Chapter 4

Explicit Computations

This chapter is devoted to explicit computations for the height $\langle \Delta, \Delta \rangle$ of the following plane curve over $\mathbb{Q}$, using SageMath and Magma. As far as we know, this is the first attempt to numerically compute $\langle \Delta, \Delta \rangle$ for a non-hyperelliptic curve of genus $g > 2$.

Main Curve : $-X^3Y + X^2Y^2 - XY^2Z + Y^3Z + X^2Z^2 + XZ^3 = 0$

We denote this plane curve over $\mathbb{Z}$ by $\mathfrak{C}$ and we use subscripts to distinguish its base changes, for example $\mathfrak{C}_{\mathbb{Z}_p}$, $\mathfrak{C}_{\mathbb{Q}}$ and so on. We denote the affine patch $Z = 1$ of $\mathfrak{C}$ by $U_{\mathfrak{C}}$. We denote the polynomial on the left hand side by $\mathfrak{F}$ and we write $\mathfrak{f}$ for $\mathfrak{F}(x, y, 1)$.

There are several reasons for choosing this curve. First, the curve $\mathfrak{C}$ is a stable model of $\mathfrak{C}_{\mathbb{Q}}$ over $\mathbb{Z}$. Second, all its residue fields at singular points are in the type $\mathbb{F}_p$ for some prime p (instead of the type $\mathbb{F}_{p^m}$ for some integer $m > 1$), which makes it easy to compute its thicknesses (Subsection 4.3). Third, it has no bad hyperelliptic reduction, thus we do not need to compute the hyperelliptic multiplicity in Corollary 4.4.4. The thicknesses and the hyperelliptic multiplicity are the main restrictions of our computation method. Other parts of our computation (like these numerical approximations in Sections 4.5-4.7) can be used for a general curve.

We sketch our plan of the computation as follows.

In Section 4.1, 4.2 and 4.3, we prove that $\mathfrak{C}_{\mathbb{Q}}$ has semistable reduction over $\mathbb{Q}$ and that $\mathfrak{C}$ is a regular stable model for it. Thus we can apply Theorem 3.3.2.

The reduction types of $\mathfrak{C}$ are summarized in Proposition 4.2.4 and Corollary 4.3.9. In Section 4.4, we show that all invariants except the infinite $\lambda(\mathfrak{C}_{\mathbb{C}})$ in Theorem 3.3.2

are computable. By Remark 1.5.7, to compute $\lambda(\mathfrak{C}_{\mathbb{C}})$ we only need to compute $\varphi(\mathfrak{C}_{\mathbb{C}})$ and $\delta(\mathfrak{C}_{\mathbb{C}})$.

In Section 4.5, we show how to evaluate the theta function $\|\theta\|_{g-1}$ on $\mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}})$. Using $\|\theta\|_{g-1}$, we numerically compute $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in Section 4.6. In Subsection 4.7.1, we compute another invariant $T(\mathfrak{C}_{\mathbb{C}})$. With $\log(S(\mathfrak{C}_{\mathbb{C}}))$ and $T(\mathfrak{C}_{\mathbb{C}})$, the invariant $\delta(\mathfrak{C}_{\mathbb{C}})$ can be computed by Theorem 4.7.3. Now, it remains to compute $\varphi(\mathfrak{C}_{\mathbb{C}})$.

By Theorem 4.7.7, we reduce the problem to the computation of $H(\mathfrak{C}_{\mathbb{C}})$. In the second half of Subsection 4.7.2, we explain the strategy for computing $H(\mathfrak{C}_{\mathbb{C}})$.

The results of our computation are summarized in Section 4.8. In Section 4.9, we explain the reliability of our results.

Longer sections of the code in this chapter can be found in Appendix I-IX.

4.1 Smoothness and bad reduction of $\mathfrak{C}$

In Subsection 4.1.1, we will show that $\mathfrak{C}_{\mathbb{Q}}$ is a smooth curve over $\mathbb{Q}$. In Subsection 4.1.2, we show $\mathfrak{C}_{\mathbb{Z}_p}$ has bad reduction at $p = 29, 163$ and good reduction at other primes.

4.1.1 Smoothness at the infinite place

By the Jacobian criterion for smoothness, we need to show that:

$$\sqrt{(\mathfrak{F}, \mathfrak{F}_X, \mathfrak{F}_Y, \mathfrak{F}_Z)} = (X, Y, Z).$$

This can be checked in SageMath by the following lines, thus $\mathfrak{C}_{\mathbb{Q}}$ is a smooth curve over $\mathbb{Q}$.

```
R.<x,y,z>=PolynomialRing(QQ)
f=-x^3*y + x^2*y^2 - x*y^2*z + y^3*z + x^2*z^2 + x*z^3
I=R.ideal(f,derivative(f,x),derivative(f,y),derivative(f,z))
I.radical()
Ideal (z, y, x) of Multivariate Polynomial Ring in x, y, z over Rational
    Field
```

4.1.2 Bad reduction at finite places

We first consider the reduction of the affine patch $U_{\mathfrak{C}}$. Since $U_{\mathfrak{C}} \simeq \mathrm{Spec}(\frac{\mathbb{Z}[x,y]}{(\mathfrak{f})})$ has smooth generic fiber, by the Jacobian criterion, the ideal $I_{\mathbb{Q}} = (\mathfrak{f}, \frac{\partial \mathfrak{f}}{\partial x}, \frac{\partial \mathfrak{f}}{\partial y})$ is the unit ideal in $\mathbb{Q}[x,y]$. This means that if we consider $I = (\mathfrak{f}, \frac{\partial \mathfrak{f}}{\partial x}, \frac{\partial \mathfrak{f}}{\partial y})$ as an ideal in $\mathbb{Z}[x,y]$, then $I \cap \mathbb{Z} = (n)$ for some positive integer n. Let p be a prime, then we can see that $p \nmid n$ if and only if the reduction of I to $\mathbb{F}_p[x,y]$ contains a unit $\overline{n} \in \mathbb{F}_p^*$ which is equivalent to saying that $U_{\mathfrak{C}}$ has good reduction at p. Thus $U_{\mathfrak{C}}$ has bad reduction exactly at the prime divisors of n.

By the above discussion, we have positive integers n_1, n_2 and n_3 for three affine patches. The curve $\mathfrak{C}$ has bad reduction exactly at the prime divisors of $n_1 n_2 n_3$. The following SageMath code can be used for computing the primes of bad reduction.

```
sage:PP.<x,y,z> = ProjectiveSpace(QQ,2)
sage:C = Curve(-x^3*y+x^2*y^2-x*y^2*z+y^3*z+x^2*z^2+x*z^3, PP)
sage:def MyBadPrimes(C):     #finding bad reduction primes
sage:  f = C.defining_polynomial()
sage:  RZ.<xZ,yZ,zZ> = PolynomialRing(ZZ, 3)
sage:  coeffs = f.coefficients()
sage:  dens = [c.denominator() for c in coeffs]
sage:  den = lcm(dens)
sage:  F = RZ(f*den)
sage:  Fx = F.derivative(xZ)
sage:  Fy = F.derivative(yZ)
sage:  Fz = F.derivative(zZ)
sage:  NaiveDisc = 1
sage:  for P in [[xZ,yZ,1],[xZ,1,zZ],[1,yZ,zZ]]:
sage:     I = ideal([g(P) for g in [F,Fx,Fy,Fz]])
sage:     G = I.groebner_basis()
sage:     n = G[len(G)-1]
sage:     NaiveDisc = lcm(n, NaiveDisc)
sage:  return [a[0] for a in factor(NaiveDisc)]
```

Bad reduction

Remark 4.1.1. *There exists an explicit formula for the discriminant of a plane quartic curve (Page 9 in [55]), and we can also find out the primes of bad reduction by factoring it. This computation is implemented in Magma (http://magma.maths.usyd.edu.au/magma/handbook/text/1547#17791).*

With the code above, we can obtain the following proposition.

Proposition 4.1.2. *The main curve $\mathfrak{C}$ has bad reduction at 29 and 163, and good reduction at every other prime.*

4.2 Semistability of $\mathfrak{C}$

In Subsection 4.2.1, we develop an algorithm for checking whether a singular point on a generically smooth plane curve over $\mathbb{Z}_p$ is a nodal singularity. In Subsection 4.2.2, we prove that $\mathfrak{C}_{\mathbb{Z}_p}$ is semistable over $p = 29$ or 163. Notions can be found in Section 1.1.

4.2.1 Algorithm for checking nodal singularities

By Definition 1.1.14, a generically smooth curve of genus $g \geq 1$ over $\mathbb{Z}_p$ is semistable if its geometric fiber at $\overline{\mathbb{F}}_p$ has only nodal singularities and all components of arithmetic genus 0 intersect other components in at least 2 points.

Let C be a generically smooth plane curve over $\mathbb{Z}$. Similar to Subsection 4.1.2, we check the nodal singularities on one affine patch at one time. We assume the curve is defined by $f(x, y) = 0$ on the affine patch $Z = 1$ of $\operatorname{Proj} \mathbb{Z}[X, Y, Z]$, and we denote the reduction of C and f at p by $C_{\mathbb{F}_p}$ and f_p. We sketch our strategy of checking nodal singularities of $C_{\mathbb{F}_p}$ as follows:

(1) We first check that the singular locus of $C_{\mathbb{F}_p}$ is 0-dimensional. It is possible that a curve over $\mathbb{Z}_p$ is smooth over $\mathbb{Q}_p$ but totally singular over $\mathbb{F}_p$, for example, the plane curve defined by $X^2 + pXY + pY^2 = 0$.

(2) If (1) is true, then we find out the coordinates of singular points in $C_{\overline{\mathbb{F}}_p}$. Since we start from base field $\mathbb{F}_p$, we will extend it to a field $\mathbb{F}_{p^D}$ such that all singular points have coordinates in $\mathbb{F}_{p^D}$.

(3) Fixing an arbitrary singular point of $C_{\overline{\mathbb{F}}_p}$, for example $p_s = (x_0, y_0)$, we translate p_s to the origin of the affine patch $Z = 1$. This induces a new polynomial $g(x, y) = f(x + x_0, y + y_0)$.

(4) After (3), we will check the singularity type of $g(x, y)$ at $O = (0, 0)$ in $\mathbb{A}^2$ by its non-zero homogeneous part of lowest degree.

For step (1), computing the height of an ideal is implemented in SageMath.

For step (2), computing associated primes is implemented in SageMath and the following lemma implies that the associated prime ideals of $\frac{\mathbb{F}_{p^D}[x,y]}{(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})}$ are exactly the singular points of $C_{\overline{\mathbb{F}}_p}|_{Z=1}$.

Lemma 4.2.1. *Let A be a Noetherian ring. Then the minimal prime ideals of A belong to $Ass(A)$.*

Proof. See Corollary 7.1.3 in [48]. □

For step (4), we first point out $g \in \mathbb{F}_{p^D}[x,y]$ is square-free since otherwise the singular locus is of dimension 1 (contradicting (1)). Its singularity type at O is determined by its non-zero homogeneous part of lowest degree g_{min}. To be more precise, O is a smooth point if $\deg(g_{min}) = 1$, and O is a nodal (resp. cusp) point if $\deg(g_{min}) = 2$ and g_{min} can be factored into a product of two different (resp. the same) straight lines ([27] Page 66). This is also where we use the condition that C is a plane curve.

Example 4.2.2. *Let k be a field with* $\operatorname{char} k \neq 2, 3$. *The two plane curves $E_n : y^2 - x^3 - x^2 = 0$ and $E_c : y^2 - x^3 = 0$ have nodal and cusp points at the origin respectively. This can be observed by their homogeneous parts of lowest degree, which are $(y-x)(y+x)$ and y^2.*

The following is the pseudocode for our algorithm. The SageMath code can be found in Appendix IV.

Algorithm 1 (1) Checking the singular dimension

Input: f: a polynomial in $\mathbb{F}_p[x,y]$
Output: d: the dimension of singular locus of $f = 0$

```
Taking partial derivative f_x and f_y of f.
I = (f, f_x, f_y)
d = dimension of I = 2 − ht(I)
return d
```

Algorithm 1 (2) Finding out singular points

Input: $I = (f, f_x, f_y)$: an ideal in $\mathbb{F}_p[x,y]$ of height 2
Output: LST: list of singular points

```
Fieldext = 1
find = False
while find = False do
    primeideals = associated prime ideals of I
    for P in primeideals do
        if elements in Gröbner basis of P are not of degree 1 then
            break the for iteration
        end if
        find = True
        D = Fieldext
    end for
    Fieldext = Fieldext + 1
end while
change base field to F_{p^D}
LST = list of associated prime ideals of I in F_{p^D}[x, y]
```

return LST

Algorithm 1 (3) Checking the singularity type

Input: $\mathfrak{m} = (x - a, y - b)$: an element in LST with a and b in $\mathbb{F}_{p^D}$
Output: local behaviour of f at the point $\mathfrak{m}$

```
G(x, y) = F(x + a, y + b)
take H to be the non-zero homogeneous part of G in lowest degree
if degree(H) > 2 then
    result=Higher singularity
else
    result=the factorization of H over F_{p^{2D}}
end if
return result
```

From the output, we check the factorization of H manually for its singularity type.

