Explicit computation of the height of a Gross-Schoen Cycle

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Chapter 3

Arakelov geometry in genus 3

In this chapter, we discuss Arakelov geometry with an emphasis on genus 3 curves. In Section 3.1, based on the work of Z. Cinkir, we show a classification of pm-graphs of genus 3 and compute admissible invariants associated to it. In Section 3.2, based on the work of K. Yamaki, we show that for a genus 3 graphically hyperelliptic curve over a function field, the height $\langle\Delta, \Delta\rangle$ vanishes if and only if the curve is hyperelliptic. In Section 3.3, based on the work of R. de Jong, we show a unboundedness result of $\langle\Delta, \Delta\rangle$ for genus 3 curves over number fields.

We get a result on hyperelliptic graphs and apply it to genus 3 polarized graphs (Proposition 3.2.20). In Theorem 3.3.12, we give a criterion for the unboundedness of the heights of a family of curves over $\mathbb{Q}$. To the best of the author's knowledge, these are new results.

3.1 Admissible invariants for genus 3 curves

Subsection 3.1.1 is about the explicit computation for genus 3 pm-graphs. We refer to Section 1.4 for terminology on pm-graphs. Subsection 3.1.2 contains two tables for the invariants on genus 3 pm-graphs with the first Betti number $b_1 = 0$ or 1.

We will use results in this section to compute the admissible invariants of our main curve $\mathfrak{C}_{\mathbb{Q}}$ in Theorem 4.4.1.

3.1.1 Computation for genus 3 curves

In this subsection, we explain how to explicitly compute the six invariants discussed in Theorem 1.4.39. To begin with, we specialize Theorem 1.4.39 to the case $g = 3$.

Proposition 3.1.1. *Let $\overline{\Gamma}$ be a pm-graph of genus 3. Then we have*

$$\begin{aligned}\varphi(\overline{\Gamma}) &= \frac{13}{3}\tau(\overline{\Gamma}) + \frac{\theta(\overline{\Gamma})}{12} - \frac{\delta(\overline{\Gamma})}{4},\\ \lambda(\overline{\Gamma}) &= \frac{3}{7}\tau(\overline{\Gamma}) + \frac{\theta(\overline{\Gamma})}{56} + \frac{\delta(\overline{\Gamma})}{14},\\ \epsilon(\overline{\Gamma}) &= \frac{8}{3}\tau(\overline{\Gamma}) + \frac{\theta(\overline{\Gamma})}{6}.\end{aligned}$$

Proof. Substitute $g = 3$ to Theorem 1.4.39. □

Now we show how to compute the six invariants of $\overline{\Gamma}_{ex}$ (Figure 3.1.1), a genus 3 pm-graph with no eliminable points. This pm-graph is non-irreducible and contains 1 cycle. We would like to use this example to show that it is possible to compute the invariants by techniques described. This method is also used by Z. Cinkir in [9]. Letters are the lengths of edges and integers are the polarization.

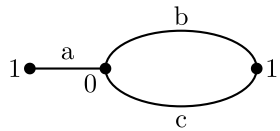


Figure 3.1.1: $\overline{\Gamma}_{ex}$

Proposition 3.1.2. *For the pm-graph $\overline{\Gamma}_{ex}$, we have*

$$\begin{aligned}\delta(\overline{\Gamma}_{ex}) &= a + b + c,\\ \tau(\overline{\Gamma}_{ex}) &= \frac{\delta(\overline{\Gamma}_{ex})}{12} + \frac{a}{6},\\ \theta(\overline{\Gamma}_{ex}) &= 6a + \frac{8bc}{b+c},\\ \varphi(\overline{\Gamma}_{ex}) &= \frac{\delta(\overline{\Gamma}_{ex})}{9} + \frac{6bc + 11a(b+c)}{9(b+c)},\\ \lambda(\overline{\Gamma}_{ex}) &= \frac{3\delta(\overline{\Gamma}_{ex})}{28} + \frac{4bc + 5a(b+c)}{28(b+c)},\\ \epsilon(\overline{\Gamma}_{ex}) &= \frac{2\delta(\overline{\Gamma}_{ex})}{9} + \frac{12bc + 13a(b+c)}{9(b+c)}.\end{aligned}$$

Proof. As a metrized graph, Γ_{ex} can be written as the wedge sum of two irreducible components $\Gamma_a \vee \Gamma_{bc}$, where Γ_a is obtained by contracting the edges of length b and c and Γ_{bc} is obtained by contracting the edge of length a. By the additivity of the six

Figure 3.1.2: Irreducible components

invariants (Remark 1.4.41), we just need to compute the invariants on $\overline{\Gamma}_a$ and $\overline{\Gamma}_{bc}$, where the polarization is induced from that on $\overline{\Gamma}_{ex}$. Figure 3.1.2 is an illustration for this.

For δ, it is trivial that $\delta(\overline{\Gamma}_a) = a$ and $\delta(\overline{\Gamma}_{bc}) = b + c$, thus

$$\delta(\overline{\Gamma}_{ex}) = a + b + c.$$

For θ, by Equation (1.17), we have

$$\begin{aligned} \theta(\overline{\Gamma}_a) &= 2 \times (1 - 2 + 2) \times (1 - 2 + 4) \times a = 6a, \\ \theta(\overline{\Gamma}_{bc}) &= 2 \times (2 - 2 + 2) \times (2 - 2 + 2) \times \frac{bc}{b+c} = \frac{8bc}{b+c}, \end{aligned}$$

thus

$$\theta(\overline{\Gamma}_{ex}) = 6a + \frac{8bc}{b+c}.$$

Recall the interpretation of τ in Definition 1.4.37. For $\overline{\Gamma}_a$, we take y to be a vertex p, and then we get $r(x,p) = d(x,p)$, where $d(\cdot,\cdot)$ is the path distance function. Thus we have

$$\tau(\overline{\Gamma}_a) = \frac{1}{4}\int_{\Gamma_a} r_x(x,p)^2 dx = \frac{1}{4}\int_0^a dx = \frac{a}{4}.$$

For $\overline{\Gamma}_{bc}$, by the formula of electrical resistance in a parallel connection, we get

$$r(x,y) = \frac{d(x,y)(b + c - d(x,y))}{b+c}.$$

Taking y to be a vertex p, we have

$$\tau(\overline{\Gamma}_{bc}) = \frac{1}{4}\int_{\Gamma_{bc}} r_x(x,p)^2 dx = \frac{1}{4}\int_0^{b+c} \left(\frac{b+c-2x}{b+c}\right)^2 dx = \frac{b+c}{12}.$$

By the additivity of τ, we get

$$\tau(\overline{\Gamma}_{ex}) = \frac{\delta(\overline{\Gamma}_{ex})}{12} + \frac{a}{6}.$$

According to Proposition 3.1.1, we get

$$\begin{aligned}
\varphi(\overline{\Gamma}_{ex}) &= \frac{a+b+c}{9} + \frac{6bc+11a(b+c)}{9(b+c)}, \\
\lambda(\overline{\Gamma}_{ex}) &= \frac{3(a+b+c)}{28} + \frac{4bc+5a(b+c)}{28(b+c)}, \\
\epsilon(\overline{\Gamma}_{ex}) &= \frac{2(a+b+c)}{9} + \frac{12bc+13a(b+c)}{9(b+c)}.
\end{aligned}$$

□

The whole list of genus 3 pm-graphs without eliminable points and their invariants can be found in [9]. In this thesis, we copy part of this list (containing those pm-graphs with the first Betti number $b_1 = 0$ or 1) in Table 3.1 and Table 3.2.

