887/3480346>

Version: Publisher's Version

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Downloaded from: <https://hdl.handle.net/1887/3480346>

Note: To cite this publication please use the final published version (if applicable).

Chapter 5

Appendix

I Solving cubic equations

Given a cubic equation

$$ay^3 + by^2 + cy + d = 0,$$

we solve it in following way.

Algorithm 4 Solutions of a cubic equation

Input: a, b, c, d : coefficients of the polynomial

i : the index of expected solution, taking value in $\{1, 2, 3\}$

Output: the i -th solution of the polynomial

1: $r_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

2: $u = \frac{9abc - 27a^2d - 2b^3}{54a^3}$

3: $v = \frac{(12ac^3 - 3b^2c^2 - 54abcd + 81a^2d^2 + 12b^3d)^{1/2}}{18a^2}$

4: **if** $|u + v| \geq |u - v|$ **then**

5: $m = (u + v)^{(1/3)}$

6: **else**

7: $m = (u - v)^{(1/3)}$

8: **end if**

9: **if** $m = 0$ **then**

10: $n = 0$

11: **else**

12: $n = \frac{b^2 - 3ac}{9am}$

13: **end if**

14: **return** $r_3^{i-1}m + r_3^{2i-2}n - \frac{b}{3a}$

When we say the *1st* root of a cubic function, we mean the output of the algorithm above when $i = 1$, similarly for other indices i . The Magma code for our curve $\mathcal{C}_{\mathbb{C}}$ is given as follows:

```

1      a:=1;
      b:=x^2-x;
3      c:=-x^3;
      d:=x^2+x;
5      cubic_unit:=-1/2+3^(1/2)/2*I;
      root:=function(u,v,i);
7      numa:=a;
      numb:=Evaluate(b,[u+v*I,0]);
9      numc:=Evaluate(c,[u+v*I,0]);
      preuu:=(9*a*b*c-27*a^2*d-2*b^3)/(54*a^3);
11     uu:=Evaluate(preeu,[u+v*I,0]);
      prevv:=Evaluate((3*(4*a*c^3-b^2*c^2-18*a*b*c*d+27*a^2*d^2+4*b^3*d)),[u+
      v*I,0]);
13     vv:=(prevv^(1/2))/(18*a^3);
      m:=(uu-vv)^(1/3);
15     if Abs(uu+vv) ge Abs(uu-vv) then
      m:=(uu+vv)^(1/3);
17     end if;
      n:=0;
19     if m ne 0 then
      n:=(numb^2-3*numa*numc)/(9*numa*m);
21     end if;
      result:=(cubic_unit^(i-1))*m+n*cubic_unit^(2*i-2)-(numb)/(3*numa);
23     return result;
      end function;

```

In Chapter 4, we need to integrate some functions defined on a plane quartic curve. It turns out that, for these integrations, we need to label the different y 's for any fixed x in the way above.

II List of Weierstrass points

The set of Weierstrass points of $\mathfrak{C}_{\mathbb{C}}$ is:

$$\begin{aligned} \mathfrak{W} = \{ & \\ & (0.0000000000, 0.0000000000), \\ & (-2.104587156, -0.3159811808), \\ & (-1.241974125, -1.824965133), \\ & (0.3539677313, -0.7304015296), \\ & (-1.959274470, -1.410535295), \\ & (-1.0000000000, 0.0000000000), \\ & (-1.455757161 - 1.277524807 * I, -0.5258271579 - 3.390009274 * I), \\ & (-1.455757161 + 1.277524807 * I, -0.5258271579 + 3.390009274 * I), \\ & (1.266455202 - 1.314861880 * I, 0.1352071928 + 2.966892635 * I), \\ & (1.266455202 + 1.314861880 * I, 0.1352071928 - 2.966892635 * I), \\ & (0.2089717154 - 1.597843531 * I, 0.1450907026 - 2.034168656 * I), \\ & (0.2089717154 + 1.597843531 * I, 0.1450907026 + 2.034168656 * I), \\ & (-1.319681529 - 0.4985373090 * I, -0.4698939284 - 0.6348619710 * I), \\ & (-1.319681529 + 0.4985373090 * I, -0.4698939284 + 0.6348619710 * I), \\ & (1.691125343 - 0.6842628628 * I, 0.6312556990 + 0.6046227212 * I), \\ & (1.691125343 + 0.6842628628 * I, 0.6312556990 - 0.6046227212 * I), \\ & (-1.125631934 - 0.2207052168 * I, -0.6596966755 + 0.07228789093 * I), \\ & (-1.125631934 + 0.2207052168 * I, -0.6596966755 - 0.07228789093 * I), \\ & (-0.5282412488 - 1.005235134 * I, -0.4177679369 + 0.3210649478 * I), \\ & (-0.5282412488 + 1.005235134 * I, -0.4177679369 - 0.3210649478 * I), \\ & (-0.3349732907 - 0.4139797966 * I, -0.4206274002 - 0.9204898342 * I), \\ & (-0.3349732907 + 0.4139797966 * I, -0.4206274002 + 0.9204898342 * I), \\ & (0.07366689191 - 0.3721976724 * I, 0.7232010036 + 0.2545999915 * I), \\ & (0.07366689191 + 0.3721976724 * I, 0.7232010036 - 0.2545999915 * I) \} \end{aligned}$$