Remark 4.2.3. *We can also check if a plane curve is a nodal curve by Magma (http : //magma.maths.usyd.edu.au/magma/handbook/text/1411#15882).*

4.2.2 Semistability of $\mathfrak{C}$

With the algorithm in Subsection 4.2.1, we can get the following result of $\mathfrak{C}$.

Proposition 4.2.4. *$\mathfrak{C}_{\mathbb{Z}_{29}}$ has exactly one singular point at $(X + 3Z, Y - 2Z, 29)$ which is a nodal point. And $\mathfrak{C}_{\mathbb{Z}_{163}}$ has exactly one singular point at $(X - 49Z, Y - 36Z, 163)$ which is a nodal point. All other fibers are smooth.*

Remark 4.2.5. *The degree 2 parts of $\mathfrak{C}$ at these two points (H in the last algorithm) are $-6x^2 + 3xy - 11y^2$ (for 29) and $80x^2 - 56xy + 15y^2$ (for 163) respectively.*

Corollary 4.2.6. *$\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is stable and geometrically irreducible for every prime p. The geometric genus of $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ at $p = 29, 163$ is 2. Thus $\mathfrak{C}$ is a stable curve over $\mathbb{Z}$.*

Proof. When $p \neq 29$ or 163, the curve $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is a smooth plane quartic curve and thus stable and geometrically irreducible.

For $p = 29$ or 163, if $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ has multiple irreducible components, then each component corresponds to a polynomial $\mathfrak{F}_i$ such that $\mathfrak{F} = \prod \mathfrak{F}_i$. The polynomials $\mathfrak{F}_i$ are different otherwise $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ can not have only nodal singularities. If all $\mathfrak{F}_i$ are of degree 1, then the curve $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is the union of 4 straight lines in $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$, which can not be nodal and have exactly 1 singular point at the same time. When $\deg(\mathfrak{F}_{i_0}) > 1$ for some $\mathfrak{F}_{i_0}$, Bézout's theorem shows that there are more than 1 singular points which contradicts the fact that $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ has only 1 nodal point at $p = 29$ and 163. This shows that $\mathfrak{C}_{\overline{\mathbb{F}}_{29}}$ and $\mathfrak{C}_{\overline{\mathbb{F}}_{163}}$ are geometrically irreducible. Furthermore, their normalizations are curves of genus 2 which means that $\mathfrak{C}_{\overline{\mathbb{F}}_{29}}$ and $\mathfrak{C}_{\overline{\mathbb{F}}_{163}}$ are stable curves. $\square$

4.3 Thickness of $\mathfrak{C}$ at nodal points

In this section, we show that the thickness of $\mathfrak{C}$ at the two singular points is 1. This implies that $\mathfrak{C}$ is a regular stable model for $\mathfrak{C}_{\mathbb{Q}}$. We introduce the Fitting ideal in Subsection 4.3.1 and compute the thickness of $\mathfrak{C}$ in Subsection 4.3.2. The content of Subsection 4.3.3 is not used in our computation for $\mathfrak{C}$ but might be helpful for other curves.

4.3.1 Fitting ideal

In this section, we introduce the Fitting ideal and state its relation to thickness. Details can be found in Tag 0C3C and Subsections 2.2-2.4 of [5].

Definition 4.3.1. *If R is a commutative ring with 1 and M is a finitely generated R-module, then we have a free resolution of M*

$$\bigoplus_{l\in\mathfrak{L}} R \xrightarrow{\phi} \bigoplus_{j\in\mathfrak{J}} R \longrightarrow M \to 0 \tag{4.1}$$

where $\mathfrak{J}$ is a finite index set and $\mathfrak{L}$ can be infinite. The map ϕ corresponds to a $\#\mathfrak{J}\times\#\mathfrak{L}$ matrix N (might be an infinite matrix) and we define the k-th Fitting ideal $Fit_k^R(M)\subset R$ to be the ideal generated by all the $(\#\mathfrak{J}-k)\times(\#\mathfrak{J}-k)$ minors of N.

Remark 4.3.2. *The Fitting ideals are independent of the choice of the resolution (see Tag 07Z8).*

There is also a scheme version for the Fitting ideal. Lemma 4.3.3 shows that the Fitting ideal behaves well under localization and gluing.

Lemma 4.3.3. *Let X be a scheme. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type, then for each non-negative integer i, there exists a unique quasi-coherent sheaf of ideals $Fit_i^X(\mathcal{F})$ such that on each affine $U=\operatorname{Spec}(A)$ étale over X, we have*

$$Fit_i^X(\mathcal{F})(U) = Fit_i^A(\mathcal{F}(U))\subset A.$$

Proof. Tag 0CZ3. $\square$

Lemma 4.3.4. *If $X\to S$ is a scheme morphism of finite type and $S'\to S$ is an affine morphism, then for any non-negative integer i, we have*

$$Fit_i^X(\Omega_{X/S})\otimes_S S' \simeq Fit_i^{X_{S'}}(\Omega_{X_{S'}/S'})$$

as $O_{S'}$-modules.

Proof. According to Lemma 4.3.3, we only need to consider the affine case. Let's assume that B is a finitely generated A-algebra, then we have a resolution as in Definition 4.3.1

$$\oplus_{i\in\mathfrak{L}}B \xrightarrow{\phi} \oplus_{j\in\mathfrak{J}}B \longrightarrow \Omega_{B/A} \to 0.$$

For a ring homomorphism $A \to A'$, since $\otimes_A A'$ is a right exact functor and the cotangent bundle is stable under base change, we have

$$\oplus_{i\in\mathfrak{L}}B_{A'} \xrightarrow{\phi'} \oplus_{j\in\mathfrak{J}}B_{A'} \longrightarrow \Omega_{B\otimes_A A'/A'} \to 0.$$

Since the matrix defining $Fit_i^B(\Omega_{B/A})$ is not changed after applying the functor $\otimes_A A'$, we have

$$Fit_i^B(\Omega_{B/A}) \otimes_A A' \simeq Fit_i^{B\otimes_A A'}(\Omega_{(B\otimes_A A')/A'})$$

as A'-modules. □

For a semistable curve X over S, the first Fitting ideal of the cotangent bundle $\Omega_{X/S}$ cuts out a closed subscheme of X which we denote by $\mathrm{Sing}(X/S)$. The complement of $\mathrm{Sing}(X/S)$ is exactly the smooth locus of $X \to S$. Remark 2.14 in [5] shows the following relation between the thicknesses of singular points on X and $\mathrm{Sing}(X/S)$.

Lemma 4.3.5. *Let S be the spectrum of a strict Henselian discrete valuation ring A with a uniformizer t. If $X \to S$ is a semistable curve with smooth generic fiber, then*

$$\mathrm{Sing}(X/S) \simeq \mathrm{Spec}\left(\prod_{e\in\mathfrak{N}} A/(t^{\alpha(e)})\right)$$

where $\mathfrak{N}$ is the set of nodal points on $X \otimes_S \mathrm{Spec}(A/(t))$ and $\alpha(e)$ is the thickness at e.

Example 4.3.6. *Let $\mathbb{Z}_p^{un}$ be the unramified closure of $\mathbb{Z}_p$. For the semistable elliptic curve*

$$C : Y^2Z - X^3 - aX^2Z - cZ^3 = 0$$

over $\mathbb{Z}_p^{un}$, where $p > 3$, $a \in (\mathbb{Z}_p^{un})^*$ and $c \in p\mathbb{Z}_p^{un}\backslash\{0\}$, we have

$$\mathrm{Sing}(C/S) \simeq \mathrm{Spec}(\mathbb{Z}_p^{un}/p^{\mathrm{ord}_p(c)}\mathbb{Z}_p^{un}).$$

Thus we conclude that C has a nodal singularity of thickness $\mathrm{ord}_p(c)$ at (x, y, p) on the affine patch $Z = 1$.

Proof. It is easy to show that there is only one nodal point (x, y, p) on the affine patch $Z = 1$. We denote $\frac{\mathbb{Z}_p^{un}[x,y]}{y^2-x^3-ax^2-c}$ by R and denote $y^2 - x^3 - ax^2 - c$ by f. By Definition 4.3.1 and Lemma 4.3.3, we make the following resolution of Ω_C on $Z = 1$:

$$R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} \Omega_{R/\mathbb{Z}^{un}} \longrightarrow 0$$

where ϕ sends 1 to $f_x \oplus f_y$ and ψ sends $u \oplus v$ to $udx + vdy$. This resolution just comes from the construction of the cotangent bundle, and thus is exact. By Lemma 4.3.3, the first Fitting ideal of this curve on $Z = 1$ is given by the ideal $I = (f_x, f_y)$ in R. Then we have

$$R/I \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{(y^2 - x^3 - ax^2 - c, 2y, -3x^2 - 2ax)} \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{(x,y,c)} \simeq \mathbb{Z}_p^{un}/(c).$$

By Lemma 4.3.5, we have

$$\mathrm{Sing}(C/\mathbb{Z}_p^{un}) \simeq \mathrm{Spec}(\mathbb{Z}_p^{un}/p^{\mathrm{ord}_p(c)}\mathbb{Z}_p^{un}),$$

and this shows the result. □

4.3.2 Thickness of $\mathfrak{C}$

Recall that in Section 4.2, we showed the following result for $\mathfrak{C}$:

- $\mathfrak{C}_{\mathbb{Z}_{29}}$ has exactly one nodal point at $(X+3, Y-2, 29)$ on the affine patch $Z = 1$ with residue field $\mathbb{F}_{29}$.
- $\mathfrak{C}_{\mathbb{Z}_{163}}$ has exactly one nodal point at $(X-49, Y-36, 163)$ on the affine patch $Z = 1$ with residue field $\mathbb{F}_{163}$.
- $\mathfrak{C}_{\mathbb{Z}}$ has no other singular points.

We start our computation on $\mathfrak{C}$ by the observation that all nodal points are on $U_{\mathfrak{C}}$. By Lemma 4.3.3, we can compute the Fitting ideal $Fit_1^{\mathfrak{C}}(\Omega_{\mathfrak{C}/\mathbb{Z}})(U_{\mathfrak{C}})$ on $U_{\mathfrak{C}}$ by the following resolution:

$$R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} \Omega_{\mathfrak{C}/\mathbb{Z}}(U_{\mathfrak{C}}) \longrightarrow 0 \tag{4.2}$$

where $R = \frac{\mathbb{Z}[x,y]}{(\mathfrak{f})}$, the map ϕ sends 1 to $\mathfrak{f}_x \oplus \mathfrak{f}_y$ and ψ sends $u \oplus v$ to $udx + vdy$.

This computation of the ideal $I = (\mathfrak{f}, \mathfrak{f}_x, \mathfrak{f}_y)$ can be carried out in SageMath by the following code:

```
R.<x,y>=PolynomialRing(ZZ)
f=-x^3*y+x^2*y^2-x*y^2+y^3+x^2+x
fx=derivative(f,x)
fy=derivative(f,y)
I=R.ideal([f,fx,fy])
B=I.groebner_basis()
B
[x + 3048, y + 2898, 4727]
```

```
factor(4727)
29 * 163
```

Thickness

Thus we have

$$\frac{R}{Fit_1^R(\Omega_{R/\mathbb{Z}})} \simeq \frac{\mathbb{Z}[x,y]}{(\mathfrak{f},\mathfrak{f}_x,\mathfrak{f}_y)} \simeq \frac{\mathbb{Z}[x,y]}{(x+3048, y+2898, 29\times 163)}. \tag{4.3}$$

By Lemma 4.3.4, we can tensor Equation (4.3) with $\otimes_{\mathbb{Z}}\mathbb{Z}_p^{un}$ for $p = 29$ or 163 and get

$$\begin{aligned} \mathrm{Sing}(\mathfrak{C}_{\mathbb{Z}_{29}^{un}}) &\simeq \mathbb{Z}_{29}^{un}/(29\cdot\mathbb{Z}_{29}^{un}), \\ \mathrm{Sing}(\mathfrak{C}_{\mathbb{Z}_{163}^{un}}) &\simeq \mathbb{Z}_{163}^{un}/(163\cdot\mathbb{Z}_{163}^{un}). \end{aligned}$$

Since $\mathbb{Z}_p^{un}$ is a strict Henselian discrete valuation ring, we conclude that the thickness of $\mathfrak{C}$ at these two points are both 1 by Lemma 4.3.5. We can summarize our computation into the following proposition.

Proposition 4.3.7. *$\mathfrak{C}_{\mathbb{Z}_{29}}$ has thickness 1 at the only nodal point, and $\mathfrak{C}_{\mathbb{Z}_{163}}$ has thickness 1 at the only nodal point.*

Corollary 4.3.8. *$\mathfrak{C}$ is the regular stable model of $\mathfrak{C}_{\mathbb{Q}}$ over $\mathbb{Z}$.*

Proof. By Corollary 4.2.6, the curve $\mathfrak{C}$ is stable. By Proposition 4.3.7, all singular points on $\mathfrak{C}$ have thickness 1, which means that $\mathfrak{C}$ is regular. □

Corollary 4.3.9. *The dual graphs of $\mathfrak{C}$ at 29 or 163 are of type $1I$ in Table 3.1 with the edge weighted by 1.*

Proof. An application of Corollary 4.2.6 and Proposition 4.3.7. □

4.3.3 Further discussion of thickness

For a polynomial f, we write $f^{\deg\leq i}$ (resp. $f^{\deg>i}$) for the polynomial containing monomials of f in degree not bigger (resp. bigger) than i. For example, if $f = x^4+x^3y^2+5x^2+xy+y^3$, then $f^{\deg\leq 3} = 5x^2+xy+y^3$.