We can find from Table 3.2 that $\lambda(\overline{\Gamma}) \geq \frac{3\delta(\overline{\Gamma})}{28}$ and $\epsilon(\overline{\Gamma}) \geq \frac{2\delta(\overline{\Gamma})}{9}$. These two bounds actually hold for all pm-graphs of genus 3.

The invariant $\varphi(\overline{\Gamma})$ is more complicated. When $b_1 \leq 1$, we can find from Table 3.2 that $\varphi(\overline{\Gamma}) \geq \frac{1}{9}\delta(\overline{\Gamma})$. This bound does not hold for a general genus 3 pm-graph. By a technical analysis of inequalities, Z. Cinkir proved the following proposition which is conjectured by X. Faber in Remark 5.1 in [22].

Proposition 3.1.3. *For a pm-graph $\overline{\Gamma}$ of genus 3, we have $\varphi(\overline{\Gamma}) \geq \frac{17\delta(\overline{\Gamma})}{288}$.*

Proof. See the proof of Claim on Page 332 in [9]. □

Remark 3.1.4. *Proposition 3.1.3 is not a corollary of Theorem 1.4.34 since $c(3) \leq \frac{17}{288}$.*

3.1.2 Tables for genus 3 pm-graphs

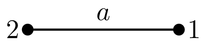
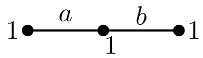
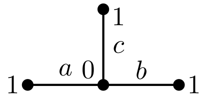
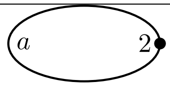
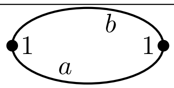
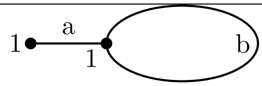
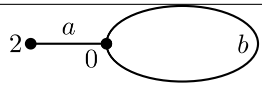
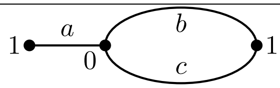
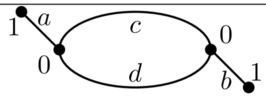
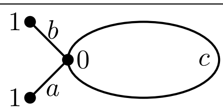
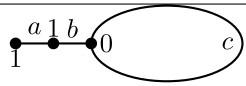
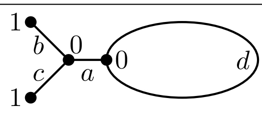
	$\overline{\Gamma}$	$\delta(\overline{\Gamma})$	$\tau(\overline{\Gamma})$	$\theta(\overline{\Gamma})$
$0I$		0	0	0
$0II$		a	$\frac{\delta(\overline{\Gamma})}{4}$	$6\delta(\overline{\Gamma})$
$0III$		$a+b$	$\frac{\delta(\overline{\Gamma})}{4}$	$6\delta(\overline{\Gamma})$
$0IV$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{4}$	$6\delta(\overline{\Gamma})$
$1I$		a	$\frac{\delta(\overline{\Gamma})}{12}$	0
$1II$		$a+b$	$\frac{\delta(\overline{\Gamma})}{12}$	$\frac{8ab}{a+b}$
$1III$		$a+b$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a}{6}$	$6a$
$1IV$		$a+b$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a}{6}$	$6a$
$1V$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a}{6}$	$6a+\frac{8bc}{b+c}$
$1VI$		$a+b+c+d$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b}{6}$	$6(a+b)+\frac{8cd}{c+d}$
$1VII$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b}{6}$	$6(a+b)$
$1VIII$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b}{6}$	$6(a+b)$
$1IX$		$a+b+c+d$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b+c}{6}$	$6(a+b+c)$

Table 3.1: Table of $\overline{\Gamma}$, $\delta(\overline{\Gamma})$, $\theta(\overline{\Gamma})$ and $\tau(\overline{\Gamma})$

	$\varphi(\overline{\Gamma})$	$\lambda(\overline{\Gamma})$	$\epsilon(\overline{\Gamma})$
$0I$	0	0	0
$0II$	$\frac{4\delta(\overline{\Gamma})}{3}$	$\frac{2\delta(\overline{\Gamma})}{7}$	$\frac{5\delta(\overline{\Gamma})}{3}$
$0III$	$\frac{4\delta(\overline{\Gamma})}{3}$	$\frac{2\delta(\overline{\Gamma})}{7}$	$\frac{5\delta(\overline{\Gamma})}{3}$
$0IV$	$\frac{4\delta(\overline{\Gamma})}{3}$	$\frac{2\delta(\overline{\Gamma})}{7}$	$\frac{5\delta(\overline{\Gamma})}{3}$
$1I$	$\frac{\delta(\overline{\Gamma})}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}$
$1II$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{2ab}{3(a+b)}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{ab}{7(a+b)}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{4ab}{3(a+b)}$
$1III$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{11a}{9}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{5a}{28}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{13a}{9}$
$1IV$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{11a}{9}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{5a}{28}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{13a}{9}$
$1V$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{6bc+11a(b+c)}{9(b+c)}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{4bc+5a(b+c)}{28(b+c)}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{12bc+13a(b+c)}{9(b+c)}$
$1VI$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{6cd+11(a+b)(c+d)}{9(c+d)}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{4cd+5(a+b)(c+d)}{28(c+d)}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{12cd+13(a+b)(c+d)}{9(c+d)}$
$1VII$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{11(a+b)}{9}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{5(a+b)}{28}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{13(a+b)}{9}$
$1VIII$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{11(a+b)}{9}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{5(a+b)}{28}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{13(a+b)}{9}$
$1IX$	$\frac{\delta(\overline{\Gamma})}{9} + \frac{11(a+b+c)}{9}$	$\frac{3\delta(\overline{\Gamma})}{28} + \frac{5(a+b+c)}{28}$	$\frac{2\delta(\overline{\Gamma})}{9} + \frac{13(a+b+c)}{9}$

Table 3.2: Table of $\varphi(\overline{\Gamma})$, $\lambda(\overline{\Gamma})$ and $\epsilon(\overline{\Gamma})$

3.2 Graphically hyperelliptic curves over function fields

In this section, B is a smooth curve over an algebraically closed field k with function field K. Subsections 3.2.1-3.2.2 are still about pm-graphs of genus 3 (with an application to the height $\langle\Delta,\Delta\rangle$). In Subsection 3.2.3, we show that the height $\langle\Delta,\Delta\rangle$ of a graphically hyperelliptic genus 3 curve over K vanishes if and only the curve is hyperelliptic. We refer to Section 1.4 for the terminology on pm-graphs and Subsection 1.5.2 for the theory of Gross-Schoen cycles.