III Code for canonical form

```

1      from sage.schemes.riemann_surfaces.riemann_surface import
        RiemannSurface
2      R.<x,y> = QQ[]
3      f = -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x

5      S = RiemannSurface(f)
        M = S.period_matrix()
7      Omega1 = M[[0,1,2],[0,1,2]]
        Omega2 = M[[0,1,2],[3,4,5]]
9      A = Omega1.apply_map(real)
        B = Omega1.apply_map(imag)
11     tau = S.riemann_matrix()
        Omega1bar= A-B*I
13     Imtau = tau.apply_map(imag)
        RieSecBil = Omega1bar*Imtau*Omega1.transpose()
15     RieSecBil
        [ 8.14058999836226 - 4.44089209850063e-16*I -2.71111012021317 -
          9.21762666195036e-14*I -5.21995559097122 - 5.85642645489770e-14*I
          ]
17     [-2.71111012021315 + 6.52811138479592e-14*I  8.66278158093292 +
          8.32667268468867e-14*I  5.65493670273040 + 1.56541446472147e-13*I
          ]
        [-5.21995559097118 + 1.45994327738208e-14*I  5.65493670273037 -
          1.75831571525009e-14*I  10.7557438794561 + 1.18932641512970e-14*I
          ]

19     w11 = RieSecBil[0][0]
21     w1y = RieSecBil[0][1]
        w1x = RieSecBil[0][2]
23     wyy = RieSecBil[1][1]
        wyx = RieSecBil[1][2]
25     wxx = RieSecBil[2][2]
        k = (wyx-w1x*w1y/w11)/(wyy-w1y*w1y/w11)
27     norm1 = RieSecBil[0][0]**(1/2)
        norm2 = (wyy-(2*w1y*w1y/w11)+w1y*w1y/w11)**(1/2)
29     norm3 = (wxx+w1x*w1x/w11+k*k*wyy+k*k*w1y*w1y/w11-2*w1x*w1x/w11-2*k*
          wyx+2*k*w1y*w1x/w11+2*k*w1x*w1y/w11-2*k*w1x*w1y/w11-2*k*k*w1y*w1y
          /w11)**(1/2)

```

Gram-Schmidt process

In the code, the symbol w_{mn} denotes the integration of 1-1 forms, for example, w_{1y} represents $\frac{i}{2} \int_X \omega_1 \wedge \bar{\omega}_y$. The Gram-Schmidt process is actually done by hand, and we just use SageMath for evaluation.

IV Code for semistability

```

1  from collections import defaultdict
   PP.<x,y,z> = ProjectiveSpace(QQ,2)
3  Poly.<U,V> = PolynomialRing(ZZ, 2)
   RR.<X, Y, Z> = PolynomialRing(ZZ, 3)
5  C = Curve(-x^3*y+x^2*y^2-x*y^2*z+y^3*z+x^2*z^2+x*z^3, PP)
   inj1 = Poly.hom([X, Y], RR)
7
9  def IsDegree1(I):    #checking the ideal I has degree 1 generators
   out = True
11  for f in I.gens():
   if not f.degree() == 1:
13  out = False
   return out
15
17  def MyBadPrimes(C):    #finding out primes with bad reduction
   f = C.defined_polynomial()
19  RZ.<xZ,yZ,zZ> = PolynomialRing(ZZ, 3)
   coeffs = f.coefficients()
21  dens = [c.denominator() for c in coeffs]
   den = lcm(dens)
23  F = RZ(f*den)
   Fx = F.derivative(xZ)
25  Fy = F.derivative(yZ)
   Fz = F.derivative(zZ)
27  NaiveDisc = 1
   for P in [[xZ,yZ,1],[xZ,1,zZ],[1,yZ,zZ]]:
29  I = ideal([g(P) for g in [F,Fx,Fy,Fz]])
   G = I.groebner_basis()
31  n = G[len(G)-1]
   NaiveDisc = lcm(n, NaiveDisc)
33  return [a[0] for a in factor(NaiveDisc)]
35
37  def MyBadPoints(C, default_prime):    #CORE FUNCTION: finding out
   all singular points over certain prime number and the tangent
   lines
   f = C.defined_polynomial()
39  RZ.<xZ,yZ,zZ> = PolynomialRing(ZZ, 3)
   coeffs = f.coefficients()
   dens = [c.denominator() for c in coeffs]
41  den = lcm(dens)
   F = RZ(f*den)

```

```

43   Fx = F.derivative(xZ)
44   Fy = F.derivative(yZ)
45   Fz = F.derivative(zZ)
46   Out = ['Dimension of singular locus is positive.']
47   GG = [] #list of associate primes (after field
         extension)
         sing_set = [] #list of degree 2 parts of singular points
         after transition
49   factor_sing = [] #list of factorizations of degree 2 parts
         degree = 1
51   SingDim = 0
         found = False
53   while found == False:
         Rp = RZ.change_ring(GF(default_prime^degree))
55   win = True
         S.<u, v> = PolynomialRing(GF(default_prime^degree), 2)
57   for P in [ [u, v, 1], [u, 1, v], [1, u, v] ]:
         affine_patch_map = RZ.hom(P, S)
59   I_2 = Ideal(S,[ affine_patch_map(g) for g in [F, Fx, Fy, Fz]])
         G_2 = I_2.associated_primes()
61   for I in G_2:
         VV = I.gens()
63   if Set(VV).cardinality() == 1:
         SingDim = 1
65   if SingDim == 1:
         break
67   for I in G_2:
         if IsDegree1(I) == False:
69   win = False #This means that the singular points don't have
         coordinates in this base field and we still need to extend it
         if SingDim == 1:
71   break
         Out = []
73   if win == True:
         for P in [ [u, v, 1], [u, 1, v], [1, u, v] ]:
75   affine_patch_map = RZ.hom(P, S)
         I_2 = Ideal(S,[affine_patch_map(g) for g in [F, Fx, Fy, Fz]])
77   G_2 = I_2.associated_primes()
         GG += G_2
79   for AP in G_2: #in this for loop, we find out the coordinate
         of singular points
         T.<x, y> = S.quotient(ideal(S, [w for w in AP.gens()]))
81   Quot = T.cover()
         if Quot(1) == 0:
83   continue
         for m in GF(default_prime^degree):
85   if Quot(u-m)==0:

```

```

SingX = m
87  break
    for n in GF(default_prime^degree):
89  if Quot(v-n)==0:
SingY = n
91  break
    Trans = S.hom([u+SingX, v+SingY])
93  change_to_S = RZ.hom(P, S)
    TranP = Trans(change_to_S(F))    #our curve after transition
95  homog_part = defaultdict(TranP.parent())    #the following 4 lines
        help us to get the degree 2 part
    for coeff, monom in TranP:
97  homog_part[monom.degree()] += coeff * monom
    sing_set.append(homog_part[2])
99  found = True
    Out += GG
101  for w in sing_set:    #find out the singularity type
    SS = S.change_ring(GF(default_prime^(2*degree)))
103  if w!=0:
    factor_sing.append(factor(SS(w)))
105  else:
    factor_sing.append('HIGHER_SINGULARITY')
107  else: degree += 1
    return Out, factor_sing