Proposition 4.3.10. *For an odd prime p, we choose $U \simeq \mathrm{Spec}(A)$ to be an affine open subscheme of a semistable curve C over $\mathbb{Z}_p^{un}$ where $A = \mathbb{Z}_p^{un}[x,y]/(f)$. If U has only 1 nodal point $O_A = (x,y,p)$ and*

$$f^{\deg\leq 2} = ax^2 + bxy + cy^2 + d$$

for a, b, c in $\mathbb{Z}_p^{un}$ and d in $p\mathbb{Z}_p^{un}\backslash\{0\}$, then the thickness of C at O_A is equal to the thickness of $V = \mathrm{Spec}(B)$ at $O_B = (x,y,p)$ where $B = \frac{\mathbb{Z}_p[x,y]}{(f^{\deg\leq 2}(x,y))}$.

Proof. By the criterion of singularity type for plane curves (Page 66 in [27]), point O_A is a nodal point on U if and only if O_B is a nodal point on V. The geometric fiber $V_{\overline{\mathbb{F}}_p}$ is the union of two straight lines l_1 and l_2 on the affine plane. The two straight lines are not parallel otherwise the origin can not be a nodal point. This means that O_B is the only singular point on V.

We denote the partial derivative of $f^{\deg>2}$ with respect to x and y by $(f^{\deg>2})_x$ and $(f^{\deg>2})_y$. By Lemma 4.3.5, we have

$$\frac{\mathbb{Z}_p^{un}}{(p^{\alpha(O_A)})} \simeq \frac{A}{(Fit_1^A(\Omega_{A/\mathbb{Z}_p^{un}}))} \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{I_A} =: R_U \tag{4.4}$$

$$\frac{\mathbb{Z}_p^{un}}{(p^{\alpha(O_B)})} \simeq \frac{B}{(Fit_1^B(\Omega_{B/\mathbb{Z}_p^{un}}))} \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{I_B} =: R_V \tag{4.5}$$

where

$$I_A = (f, 2ax + by + (f^{\deg>2})_x, bx + 2cy + (f^{\deg>2})_y),$$
$$I_B = (f^{\deg\leq 2}, 2ax + by, bx + 2cy).$$

These isomorphisms shows that R_U and R_V are local rings.

By the definition of nodal singularity, the image of $b^2 - 4ac$ in $\frac{\mathbb{Z}_p^{un}}{p\mathbb{Z}_p^{un}} \simeq \overline{\mathbb{F}}_p$ does not vanish, which means that $b^2 - 4ac \in (\mathbb{Z}_p^{un})^*$. We can simplify I_A and I_B to be

$$I_A = (f, x + l(x,y), y + m(x,y)),$$
$$I_B = (f^{\deg\leq 2}, x, y) = (x, y, d),$$

where

$$l(x,y) = \frac{b(f^{\deg>2})_y - 2c(f^{\deg>2})_x}{b^2 - 4ac},$$

$$m(x,y) = \frac{b(f^{\deg>2})_x - 2a(f^{\deg>2})_y}{b^2 - 4ac},$$

are polynomials in $\mathbb{Z}_p^{un}[x,y]$.

According to isomorphisms in Equation (4.4) and (4.5), in order to show $\alpha(O_A) = \alpha(O_B)$, we just need to show $R_U \simeq R_V$. Since the completion of $\frac{\mathbb{Z}_p^{un}}{(p^t)}$ with respect to the maximal ideal is still itself, we just need to show $\widehat{R_U} \simeq \widehat{R_V}$. By Equation (4.4) and (4.5), we get

$$\widehat{R_U} = \lim_n \frac{\mathbb{Z}_p^{un}[[x,y]]}{I_A + (x,y,p)^n},$$
$$\widehat{R_V} = \lim_n \frac{\mathbb{Z}_p^{un}[[x,y]]}{I_B + (x,y,p)^n}.$$

Claim 4.3.11. *We have the following equality*

$$I_A + (x, y, p)^n = I_B + (x, y, p)^n \tag{4.6}$$

for every positive integer n.

PROOF OF CLAIM: When $n = 1$, we can see $I_A \subset (x, y, p)$ and $I_B \subset (x, y, p)$ and thus the claim is trivial.

For $n > 1$, we first note that

$$I_B + (x, y, p)^n = (x, y, d, p^n) = (x, y, p^{\min(n, \mathrm{ord}_p d)}),$$

and Equation (4.6) is equivalent to $(x, y) \subset I_A + (x, y, p)^n$.

Now we show that $(x, y)^n \subset I_A + (x, y, p)^n$ implies $(x, y)^{n-1} \subset I_A + (x, y, p)^n$. We will show that $x^i y^{n-i-1} \in I_A + (x, y, p)^n$ for every integer i in $[0, n-1]$. Since either x's or y's exponent is positive, without loss of generality, we can assume $i \geq 1$. Then

$$x^i y^{n-i-1} = (x + l(x, y))x^{i-1}y^{n-i-1} - l(x, y) \cdot x^{i-1}y^{n-i-1},$$

where the degree of l is either equal to 0 or strictly bigger than 1. Since $x + l(x, y) \in I_A$ and $l(x, y) \cdot x^{i-1}y^{n-i-1} \in (x, y)^n \subset (x, y, p)^n$, we have

$$x^i y^{(n-i-1)} \in I_A + (x, y, p)^n$$

thus

$$(x, y)^{n-1} \subset I_A + (x, y, p)^n.$$

This procedure does not use the powers of p in the ideal. Repeating this procedure, we can finally show that (x, y) is contained in both sides in Equation (4.6), which implies

$$I_A + (x, y, p)^n = (x, y, p^{\min(n, \mathrm{ord}_p d)}) = I_B + (x, y, p)^n.$$

CLAIM PROVEN

By the claim, we have $\widehat{R_U} \simeq \widehat{R_V}$ which implies $\alpha(O_A) = \alpha(O_B)$. □

Corollary 4.3.12. *If $p^e \| d$ in $\mathbb{Z}_p^{un}$, the thickness of the curve C in Proposition 4.3.10 at the point $O_A = (x, y, p)$ is e.*

Proof. By Proposition 4.3.10, we only need to compute the thickness of $f^{\deg \leq 2}$ at the point $O_B = (x, y, p)$. By Lemma 4.3.5, we have the following on V

$$\frac{\mathbb{Z}_p^{un}}{(p^{\alpha(O_A)})} \simeq \frac{\mathbb{Z}_p^{un}[x, y]}{(f^{\deg \leq 2}, (f^{\deg \leq 2})_x, (f^{\deg \leq 2})_y)} \tag{4.7}$$

$$\simeq \frac{\mathbb{Z}_p^{un}[x, y]}{(ax^2 + bxy + cy^2 + d, 2ax + by, bx + 2cy)}. \tag{4.8}$$

By semistability of C, we have $b^2 - 4ac \in (\mathbb{Z}_p^{un})^*$, thus $(2ax + by, bx + 2cy) = (x, y)$. Substituting $(2ax + by, bx + 2cy) = (x, y)$ to Equation (4.8), we get

$$\frac{\mathbb{Z}_p^{un}[x, y]}{(ax^2 + bxy + cy^2 + d, 2ax + by, bx + 2cy)} \simeq \frac{\mathbb{Z}_p^{un}}{(d)},$$

which gives $\alpha(O_A) = e$. □

Example 4.3.13. *In Proposition 4.3.10, f has no linear terms, and now we show that this requirement is essential. We assume that $U \simeq \operatorname{Spec}\left(\frac{\mathbb{Z}_p^{un}[x,y]}{(f)}\right)$ is an open subscheme of C where*

$$f = x^d + xy + p^m x - p^n y + p^l \tag{4.9}$$

for integers $m > 0$, $l > 0$, $n > 0$ and $d > 2$. Then

$$f^{\deg\leq 2} = xy + p^m x - p^n y + p^l. \tag{4.10}$$

We will compute the thickness of f and $f^{\deg\leq 2}$ at the origin (x, y, p).

(1) by substituting

$$\begin{aligned} x &\to x' + p^n \\ y &\to y' - dp^{n(d-1)} - p^m, \end{aligned}$$

into Equation (4.9), we get

$$\begin{aligned} f_1(x', y') &= f(x' + p^n, y' - dp^{n(d-1)} - p^m) \\ &= f_1^{\deg>2} + \frac{d(d-1)p^{n(d-2)}}{2} x'^2 + x'y' + p^{nd} + p^{m+n} + p^l. \end{aligned}$$

(2) by substituting

$$\begin{aligned} x &\to x'' + p^n \\ y &\to y'' - p^m, \end{aligned}$$

into Equation (4.10), we get

$$\begin{aligned} f_2(x'', y'') &= f^{\deg\leq 2}(x'' + p^n, y'' - p^m) \\ &= x''y'' + p^{m+n} + p^l. \end{aligned}$$

Now we can apply Corollary 4.3.12 for computing the thickness at the origin. Taking $n = 2$, $d = 3$ and $m = l = 10$, we get the thickness is 6 in (1) and is 10 in (2).

4.4 $\langle\Delta, \Delta\rangle$ for $\mathfrak{C}_{\mathbb{Q}}$

Recall that in Theorem 3.3.2, we already decomposed $\langle\Delta, \Delta\rangle$ into a sum of contributions from Archimedean places and non-Archimedean places. In this section, we will show that all terms but the infinite λ invariants in Theorem 3.3.2 are computable now.

1 For a finite place v, by Proposition 4.2.4, Corollary 4.2.6 and Proposition 4.3.7, the dual graph of $\mathfrak{C}$ at v is known. We can get its admissible invariants (including $\lambda(\mathfrak{C}_v)$) from Table 3.1 and Table 3.2. There are only finitely many primes with bad reduction, and invariants at primes with good reduction contribute 0 to the height.

2 For a finite place v, by Proposition 3.3.3 and Proposition 3.3.5, the number $\mathrm{ord}_v(\chi'_{18})$ can be computed if we know the dual graph and the location of $\mathfrak{C}_v$ in the moduli space $\overline{\mathcal{M}}_3$.

3 For the infinite place $v : \mathbb{Q} \to \mathbb{C}$, in Equation (2.3) we have an explicit expression for χ'_{18} for v, and the Hodge metric is determined by the period matrix of $\mathfrak{C}_{\mathbb{C}}$ (Equation (2.7)).

4 For the infinite place $v : \mathbb{Q} \to \mathbb{C}$, the invariant $\lambda(\mathfrak{C}_v)$ is the most difficult one, we will show how to compute it in later sections. In fact, all the remaining sections are necessary for the computation of $\lambda(\mathfrak{C}_v)$.

4.4.1 λ at finite places

By Proposition 4.1.2 and Corollary 4.3.9, the only non-trivial dual graphs of $\mathfrak{C}$ come from 29 and 163, which are the type $1I$ graph in Table 3.1. We can get the admissible invariants of $\overline{\Gamma}_v$ for $v = 29$ and 163 from Table 3.1 and Table 3.2, which can be summarized as follows.

Proposition 4.4.1. *For $v = 29$ or 163, we have $\delta_0(\mathfrak{C}_v) = 1$, $\delta_1(\mathfrak{C}_v) = 0$, $\tau(\mathfrak{C}_v) = \frac{1}{12}$, $\theta(\mathfrak{C}_v) = 0$, $\varphi(\mathfrak{C}_v) = \frac{1}{9}$, $\lambda(\mathfrak{C}_v) = \frac{3}{28}$ and $\epsilon(\mathfrak{C}_v) = \frac{2}{9}$.*

Corollary 4.4.2. *The λ invariants from non-Archimedean places contribute*

$$\begin{aligned} -21 \sum_{v \in M(\mathbb{Q})_0} \lambda(\mathfrak{C}_v) N(v) =& -\frac{21 \times 3}{28}(\log 29 + \log 163) \\ \approx& -19.0373535692 \end{aligned}$$

to the height $\langle\Delta, \Delta\rangle$ of $\mathfrak{C}_{\mathbb{Q}}$.

Proof. Substitute Proposition 4.4.1 into Theorem 3.3.2. □

4.4.2 χ'_{18} at finite places

The closure of the hyperelliptic locus H (denoted by $\overline{H}$) in $\overline{\mathcal{M}_3}$ is a divisor. Proposition 3.3.3 and Proposition 3.3.5 relate $\overline{H}$ and $\mathrm{ord}_v(\chi'_{18})$. Thus we need to study the integer $\mathrm{mult}_v(\overline{H})$ for our curve $\mathfrak{C}$. Note that $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is geometrically irreducible and of geometric genus 2 (Corollary 4.2.6) at $p = 29$ and 163.

Let D be a smooth curve of genus 2 over an algebraically closed field. It is well-known that D is hyperelliptic and has a unique hyperelliptic involution, i.e. a non-trivial element $\sigma \in \mathrm{Aut}(D)$ such that $\sigma^2 = Id_D$ and $D/<\sigma>\simeq \mathbb{P}^1$. We say points $p \neq q$ on D are *conjugate* if $\sigma(p) = q$.