The number $h(\overline{\Gamma})$ introduced in Equation (3.1) will be used in Theorem 3.3.12 and Proposition 4.4.5.

3.2.1 An inequality for $\langle\Delta,\Delta\rangle$

By *a polarized graph*, we mean a pm-graph without the metric, in other words, it is a pair $\overline{G} = (G, \mathfrak{q})$ where $G = (V, E)$ is a graph and $\mathfrak{q}$ is a polarization making the canonical divisor (Definition 1.4.8) effective.

The polarized graphs $\overline{\mathbf{H}} = (\mathbf{H}, 0)$ and $\overline{\mathbf{N}} = (\mathbf{N}, 0)$ in Figure 3.2.1 are two irreducible polarized graphs (the polarization is the constant function 0) without eliminable vertices, and we call the two graphs *maximal models*.

Definition 3.2.1. *We say $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ is a model for a polarized graph $\overline{G}$ if $\overline{G}$ is equivalent to a contraction of $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ with the induced polarization.*

Remark 3.2.2. *For simplicity, we use the same notations for pm-graphs and polarized graphs (like the contraction $\overline{G}_S$ and $\overline{G}^S$). We also use Table 3.1 for the types of genus 3 polarized graphs when $b_1 = 0$ or 1.*

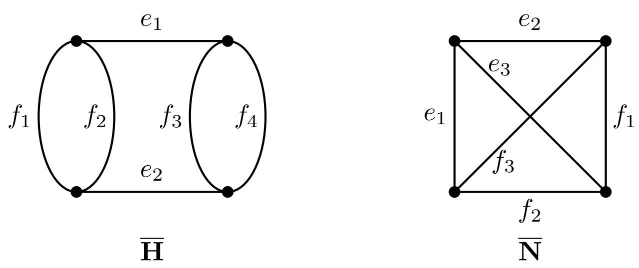


Figure 3.2.1: Maximal models

Lemma 3.2.3. *Every polarized graph $\overline{G}$ of genus 3 with only edges of type 0 is equivalent to a polarized graph having $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ as a model. If we assume further that $\overline{G}$ is not equivalent to $\overline{\mathbf{N}}$, then it has $\overline{\mathbf{H}}$ as a model.*

Proof. This can be proven by a combinatorial checking. □

Definition 3.2.4. *Let $\overline{G} = (G, \mathfrak{q})$ be a polarized graph of genus 3 with no eliminable vertices. We say a pair of edges $\{e, e'\}$ of $\overline{G}$ is of h-type if $\overline{G}^{\{e,e'\}}$ is of type $1II$ in Table 3.1.*

Example 3.2.5. *In Figure 3.2.1, $\{e_1, e_2\}$ is the only pair of h-type edges in $\overline{\mathbf{H}}$ while $\overline{\mathbf{N}}$ has no edges of h-type.*

Lemma 3.2.6. *A polarized graph $\overline{G}$ of genus 3 without eliminable vertices has at most one pair of edges of h-type.*

Proof. If $\{e_1, e_2\}$ is a pair of edges of h-type, then e_1 and e_2 sit in the same irreducible component otherwise $\overline{G}^{\{e_1,e_2\}}$ is reducible.

Let $\{e_3, e_4\}$ be another pair of edges of h-type. The two pairs lie on the same irreducible component of $\{e_1, e_2\}$, otherwise $\overline{G}^{\{e_1,e_2,e_3,e_4\}}$ can not be a graph without eliminable vertices, which contradicts Lemma 1.4.17. We denote this irreducible component with induced polarization by $\overline{G}_1$. By Lemma 3.2.3, the polarized graph $\overline{G}_1$ is equivalent to a certain contraction of $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ with the induced polarization. Since $\overline{\mathbf{H}}$ and $\overline{\mathbf{N}}$ have at most 1 pair of edges of h-type (Example 3.2.5), so does their contraction. Thus $\{e_1, e_2\} = \{e_3, e_4\}$. □

Since pm-graphs are polarized graphs with metrics, our discussion so far can be extended to pm-graphs easily.

Let $\overline{\Gamma} = (G, w, \mathfrak{q})$ be a genus 3 pm-graph with no eliminable vertices. If there exists a pair of edges of h-type $\{e_1, e_2\}$ on $\overline{\Gamma}$, we define

$$h(\overline{\Gamma}) \coloneqq \min\{w(e_1), w(e_2)\}, \tag{3.1}$$

otherwise we set $h(\overline{\Gamma}) = 0$. For a general pm-graph $\overline{\Gamma}$ which is equivalent to $\overline{\Gamma}_0$ with no eliminable vertices, we define

$$h(\overline{\Gamma}) \coloneqq h(\overline{\Gamma}_0).$$

Lemma 3.2.7. *$h(\cdot)$ is additive on pm-graphs of genus 3.*

Proof. This is trivial from the definition of $h(\cdot)$. □

Recall the definition of $\psi(\overline{\Gamma})$ in Corollary 1.5.8. For a genus 3 pm-graph $\overline{\Gamma}$ with only edges of type 0, we define

$$\Phi(\overline{\Gamma}) := \frac{1}{3}\delta_0(\overline{\Gamma}) + \frac{4}{3}h(\overline{\Gamma}) - \psi(\overline{\Gamma}). \tag{3.2}$$

Lemma 3.2.8. *The invariant Φ is additive for pm-graphs with only type 0 edges.*

Proof. The function ψ is additive since it is a linear combination of admissible invariants (Corollary 1.5.8). On pm-graphs with only edges of type 0, the invariant δ_0 is additive since $\delta_0 = \delta$. The additivity of h is trivial according to Lemma 3.2.6. □

Lemma 3.2.9. *For a tree pm-graph $\overline{\Gamma}$ of genus g, we have*

$$\psi(\overline{\Gamma}) = \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \left(\frac{12i(g-i)}{2g+1} - 1 \right) \delta_i(\overline{\Gamma}).$$

Proof. In this case, we have $\delta_0(\overline{\Gamma}) = 0$ and $h(\overline{\Gamma}) = 0$, thus ψ is a linear combination of ϵ and φ (Equation (1.20)). Theorem 1.4.39 implies that we can reduce the problem to the computation of τ, θ and δ. Since $\overline{\Gamma}$ is a tree, the underlying graph Γ is the wedge sum of segments. By the additivity of these invariants, we only need to compute them for the pm-graph with one segment and two endpoints polarized by i and $g-i$ for $0 < i \leq \lfloor \frac{g}{2} \rfloor$. □

Now we give a lower bound of $\langle \Delta, \Delta \rangle$ for non-hyperelliptic curves by Lemma 3.2.11. For a semistable curve $f : \mathcal{X} \to B$, we denote the dual graph at a closed point $s \in B$ by $\overline{\Gamma}_s$. If the genus of f is 3, all edges in $\overline{\Gamma}_s$ are of type 0 or 1. We denote the contraction of all type 1 (resp. 0) edges in $\overline{\Gamma}_s$ with the induced polarization by $\overline{\Gamma}_s^\circ$ (resp. $\overline{\Gamma}_s^+$).