```


V Code for semi-canonical divisor

In Magma, a base point P_{bs} is already chosen and $AbelJacobi(D)$ means $AJ(D-\deg(D)[P_{bs}])$.

```

PeriodMatrix:=Matrix([
2  [-0.0756487726827658975 + 0.82850793518670070913*I,
4  0.48656585589140586329 - 0.07531944436044814614*I,
6  0.53238504944901350223 + 0.23143384347205016268*I],
8  [0.48656585589140586329 - 0.07531944436044814614*I,
10 0.500584921111822623585 + 0.48874080037323974253*I,
   0.137880100924435244693 - 0.05703730011758514006*I],
   [0.53238504944901350223 + 0.23143384347205016268*I,
    0.137880100924435244693 - 0.05703730011758514006*I,
    0.689717878372433609278 + 0.69264774350470313589*I]])

```

```

<I> := ComplexField(30);
2  Qxy<x, y> := PolynomialRing(Rationals(), 2);
4  f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x; //defining polynomial
6  X := RiemannSurface(f); //create the riemann surface
8  tau:=SmallPeriodMatrix(X);

10 Pt := X ! [0,0]; // a point on X
12 Div:=Divisor([Pt],[2]); //effective divisor of degree 2
14 Div2:=Divisor([BasePoint(X)],[2]);
16 Div3:=Divisor([BasePoint(X),Pt],[1,1]);

18 asd := InfinitePoints(X);
20 U:=[asd[1], asd[2], asd[3]];
22 V:=[2,1,1];
24 CanDiv:=Divisor(U,V); //canonical divisor

26 MatrixIm := function(M) //function to take the imaginary part of a
   matrix.
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

Columns := function(M) //function to extract columns as a list of
   matrices

```

```

28   out := [];
    for i in [1..Ncols(M)] do
30   out := out cat [ColumnSubmatrix(M, i, 1)];
    end for;
32   return out;
    end function;

34

    char:=Matrix([[0],[0],[0],[0],[0],[0]]);
36   Imtau:=MatrixIm(tau);
    detimtau:=Determinant(Imtau);
38   LeftPeriod:=RemoveColumn(RemoveColumn(RemoveColumn(BigPeriodMatrix(X)
        ,4),4),4);
    ID3 := DiagonalMatrix(C, [1,1,1]);
40   columnlist := Columns(ID3) cat Columns(tau);
    AJCD := Matrix(AbelJacobi(CanDiv)); //the image of canonical
        divisor under AJ map

42

    AJ_2_torsion := [];
44   for v in CartesianPower([0,1], 6) do
    new := ZeroMatrix(C, 3, 1);
46   for i in [1..6] do
    new := new + v[i] * columnlist[i];
48   end for;
    AJ_2_torsion := AJ_2_torsion cat [(1/2) * new];
50   end for;

52   translates_to_try := []; //this will be a list of all 64 square
        roots of CanDiv
    for V in AJ_2_torsion do
54   translates_to_try := translates_to_try cat [(1/2) * AJCD + V];
    end for;

56

    torus_theta_function := function(D)// D is a divisor of degree g-1
58   myinput := Matrix(AbelJacobi(D));
    myabsttheta:=Abs(Theta(char, myinput, tau));
60   myinput:= MatrixIm(myinput);
    myexponent:=-3.141592653589*Transpose(myinput)*(Imtau^(-1))*
        myinput;
62   return Exp(myexponent[1,1])*myabsttheta*detimtau^(1/4);
    end function;

64

    find_correct_theta_translate := function(D) //find out the correct
        translation
66   P := AbelJacobi(D);
    out := [];
68   for V in translates_to_try do
    myinput := Matrix(P) - V;

```

```
70   myabsttheta:=Abs(Theta(char , myinput , tau));
      myiminput:= MatrixIm(myinput);
72   myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^(-1))*
      myiminput;
      if Abs(Exp(myexponent[1,1])*myabsttheta*detimtau^(1/4)) le 0.00000001
      then
74     return V;
      end if;
76   end for;
      end function;

78
      //we check the result by choosing 3 different effective divisors
80   ThetaTranslate1 := find_correct_theta_translate(Div);
      ThetaTranslate1;
82   ThetaTranslate2 := find_correct_theta_translate(Div2);
      ThetaTranslate2;
84   ThetaTranslate3 := find_correct_theta_translate(Div3);
      ThetaTranslate3;
```