Proposition 4.4.3. *Let H_0 be a smooth curve of genus 2 over an algebraically closed field K. Let p and q be conjugate points on H_0. The curve C given by identifying p and q (glue them into a nodal singularity) on H_0 is not a plane quartic curve.*

Proof. We assume that C is a plane quartic defined by $f(x, y) = 0$ on certain affine patch U_{xy}. Without losing generality, we can assume the nodal point is $(0, 0)$, then the equation becomes

$$f(x, y) = f^{\deg \geq 3}(x, y) + f^{\deg = 2}(x, y), \tag{4.11}$$

where $f^{\deg=2}(x, y)$ is a non-degenerate quadratic form.

Now we blow up the curve C at $(0, 0)$ by substituting $y = xt$, then we get an affine open set U_{xt} of H_0 given by

$$f^{\deg \geq 3}(x, xt)/x^2 + f^{\deg=2}(x, xt)/x^2 = 0. \tag{4.12}$$

By the non-degeneracy of $f^{\deg=2}(x, y)$, we know that $f^{\deg=2}(x, xt)/x^2$ is a polynomial in t with distinct roots t_1 and t_2. After the blow up, we get the original smooth curve H_0 and the nodal point $(0, 0)$ on C is resolved into two distinct points $(0, t_1)$ and $(0, t_2)$ on U_{xt}. These two points are exactly p and q.

In Equation (4.12), we have $f^{\deg\geq 3}(x, xt)/x^2 = xf_1(t) + x^2 f_2(t)$. At least, we know that f_2 is non-zero (we assumed that C is a plane quartic), and $f^{\deg=2}(x, xt)/x^2$ is a polynomial in t. In other words, t gives a 2-1 map from H_0 to $\mathbb{P}^1_K$. Since H_0 is hyperelliptic, it has a natural 2-1 map Quo to $\mathbb{P}^1_K$. By the uniqueness of the hyperelliptic 2-1 map for a hyperelliptic curve, we have a unique automorphism η of $\mathbb{P}^1_K$ that makes the following diagram commute:

$$\begin{tikzcd} H_0 \arrow[r, "Quo"] \arrow[d, "t"] & \mathbb{P}^1 \arrow[dl, "\eta"] \\ \mathbb{P}^1 & \end{tikzcd}$$

where conjugate points are mapped to the same point by *Quo*.

The images of p and q are 2 different points along t since they correspond to $(0, t_1)$ and $(0, t_2)$. This is impossible since they are conjugate and are mapped to the same point by *Quo*. Thus the curve C can not be a plane quartic. $\square$

Corollary 4.4.4. *When* $p = 29$ *or* 163*, we have* $\mathrm{mult}_p(\overline{H}) = 0$ *for* $\mathfrak{C}_{\mathbb{Z}_p}$*.*

Proof. According to Proposition 4.2.6, the reduction of $\mathfrak{C}$ at 29 (and 163) is an irreducible plane quartic with exactly 1 singular point. By Proposition 4.4.3, the reduction of $\mathfrak{C}$ at $p = 29$ and 163 are not obtained by gluing conjugate points on a genus 2 curve.

The singular curve $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ lies on the closure of the hyperelliptic locus if and only if $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ has an involution ι and the quotient map $\mathfrak{C}_{\overline{\mathbb{F}}_p}/\langle\iota\rangle$ is a tree of $\mathbb{P}^1$ connected by nodal points (Page 101 in [3]). The normalization of $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is a genus 2 curve (denoted by $\widetilde{\mathfrak{C}}_{\overline{\mathbb{F}}_p}$) thus is hyperelliptic. Suppose $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is hyperelliptic, then $\mathfrak{C}_{\overline{\mathbb{F}}_p}/\langle\iota\rangle \simeq \mathbb{P}^1$ since we already know that $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is irreducible (Proposition 4.2.6).

We write q_ι for the quotient map induced by ι and write n for the normalization map of $\mathfrak{C}_{\overline{\mathbb{F}}_p}$, that is

$$\widetilde{\mathfrak{C}}_{\overline{\mathbb{F}}_p} \xrightarrow{n} \mathfrak{C}_{\overline{\mathbb{F}}_p} \xrightarrow{q_\iota} \mathbb{P}^1.$$

According to the first paragraph, we know that n identifies two non-conjugate points. The composition of $q_\iota \circ n$ is a 2-1 map from $\widetilde{\mathfrak{C}}_{\overline{\mathbb{F}}_p}$ to $\mathbb{P}^1$. This is impossible since we already know that n identifies non-conjugate points. Thus $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is non-hyperelliptic. The multiplicity of $\mathfrak{C}_{\mathbb{Z}_p}$ at the hyperelliptic locus in $\overline{\mathcal{M}}_3$ is 0. $\square$

Proposition 4.4.5. *For* $v = 29$ *or* 163*, we have* $\mathrm{ord}_v(\chi'_{18})(\mathfrak{C}_{\mathbb{Z}_v}) = 2$*. And*

$$\mathrm{ord}_v(\chi'_{18})(\mathfrak{C}_{\mathbb{Z}_v}) = 0$$

for all other finite places v*.*

Proof. Combining Proposition 3.3.3 and Proposition 3.3.5 we have

$$\mathrm{ord}_v(\chi'_{18}) = 2\mathrm{mult}_v(\overline{H}) + 6\delta_1(\overline{\Gamma}_v) + 2\delta_0(\overline{\Gamma}_v).$$

Then we get the result by combining Proposition 4.2.4, Proposition 4.4.1 and Corollary 4.4.4. $\square$

Remark 4.4.6. $\mathrm{ord}_v(\chi'_{18})$ *vanishes at finite places of good reduction.*

Corollary 4.4.7. *For finite places,* χ'_{18} *contributes*

$$\frac{21}{18}(2\log 29 + 2\log 163) \approx 19.7424407385$$

to $\langle\Delta, \Delta\rangle$*.*

Proof. Substituting Proposition 4.4.5 into Theorem 3.3.2. $\square$

4.4.3 χ'_{18} at the infinite place

Recall notations introduced in Equation (2.2), Equation (2.3) and Remark 2.1.10. Using the metric given by Equation (2.7), we get that the contribution of $\log\|\chi'_{18}\|_{\mathrm{Hdg}}$ in Theorem 3.3.2 is:

$$\begin{aligned}
&-\frac{21}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\tau)\\
=&-\frac{21}{18}\log(\|2^{-28}(2\pi i)^{54}\tilde{\chi}_{18}(\tau)(dz_1\wedge dz_2\wedge dz_3)^{\otimes 18}(\tau)\|_{\mathrm{Hdg}})\\
=&-\frac{21}{18}\log|2^{-28}(2\pi)^{54}\prod_{\epsilon\in S_3}\theta_\epsilon(0,\tau)(\det\mathrm{Im}\tau)^9|.
\end{aligned}$$

All components except the list of even characteristics are implemented in Magma, while the list of even theta characteristics for dimension 3 is easy to compute by hand. Magma code for this computation can be found in Appendix VI. With our calculation, we get the following proposition.

Proposition 4.4.8. *At the infinite place, the χ'_{18} modular form contributes*

$$-\frac{21}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}\approx -81.0426321447$$

to $\langle\Delta,\Delta\rangle$.

4.5 Evaluation of $\|\theta\|_{g-1}$

In this section, we will define and show how to evaluate $\|\theta\|_{g-1}$ at points in $\mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}})$. At the end of Subsection 4.5.1, we summarize our strategy. Subsection 4.5.2 is about the computation of a canonical divisor of $\mathfrak{C}_{\mathbb{C}}$. We can evaluate $\|\theta\|_{g-1}$ with Proposition 4.5.11 in Subsection 4.5.3.

To avoid confusion, in this section, we still use g in some notations even though we know $g=3$ for $\mathfrak{C}_{\mathbb{C}}$, for example we use $\|\theta\|_{g-1}$ rather than $\|\theta\|_2$.

4.5.1 Strategy

Fixing a base point P_{bs}, a basis of holomorphic forms $\{\omega_i\}_{1\leq i\leq g}$ and a symplectic homology basis $\{\eta_i\}$ of the genus g Riemann surface C, we have a period matrix $\Omega=(\Omega_1,\Omega_2)$ associated to these datum. Then we have an element $\tau=\Omega_1^{-1}\Omega_2$ in the Siegel upper half-space $\mathbb{H}_g$. Taking $\{\eta_i\}_{1\leq i\leq g}=\{\omega_i\}_{1\leq i\leq g}\cdot {}^t\Omega_1^{-1}$, we have the following map

$$\mathrm{Div}^{g-1}(C)\to\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g,\quad \sum_n n_kP_k\to\sum_n n_k\int_{P_{bs}}^{P_k}(\eta_1,\ldots,\eta_g),$$

which induces a bijective map:

$$u : \mathrm{Pic}^{g-1}(C) \to \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g. \tag{4.13}$$

Remark 4.5.1. *The 'Abel-Jacobi' map above is well-defined for a chosen base point P_{bs}. We will write AJ for the Abel-Jacobi map from $\mathrm{Div}^0(C)$ to $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, which we can define without a base point.*

The zero locus of Riemann's theta function

$$\theta(z;\tau) := \sum_{n\in\mathbb{Z}^g} \exp(\pi i^t n\tau n + 2\pi i^t nz) \tag{4.14}$$

defines a divisor Θ_0 on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Recall that the theta divisor Θ in $\mathrm{Pic}^{g-1}(C)$ corresponds to classes of line bundles admitting a global section. The following theorem of Riemann (Theorem 1.4.2 in [12]) links Θ_0 and Θ.

Theorem 4.5.2. *We denote t_κ to be the translation map of the tori with respect to $\kappa \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, that is, an endomorphism of $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ sending $x \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ to $x + \kappa \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. There is a unique element $\kappa = \kappa(P_{bs})$ in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ such that $(t_\kappa \circ u)^*\Theta = \Theta_0$ which also induces a canonical isomorphism of line bundles $(t_\kappa \circ u)^*\mathcal{O}(\Theta_0) \xrightarrow{\sim} \mathcal{O}(\Theta)$ on $\mathrm{Pic}^{g-1}(C)$. Furthermore, we have $(t_\kappa \circ u)(K_C - D) = -(t_\kappa \circ u)(D)$ for any divisor D of degree $g-1$.*

By a *semi-canonical divisor* on C, we mean a divisor s on C of degree $g-1$ such that $2s \sim \Omega_C$. For a compact Riemann surface C of genus $g > 0$, there are 2^{2g} semi-canonical elements in $\mathrm{Pic}^{g-1}(C)$. These semi-canonical divisors are equal up to a 2-torsion point of $\mathrm{Jac}(C)$.

Corollary 4.5.3. *The map $t_\kappa \circ u$ identifies the set of classes of semi-canonical divisor on C with the set of 2-torsion points on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$.*

By Riemann's theorem, we can translate a metric on $\mathcal{O}(\Theta_0)$ to $\mathcal{O}(\Theta)$ along the map $t_\kappa \circ u$. The following paragraph shows how we choose the metric on $\mathcal{O}(\Theta_0)$.

We write s for the canonical section of $\mathcal{O}(\Theta_0)$ and fix a $(1,1)$-form on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ by

$$\nu := \frac{i}{2} \sum_{1\le k,l\le g} (\mathrm{Im}\tau)^{-1}_{k,l}\, dz_k \wedge d\bar{z}_l. \tag{4.15}$$

The $2g$-form $\frac{1}{g!}\nu^g$ gives the Haar measure on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. We choose $\|\cdot\|_{\Theta_0}$ to be the metric on $\mathcal{O}(\Theta_0)$ uniquely determined by:

(i) the curvature form of $\|\cdot\|_{\Theta_0}$ is equal to ν,

(ii) $\frac{1}{g!}\int_{\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g} \|s\|^2_{\Theta_0}\nu^g = 2^{-g/2}$.

For simplicity, we write $\|\theta\|$ for $\|(t_\kappa \cdot u)^* s\|_\Theta$ or $\|s\|_{\Theta_0}$. Then we have the following expression of $\|\theta\|$.

Proposition 4.5.4. *Let $z \in \mathbb{C}^g$ and $\tau \in \mathbb{H}_g$. Then the formula*

$$\|\theta\|(z;\tau) = (\det \mathrm{Im}\tau)^{1/4}\exp(-\pi{}^t y \cdot (\mathrm{Im}\tau)^{-1} \cdot y) \cdot |\theta(z;\tau)|$$

holds, where $y = \mathrm{Im}z$ and θ is defined in Equation (4.14).

Proof. See Page 413 in [23]. □

Notation 4.5.5. *We write $\|\theta\|$ for the metric of the canonical section of $\mathcal{O}(\Theta_0)$ on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, and write $\|\theta\|_{g-1}$ for the metric of the canonical section of $\mathcal{O}(\Theta)$ on $\mathrm{Pic}^{g-1}(C)$ induced by $\|\theta\|$.*

By Theorem 4.5.2, there exist a unique $\Delta' \in \mathrm{Pic}^{g-1}(C)$ such that for all $D \in \mathrm{Pic}^{g-1}(C)$, we have:

$$2\Delta' = K_C, \tag{4.16}$$
$$\|\theta\|_{g-1}(D) = \|\theta\|(AJ(D - \Delta')), \tag{4.17}$$

where AJ is the Abel-Jacobi map from $\mathrm{Pic}^0(C)$ to $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$.