Remark 3.2.10. *The pm-graph $\overline{\Gamma}_s^\circ$ is the wedge sum of irreducible components in $\overline{\Gamma}_s$ which are not isomorphic to segments. And $\overline{\Gamma}_s^+$ is a wedge sum of segment components in $\overline{\Gamma}_s$. Every edge in $\overline{\Gamma}_s$ corresponds to an edge in either $\overline{\Gamma}_s^\circ$ or $\overline{\Gamma}_s^+$. If F is an additive function on pm-graphs, then $F(\overline{\Gamma}_s) = F(\overline{\Gamma}_s^\circ) + F(\overline{\Gamma}_s^+)$.*

Lemma 3.2.11. *Let $f : \mathcal{X} \to B$ be a semistable curve of genus 3 with smooth non-hyperelliptic generic fiber. We have*

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq \sum_{s \in B} \left(\frac{\delta_0(\overline{\Gamma}_s)}{3} + 3\delta_1(\overline{\Gamma}_s) + \frac{4h(\overline{\Gamma}_s)}{3} \right),$$

where $\overline{\Gamma}_s$ is the dual graph over $s \in B$.

Proof. See Corollary 3.8 in [66]. □

Proposition 3.2.12. *Let $f : \mathcal{X} \to B$ be a semistable curve of genus 3 with smooth non-hyperelliptic generic fiber C. Then we have*

$$\langle \Delta, \Delta \rangle \geq \frac{7}{4} \sum_s \Phi(\overline{\Gamma}_s^{\circ}) + \delta_1(C),$$

where $\delta_1(C) := \sum_s \delta_1(\overline{\Gamma}_s)$.

Proof. Every irreducible component of $\overline{\Gamma}_s$ is an irreducible component of $\overline{\Gamma}_s^{\circ}$ or $\overline{\Gamma}_s^{+}$ and vice versa (Remark 3.2.10). Thus by the additivity of ψ, we have

$$\psi(\overline{\Gamma}_s) = \psi(\overline{\Gamma}_s^{\circ}) + \psi(\overline{\Gamma}_s^{+}).$$

From Corollary 1.5.8 and Lemma 3.2.11, we get

$$\begin{aligned}\langle \Delta, \Delta \rangle =& \frac{7}{4} \left((\omega_{X/B}, \omega_{X/B}) - \sum_{s \in B} \psi(\overline{\Gamma}_s) \right) \\ \geq& \frac{7}{4} \left(\sum_{s \in B} \Phi(\overline{\Gamma}_s^{\circ}) + \sum_{y \in B} \left(3\delta_1(\overline{\Gamma}_s^{+}) - \psi(\overline{\Gamma}_s^{+}) \right) \right).\end{aligned}$$

Since $\overline{\Gamma}_s^{+}$ is a tree, by Lemma 3.2.9, we have $\psi(\overline{\Gamma}_s^{+}) = \frac{17}{7}\delta_1(\overline{\Gamma}_s^{+})$. Thus

$$3\delta_1(\overline{\Gamma}_s^{+}) - \psi(\overline{\Gamma}_s^{+}) = \frac{4\delta_1(\overline{\Gamma}_s^{+})}{7} = \frac{4\delta_1(\overline{\Gamma}_s)}{7}. \tag{3.3}$$

We get the result by substituting Equation (3.3) into the inequality above. □

Proposition 3.2.12 reduces the positivity of $\langle \Delta, \Delta \rangle$ to the computation for Φ and δ_1 at special fibers. In [67], K. Yamaki shows the following result for Φ.

Theorem 3.2.13. *Let $\overline{\Gamma} = (G, w, \mathfrak{q})$ be a pm-graph of genus 3 without eliminable vertices. Suppose that $\overline{\mathbf{H}}$ is a model of $\overline{\Gamma}$, then we have $\Phi(\overline{\Gamma}) \geq 0$. Moreover, $\Phi(\overline{\Gamma}) = 0$ if and only if one of the following cases occurs:*

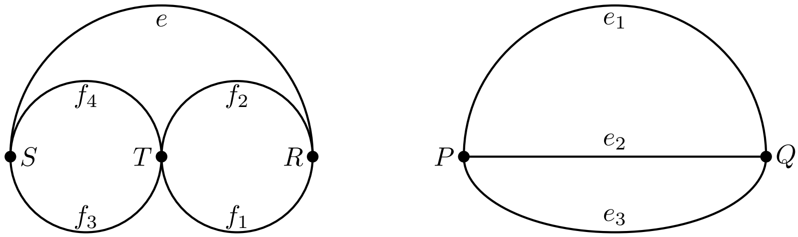
(1) $\overline{\Gamma}$ *is the trivial pm-graph.*

(2) $\overline{\Gamma}$ *is isomorphic to* $\mathbf{E}_1$ *in Figure 3.2.2 with the weight condition*

$$w(f_1) = w(f_2), \quad w(f_3) = w(f_4), \quad w(e) = w(f_1) + w(f_3).$$

(3) $\overline{\Gamma}$ *is isomorphic to* $\mathbf{E}_2$ *in Figure 3.2.2 with the weight condition*

$$w(e_1) = w(e_2) = w(e_3).$$



$\mathbf{E}_1$ $\mathbf{E}_2$

Figure 3.2.2: Two polarized graphs with model $\overline{\mathbf{H}}$

Proof. See Theorem 2.7 in [67]. $\square$

Corollary 3.2.14. *Let $f : \mathcal{X} \to B$ be a semistable curve of genus 3 with smooth non-hyperelliptic generic fiber X. If $\Phi(\overline{\Gamma}_s) \geq 0$ for all s that $\overline{\Gamma}_s$ is equivalent to $\overline{\mathbf{N}}$, then $\langle \Delta, \Delta \rangle \geq 0$. In addition, if there exists s such that $\overline{\Gamma}_s$ is not equivalent to one of the pm-graphs in Theorem 3.2.13, then we have $\langle \Delta, \Delta \rangle > 0$.*

Proof. This is a consequence of Lemma 3.2.3 and Theorem 3.2.13. $\square$

3.2.2 Hyperelliptic polarized graph

Definition 3.2.15. *A hyperelliptic graph $G = (V, E)$ is either the one-point graph, or a graph with an order 2 automorphism ι on G satisfying the following properties:*

(1) G has no self-loops.

(2) $\iota(e) \neq e$ for any $e \in E$.

(3) The quotient graph $G/\langle \iota \rangle$ is a tree.

(4) If a vertex $n \in V$ is not fixed by ι, then the valence satisfies $v(n) \geq 3$.