VI Code for $\|\chi'_{18}\|_{\text{Hdg}}$

```

1      C<I> := ComplexField(50);
      Qxy<x, y> := PolynomialRing(Rationals(), 2);
3      f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x; //defining polynomial
      X := RiemannSurface(f); //create the riemann surface
5
      tau1:=SmallPeriodMatrix(X); //period matrix implemented in Magma
7      tau2:=Matrix(C,3,[ 0.855638485763810 + 1.03033263075936*I,
      0.380495294693576 + 0.00436338826915927*I, 0.506949568788359 -
      0.178115589372895*I,
      0.380495294693584 + 0.00436338826915617*I , 0.401072147476828 +
      0.694848752465586*I , -0.580339267402228 - 0.147446853346675*I ,
9      0.506949568788374 - 0.178115589372884*I, -0.580339267402254 -
      0.147446853346687*I, 0.219761917471621 + 0.625482641303850*I
      ]); //period matrix implemented in SageMath
11
      //the following set S contains all the even characteristics
13      S:=[Matrix(RationalField(),1,[0,0,0,0,0,0]),Matrix(RationalField()
      ,1,[0,0,0,0,0,1]),Matrix(RationalField(),1,[0,0,0,0,1,0]),Matrix(
      RationalField(),1,[0,0,0,0,1,1]),Matrix(RationalField()
      ,1,[0,0,0,1,0,0]),Matrix(RationalField(),1,[0,0,0,1,0,1]),Matrix(
      RationalField(),1,[0,0,0,1,1,0]),Matrix(RationalField()
      ,1,[0,0,0,1,1,1]),Matrix(RationalField(),1,[0,0,1,0,0,0]),Matrix(
      RationalField(),1,[0,0,1,1,0,0]),Matrix(RationalField(),1,[0,0,1,1,1,0]),Matrix(
      RationalField(),1,[0,1,0,0,0,0]),Matrix(RationalField()
      ,1,[0,1,0,0,0,1]),Matrix(RationalField(),1,[0,1,0,1,0,0]),Matrix(
      RationalField(),1,[0,1,0,1,0,1]),Matrix(RationalField()
      ,1,[0,1,1,0,0,0]),Matrix(RationalField(),1,[0,1,1,1,0,0]),Matrix(
      RationalField(),1,[0,1,1,1,1,1]),Matrix(RationalField()
      ,1,[0,1,1,0,1,1]),Matrix(RationalField(),1,[1,0,0,0,0,0]),Matrix(
      RationalField(),1,[1,0,0,0,0,1]),Matrix(RationalField()
      ,1,[1,0,0,0,1,0]),Matrix(RationalField(),1,[1,0,0,0,1,1]),Matrix(
      RationalField(),1,[1,0,1,0,0,0]),Matrix(RationalField()
      ,1,[1,0,1,0,1,0]),Matrix(RationalField(),1,[1,0,1,1,0,1]),Matrix(
      RationalField(),1,[1,0,1,1,1,1]),Matrix(RationalField()
      ,1,[1,1,0,0,0,0]),Matrix(RationalField(),1,[1,1,0,0,0,1]),Matrix(
      RationalField(),1,[1,1,0,1,1,0]),Matrix(RationalField()
      ,1,[1,1,0,1,1,1]),Matrix(RationalField(),1,[1,1,1,0,0,0]),Matrix(
      RationalField(),1,[1,1,1,0,1,1]),Matrix(RationalField()
      ,1,[1,1,1,1,1,0]),Matrix(RationalField(),1,[1,1,1,1,0,1])];
15
      MatrixIm := function(M) //imaginary part of a matrix
      N := ZeroMatrix(C, Nrows(M), Ncols(M));
17      for i in [1..Nrows(M)] do

```

```
19   for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
21   end for;
return N;
23   end function;

25   compute:=function(period) //compute ||theta||
chitilde:=1;
27   zerovector:=Matrix(C,1,[0,0,0]);
for ele in S do
29   chitilde:=chitilde*Theta((1/2)*ele,zerovector,period);
end for;
31   abschitilde:=Abs(chitilde);
Imtau:=MatrixIm(period);
33   det:=Determinant(Imtau);
prepreresult:=Log((det^9)*abschitilde);
35   preresult:=26*Log(2)+54*Log(Pi(C))+prepreresult;
result:=-((21/18)*preresult); //contribution of \chi_18 at the
infinite place
37   return result;
end function;

39   //we compute the result using two period matrices implemented in
Magma and SageMath
41   compute(tau1);
compute(tau2);
```

VII Code for the integration part in $\log(S(\mathcal{C}_{\mathbb{C}}))$

```

C<I> := ComplexField(30);
2 Cxy<x, y> := PolynomialRing(RationalField(), 2);
f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x; //defining polynomial
4 X := RiemannSurface(f : Precision:=30); //create riemann surface
tau:=SmallPeriodMatrix(X);
6 BP:=BasePoint(X);

8 MatrixIm := function(M) //take the imaginary part of a matrix
N := ZeroMatrix(C, Nrows(M), Ncols(M));
10 for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
12 N[i, j] := Imaginary(M[i, j]);
end for;
14 end for;
return N;
16 end function;

18 Imtau:=MatrixIm(tau);
detimtau:=Determinant(Imtau);
20 ThetaTranslate:=Matrix([[0.479250542651680186758281300774 -
0.00334176833187451614116524746349*I],
[0.698684877508432322290935180841 + 0.199495723882563563098727841525*
22 I],
[0.00722266620787249384535143960054 -
0.0430102069343208149623250436805*I]]);

24 normalizedtheta := function(D); //compute ||theta||_{g-1} at D
P := AbelJacobi(D);
26 myinput := Matrix(P) - ThetaTranslate;
char:=Matrix([[0],[0],[0],[0],[0],[0]]);
28 myabsttheta:=Abs(Theta(char, myinput, tau));
myiminput:= MatrixIm(myinput);
30 myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^-1)*myiminput
;
return Abs(Exp(myexponent[1, 1])*myabsttheta*detimtau^(1/4));
32 end function;

34 a:=1;
b:=x^2-x;
36 c:=-x^3;
d:=x^2+x;
38 cubic_unit:=-1/2+3^(1/2)/2*I;

```

```

40   root:=function(u,v,i);    //solve the cubic equation with
      coefficients a, b, c and d.
      numa:=a;
42   numb:=Evaluate(b,[u+v*I,0]);
      numc:=Evaluate(c,[u+v*I,0]);
44   preuu:=(9*a*b*c-27*a^2*d-2*b^3)/(54*a^3);
      uu:=Evaluate(preeu,[u+v*I,0]);
46   prevv:=Evaluate((3*(4*a*c^3-b^2*c^2-18*a*b*c*d+27*a^2*d^2+4*b^3*d)),[
      u+v*I,0]);
      vv:=(prevv^(1/2))/(18*a^2);
48   m:=(uu-vv)^(1/3);
      if Abs(uu+vv) ge Abs(uu-vv) then
50   m:=(uu+vv)^(1/3);
      end if;
52   n:=0;
      if (Abs(m) ge 0.00001) then
54   n:=(numb^2-3*numa*numc)/(9*numa*m);
      else n:=0;
56   end if;
      result:=(cubic_unit^(i-1))*m+n*cubic_unit^(2*i-2)-(numb)/(3*numa);
58   return result;
      end function;

60   embedding:=function(u,v,i);    //find out the embedding index in
      Magma
62   zz,pt1:=IsPoint(X,<[u,v],1>);
      zz,pt2:=IsPoint(X,<[u,v],2>);
64   zz,pt3:=IsPoint(X,<[u,v],3>);
      if (Abs(Coordinates(pt1)[2]-root(u,v,i)) le 0.0001) then result:=1;
66   elif (Abs(Coordinates(pt2)[2]-root(u,v,i)) le 0.0001) then result:=2;
      elif (Abs(Coordinates(pt3)[2]-root(u,v,i)) le 0.0001) then result:=3;
68   end if;
      return result;
70   end function;

72   theta_wrt_yi:=function(u,v,i);    //for a fixed point BP, evaluate
      theta_{g-1}(gP-Q) at the point Q=(Rex=u, Imx=v, index=i)
      zz, pt_in_cover_i:=IsPoint(X,<[u,v],embedding(u,v,i)>);
74   targetdivisor:=Divisor([BP,pt_in_cover_i],[3,-1]);
      return normalizedtheta(targetdivisor);
76   end function;

78   omegal := function(u,v,i)    // first differential form, i denotes
      index of y
      u:=u+0.000001;
80   v:=v+0.000001;
      x:=u+v*I;