Now we explain our strategy for evaluating $\|\theta\|_{g-1}(D)$ where $D \in \mathrm{Pic}^{g-1}(C)$. Recall that we fixed an isomorphism $\mathrm{Pic}^{g-1}(C) \xrightarrow{\sim} \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\tau$ in Equation (4.13). By Proposition 4.5.4 and Equation (4.17), we reduce the problem to computing $AJ(D - \Delta')$. By the equality

$$AJ(D - \Delta') = AJ(D - (g-1)P_{bs}) - AJ(\Delta' - (g-1)P_{bs}), \tag{4.18}$$

we only need to compute $AJ(\Delta' - (g-1)P_{bs})$. Since Pic^{g-1} is a 3-dimensional abelian variety, there are 64 elements in $\mathrm{Pic}^{g-1}(C)$ satisfying the Equation (4.16). These elements give 64 possibilities for $AJ(\Delta' - (g-1)P_{bs})$. In Subsection 4.5.2, we compute the canonical divisor K_C. In Subsection 4.5.3, we explain our algorithm for finding the correct $AJ(\Delta' - (g-1)P_{bs})$ among the 64 possibilities.

Remark 4.5.6. *θ and AJ are implemented in Magma. Using Proposition 4.5.4, we can evaluate $\|\theta\|$.*

4.5.2 Canonical divisor of $\mathfrak{C}_{\mathbb{C}}$

In this subsection, we compute a canonical divisor of $\mathfrak{C}_{\mathbb{C}}$. Recall that a canonical divisor of a Riemann surface is the divisor of a non-zero meromorphic differential form.

We write the equation of $\mathfrak{C}_{\mathbb{C}}$ on the affine patch $U = \mathfrak{C}_{\mathbb{C}}|_{X=1}$ as

$$f_0 = -y_X + y_X^2 - y_X^2 z_X + y_X^3 z_X + z_X^2 + z_X^3,$$

then the differential form on U

$$\omega_0 = \frac{z_X dz_X}{(f_0)_{y_X}} = \frac{z_X dz_X}{3y_X^2 z_X - 2z_X y_X + 2y_X - 1} \tag{4.19}$$

can be extended to a global holomorphic form ω (Theorem 4.6.10). Thus we just need to compute div (ω). Since ω is a holomorphic form, we only need to consider zeroes of ω.

The locally defined function z_X is a local parameter for all but finite points on U. We write U_1 for the open subset of U where z_X is a local parameter. The numerator z_X of ω_0 on U vanishes to order 1 at the points

$$P_1 = (1 : 0 : 0),$$
$$P_2 = (1 : 1 : 0),$$

while the denominator does not. Thus z_X is a local parameter near P_1 and P_2, and ω_0 has simple zeroes at P_1 and P_2. So we obtain

$$\operatorname{div}(\omega)|_{U_1} = [P_1] + [P_2].$$

It can be shown that points in $U \backslash U_1$ are not in the support of $\operatorname{div}(\omega)|_U$. Thus we have

$$\operatorname{div}(\omega)_U = [P_1] + [P_2].$$

There are two points of $\mathfrak{C}_{\mathbb{C}}$ not lying on U:

$$P_3 = (0 : 1 : 0),$$
$$P_4 = (0 : 0 : 1).$$

$P_3 = (0 : 1 : 0)$ lies on the affine patch $V = \mathfrak{C}_{\mathbb{C}}|_{Y=1}$. Substituting

$$z_X \to \frac{z_Y}{x_Y}$$
$$y_X \to \frac{1}{x_Y}$$

into Equation (4.19) and the defining polynomial of $\mathfrak{C}_{\mathbb{C}}$, we get

$$\omega|_V = \frac{z_Y x_Y dz_Y - z_Y^2 dx_Y}{3z_Y - 2z_Y x_Y + 2x_Y^2 - x_Y^3}, \tag{4.20}$$

and

$$z_Y + x_Y^2 = x_Y^3 + z_Y x_Y - z_Y^3 x_Y - z_Y^2 x_Y^2. \tag{4.21}$$

The coordinate of P_3 in this affine patch is $(0,0)_V$, and either x_Y or z_Y is the local parameter for P_3. From Equation (4.21), we can see x_Y is a local parameter at P_3 ($\mathrm{ord}_{P_3}(x_Y) = 1$) and thus $\mathrm{ord}_{P_3}(z_Y) = 2$. Then the right hand side of Equation (4.21) has order strictly bigger than 2 (that is, $\mathrm{ord}_{P_3}(z_Y + x_Y^2) > 2$). Substituting

$$z_Y = -x_Y^2 + \text{higher degree terms}$$

into Equation (4.20), we get $\mathrm{ord}_{P_3}(\omega) = 2$.

It remains to compute the order of ω at P_4. Substituting the order of ω at P_1, P_2 and P_3 into the equations below

$$4 = \deg(K_{\mathfrak{C}_\mathbb{C}}) = \mathrm{ord}_{P_1}(\omega) + \mathrm{ord}_{P_2}(\omega) + \mathrm{ord}_{P_3}(\omega) + \mathrm{ord}_{P_4}(\omega),$$

we get $\mathrm{ord}_{P_4}(\omega) = 0$. In conclusion, we get the following proposition.

Proposition 4.5.7.

$$\begin{aligned} K_{\mathfrak{C}_\mathbb{C}} = \mathrm{div}(\omega) &= [P_1] + [P_2] + 2[P_3] \\ &= [(1:0:0)] + [(1:1:0)] + 2[(0:1:0)]. \end{aligned}$$

Remark 4.5.8. *We can also compute the canonical divisor in Magma.*

4.5.3 2-translation

Given the canonical divisor K_C of C, we get 64 possibilities for $AJ(\Delta' - (g-1)P_{bs})$. In this subsection, we explain how to find the correct one among the 64.

We use the base point P_{bs} fixed in Subsection 4.5.1, and write T_C for the torus $\mathrm{Jac}(C) = \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\tau$. We use the isomorphism $u : \mathrm{Pic}^{g-1}(C) \xrightarrow{\sim} T_C$ given in Equation (4.13). According to the last paragraph in Subsection 4.5.1, there is a subset V of T_C, containing 64 elements, such that each element v_Δ in V satisfies

$$2v_\Delta = AJ(K_C - 2(g-1)P_{bs}).$$

We want to find the one that makes Equation (4.17) hold.

The difference of any two elements in V is a 2-torsion point of T_C. If we fix a $v_{\Delta'}$ in V, then we just need to figure out the correct translation by a 2-torsion point η in T_C that makes the equation

$$\|\theta\|_{g-1}(D) = \|\theta\|(AJ(D - (g-1)P_{bs}) - (v_{\Delta'} + \eta)) \tag{4.22}$$

hold.

Lemma 4.5.9. $\|\theta\|_{g-1}$ *vanishes at points in* $\mathrm{Pic}^{g-1}(C)$ *that have effective representative divisors.*

Proof. This follows from Theorem 4.5.2 and the definition of Θ (the paragraph before Theorem 4.5.2). □

We use the lemma above to compute the 2-torsion point η. For example, $\|\theta\|_{g-1}$ vanishes at $(g-1)P_{bs}$. We sketch the algorithm as follows. Magma code and the period matrix τ chosen by Magma can be found in Appendix V.

Algorithm 2 Computation of the correct $v_{\Delta'} + \eta$

Input: C: a plane quartic over $\mathbb{C}$
P_{bs}: a default base point of C
Output: $v_{\Delta'} + \eta$ in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$

Compute the small period matrix τ of C.
Generate a set S consisting all vectors of the form $\sum c_i \cdot v_i$ where v_i are column vectors of $(1|\tau)$ and $c_i \in \{0, \frac{1}{2}\}$
Compute $v_{K_C} = AJ(K_C - 2[P_{bs}])$
$v_{\Delta'} = v_{K_C}/2$
for η in S **do**
  if $\|\theta\|(AJ(2[P_{bs}]) - v_{\Delta'} - \eta) \leq 0.00000001$ **then**
    return $v_{\Delta'} + \eta$
  end if
end for

Remark 4.5.10. *(1) Since we only check the vanishing of* $\|\theta\|_{g-1}$ *at one specific effective divisor* $2P_{bs}$*, it can happen that more than one* 2*-torsion point of* T_C *makes this specific theta value vanish. For our curve* $\mathfrak{C}_{\mathbb{C}}$*, the computation result of our code shows that this does not happen (only one the 64 choices makes the function vanish).*

(2) The correctness of the 2-translation can be checked by using a different effective divisor of degree $g-1$*, since they should give the same answer.*

Proposition 4.5.11. *Using the default base point* P_{bs} *and the (co)homology basis chosen by Magma, the point* $AJ(\Delta' - (g-1)P_{bs})$ *in* $T_{\mathfrak{C}_{\mathbb{C}}}$ *that makes Equation (4.17) hold is given by:*

$$z_1 = 0.47925054265168018676 - 0.00334176833187451614 * I$$
$$z_2 = 0.69868487750843232229 + 0.19949572388256356310 * I$$
$$z_3 = 0.00722266620787249385 - 0.04301020693432081496 * I.$$

With this proposition, we can evaluate $\|\theta\|_{g-1}$ by Equation (4.22). Note that the Abel-Jacobi map AJ, the modified theta function $\|\theta\|$ and the addition of divisors on $\mathfrak{C}_{\mathbb{C}}$ can be implemented in Magma.

Remark 4.5.12. *The point P_{bs} chosen by Magma is*

$$(-2.000000000 : -4.214319743 : 1).$$

4.6 Computation of the Green's function

In this section, we compute the Green's function on $\mathfrak{C}_{\mathbb{C}}$. The invariant $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in the Green's function will be used in the computation of $\lambda(\mathfrak{C}_{\mathbb{C}})$. In Subsection 4.6.1, we compute the Weierstrass points of $\mathfrak{C}_{\mathbb{C}}$. In Subsection 4.6.2, we compute the volume form of $\mathfrak{C}_{\mathbb{C}}$. In Subsection 4.6.3, we explain our algorithm for computing $\log(S(\mathfrak{C}_{\mathbb{C}}))$. We refer to Subsection 1.2.1 and [12] for definitions and theorems.

Instead of constructing $G(x, y)$ from Definition 1.2.2, we give an explicit formula of the Green's function discovered by R. de Jong in [12].

Following Proposition 2.2.6 in [12], we write $\mathfrak{W}$ for the set of Weierstrass points counted with weights and define the invariant $S(X)$ of a compact Riemann surface X of genus g as

$$\begin{aligned}\log(S(X)) = &- g^2 \cdot \int_X \log\|\theta\|_{g-1}(gP - Q) \cdot \mu(Q) \\ &+ \frac{1}{g} \cdot \sum_{W \in \mathfrak{W}} \log\|\theta\|_{g-1}(gP - W). \end{aligned} \tag{4.23}$$

Theorem 4.6.1. *If P and Q are distinct points on a compact Riemann surface X of genus $g > 1$ and P is not a Weierstrass point, then we have*

$$G(P, Q)^g = S(X)^{1/g^2} \cdot \frac{\|\theta\|_{g-1}(gP - Q)}{\prod_{W \in \mathfrak{W}} \|\theta\|_{g-1}(gP - W)^{1/g^3}}.$$

Proof. See the proof of Theorem 2.1.2 and Proposition 2.2.6 in [12]. □

By the computation in the last subsection, we are now able to evaluate

$$\|\theta\|_{g-1} : \mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}}) \to \mathbb{R}$$

with $\|\theta\|$ (Proposition 4.5.4). With this explicit formula for the Green's function, our goal is reduced to the computation of the Weierstrass points and the invariant $S(\mathfrak{C}_{\mathbb{C}})$.

4.6.1 Weierstrass points of plane quartic curves

We will first recall definitions related to Weierstrass points and some fundamental properties, further results can be found in [2], Page 41.

Definition 4.6.2. *Let X be a compact Riemann surface of genus $g > 1$ with canonical divisor K. An effective divisor D on X is called special if $h^0(\mathcal{O}(K - D)) > 0$. A point P is called a Weierstrass point if gP is a special divisor.*

Definition 4.6.3. *If we write $Gap(P)$ for*

$$Gap(P) \coloneqq \{n \in \mathbb{Z}_{>0} : h^0(\mathcal{O}(nP)) = h^0(\mathcal{O}((n-1)P))\},$$

the weight of a point P is defined as $w(P) \coloneqq \sum_{n \in \mathrm{Gap}(P)} n - g(g-1)/2$.

Example 4.6.4. *For a hyperelliptic curve of genus g, the Weierstrass points are exactly the $2g + 2$ ramification points of a hyperelliptic 2-1 map with equal weight $\frac{g(g-1)}{2}$.*

We know that $w(P) = 0$ for all but finitely many points, thus the divisor

$$W_X \coloneqq \sum_{P \in X} w(P)P$$

is well-defined. It is well-known (Proposition 1.12 in [64]) that this effective divisor is of degree $g(g-1)(g+1)$. Thus for a plane quartic curve C, we have $\deg(W_C)$ =24.