Lemma 3.2.16. *Let (G, ι) be a non-trivial hyperelliptic graph with $\iota(e_1) = e_2$. The graph $G_{\{e_1, e_2\}}$ given by contracting edges e_1 and e_2 is either a one-point graph or a non-trivial hyperelliptic graph with the induced automorphism ι_0 of order 2.*

Proof. We assume that $G_{\{e_1,e_2\}}$ is not a one-point graph, thus it has the induced automorphism ι_0 of order 2.

Condition (2) in Definition 3.2.15 is trivial.

The quotient graph $(G_{\{e_1,e_2\}}, \iota_0)/\langle \iota_0 \rangle$ is given by contracting an edge from the tree $G/\langle \iota \rangle$, thus is also a tree. So Condition (3) in Definition 3.2.15 is verified.

If c is a self-loop in $G_{\{e_1,e_2\}}$, then Condition (2) says that $\iota(c)$ is a different self-loop. Thus the quotient $(G_{\{e_1,e_2\}}, \iota_0)/\langle \iota_0 \rangle$ must have at least 1 self-loop, which contradicts Condition (3) we just proved. So Condition (1) in Definition 3.2.15 is verified.

We assume p to be a vertex in $G_{\{e_1,e_2\}}$ that is not fixed by ι_0. If p is the contraction point of $e_1 \in E$, then by the assumption, neither of e_1's endpoints are fixed by ι. It can be checked that e_1 and e_2 can not share the endpoints, otherwise $G_{\{e_1,e_2\}}$ contains a self-loop. Then we obtain $v(p) \geq 3 + 3 - 2 = 4$ by Condition (4) in Definition 3.2.15.

If p does not belong to the endpoints of contracting edges, then its valence is the same as that of the original graph (we write p' for this point in G). Since p is not fixed by ι_0, p' cannot be fixed by ι and thus $v(p) \geq 3$. So Condition (4) in Definition 3.2.15 is verified.

In conclusion, $(G_{\{e_1,e_2\}}, \iota_0)$ is a hyperelliptic graph. □

This lemma says that the hyperelliptic graph behaves well under the quotient map.

Proposition 3.2.17. *A hyperelliptic graph G does not have vertices with valence 1.*

Proof. Let p be a vertex of G with valence 1. By Condition (4) in Definition 3.2.15, it is fixed by ι. Thus the only edge related to it is fixed by ι, which contradicts Condition (2). □

Lemma 3.2.18. *Let G be a non-trivial graph. If ι and ι' are two order 2 automorphisms of G that make G a hyperelliptic graph, then $\iota = \iota'$.*

Proof. See Lemma 3.1 in [67]. □

Definition 3.2.19. *A polarized graph $\overline{G} = (V, E, \mathfrak{q})$ is called a hyperelliptic polarized graph if G is a one-point graph or the following are satisfied:*

(1) G is a non-trivial hyperelliptic graph with the order 2 automorphism ι.

(2) ι preserves the polarization $\mathfrak{q}$.

(3) $\mathfrak{q}(n) = 0$ for any $n \in V$ with $\iota(n) \neq n$.

If $w : E \to \mathbb{R}_{>0}$ is a weight function on the edges of $\overline{G}$ with the property $w(e) = w(\iota(e))$ for all $e \in E$, then we call $(\overline{G}, w)$ a hyperelliptic weighted polarized graph or a hyperelliptic pm-graph.

Proposition 3.2.20. *(1) Let e be an edge on a non-trivial hyperelliptic graph G (with the order 2 automorphism ι). Then $\{e, \iota(e)\}$ is a pair of edges of h-type.*

(2) All hyperelliptic polarized graphs of genus 3 *without eliminable points are of type* $1II$ *in Table 3.1.*

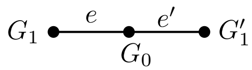
Proof. By applying Lemma 3.2.16 repeatedly, we find $G^{\{e,\iota(e)\}}$ is a hyperelliptic graph. Thus $\{e, \iota(e)\}$ has to be a pair of edges of h-type.

By Lemma 3.2.6, a polarized graph $\overline{G}$ of genus 3 without eliminable edges has at most 1 pair of edges of h-type. By the first assertion, the polarized graph $\overline{G}$ is equivalent to the type $1II$ graph in Table 3.1. □

Question 3.2.21. *Can we have a clearer description of hyperelliptic polarized graphs? What can we say for higher genus?*

Proposition 3.2.22. *A hyperelliptic polarized graph* $\overline{G}$ *only has edges of type* 0*.*

Proof. If there is an edge e of positive type, then $e' := \iota(e)$ is also an edge of positive type. Thus $\overline{G}$ looks like the following figure, where G_0, G_1 and G_1' are subgraphs instead of vertices.



The automorphism ι induces an isomorphism between G_1 and G_1', and an automorphism of G_0. By Condition (3) in Definition 3.2.15, the quotient graph $\overline{G}/\langle\iota\rangle$ is a tree, thus G_1 and G_1' are non-trivial trees or the one-point graph. Both of the two cases will lead to a vertex with valence 1, while this can not be true by Proposition 3.2.17. □

3.2.3 Graphically hyperelliptic curves

In this subsection, we take X to be a smooth curve over K of genus $g > 1$ with a semistable model $f : \mathcal{X} \to B$. Similarly to in Section 1.4.1, we denote $\overline{\Gamma}_s = (G_s, w_s, \mathfrak{q}_s)$ the dual graph of $\mathcal{X}$ at a closed point $s \in B$. By $\overline{\Gamma}_s^\circ$, we mean the induced pm-graph given by contracting edges of positive type in $\overline{\Gamma}_s$.

Definition 3.2.23. *If* $\overline{\Gamma}_s^\circ$ *is equivalent to a hyperelliptic pm-graph for all closed points* $s \in B$*, we call* X *or* f *a graphically hyperelliptic curve.*

Theorem 3.2.24. *Let* X *be a graphically hyperelliptic smooth genus 3 curve over* K *with a semistable model* $\mathcal{X} \to B$*. If* $\langle\Delta, \Delta\rangle = 0$ *and there is at least one closed point* $s \in B$ *such that* $\overline{\Gamma}_s^\circ$ *is non-trivial, then* X *is a hyperelliptic curve.*

Proof. Since $\overline{\Gamma}_s^\circ$ is a hyperelliptic graph, it cannot be of the form $\mathbf{N}$, $\mathbf{E}_1$ or $\mathbf{E}_2$. By Corollary 3.2.14, the curve X cannot be non-hyperelliptic, otherwise we have $\langle\Delta, \Delta\rangle > 0$. □

3.3 Non-hyperelliptic curves over number fields

In Subsection 3.3.1, we decompose $\langle\Delta,\Delta\rangle$ into the sum of contributions from (in)finite places (Theorem 3.3.2). In Subsection 3.3.2, we give a lower bound for $\mathrm{ord}_v(\chi'_{18})$ (Proposition 3.3.4). In Subsection 3.3.3, we prove an unboundedness result of $\langle\Delta,\Delta\rangle$. We refer to Section 1.5, Section 2.2 and Section 3.1 for terminology and theorems.