```

```

82     y:=root(u,v,i);
      return 1/(-x^3+2*x^2*y-2*x*y+3*y^2);
84     end function;

86     omegax := function(u,v,i)    //second differential form, i denotes
      index of y
      u:=u+0.000001;
88     v:=v+0.000001;
      x:=u+v*I;
90     y:=root(u,v,i);
      return x/(-x^3+2*x^2*y-2*x*y+3*y^2);
92     end function;

94     omegay := function(u,v,i)    //third differential form, i denotes
      index of y
      u:=u+0.000001;
96     v:=v+0.000001;
      x:=u+v*I;
98     y:=root(u,v,i);
      return y/(-x^3+2*x^2*y-2*x*y+3*y^2);
100    end function;

102    canonicalform:=function(u,v,i)    //the volume form,i denotes index
      of y
      ort_nor_w1:=0.350487116953118*omegal(u,v,i);
104    ort_nor_wy:=(0.358981759779085*omegay(u,v,i)+0.119553875346235*omegal
      (u,v,i));
      ort_nor_wx:=(0.203008239643111*omegal(u,v,i)-0.216555180015011*omegay
      (u,v,i)+0.429067210690657*omegax(u,v,i));
106    result:=(ort_nor_w1*Conjugate(ort_nor_w1)+ort_nor_wy*Conjugate(
      ort_nor_wy)+ort_nor_wx*Conjugate(ort_nor_wx))/3;
      return result;
108    end function;

110    ff:=function(u,v)    //compute three i in one time
      result:= Log(theta_wrt_yi(u,v,1))*(canonicalform(u,v,1))+Log(
      theta_wrt_yi(u,v,2))*(canonicalform(u,v,2))+Log(theta_wrt_yi(u,v
      ,3))*(canonicalform(u,v,3));
112    return result;
      end function;

114
116    SX:=0;    //take Riemann sum
      scale:=0.01;
      rex_start:=-10;
118    imx_start:=-10;
      for p in [1..200] do
120    for q in [1..2000] do

```



```
122     SX:=SX+ff(p*scale+rex_start+0.005,q*scale+imx_start+0.005)*scale*  
124         scale*(-9);  
     end for ;  
     end for ;  
     SXn10:=SX;  
     SXn10;
```

We can change the *rex_start* and *imx_start* in the code to get Riemann sums for a selected area. The edge length of small squares is *scale*. We choose *scale* = 0.01 when $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$, and choose *scale* = 0.1 for other region in $-50 \leq x \leq 50$ and $-50 \leq y \leq 50$.

In this code, we only compute the Riemann sums for points with coordinates in $-10 \leq x \leq -8$ and $-10 \leq y \leq 10$ (we choose *scale* = 0.01). We need to change *rex_start* and *imx_start* manually to get the Riemann sums of other region.

The reason of doing this is that we can split the computation into small pieces, so that we can parallelly compute them in Magma. This can reduce the computing time significantly. The codes above takes around 40 hours. If we carry out all the computation ($-50 \leq x \leq 50$ and $-50 \leq y \leq 50$) in one time, it will take more than 600 hours. We also use this trick in the computation of $H(\mathfrak{C}_{\mathbb{C}})$.

VIII Code for $T(X)$

```

1      C<I> := ComplexField(50);    //we need high precision here, since
      components of T(X) can be very small
      Cxy<x, y> := PolynomialRing(RationalField(), 2);
3      f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;    //defining polynomial
      X := RiemannSurface(f : Precision:=15);    //create riemann surface
5      tau:=SmallPeriodMatrix(X);

7      MatrixIm := function(M)    //function to take the imaginary part of a
      matrix with entries in the field C defined just above.
      N := ZeroMatrix(C, Nrows(M), Ncols(M));
9      for i in [1..Nrows(M)] do
      for j in [1..Ncols(M)] do
11     N[i, j] := Imaginary(M[i, j]);
      end for;
13     end for;
      return N;
15     end function;

17     Imtau:=MatrixIm(tau);
      detimtau:=Determinant(Imtau);
19     ThetaTranslate:=Matrix([[0.479250542651680186758281300774 -
      0.00334176833187451614116524746349*I],
      [0.698684877508432322290935180841 + 0.199495723882563563098727841525*
21     I],
      [0.00722266620787249384535143960054 -
      0.0430102069343208149623250436805*I]]);

23     normalizedtheta := function(D);    // ||theta ||_{g-1}
      P := AbelJacobi(D);
25     myinput := Matrix(P) - ThetaTranslate;
      char:=Matrix([[0],[0],[0],[0],[0],[0]]);
27     myabsttheta:=Abs(Theta(char, myinput, tau));
      myiminput:= MatrixIm(myinput);
29     myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^-1)*myiminput
      ;
      return Abs(Exp(myexponent[1,1])*myabsttheta*detimtau^(1/4));
31     end function;

33     a:=1;    //the following lines are the coefficient of the defining
      polynomial in y
      b:=x^2-x;
35     c:=-x^3;
      d:=x^2+x;
37     cubic_unit:=-1/2+3^(1/2)/2*I;