Let C be a smooth plane curve. For $x \in C$, we write T_x for the tangent line of C at x. A point $p \in C$ is called a *flex point* if p is a smooth point and $I(p, T_p \cap C) \geq 3$, where $I(p, T_p \cap C)$ is the intersection multiplicity of C and T_p at p. A flex point is called an *ordinary flex point* if $I(p, T_p \cap C) = 3$, otherwise it is called a *hyperflex*.

Definition 4.6.5. *The Hessian of a polynomial $F(X, Y, Z) \in K[X, Y, Z]$ is the following matrix*

$$\mathrm{Hess}(F) \coloneqq \begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}.$$

Proposition 4.6.6. *Let K be an algebraically closed field with* $\operatorname{char} K = 0$*. Let C be a smooth plane curve in $\mathbb{P}^2_K$ defined by $F(X, Y, Z) = 0$. We write C_H for the plane curve defined by* $\det \mathrm{Hess}(F) = 0$*. Then*

1. *$P \in C \cap C_H$ if and only if P is a flex point.*

2. *$I(P, C \cap C_H) = 1$ if and only if P is an ordinary flex.*

Proof. See Page 116 in [27]. □

Weierstrass points on a smooth plane quartic curve are of weight 1 or 2, and they correspond to ordinary flex points and hyperflex points respectively. Thus we have an equality (see [64] Page 13, Theorem 2.2):

$$\#\{\text{ordinary flex points on } C\} + 2 \times \#\{\text{hyperflex points on } C\} = 24.$$

With the discussion above, we can calculate Weierstrass points of $\mathfrak{C}_{\mathbb{C}}$ in SageMath. The result is attached to the Appendix II.

```
x,y,z=var('x,y,z')
C=-x^3*y+x^2*y^2-x*y^2*z+y^3*z+x^2*z^2+x*z^3
M=C.hessian()
det= M.determinant()
solve([C == 0,z==1,det==0],x,y,z)
```

Weierstrass points

The code above returns the intersection points of $\mathfrak{C}_{\mathbb{C}}$ and $\mathfrak{C}_{\mathbb{C},H}$ in the affine patch $Z = 1$. Since it contains 24 points, we can conclude that these are all the Weierstrass points, and all of them are of weight 1.

Proposition 4.6.7. *If we choose* $P = (-2.000000000 : -4.214319743 : 1)$*, then we have*

$$\frac{1}{3}\sum_{W\in\mathfrak{W}} \log\|\theta\|_{g-1}(gP - W) \approx -6.817611049$$

for the curve $\mathfrak{C}_{\mathbb{C}}$.

The reason we choose this point P is that this is the default base point for $\mathfrak{C}_{\mathbb{C}}$ chosen by Magma. In the following sections, we will use this point several times.

Remark 4.6.8. *For a general smooth plane quartic curve C, the weights of Weierstrass points are computable. We give a sketch of the procedure, and details can be found in [64] Pages 7-8. Let $\{\omega_k\}_{1\leq k\leq 3}$ be a basis for $H^0(C, \Omega_C)$. Then locally we can write ω_k as $f_k dz$ where z is a local parameter near point P. Now we have the Wronskian determinant around P:*

$$W_z(\omega_1, \omega_2, \omega_3) := \det \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''/2 & f_2''/2 & f_3''/2 \end{pmatrix}, \tag{4.24}$$

where the superscript indicates the order of differentiation with respect to z. This locally gives a non-zero rational section in $\Omega_C^{\otimes 6}$ near P by $W(\omega_1, \omega_2, \omega_3)dz^{\otimes 6}$, which can be extended to a global section. For this global section of $\Omega_C^{\otimes 6}$, we have

$$\operatorname{div} W(\omega_1, \omega_2, \omega_3) = \sum_{x \in C} w(x)x,$$

where $w(x)$ is the weight of x.

Remark 4.6.9. *If we have a sequence of positive integers $\{a_i\}_{1 \leq i \leq t}$ such that*

$$\sum_{i=1}^{t} a_i = g(g-1)(g+1),$$

can we always find a compact Riemann surface of genus g whose Weierstrass points have weights $\{a_i\}_{1 \leq i \leq t}$? The answer is no. In [64] Theorem 7.1, A.M. Vermeulen showed that there exist genus 3 curves with 0, 1, 2 *hyperflex points, but there is no genus 3 curve with* 10, 11 *hyperflex points.*

4.6.2 Computation of the volume form

Let C be a smooth plane curve of genus $g \geq 1$ defined by a homogeneous polynomial $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ of degree $d \geq 3$. For simplicity, we write $f(x, y) = F(x, y, 1)$ and $U = C|_{Z=1}$. Then we can construct an explicit basis of $H^0(C, \Omega_C^1)$ by the following theorem.

Theorem 4.6.10. *Let U_0 be the open subset of U where $\frac{\partial f}{\partial y}(x, y) \neq 0$. Then the restriction of a global holomorphic differential of C on U_0 can be written in the form $\frac{\phi(x,y)dx}{\frac{\partial f}{\partial y}(x,y)}$, where $\phi(x, y)$ is a polynomial of degree at most $d - 3$.*

Proof. See Chapter 9, Theorem 1 in [6]. □

For our curve $\mathfrak{C}_{\mathbb{C}}$, we have

$$\frac{\partial \mathfrak{f}(x, y)}{\partial y} = -x^3 + 2x^2y - 2xy + 3y^2.$$

By Theorem 4.6.10, we get a basis of $H^0(X, \Omega^1_{\mathfrak{C}_{\mathbb{C}}})$ as follows:

$$\left\{ \frac{dx}{-x^3 + 2x^2y - 2xy + 3y^2}, \frac{ydx}{-x^3 + 2x^2y - 2xy + 3y^2}, \frac{xdx}{-x^3 + 2x^2y - 2xy + 3y^2} \right\}. \tag{4.25}$$

We abbreviate these to ω_1, ω_y and ω_x respectively.

Now we can apply the Gram-Schmidt process to obtain an orthonormal basis with respect to the inner product

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \overline{\eta}.$$

The following theorem of Riemann gives us the inner product of every pair of basis elements, which will simplify our computation.

Theorem 4.6.11. *Let X be a compact Riemann surface of genus $g \geq 1$. Fix a symplectic basis for the homology $H_1(X, \mathbb{Z})$ and a basis $\omega_1, \ldots, \omega_g$ of the holomorphic differentials $H^0(X, \Omega_X^1)$. We have a period matrix $\Omega = (\Omega_1 | \Omega_2)$ given by these data. Then the following matrix identity*

$$\left(\frac{i}{2} \int_X \omega_k \wedge \overline{\omega}_l \right)_{1 \leq k,l \leq g} = \frac{i}{2} (\overline{\Omega}_2 \, {}^t\Omega_1 - \overline{\Omega}_1 \, {}^t\Omega_2) = \overline{\Omega}_1 (\mathrm{Im}\tau) \, {}^t\Omega_1$$

holds.

Proof. See Pages 231-232 in [29]. □

Remark 4.6.12. *The choice of the homology basis does not affect the matrix in Theorem 4.6.11.*

The calculation of the period matrix implemented in SageMath uses the ordered basis $[\omega_1, \omega_y, \omega_x]$, which is exactly what we constructed in Equations (4.25). Thus we can carry out the Gram-Schmidt process in SageMath. The code can be found in Appendix III, and we summarize our computation as the following proposition.

Proposition 4.6.13. *We have the following orthonormal basis of differential forms:*

$$\begin{aligned}
\omega_{on1} &= 0.350487116953118 * \omega_1 \\
\omega_{on2} &= 0.358981759779085 * \omega_y + 0.119553875346235 * \omega_1 \\
\omega_{on3} &= 0.429067210690657 * \omega_x - 0.216555180015011 * \omega_y \\
&\quad + 0.203008239643111 * \omega_1.
\end{aligned}$$

We write

$$\mu_{Ar} = \frac{i}{2 \cdot 3} \sum_{j=1}^{3} \omega_{onj} \wedge \overline{\omega}_{onj} \tag{4.26}$$

for the volume form of $\mathfrak{C}_{\mathbb{C}}$.

Remark 4.6.14. *We can get a different period matrix with Magma which also leads to a (1-1) form μ'_{Ar}. It can be checked by evaluation that the two (1-1) forms μ'_{Ar} and μ_{Ar} are identical.*

4.6.3 Computation of $\log(S(\mathfrak{C}_{\mathbb{C}}))$

Recall that in Proposition 4.6.7, we already computed the discrete sum part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$ (see Equation (4.23)) for a chosen point P. In this subsection, we approximate the integral part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in Equation (4.23) by using the Riemann sums.

First, we show that the integrated function has few singular points, otherwise the Riemann sums can have big error terms. In the expression of $\log(S(X))$ in Equation (4.23), the only possible singularities of the integration come from the zero locus of $\|\theta\|_{g-1}$. Recall that $\|\theta\|_{g-1}(D)$ vanishes if and only if D is rationally equivalent to an effective divisor of degree $g-1$. The following proposition implies that there is only 1 singular point in the integration.

Proposition 4.6.15. *Let X be a non-hyperelliptic compact Riemann surface of genus 3, i.e. a plane quartic curve. We choose two points P and Q on X. If P is a non-Weierstrass point, then $\|\theta\|_{g-1}(gP-Q)=0$ if and only if $Q=P$.*

Proof. We just need to show that $3P-Q$ is equivalent to an effective divisor exactly when $P=Q$. By Riemann-Roch, we get

$$h^0(\mathcal{O}(3P)) - h^0(\mathcal{O}(K_X - 3P)) = \chi(\mathcal{O}_X) + \deg(3P) = 1. \tag{4.27}$$

Since P is not a Weierstrass point, by Definition 4.6.2, we have $h^0(K_X - 3P) = 0$. Thus $h^0(\mathcal{O}(3P)) = 1$, which means that $\Gamma(\mathcal{O}(3P), X) = \mathbb{C}$. Thus the equality $3P \sim Q+U+V$ implies that $P=Q=U=V$, otherwise we should have $h^0(\mathcal{O}(3P)) \geq 2$. □

The defining polynomial $\mathfrak{f}$ of $\mathfrak{C}_{\mathbb{C}}$ on the affine patch $U_{\mathbb{C}}$ is given by

$$y^3 + (x^2 - x)y^2 - x^3 y + x^2 + x = 0.$$

For a generic $x \in \mathbb{C}$, there are three solutions of y such that $\mathfrak{f}(x,y)=0$. The following remark explains how we label the three solutions.

Remark 4.6.16. *(Important) We will label these y_i's by the cubic roots formula in Appendix I. This labelling is well-defined except at finitely many ramification points of the map $(x,y) \to x$. This finite set does not influence our numerical approximation. We will use this label frequently in the computation.*

Second, we show that the volume form decreases quickly, thus we can reasonably carry out the Riemann sums in a finite region. This can be summarized as the following proposition.

Proposition 4.6.17. *As $|x| \to \infty$, we have the following asymptotic approximation*

$$\mu_{Ar}\big|_{(x,y)} = O\left(\frac{1}{x^4}\right).$$

Sketch of proof : We first write $\mathfrak{f} = 0$ in the form

$$y^3 + (x^2 - x)y^2 - x^3 y + x^2 + x = 0.$$

Using the cubic equation formula in Appendix I, we can get asymptotic approximations for roots. Taking $i = 1$ in Appendix I as an example, we can subsequently get

$$\begin{aligned} u &= -\frac{x^6}{27} + \text{lower degree terms}, \\ m &= -\frac{x^2}{3} + \text{lower degree terms}, \\ n &= -\frac{x^2}{3} + \text{lower degree terms}, \\ y_1 &= -x^2 + \text{lower degree terms}. \end{aligned}$$

Substituting this into the basis of differential forms (4.25), we can get $\mu_{Ar}|_{(x,y_1)} = O(\frac{1}{x^4})$.

For $i = 2$ or 3, the quadratic term of y_i gets cancelled, and we can show

$$y_i = c_i x + \text{lower degree terms},$$

where c_i are constants that can be explicitly computed. Substituting this to Equations (4.25), we get $\mu_{Ar}|_{(x,y_i)} = O(\frac{1}{x^4})$. QED

Finally, we can numerically compute $\log(S(\mathfrak{C}_{\mathbb{C}}))$ with the above two propositions. Recall the expression of $\log(S(X))$ in Equation (4.23). Since $\mathfrak{C}_{\mathbb{C}} \backslash U_{\mathbb{C}}$ is a 0-measure set, we only need to compute

$$\int_{U_{\mathbb{C}}} \log\|\theta\|_{g-1}(3P - Q) \cdot \mu_{Ar}(Q).$$

With Remark 4.6.16, we denote the i-th y-coordinate over x by $y_i(x)$. We denote by $U^0_{\mathbb{C}}$ the set where the index i is well-defined ($U_{\mathbb{C}} \backslash U^0_{\mathbb{C}}$ is a 0-measure set), then our computation is reduced to

$$\sum_{i=1}^{3} \int_{(x,y_i(x)) \in U^0_{\mathbb{C}}} \log\|\theta\|_{g-1}(3P - Q_i) \cdot \mu_{Ar}(Q_i), \tag{4.28}$$

where $Q_i = (x, y_i(x))$.