Theorem 3.3.2 will be the main tool for our computation of $\mathfrak{C}_{\mathbb{Q}}$ in Chapter 4. The Horikawa index will be used for the computation of $\mathrm{ord}_v(\chi'_{18})$ at a finite place v.

3.3.1 $\langle\Delta,\Delta\rangle$ for non-hyperelliptic curves of genus 3

Let k be a number field with $M(k)_0$ (resp. $M(k)_\infty$) its finite (resp. infinite) places and let $M(k)$ be the union $M(k)_0\cup M(k)_\infty$. We denote $\mathrm{Spec}(O_k)$ by S. Let X be a smooth curve of genus $g\geq 2$ over k which also has semistable reduction over k.

By Theorem 1.5.6, the height of a canonical Gross-Schoen cycle of X is

$$\langle\Delta,\Delta\rangle=\frac{2g+1}{2g-2}(\hat{\omega},\hat{\omega})_{ad}-\sum_{v\in M(k)}\varphi(X)\log Nv. \tag{3.4}$$

Let $f:\mathcal{X}\to S$ be a stable model of X and let $\omega_{\mathcal{X}/S}$ be the relative dualizing sheaf on $\mathcal{X}$. We endow the line bundle $\det f_*\omega_{\mathcal{X}/S}$ with the metric induced by Equation (1.6) at infinite places of k, and denote the metrized line bundle by $\det f_*\overline{\omega}_{\mathcal{X}/S}$.

By Corollary 1.3.11 and Theorem 1.5.3, we get

$$\begin{aligned}(\hat{\omega},\hat{\omega})_{ad}=&12\deg\det f_*\overline{\omega}_{\mathcal{X}/S}-\sum_{v\in M(k)_0}\delta(\overline{\Gamma}_v)\log(Nv)+\sum_{\sigma\in k(\mathbb{C})}\delta(\mathcal{X}_\sigma)\\&+\sum_{v\in M(k)_0}\epsilon(\overline{\Gamma}_s)\log(Nv)+4g[k:\mathbb{Q}]\log(2\pi).\end{aligned}$$

Substituting the equation above, Equation (1.15) and Equation (1.18) to Equation (3.4), we get the following proposition (Corollary 4.2 in [13]).

Proposition 3.3.1. *Let X be a smooth curve of genus $g\geq 2$ defined over the number field k and also has semistable reduction over k. Let $\Delta\in\mathrm{CH}^2(X^3)_{\mathbb{Q}}$ be a canonical Gross-Schoen cycle on X^3. Then the equality*

$$\langle\Delta,\Delta\rangle=\frac{6(2g+1)}{g-1}\left(\deg\det f_*\overline{\omega}_{\mathcal{X}/S}-\sum_{v\in M(k)}\lambda(X_v)\log Nv\right)$$

holds.

Let $\overline{\pi} : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal stable curve of genus $g \geq 2$ and let $\Omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ be the universal relative dualizing sheaf. For the stable curve $\mathcal{X} \to S$, the pull-back of $\mathcal{L}_{\overline{\pi}} := \det \overline{\pi}_* \Omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ along the classifying map $S \to \overline{\mathcal{M}}_g$ gives the metrized line bundle $\det f_* \overline{\omega}_{\mathcal{X}/S}$ on S.

Recall that we introduced a geometric Siegel modular form $\chi'_{18} \in S_{3,18}(\mathbb{Z})$ in Subsection 2.2.1, which corresponds to an element in $T_{3,18}(\mathbb{Z})$ (also denoted by χ'_{18}). Thus χ'_{18} can be considered as a rational section of $\mathcal{L}_{\overline{\pi}}^{\otimes 18}$. Now we assume that the generic fiber of $\mathcal{X} \to S$ is non-hyperelliptic and also of genus 3. Then the pull-back of χ'_{18} along the classifying map $S \to \overline{\mathcal{M}_3}$ gives a non-zero rational section of $\mathcal{L}_f^{\otimes 18}$ (Lemma 2.2.1). Over $\mathbb{C}$, the pullback of the Hodge metric (Equation (2.7)) on $\mathcal{L}_\pi$ coincides with the metric derived from Equation (1.5). Thus we have the following formula for $\deg \det f_* \overline{\omega}_{\mathcal{X}/S}$:

$$18 \deg \det f_* \overline{\omega}_{\mathcal{X}/S} = \sum_{v \in M(k)_0} \mathrm{ord}_v(\chi'_{18}) \log N v - \sum_{v \in M(k)_\infty} \log \|\chi'_{18}\|_{\mathrm{Hdg},v}. \tag{3.5}$$

Applying this to Proposition 3.3.1, we get the following result (Theorem 8.2 in [13]).

Theorem 3.3.2. *Let X be a smooth non-hyperelliptic curve of genus 3 defined over the number field k which has semistable reduction over k. Let $f : \mathcal{X} \to \mathrm{Spec}(O_k)$ be the stable model of X over O_k and consider χ'_{18} as a rational section of the line bundle $\mathcal{L}_f^{\otimes 18}$. Then the height of a canonical Gross-Schoen cycle Δ on X^3 satisfies*

$$\begin{aligned} \frac{\langle \Delta, \Delta \rangle}{21} = & \sum_{v \in M(k)_0} \left(\frac{1}{18} \mathrm{ord}_v(\chi'_{18}) - \lambda(X_v) \right) \log N v \\ & + \sum_{v \in M(k)_\infty} \left(-\frac{1}{18} \log \|\chi'_{18}\|_{\mathrm{Hdg},v} - \lambda(X_v) \right). \end{aligned}$$

3.3.2 The Horikawa index

In this subsection, S is the spectrum of a discrete valuation ring R with the closed point s and fraction field $K(S)$. Let $f : \mathcal{X} \to S$ be a stable curve of genus 3 with smooth non-hyperelliptic generic fiber.

Using the notation defined in the beginning of Subsection 2.2.1, the bundles $\mathcal{E}_f$ and $\mathcal{G}_f$ are locally free and the morphism $\nu_f : \mathrm{Sym}^2 \mathcal{E}_f \to \mathcal{G}_f$ given by $\eta_1 \cdot \eta_2 \to \eta_1 \otimes \eta_2$ is generically surjective (both are $K(S)$-linear spaces of dimension 6 at the generic fiber of S). Since R is a discrete valuation ring, we know $\mathrm{Sym}^2 \mathcal{E}_f$ and $\mathcal{G}_f$ can be viewed as free R-modules of rank 6. The generic surjectivity of ν_f also guarantees its global injectivity. This induces a short exact sequence:

$$0 \to \mathrm{Sym}^2 \mathcal{E}_f \to \mathcal{G}_f \to \mathcal{Q}_f \to 0. \tag{3.6}$$

Since $\mathrm{Sym}^2\mathcal{E}_f$ and $\mathcal{G}_f$ are isomorphic to $R^{\oplus 6}$, we find that $\mathcal{Q}_f$ is of finite length over R. We define $\mathrm{length}_{O_S}\mathcal{Q}_f$ as the *Horikawa index* of f at s, denoted by $\mathrm{Ind}_s(f)$.