```

```

39   root:=function(u,v,i); //solve equation with coefficients a,b,c,d
      numa:=a;
41   numb:=Evaluate(b,[u+v*I,0]);
      numc:=Evaluate(c,[u+v*I,0]);
43   preuu:=(9*a*b*c-27*a^2*d-2*b^3)/(54*a^3);
      uu:=Evaluate(preeu,[u+v*I,0]);
45   prevv:=Evaluate((3*(4*a*c^3-b^2*c^2-18*a*b*c*d+27*a^2*d^2+4*b^3*d)),[
      u+v*I,0]);
      vv:=(prevv^(1/2))/(18*a^2);
47   m:=(uu-vv)^(1/3);
      if Abs(uu+vv) ge Abs(uu-vv) then
49   m:=(uu+vv)^(1/3);
      end if;
51   n:=0;
      if m ne 0 then
53   n:=(numb^2-3*numa*numc)/(9*numa*m);
      end if;
55   result:=(cubic_unit^(i-1))*m+n*cubic_unit^(2*i-2)-(numb)/(3*numa);
      return result;
57   end function;

59   wronskisquare:=function(x,y)
      fy:=-x^3+2*x^2*y-2*x*y+3*y^2; //partial derivative of defining
      polynomial wrt y
61   dy:=(3*x^2*y-2*x*y^2+y^2-2*x-1)/(-x^3+2*x^2*y-2*x*y+3*y^2); //
      implicit derivative of y wrt x
      ddy:=(6*x*y+3*x^2*dy+3*x^2*dy-2*y^2-4*x*y*dy-4*x*y*dy-2*x^2*dy^2+2*y*
      dy+2*y*dy+2*x*dy^2-6*y*dy^2-2)/(-x^3+2*x^2*y-2*x*y+3*y^2); //
      second derivative
63   dfy:=-3*x^2+4*x*y+2*x^2*dy-2*y-2*x*dy+6*y*dy; //implicit
      derivative of fy wrt x
      ddfy:=-6*x+4*y+4*x*dy+4*x*dy+2*x^2*ddy-2*dy-2*dy-2*x*ddy+6*dy^2+6*y*
      ddy; //second derivative
65   wl:=1/fy; //differential forms
      wy:=y/fy;
67   wx:=x/fy;
      dwl:=-dfy/fy^2; //implicit derivative of wl wrt x
69   dwy:=(dy*fy-dfy*y)/(fy^2);
      dwx:=(fy-x*dfy)/(fy^2);
71   ddwl:=-(ddfy*fy^2-2*fy*dfy^2)/(fy^4); //second derivative
      ddwy:=((ddy*fy+dy*dfy-ddfy*y-dfy*dy)*fy^2-2*fy*dfy*(dy*fy-dfy*y))/(fy
      ^4);
73   ddwx:=((dfy-dfy-x*ddfy)*fy^2-2*fy*dfy*(fy-x*dfy))/(fy^4);
      c11:=0.350487116953118; //orthogonalizing matrix
75   c21:=0.119553875346235;
      c22:=0.358981759779085;

```

```

77     c31:=0.203008239643111;
       c32:=-0.216555180015011;
79     c33:=0.429067210690657;
       W11:=c11*w1;    //following lines gives the wronskian matrix wrt a
                          orthonormal basis of differential form.
81     W12:=c21*w1+c22*wy;
       W13:=c31*w1+c32*wy+c33*wz;
83     W21:=c11*dw1;
       W22:=c21*dw1+c22*dwy;
85     W23:=c31*dw1+c32*dwy+c33*dwz;
       W31:=c11*ddw1;
87     W32:=c21*ddw1+c22*ddwy;
       W33:=c31*ddw1+c32*ddwy+c33*ddwz;
89     Wronski:=Matrix ([[W11,W12,W13],[W21,W22,W23],[W31/2,W32/2,W33/2]]);
       detwro:=Abs(Determinant(Wronski));
91     return detwro^2;
       end function;

93
       embedding:=function(u,v,i);    //this gives the embedding index of
           point (u+v*I,y_i) on \cC implemented in Magma
95     zz,pt1:=IsPoint(X,<[u,v],1>);
       zz,pt2:=IsPoint(X,<[u,v],2>);
97     zz,pt3:=IsPoint(X,<[u,v],3>);
       if (Abs(Coordinates(pt1)[2]-root(u,v,i)) le 0.01) then result:=1;
99     elif (Abs(Coordinates(pt2)[2]-root(u,v,i)) le 0.01) then result:=2;
       elif (Abs(Coordinates(pt3)[2]-root(u,v,i)) le 0.01) then result:=3;
101    end if;
       return result;
103    end function;

105    WeierPts_coor:=[[ 0, 0], [ -2.104587155963303, -0.3159811807558051],
        [ (-1.455757161115023 - 1.277524806650578*I),
          (-0.5258271579143416 - 3.390009274518869*I)], [
          (-1.455757161115023 + 1.277524806650578*I), (-0.5258271579143403
          + 3.390009274518867*I)], [ -1.95927446954141,
          -1.410535295372148], [ (1.266455202041324 - 1.31486188027894*I),
          (0.1352071928093522 + 2.966892634578606*I)], [ (1.266455202041324
          + 1.31486188027894*I), (0.1352071928093521 - 2.966892634578606*I
          )], [ (0.2089717154013823 - 1.59784353074737*I),
          (0.1450907026008794 - 2.034168656250672*I)], [
          (0.2089717154013823 + 1.59784353074737*I), (0.1450907026008798 +
          2.034168656250671*I)], [ (-1.31968152920268 - 0.4985373089678906*
          I), (-0.4698939283630282 - 0.634861971042462*I)], [
          (-1.31968152920268 + 0.4985373089678906*I), (-0.4698939283630282
          + 0.6348619710424618*I)], [ (1.691125343336006 -
          0.6842628628388769*I), (0.6312556990319607 + 0.6046227212351708*I
          )], [ (1.691125343336006 + 0.6842628628388769*I),