If we consider the complex number x as a point $(\mathrm{Re}(x), \mathrm{Im}(x))$ in $\mathbb{R}^2$, then Equation (4.28) is actually an integration of a real-valued function (with possible singularities), denoted by F, over $\mathbb{R}^2$. We use Riemann sums to approximate the integral of F on $\mathbb{R}^2$.

By Proposition 4.6.15, the integrated function has only 1 singular point P. According to Proposition 4.6.17, the volume form μ_{Ar} decreases quickly as $|x|$ becomes large. Thus to obtain a reasonable approximation, we only need to take the Riemann sums of F on a finite region in $\mathbb{R}^2$, which contains P. We display our algorithm as follows, and the Magma code can be found in Appendix VII.

Algorithm 3 Integration part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in Equation (4.23)

Input: $\mathfrak{f}$: the defining polynomial of $\mathfrak{C}_{\mathbb{C}}$ on $U_{\mathbb{C}}$

X: the Riemann surface given by $f = 0$

μ_{Ar}: the volume form (considered as a function on $\mathfrak{C}_{\mathbb{C}}$)

$\|\theta\|_{g-1}$: the theta function on $\mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}})$

P_{bs}: the fixed point

Output: $-3^2 * LogS_{int}$: the integration part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$

y_i = the i-th root function in x for $i = 1, 2, 3$ (Appendix I)

define a function Pt which sends the tuple (a, b, i) to the point $(a + b * I, y_i(a + b * I))$ on $\mathfrak{C}_{\mathbb{C}}$

scale= 0.1

radius= 50

$LogS_{int} = 0$

for j in $[1..ceiling(2 * radius/scale)]$ **do**

  for k in $[1..ceiling(2 * radius/scale)]$ **do**

    $\mathrm{Re}x_0 = -radius + j * scale$

    $\mathrm{Im}x_0 = -radius + k * scale$

    $Q_i = Pt(\mathrm{Re}x_0, \mathrm{Im}x_0, i)$ for $i = 1, 2, 3$

    $LogS_{int} = LogS_{int} + \sum_{i=1}^{3} Log(\|\theta\|_{g-1}(3P - Q_i)) \cdot \mu_{Ar}(Q_i) * scale^2$

  end for

end for

return $-3^2 * LogS_{int}$

Remark 4.6.18. *In the algorithm, we take the Riemann sums on the region* $|\mathrm{Re}\, x| \leq 50$, $|\mathrm{Im}\, x| \leq 50$, *and choose the size of the grids (corresponds to scale in the code above) to be* $\frac{1}{10}$. *In practice, we use finer grids for the region* $|\mathrm{Re}\, x| \leq 10$, $|\mathrm{Im}\, x| \leq 10$ *(we choose the size to be* $\frac{1}{100}$*). This can improve the accuracy of our numerical approximation.*

Our computation can be summarized as follows.

Computation 4.6.19. $\log(S(\mathfrak{C}_{\mathbb{C}})) \approx 1.10$

Remark 4.6.20. *Actually, we carried out the computation for two different choices of the fixed point* P. *One is the default base point* P_{bs} *chosen by Magma (Proposition 4.6.7), and the other one can be represented as* $(\mathrm{Re}x = 1, \mathrm{Im}x = 2, i = 3)$. *They gave*

1.07 *and* 1.13 *respectively, and we choose their arithmetic mean* 1.10 *as an approximation of* $\log(S(\mathfrak{C}_{\mathbb{C}}))$. *In Section 4.9, we will show that this is at least enough for showing the positivity of* $\langle\Delta, \Delta\rangle$, *although this approximation for* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *is not that precise.*

Remark 4.6.21. *This is only a numerical approximation. It is difficult to give a theoretic bound for the error term of our numerical integration, since we do not know how the term* $\log(\|\theta\|_{g-1}(3P - Q))$ *varies.*

With the calculation we carried out so far, the computation of the Green's function $G(x, y)$ on $U_{\mathbb{C}}$ is an easy evaluation by Theorem 4.6.1.

4.7 Computation of $T(\mathfrak{C}_{\mathbb{C}})$ and $H(\mathfrak{C}_{\mathbb{C}})$

In this section, we compute two invariants $T(\mathfrak{C}_{\mathbb{C}})$ (in Subsection 4.7.1) and $H(\mathfrak{C}_{\mathbb{C}})$ (in Subsection 4.7.2), whose relation with $\delta(\mathfrak{C}_{\mathbb{C}})$ and $\varphi(\mathfrak{C}_{\mathbb{C}})$ can be found in Theorem 4.7.3 and Theorem 4.7.7. The main references for this section are [12] and [65].

4.7.1 Computation of $T(\mathfrak{C}_{\mathbb{C}})$

Let X be a compact Riemann surface of genus $g > 1$ and let z be a local coordinate near $P \in X$. we define

$$\|F_z\|(P) \coloneqq \lim_{Q \to P} \frac{\|\theta\|_{g-1}(gP - Q)}{|z(P) - z(Q)|^g}.$$

Remark 4.7.1. *See Page 31 in [25] for a discussion of the convergence of this limit.*

Let $W_z(\omega)(P)$ be the Wronskian determinant (Equation (4.24)) at P with respect to an orthonormal basis of holomorphic forms ω on X. We define

$$T(X)_{z,P} \coloneqq \|F_z\|(P)^{-(g+1)} \cdot \prod_{W \in \mathfrak{W}} \|\theta\|_{g-1}(gP - W)^{(g-1)/g^3} |W_z(\omega)(P)|^2, \tag{4.29}$$

where the product goes through the Weierstrass points on X, counted with weights.

Lemma 4.7.2. *The number* $T(X)_{z,P}$ *is an invariant of* X*, that is, it does not depend on the choice of* z *and* P.

Proof. See Theorem 2.1.3 and Proposition 2.2.7 in [12]. □

For simplicity, we write $T(X)$ for this invariant. The reason for computing $T(X)$ is the following theorem.

Theorem 4.7.3. *The Faltings* δ *invariant, and the constants* $T(X)$ *and* $S(X)$ *satisfy*

$$\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X).$$

Proof. Theorem 2.1.3 in [12]. □

Now we show how to compute $T(X)$. We are already able to compute the theta function appearing in $T(X)$, and thus only $W_z(\omega)(P)$ and $\|F_z(P)\|$ remain to be done.

For all but finitely many points on $U_{\mathbb{C}}$, the x-coordinate is a local coordinate. Fix a point P, by Appendix I we can write $P = (x_P, y_i(x_P))$ for a certain index i. If we choose a real vector (a, b), then the point

$$Q_{abn} = (x_P + (a + bI) \cdot 10^{-n}, y_i(x_P + (a + bI) \cdot 10^{-n}))$$

approaches P from the direction (a, b) as n goes to infinity. Taking x as the local coordinate, we can approximate $\|F_x\|(P)$ by

$$\|F_x\|(P) \approx \frac{\|\theta\|_{g-1}(gP - Q_{abn})}{|x(P) - x(Q_{abn})|^g} = \frac{\|\theta\|_{g-1}(gP - Q_{abn})}{|10^{-n} \cdot (a + bI)|^g}$$

for a properly chosen n.

Remark 4.7.4. *(1) In our computation, we can choose the vector (a, b) to be a point on the unit circle.*

(2) In our computation of $T(\mathfrak{C}_{\mathbb{C}})$, we choose 10^{-50} as the precision. For this precision, we can choose n in $\{4, 5, 6, 7\}$. The reason is that the Abel-Jacobi map implemented in Magma is not as precise as the chosen precision.

(3) The Wronskian determinant part decreases quickly as the coordinates of P goes away from the origin of the chosen affine patch (the denominator has a higher degree than the numerator). If we choose a point where the coordinates of P are big, the Wronskian determinant part can be smaller than the precision we set. This numerical issue in Magma can lead to unstable output.

(4) When the three requirements above are satisfied, we can find that the output does not depend significantly on the choice of the point P. Thus we have a reliable approximation of $\|F_x\|(P)$.

The computation of $W_z(\omega)$, defined in Equation (4.24), is just some lengthy but easy calculation. The main tool here is taking implicit differentiation. Recall that in Subsection 4.6.2, we have an ordered basis of holomorphic forms (they are not orthonormal) on $\mathfrak{C}_{\mathbb{C}}$, which can be written in the following form

$$\left\{ \frac{dx}{\mathfrak{f}_y}, \frac{ydx}{\mathfrak{f}_y}, \frac{xdx}{\mathfrak{f}_y} \right\}.$$

For a general point $P = (x_0, y_0)$ on $\mathfrak{C}_{\mathbb{C}}$, x is a local coordinate. We take

$$g_1 = \frac{1}{\mathfrak{f}_y}, g_2 = \frac{y}{\mathfrak{f}_y}, g_3 = \frac{x}{\mathfrak{f}_y}.$$

For points close to P, y is a function in x and it makes sense to take derivatives of g_i's with respect to x. These derivatives with respect x can be expressed as rational functions of x, y, y' and y''.

Taking y' as an example, we take the implicit derivative at both sides of $\mathfrak{f} = 0$ with respect to x:

$$-3x^2y - x^3y' + 2xy^2 + 2x^2yy' - y^2 - 2xyy' + 3y^2y' + 2x + 1 = 0.$$

This gives

$$y'(P) = \frac{3x_0^2y_0 - 2x_0y_0^2 + y_0^2 - 2x_0 - 1}{-x_0^3 + 2x_0^2y_0 - 2x_0y_0 + 3y_0^2}.$$

We can get the values of g_i's and their derivatives in similar way.

Finally, we use the coefficients in Proposition 4.6.13 to compute the Wronskian determinant with respect to the orthonormal forms $\{\omega_{onj}\}_{1\leq j\leq 3}$.

Magma code for this subsection can be found in Appendix VIII. Our computation yields the following end result.

Computation 4.7.5. $T(\mathfrak{C}_{\mathbb{C}}) \approx 0.002544$.

Remark 4.7.6. *In Section 4.9, we will see that our computation for $T(\mathfrak{C}_{\mathbb{C}})$ is stable among different choices of P.*

4.7.2 Computation of $H(\mathfrak{C}_{\mathbb{C}})$

For a principally polarized abelian variety (A, Θ) of dimension g with period matrix $(1|\tau)$, we define a 1-1 form

$$v_{(A,\Theta)} := \frac{i}{2}\sum_{j,k=1}^{g} (\mathrm{Im}\tau)^{-1}_{jk} dz_j \wedge d\bar{z}_k. \tag{4.30}$$

We define $H(A, \Theta)$ as

$$H(A, \Theta) := \frac{1}{g!}\int_A \log\|\theta\| v^g, \tag{4.31}$$

where $\|\theta\|$ has an explicit expression in Proposition 4.5.4. For a compact Riemann surface X, we denote $H(\mathrm{Jac}(X), \Theta_{can})$ by $H(X)$. The following theorem explains the reason we compute $H(X)$.

Theorem 4.7.7. *For any compact Riemann surface X of genus $g \geq 1$, we have*

$$\delta(X) = -24H(X) + 2\varphi(X) - 8g\log 2\pi.$$

Proof. See Theorem 5.4 in [65]. □

The following two points explain why we think it is reasonable to believe that we can approximate $H(\mathfrak{C}_{\mathbb{C}})$ to good precision by taking Riemann sums.

(1) Although $\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})$ is 6-dimensional as a real manifold, it is a relatively small torus (with respect to the default choice of the base point and the (co)homology basis implemented in Magma).

(2) The singular points of the integrated function in $H(\mathfrak{C}_{\mathbb{C}})$ are equal to the theta divisor, a compact submanifold of real codimension 2 in $\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})$. Thus it is reasonable to believe that the integration of this singular function behaves well (in an analytic sense) on $\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})$.

Now we give a description of the computation.

First, we simplify the form $\frac{1}{g!}v^g$ in Formula (4.31). This is done by

$$\begin{aligned}\frac{1}{g!}v^g &= \left(\frac{i}{2}\right)^g (\det(\mathrm{Im}\tau)^{-1}) \cdot \bigwedge_{j=1}^{g}(dz_j \wedge d\bar{z}_j)\\ &= (\det(\mathrm{Im}\tau)^{-1}) \cdot \bigwedge_{j=1}^{g}(dx_j \wedge dy_j),\end{aligned}$$

where $z_j = x_j + iy_j$.

Second, we calculate the volume of the complex torus $T_{\mathfrak{C}_{\mathbb{C}}} := \mathrm{Jac}(\mathfrak{C}_{\mathbb{C}}) = \mathbb{C}^3/\mathbb{Z}^3 + \tau\mathbb{Z}^3$. The volume of $T_{\mathfrak{C}_{\mathbb{C}}}$ is

$$\mathrm{Vol}(T_{\mathfrak{C}}) = \left|\det\begin{pmatrix} I & 0\\ \mathrm{Re}\tau & \mathrm{Im}\tau\end{pmatrix}\right| = \det\mathrm{Im}\tau.$$

Finally, we can take the Riemann sums. By splitting each edge of $T_{\mathfrak{C}_{\mathbb{C}}}$ into c parts, we get c^6 small polyhedrons. We approximate $H(\mathfrak{C}_{\mathbb{C}})$ by

$$\begin{aligned}&\sum_{i=1}^{c^6}\log\|\theta\|(v_i)(\det\mathrm{Im}\tau)^{-1}\mathrm{Vol}(T_{\mathfrak{C}})/c^6\\ =&\sum_{i=1}^{c^6}\log\|\theta\|(v_i)/c^6.\end{aligned}\tag{4.32}$$

where v_i is a chosen point in each small polyhedron.