Proposition 3.3.3. *If we consider χ'_{18} as a rational section of the line bundle $\mathcal{L}_f^{\otimes 18}$ on S, then we have the equality*

$$\mathrm{ord}_s(\chi'_{18}) = 2\mathrm{Ind}_s(f) + 2\delta(\overline{\Gamma}_s).$$

In particular, χ'_{18} is a global section of $\mathcal{L}_f^{\otimes 18}$.

Proof. See Proposition 9.3 in [13]. □

Proposition 3.3.4. *With the notation above, the inequality*

$$\mathrm{ord}_s(\chi'_{18}) \geq 2h(\overline{\Gamma}_s) + 2\delta_0(\overline{\Gamma}_s) + 6\delta_1(\overline{\Gamma}_s)$$

holds.

Proof. By Proposition 3.7 in [66], we have

$$\mathrm{Ind}_s(f) \geq h(\overline{\Gamma}_s) + 2\delta_1(\overline{\Gamma}_s),$$

where $h(\cdot)$ is defined in Equation (3.1). We prove the assertion by combining this with Proposition 3.3.3. □

Let $\overline{H}$ be the closure of the hyperelliptic locus of $\mathcal{M}_3$ in $\overline{\mathcal{M}}_3$. Pulling back the line bundle $\mathcal{O}_{\mathcal{M}_3}(\overline{H})$ and its canonical section along the classifying map $S \to \overline{\mathcal{M}}_3$, we can define the multiplicity $\mathrm{mult}_s\overline{H}$.

Proposition 3.3.5. *With the notation above, then we have*

$$\mathrm{Ind}_s(f) = \mathrm{mult}_s\overline{H} + 2\delta_1(\overline{\Gamma}_s).$$

Proof. See Proposition 9.6 in [13]. □

3.3.3 An unboundedness property of $\langle\Delta, \Delta\rangle$

In this subsection, we still write S for the spectrum of a discrete valuation ring. We denote the closed point of S by s. Recall Definition 1.4.18 and the paragraph after it for some graph-theoretic terminology.

Definition 3.3.6. *We say a genus 3 pm-graph $\overline{\Gamma} = (G, w, \mathfrak{q})$ satisfies Condition ($\mathfrak{H}$) if $\overline{\Gamma}$ is equivalent to a pm-graph $\overline{\Gamma}' = (G', w', \mathfrak{q}')$ such that*

(1) $\overline{\Gamma}'$ has no eliminable vertices,

(2) G' is the wedge sum of trees, type $1I$ or type $1II$ graphs in Table 3.1.

Proposition 3.3.7. *Let $C \to S$ be a genus 3 stable curve with smooth non-hyperelliptic generic fiber. If the dual graph $\overline{\Gamma}_s$ satisfies Condition $(\mathfrak{H})$, then we have*

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq 0,$$

where strict positivity holds if $\overline{\Gamma}_s$ is not trivial.

Proof. We mainly use the inequality in Proposition 3.3.4. Since the functions $h(\cdot)$, $\delta_0(\cdot)$ and $\delta_1(\cdot)$ are additive (Example 1.4.22 and Lemma 3.2.7), it remains to prove the assertion for trees, type $1I$ graphs and type $1II$ graphs in Table 3.1.

Claim 3.3.8. *If $\overline{\Gamma}_s$ is a tree, then $\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{1}{21}\delta(\overline{\Gamma}_s)$.*

Proof of claim: Table 3.1 contains all possible tree pm-graph of genus 3, then we have $\lambda(\overline{\Gamma}_s) = \frac{2}{7}\delta(\Gamma_s)$. A tree graph has no edges of type 0 thus $\delta_1(\overline{\Gamma}_s) = \delta(\overline{\Gamma}_s)$, so we obtain

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{6\delta(\overline{\Gamma}_s)}{18} - \frac{2}{7}\delta(\overline{\Gamma}_s) = \frac{1}{21}\delta(\overline{\Gamma}_s)$$

by Proposition 3.3.4. CLAIM PROVEN

Claim 3.3.9. *If $\overline{\Gamma}_s$ is of type $1I$ or $1II$ in Table 3.1, then $\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) > 0$.*

Proof of claim: For type $1I$ in Table 3.1, it is easy to see

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{a}{9} - \frac{3a}{28} = \frac{a}{252} > 0.$$

Now we consider $1II$ in Table 3.1. If we write m_1, $m_2 \in \mathbb{Z}_{>0}$ for the thicknesses of the two nodal points in C_s, then we get

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{m_1 + m_2}{252} + \frac{\min\{m_1, m_2\}}{9} - \frac{m_1 m_2}{7(m_1 + m_2)}$$

by Proposition 3.3.4 and Table 3.2. We assume $m_1 \geq m_2$. If we denote $\frac{m_1}{m_2}$ by m_3, then the right side of the inequality above becomes

$$\begin{aligned} & m_2 \cdot \frac{m_3^2 + 2m_3 + 1 + 28(1 + m_3) - 36m_3}{252(1 + m_3)} \\ =& m_2 \cdot \frac{m_3^2 - 6m_3 + 29}{252(1 + m_3)} \\ =& m_2 \cdot \frac{(m_3 - 3)^3 + 20}{252(1 + m_3)}, \end{aligned}$$

which proves the positivity. CLAIM PROVEN

Thus we have proved the proposition. □

The main tool we used in the proof of the last proposition is Proposition 3.3.4. This lower bound is not enough for our purposes if the dual graph contains more than 1 cycle (for example, the type $2III$ in [9]). However, even for type $2III$, we can prove

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq 0$$

where the equality holds when all edges are of the same length. Thus we would like to believe that the positivity holds in general. However, the inequality in Proposition 3.3.4 is not enough for this goal in the general case.

Conjecture 3.3.10. *Let $C \to S$ be a genus 3 stable curve whose generic fiber is non-hyperelliptic and smooth. We conjecture*

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq 0,$$

where strict positivity holds if $\overline{\Gamma}_s$ is not a one-point graph.

The theory of stable curves over Dedekind schemes can be extended to a complex manifold analogue. Let $\mathbb{D}$ be the complex unit disk. For a family of complex curves $g_{\mathbb{C}} : Y \to \mathbb{D}$ which is smooth over $\mathbb{D}^*$, there is a ramified map $\mathrm{j} : \mathbb{D} \to \mathbb{D}$ such that the pullback of $g_{\mathbb{C}}$ along j has a stable model over $\mathbb{D}$. See Proposition 7.2 in [38] and Page 173 in [49].