```

```

(0.6312556990319608 - 0.6046227212351708*I)], [
(-1.125631933711411 - 0.2207052167744392*I), (-0.6596966755302974
+ 0.07228789092962729*I)], [ (-1.125631933711411 +
0.2207052167744392*I), (-0.6596966755302974 -
0.07228789092962723*I)], [ (-0.5282412487764151 -
1.005235133687909*I), (-0.4177679368816311 + 0.3210649477940537*I
)], [ (-0.5282412487764151 + 1.005235133687909*I),
(-0.4177679368816311 - 0.3210649477940537*I)], [
-1.241974125539051, -1.824965132496513], [ 0.3539677313137965,
-0.7304015296367112], [ (-0.3349732907033383 -
0.4139797966480703*I), (-0.4206274001987688 - 0.9204898341402288*
I)], [ (-0.3349732907033383 + 0.4139797966480703*I),
(-0.4206274001987688 + 0.9204898341402289*I)], [
(0.0736668919074069 - 0.3721976723848182*I), (0.7232010036495048
+ 0.2545999914748455*I)], [ (0.0736668919074069 +
0.3721976723848182*I), (0.7232010036495049 - 0.2545999914748455*I
)], [ -1, 0]];
WeierPts:=[];
107
for i in WeierPts_coor do //make a lst of Wpoints from coordinates
109 zz,pt1:=IsPoint(X,<[Re(i[1]),Im(i[1])],1>);
zz,pt2:=IsPoint(X,<[Re(i[1]),Im(i[1])],2>);
111 zz,pt3:=IsPoint(X,<[Re(i[1]),Im(i[1])],3>);
if (Abs(Coordinates(pt1)[2]-i[2]) le 0.0001) then embed:=1;
113 elif (Abs(Coordinates(pt2)[2]-i[2]) le 0.0001) then embed:=2;
elif (Abs(Coordinates(pt3)[2]-i[2]) le 0.0001) then embed:=3;
115 end if;
zz, wpt:=IsPoint(X,<[Re(i[1]),Im(i[1])],embed>);
117 WeierPts:=Append(WeierPts,wpt); //the list of W-points
end for;
119
Fz:=function(P,n,dire) //Fz with n the 'distance' scale and dire
the relative direction
121
for i in [1,2,3] do
123 rootQ:=root(Re(Coordinates(P)[1]),Im(Coordinates(P)[1]),i);
if (Abs(rootQ-Coordinates(P)[2]) le 0.01) then //since Q is close
to P, their y-coord are close.
125 rinde:=i; //the index of the y-coordinate for Q
end if;
127 end for;
129
P_x_r:=Real(Coordinates(P)[1]);
P_x_i:=Imaginary(Coordinates(P)[1]);
131 Q_x_r:=P_x_r+dire[1]*10^(-n); //real part of the x coordinate of Q
Q_x_i:=P_x_i+dire[2]*10^(-n);
133

```

```

135   einde:=embedding(Q_x_r,Q_x_i,rinde);
zz,Q:=IsPoint(X,<[Q_x_r,Q_x_i],einde>);
137   gPQ:=Divisor([P,Q],[3,-1]);
distPQ:=Abs(10^(-n)*(dire[1]+dire[2]*I));

139   result:=normalizedtheta(gPQ)/(distPQ^3);
return result;
141   end function;

143   TX:=function(P,n,dire) //T(X)
brick1:=1;
145   for ele in WeierPts do //theta part
WDIV:=Divisor([P,ele],[3,-1]);
147   brick1:=brick1*Abs(normalizedtheta(WDIV)^(2/27));
end for;
149   brick2:=Fz(P,n,dire)^(-4);
brick3:=wronskisquare(Coordinates(P)[1],Coordinates(P)[2]);
151   result:=brick1*brick2*brick3;
return result;
153   end function;

155   avWe:=[0.1,0.1]; //avoid Weierstrass point
dire:=[-1,0]; //initializing direction
157   deltapq:=6; //initializing the 'distance' scale between P and Q
zz,PointP:=IsPoint(X,<[1+avWe[1],0+avWe[2]],1>); //initializing
the point P
159   TX(PointP,deltapq,dire)

```

IX Code for $H(X)$

```

1 C<I> := ComplexField(30);
2 Cxy<x, y> := PolynomialRing(RationalField(), 2);
3 f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x; //defining polynomial
4 X := RiemannSurface(f : Precision:=30); //riemann surface
5 tau:=SmallPeriodMatrix(X);
6 Imtau:=MatrixIm(tau);
7 detimtau:=Determinant(Imtau);

8
9 //the following rows are rows of the big period matrix
10 tau1:=[-0.0756487726827658975623807266149
11 ,0.828507935186700709139824389302,
12 0.486565855891405863290312449179 , -
13 0.0753194443604481461426310259273,
14 0.532385049449013502234878708914 , 0.231433843472050162680353039649];
15 tau2:=[0.486565855891405863290312449179, -
16 0.0753194443604481461426310259273,
17 0.500584921111822623585859133424, + 0.488740800373239742539460301357,
18 0.137880100924435244693062999773, -
19 0.0570373001175851400662331352196];
20 tau3:=[0.532385049449013502234878708914, +
21 0.231433843472050162680353039649,
22 0.137880100924435244693062999773, -
23 0.0570373001175851400662331352196,
24 0.689717878372433609278715552080, +
25 0.692647743504703135897856925333];
26
27 MatrixIm := function(M) //take the imaginary part of a matrix
28 N := ZeroMatrix(C, Nrows(M), Ncols(M));
29 for i in [1..Nrows(M)] do
30 for j in [1..Ncols(M)] do
31 N[i,j] := Imaginary(M[i,j]);
32 end for;
33 end for;
34 return N;
35 end function;

36
37 rectan:=Matrix([[1.0,0.0,0.0,0.0,0.0,0.0],
38 [0.0,0.0,1.0,0.0,0.0,0.0],
39 [0.0,0.0,0.0,0.0,1.0,0.0],
40 tau1,tau2,tau3]);
41 volume:=Abs(Determinant(rectan));

42
43 TorusTheta := function(V); //the function ||theta|| on a torus
44 char:=Matrix([[0],[0],[0],[0],[0],[0]]);