Code for this subsection can be found in Appendix IX, and the result of our computation can be summarized as follows.

Computation 4.7.8. $H(\mathfrak{C}_{\mathbb{C}}) \approx -0.70356$.

Similar to the approximation of $\log(S(\mathfrak{C}_{\mathbb{C}}))$, we lack the bound of error terms. In Section 4.9, we will verify that our numerical approximation of $\int_{\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})} \|\theta\|^2 \nu^g$ is very good.

4.8 What can we get from the computation?

Our primary goal is to compute the height of a canonical Gross-Schoen cycle of a certain non-hyperelliptic genus 3 curve $\mathfrak{C}_{\mathbb{Q}}$. Summing up all the computations in this chapter, we have the following result.

Computation 4.8.1. *For the plane curve $\mathfrak{C}$ defined by*

$$-X^3Y + X^2Y^2 - XY^2Z + Y^3Z + X^2Z^2 + XZ^3 = 0,$$

we have the following results:

(1) $\delta(\mathfrak{C}_{\mathbb{C}}) \approx -24.87$*,*

(2) $\varphi(\mathfrak{C}_{\mathbb{C}}) \approx 1.17$*,*

(3) $\deg \det f_* \overline{\omega}_{\mathfrak{C}} \approx -2.9190567336$*,*

(4) $(\overline{\omega}, \overline{\omega})_{Ar} \approx 3.43$*,*

(5) $(\hat{\omega}, \hat{\omega})_{ad} \approx 1.55$*,*

(6) $\langle \Delta, \Delta \rangle \approx 0.60$*.*

Proof. (1) By Theorem 4.6.19, Proposition 4.7.3 and Proposition 4.7.5 we obtain

$$\begin{aligned}\delta(\mathfrak{C}_{\mathbb{C}}) &= 4\left(\log(T(\mathfrak{C}_{\mathbb{C}})) - \frac{2}{9}\log(S(\mathfrak{C}_{\mathbb{C}}))\right)\\ &\approx 4 \cdot (\log(0.002544) - \frac{2}{9} \cdot 1.1)\\ &\approx -24.87.\end{aligned}$$

(2) Then by Theorem 4.7.7 and Proposition 4.7.8, we obtain

$$\begin{aligned}\varphi(\mathfrak{C}_{\mathbb{C}}) &= \frac{\delta(\mathfrak{C}_{\mathbb{C}}) + 24H(\mathfrak{C}_{\mathbb{C}}) + 24\log 2\pi}{2}\\ &\approx \frac{-24.87 - 24 \cdot 0.70356 + 24 \cdot \log 2\pi}{2}\\ &\approx 1.17.\end{aligned}$$

(3) By Equation (2.8), Proposition 4.4.5 and Proposition 4.4.8, we obtain

$$\begin{aligned}
\deg \det f_* \overline{\omega}_{\mathcal{X}/B} &= \sum_{p \text{ prime}} \frac{\mathrm{ord}_p(\chi'_{18}) \log p}{18} - \frac{\log \|\chi'_{18}\|_{\mathrm{Hdg}}(\mathfrak{C}_{\mathbb{C}})}{18} \\
&\approx \frac{2}{18} \log(29 \cdot 163) - 3.8591729592 \\
&\approx -2.9190567336.
\end{aligned}$$

(4) By Corollary 1.3.11 and Proposition 4.4.1, we obtain

$$\begin{aligned}
(\overline{\omega}, \overline{\omega})_{Ar} &= 12 \deg \det f_* \overline{\omega}_{\mathfrak{C}} - \sum_{p \text{ prime}} \delta(\mathfrak{C}_p) \log p - \delta(\mathfrak{C}_{\mathbb{C}}) + 4g \log 2\pi \\
&\approx -2.9190567336 \cdot 12 - \log(29 \cdot 163) + 24.87 + 12 \cdot \log(2\pi) \\
&\approx 3.43.
\end{aligned}$$

(5) By Theorem 1.5.3 and Proposition 4.4.1, we obtain

$$\begin{aligned}
(\hat{\omega}, \hat{\omega})_{ad} &= (\overline{\omega}, \overline{\omega})_{Ar} - \sum_{p \text{ prime}} \epsilon(\mathfrak{C}_{\mathbb{Z}_p}) \log p \\
&\approx 3.43 - \frac{2 \cdot \log(29 * 163)}{9} \\
&\approx 1.55.
\end{aligned}$$

(6) By Theorem 1.5.6 and Proposition 4.4.1, we obtain

$$\begin{aligned}
\langle \Delta, \Delta \rangle &= \frac{7}{4} (\hat{\omega}, \hat{\omega})_{ad} - \sum_{v \in M(\mathbb{Q})} \varphi(\mathfrak{C}_v) \log(N(v)) \\
&\approx \frac{7}{4} \cdot 1.55 - \log(29 \cdot 163)/9 - 1.17 \\
&\approx 0.60.
\end{aligned}$$

□

Remark 4.8.2. *(Important) The reason we use different precisions is that some invariants (like* $\mathrm{ord}_v(\chi'_{18})$*) can be computed to fairly high precision, while others (like* $\log(S(\mathfrak{C}_{\mathbb{C}}))$*) cannot. For the former ones, we use* 10 *as the precision. For the latter ones, we choose the precisions that are stable among our computations. For example, the first six digits after the decimal point of* $T(\mathfrak{C}_{\mathbb{C}})$ *are stable among different choices of* P *(Equation (4.29)).*

Remark 4.8.3. $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *is used in the computation of* $\varphi(\mathfrak{C}_{\mathbb{C}})$ *and* $\delta(\mathfrak{C}_{\mathbb{C}})$*, thus we use the precision of* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *for* $\varphi(\mathfrak{C}_{\mathbb{C}})$ *and* $\delta(\mathfrak{C}_{\mathbb{C}})$*. In* $\deg \det f_* \overline{\omega}_{\mathcal{X}/B}$*, all components can be computed to arbitrary precision, thus we use high precision.*

4.9 Why do we think these approximations are reliable?

In this subsection, we show that these approximations are stable among choices and compatible with known facts. Recall that we defined $\log(S(X))$, $T(X)$ and $H(X)$ in Equation (4.23), Equation (4.29) and Equation (4.31).

In Theorem 3.3.2, we can find that all invariants except $\lambda(\mathfrak{C}_{\mathbb{C}})$ can easily be computed to arbitrary precision. For $\lambda(\mathfrak{C}_{\mathbb{C}})$, we decomposed it into a linear sum of other invariants: $H(\mathfrak{C}_{\mathbb{C}})$, $\log(S(\mathfrak{C}_{\mathbb{C}}))$ and $T(\mathfrak{C}_{\mathbb{C}})$. However, we can only compute these invariants by approximation (numerical integration and taking a limit in the computation of $T(\mathfrak{C}_{\mathbb{C}})$).

For $T(\mathfrak{C}_{\mathbb{C}})$ and $\log(S(\mathfrak{C}_{\mathbb{C}}))$, we need to fix a point P. For $H(\mathfrak{C}_{\mathbb{C}})$, we need to split each edge of the torus $T_{\mathfrak{C}_{\mathbb{C}}}$ into c segments (see Equation (4.32)). The first thing we check is to show that our computations are stable along these choices.

Using the code for $T(\mathfrak{C}_{\mathbb{C}})$ (Appendix VIII), we can find that the output does not change significantly among different choices of the fixed point P, even though $T(\mathfrak{C}_{\mathbb{C}})$ is a product of factors which depend wildly on the choice of P. For example, the Wronskian part can be smaller than 10^{-24} and bigger than 10^{-2} for different choices of the fixed point P.

For $H(\mathfrak{C}_{\mathbb{C}})$, we compute it for $c = 19$ and 23. The outputs are quite stable, giving around -0.70356438 and -0.70355787 respectively.

As we explained in Remark 4.6.20, we choose two distinct points as the point P in Equation (4.23). Since $S(\mathfrak{C}_{\mathbb{C}})$ is an invariant of $\mathfrak{C}_{\mathbb{C}}$, it should not depend on the choice of P. It turns out that our approximation for $\log(S(\mathfrak{C}_{\mathbb{C}}))$ is less precise. The two points are the default base point P_{bs} chosen by Magma and the point represented by $(\mathrm{Re}x = 1, \mathrm{Im}x = 2, \text{index} = 3)$ (see Remark 4.6.16 for an explanation of the notation), where the index is explained in Appendix I. We get 1.07 and 1.13 for $\log(S(\mathfrak{C}_{\mathbb{C}}))$ respectively, and we take their arithmetic mean 1.10 as the approximation of $\log(S(\mathfrak{C}_{\mathbb{C}}))$.

By Equation (1.18), Theorem 3.3.2, Theorem 4.7.3 and Theorem 4.7.7, we can decompose $\langle \Delta, \Delta \rangle$ as follows

$$\langle \Delta, \Delta \rangle = -12H(\mathfrak{C}_{\mathbb{C}}) + 2\log(S(\mathfrak{C}_{\mathbb{C}})) - 9\log(T(\mathfrak{C}_{\mathbb{C}})) - 63.7966513771.$$

Although we cannot approximate $\log(S(\mathfrak{C}_{\mathbb{C}}))$ precisely, we can find that it contributes less to $\langle \Delta, \Delta \rangle$ than other terms. The invariants $H(\mathfrak{C}_{\mathbb{C}})$ and $\log(T(\mathfrak{C}_{\mathbb{C}}))$ contribute much more, but our approximations for them are also much more stable.

Remark 4.9.1. *(Risk) Note that the functions in the integration of* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *and* $H(\mathfrak{C}_{\mathbb{C}})$ *are singular. Thus it is still possible that our numerical approximation is far away from the correct answer.*

Summary 4.9.2. *The first part of our checking can be summarized as follows:*

(1) Our code gives relatively stable approximations for $H(\mathfrak{C}_{\mathbb{C}})$, $\log(T(\mathfrak{C}_{\mathbb{C}}))$ *and* $\log(S(\mathfrak{C}_{\mathbb{C}}))$.

(2) Among the three invariants, our approximation for $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *is less satisfying, but* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *contributes less to* $\langle\Delta,\Delta\rangle$.

The second part of our check is comparing our results with known facts.

In Theorem 1.5.6 (1), we can find $\varphi(\mathfrak{C}_{\mathbb{C}}) > 0$. This is compatible with $\varphi(\mathfrak{C}_{\mathbb{C}}) \approx 1.17$.

In Theorem 2.2.6, we have an equality about the discriminant of a plane quartic curve and the modular form $\tilde{\chi}_{18}$. Using Magma, we can get $\mathrm{Disc}(\mathfrak{F}) = 29 \cdot 163$, which shows that 29 and 163 are the only bad primes in particular. This is compatible with our computation since

$$\frac{\mathrm{Disc}(\mathfrak{F})^2}{(2\pi)^{54}\frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}} \approx 0.9999991$$

The code for this can be found in Appendix X.

Proposition 4.9.3. *Let* $f : C \to \mathrm{Spec}(O_k)$ *be a semistable arithmetic surface of genus* $g \geq 1$, *where* k *is a number field. Then we have the following inequality*

$$\deg \det f_*\overline{\omega} \geq -\log(\pi\sqrt{2})g \cdot [k : \mathbb{Q}]$$

Proof. See Equation (1.8) in [16]. □

This is compatible with our computation since

$$-2.9190567336 \geq -4.4739104284 \approx -3 \cdot \log(\pi\sqrt{2}).$$

Proposition 4.9.4. *Let* X *be a compact Riemann surface of genus* $g \geq 1$. *Then we have* $H(X) \leq -\frac{g}{4}\log 2$.

Proof. See Proposition 2.1 in [65]. □

This is compatible with our computation since

$$-0.70356 \leq -0.51986 \approx -\frac{3}{4}\log 2.$$

Remark 4.9.5. *Recall the definition of* $\|\theta\|$ *and* ν *(paragraphs around Equation (4.15)). We have the following identity*

$$\frac{1}{3!}\int_{\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})} \|\theta\|^2\nu^3 = 2^{-3/2}.$$

This can be used to check the correctness of our code for $H(\mathfrak{C}_{\mathbb{C}})$, *since we only need to replace the integrated function* $\log\|\theta\|$ *by* $\|\theta\|^2$. *The code is almost the same as that of* $H(\mathfrak{C}_{\mathbb{C}})$, *thus we omit it. Taking* $c = 19$, *we can find that the difference between our approximations of* $\frac{1}{3!}\int_{\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})} \|\theta\|^2 v^g$ *and* $2^{-3/2}$ *is even smaller than* 10^{-10}.

According to Corollary 5.7 in [70], the pairing $(\hat{\omega}, \hat{\omega})_{ad}$ is non-negative. This is compatible with the fact $(\hat{\omega}, \hat{\omega})_{ad} \approx 1.55$ computed in Theorem 4.8.1.

Thus we believe that our approximations are reliable!