```

```

37   myabsttheta:=Abs(Theta(char, V, tau));
    myexponent:=-3.141592653589*Transpose(MatrixIm(V))*(Imtau^-1)*
        MatrixIm(V);
39   return Abs(Exp(myexponent[1,1])*myabsttheta*detimtau^(1/4));
    end function;

41
42   A:=Matrix(C,3,1,[0,0,1]);
43   B:=Matrix(C,3,1,[0,1,0]);
44   C:=Matrix(C,3,1,[1,0,0]);
45   D:=Transpose(Matrix(tau[1]));
46   E:=Transpose(Matrix(tau[2]));
47   F:=Transpose(Matrix(tau[3]));

48
49   c:=19;    \\the Riemann sum
    result:=0;
50
51   for ii in [1..2] do
52     for jj in [1..c] do
53       for kk in [1..c] do
54         for ll in [1..c] do
55           for mm in [1..c] do
56             for nn in [1..c] do
57               vector:=(ii/c)*A+(jj/c)*B+(kk/c)*C+(ll/c)*D+(mm/c)*E+(nn/c)*F;
                result:=result+Log(TorusTheta(vector))/(c^6);
58             end for;
59           end for;
60         end for;
61       end for;
62     end for;
63   end for;
64 end for;
65 result;

```

The code above computes the Riemann sum for $2 \times c^5$ small polyhedrons. See the end of Appendix VII for further explanation.

X Code for Klein's formula

```

1      C<I> := ComplexField(50);
3
4      P<U,V,W>:= PolynomialRing(Rationals(), 3);
5      F:= -U^3*V+U^2*V^2-U*V^2*W+V^3*W+U^2*W^2+U*W^3;
6      Qxy<x, y>:= PolynomialRing(Rationals(), 2);
7      f:= -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x; //defining polynomial
8      X := RiemannSurface(f); //create the riemann surface
9      tau1:=SmallPeriodMatrix(X);//the period matrix
10
11     Omega1:=Matrix([[3.27458588878738559863054319936E-40 +
12         1.40623766693192062162024765467*I,
13         -1.79952501232403458891122141246 + 1.38907786473238785380622191148*I,
14         -2.53736487575268201398849418408 + 0.422954571624542218773280089228*I
15         ],[-3.32482770801607988312171989709E-40 +
16         2.21299876803863343807211504318*I,
17         0.520294432989390175634113180014 + 0.311971662365700775264258133464*I
18         ],
19         2.49655614620140646500159641736 - 1.72853703668803372011916427569*I
20         ],[-6.11781900650324543346854241722E-40 -
21         0.519248888908945152419998545118*I,
22         -0.778001857456629823002912917512 + 1.46695721833231037837976066119*I
23         ],
24         3.20106058440048591056866925446 - 1.29453940681916549786848960725*I
25         ]]);
26
27     //the following set S contains all the even characteristics
28     S:=[Matrix(RationalField(), 1, [0, 0, 0, 0, 0]), Matrix(RationalField()
29         , 1, [0, 0, 0, 0, 1]), Matrix(RationalField(), 1, [0, 0, 0, 1, 0]), Matrix(
30         RationalField(), 1, [0, 0, 0, 0, 1, 1]), Matrix(RationalField()
31         , 1, [0, 0, 0, 1, 0, 0]), Matrix(RationalField(), 1, [0, 0, 0, 1, 0, 1]), Matrix(
32         RationalField(), 1, [0, 0, 0, 1, 1, 0]), Matrix(RationalField()
33         , 1, [0, 0, 0, 1, 1, 1]), Matrix(RationalField(), 1, [0, 0, 1, 0, 0, 0]), Matrix(
34         RationalField(), 1, [0, 0, 1, 0, 1, 0]), Matrix(RationalField()
35         , 1, [0, 0, 1, 1, 0, 0]), Matrix(RationalField(), 1, [0, 0, 1, 1, 1, 0]), Matrix(
36         RationalField(), 1, [0, 1, 0, 0, 0, 0]), Matrix(RationalField()
37         , 1, [0, 1, 0, 0, 0, 1]), Matrix(RationalField(), 1, [0, 1, 0, 1, 0, 0]), Matrix(
38         RationalField(), 1, [0, 1, 0, 1, 0, 1]), Matrix(RationalField()
39         , 1, [0, 1, 1, 0, 0, 0]), Matrix(RationalField(), 1, [0, 1, 1, 1, 0, 0]), Matrix(
40         RationalField(), 1, [0, 1, 1, 1, 1, 1]), Matrix(RationalField()
41         , 1, [0, 1, 1, 0, 1, 1]), Matrix(RationalField(), 1, [1, 0, 0, 0, 0, 0]), Matrix(
42         RationalField(), 1, [1, 0, 0, 0, 0, 1]), Matrix(RationalField()
43         , 1, [1, 0, 0, 0, 1, 0]), Matrix(RationalField(), 1, [1, 0, 0, 0, 1, 1]), Matrix(
44         RationalField(), 1, [1, 0, 1, 0, 0, 0]), Matrix(RationalField()
45         , 1, [1, 0, 1, 0, 1, 0]), Matrix(RationalField(), 1, [1, 0, 1, 1, 0, 1]), Matrix(

```

```

RationalField() ,1,[1,0,1,1,1,1]),Matrix(RationalField()
,1,[1,1,0,0,0,0]),Matrix(RationalField() ,1,[1,1,0,0,0,1]),Matrix(
RationalField() ,1,[1,1,0,1,1,0]),Matrix(RationalField()
,1,[1,1,0,1,1,1]),Matrix(RationalField() ,1,[1,1,1,0,0,0]),Matrix(
RationalField() ,1,[1,1,1,0,1,1]),Matrix(RationalField()
,1,[1,1,1,1,1,0]),Matrix(RationalField() ,1,[1,1,1,1,0,1]);

21 MatrixIm := function(M) // Imaginary part of a matrix
N := ZeroMatrix(C, Nrows(M), Ncols(M));
23 for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
25 N[i,j] := Imaginary(M[i,j]);
end for;
27 end for;
return N;
29 end function;

31 Klein_ratio:=function(period)//compute the ratio of Klein formula
chitilde:=1;
33 zerovector:=Matrix(C,1,[0,0,0]);
for ele in S do
35 chitilde:=chitilde*Theta((1/2)*ele, zerovector, period);
end for;
37 left:=DiscriminantOfTernaryQuartic(F)^2;
right:=chitilde*(2*3.1415926)^(54)/(2^(28)*Determinant(Omega1)^(18));
39 return right/left;
end function;

41 Klein_ratio(tau1);

```