Lemma 3.3.11. *Let $f : \mathcal{Y} \to \mathbb{D}$ be a generically non-hyperelliptic stable curve of genus 3 that is smooth over $\mathbb{D}^*$. We consider χ'_{18} as a rational section of the line bundle $\mathcal{L}_f^{\otimes 18}$ on $\mathbb{D}$. Then the following asymptotics*

$$-\frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathcal{Y}_t) - \lambda(\mathcal{Y}_t) \sim -\left(\frac{1}{18}\mathrm{ord}_0(\chi'_{18}) - \lambda(\overline{\Gamma}_0)\right)\log|t|$$

holds as $t \to 0$, where the $\|\cdot\|_{\mathrm{Hdg}}$ is defined in Equation (2.7). The symbol $\sim$ here means that the difference of both sides can be extended to a continuous function on $\mathbb{D}^*$.

Proof. By Proposition 7.4 in [13], this is equivalent to

$$\lambda(\mathcal{Y}_t) \sim -\lambda(\overline{\Gamma}_0)\log|t| - \frac{1}{2}\log\det \mathrm{Im}\, \Omega(t),$$

as $t \to 0$. This asymptotic formula for λ was proven by R. de Jong and F. Shokrieh as Theorem C in [15]. □

Let $\{p_m\}_{m\in\mathbb{N}^+}$ be a family of points in the orbifold $\overline{\mathcal{M}}_3(\mathbb{C})$. Let $f : U \to \overline{\mathcal{M}}_3(\mathbb{C})$ be an étale map such that $\{p_{m+n_0}\}_{m\in\mathbb{N}^+} \subset f(U)$ for some positive integer n_0. Each point p_{m+n_0} can have several preimages on U along f. If there is a preimage p'_{m+n_0} of p_{m+n_0}

for each $m \in \mathbb{N}^+$ such that $\{p'_{m+n_0}\}_{m\in\mathbb{N}^+}$ converges to a point p'_s on U in the Euclidean topology, we say the family of points $\{p_m\}_{m\in\mathbb{N}^+}$ *converges* to the point $f(p'_s)$ on $\overline{\mathcal{M}}_3(\mathbb{C})$. If a family of points on $\overline{\mathcal{M}}_3(\mathbb{C})$ converges, then the converging point is well-defined (does not depend on the choices of f and the family of preimages).

Theorem 3.3.12. *Let $\{L_m\}_{m\in\mathbb{N}^+}$ be a family of smooth non-hyperelliptic curves of genus 3 over $\mathbb{Q}$. If the following properties hold:*

(1) considering $\{L_m \otimes_{\mathbb{Q}} \mathbb{C}\}_{m\in\mathbb{N}^+}$ as a family of points in $\mathcal{M}_3(\mathbb{C})$, this family of points lies on a curve in $\overline{\mathcal{M}}_3(\mathbb{C})$ and converges to a point in $\overline{\mathcal{M}}_3(\mathbb{C})\backslash\mathcal{M}_3(\mathbb{C})$ which has a non-trivial dual graph satisfying Condition ($\mathfrak{H}$),

(2) the dual graphs of their stable models (which exist over finite extensions of the base field $\mathbb{Q}$, see Theorem 1.1.16) over finite places satisfy Condition ($\mathfrak{H}$),

then their heights of canonical Gross-Schoen cycles $\langle\Delta_m, \Delta_m\rangle$ go to infinity.

Proof. We assume that L_m has semistable reduction over k_m with $[k_m : \mathbb{Q}] < +\infty$ for all $m \in \mathbb{N}^+$. Then we can decompose the height $\langle\Delta_m, \Delta_m\rangle$ with the formula in Theorem 3.3.2:

$$\begin{aligned}\langle\Delta_m, \Delta_m\rangle =& \frac{21}{[k_m : \mathbb{Q}]}\left(\sum_{v\in M(k_m)_0}\left(\frac{1}{18}\mathrm{ord}_v(\chi'_{18}) - \lambda(L_{m,v})\right)\log Nv\right)\\ &-\frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(L_m) - \lambda(L_m).\end{aligned}$$

Condition (2) implies that the contribution from finite places is non-negative (Proposition 3.3.7). It remains to show that the contribution from the infinite place $\mathbb{Q} \to \mathbb{C}$ goes to infinity as $m \to \infty$.

By Condition (1), after discarding finitely many curves in $\{L_m\}$, we can assume that there is a family of complex genus 3 curves $\mathfrak{f} : \mathcal{X} \to \mathbb{D}$ such that:

(1) $\mathfrak{f}$ is smooth over $\mathbb{D}^*$, and is singular at the centre of $\mathbb{D}$,

(2) there exists a series of points $\{t_m\}_{m\in\mathbb{N}^+}$ on $\mathbb{D}^*$ approaching to the centre as $m \to \infty$ such that $L_m \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathcal{X}_{t_m}$.

Taking a suitable ramified map $[n] : \mathbb{D} \to \mathbb{D}$ defined by $t \to t^n$, we can pass to a stable model $\mathfrak{f}' : \mathcal{X}' \to \mathbb{D}$ of $\mathfrak{f}$ and a family of points t'_m such that $[n](t'_m) = t_m$. Condition (1) implies that the fiber of $\mathfrak{f}'$ at the origin 0 is a singular stable curve satisfying Condition ($\mathfrak{H}$). By Lemma 3.3.11, we have

$$-\frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathcal{X}'_{t'_m}) - \lambda(\mathcal{X}'_{t'_m}) \sim -\left(\frac{1}{18}\mathrm{ord}_0(\chi'_{18}) - \lambda(\overline{\Gamma}_0)\right)\log|t'_m|.$$

Thus we get

$$-\frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathcal{X}_{t_m})-\lambda(\mathcal{X}_{t_m})\sim-\frac{1}{n}\cdot\left(\frac{1}{18}\mathrm{ord}_0(\chi'_{18})-\lambda(\overline{\Gamma}_0)\right)\log|t_m|.$$

According to Proposition 3.3.7, we can say $\langle\Delta_m,\Delta_m\rangle\to\infty$ as $m\to\infty$. □

Remark 3.3.13. *If we can prove Conjecture 3.3.10, then we can discard mentioning the Condition ($\mathfrak{H}$) in (1) and remove the condition (2) in Theorem 3.3.12.*

3.3.4 An application of Theorem 3.3.12

In this subsection, we give an explicit family of curves that satisfies the conditions in Theorem 3.3.12.

We define a family of plane curves by

$$\{C_n : y^4 = x^4-(4n-2)x^2+1\}_{n\in\mathfrak{N}},$$

where $\mathfrak{N}=\{n\in\mathbb{N}^+|n\equiv 2(\mathrm{mod}\,3), n\not\equiv 0,1(\mathrm{mod}\,2^5)\}$.

J. Guàrdia proved that the dual graphs of the stable models of these curves over K_n (Notation 3.2 in [32]) are in Table 3.1 (all pm-graphs in Table 3.1 satisfy Condition ($\mathfrak{H}$)) for all finite places. This means that Condition (2) in Theorem 3.3.12 is satisfied.

As a compact Riemann surface, the curve C_n is isomorphic to $D_{1/n}: y^4=x(x-1)(x-\frac{1}{n})$. The family of curves $D_\kappa : y^4 = x(x-1)(x-\kappa)$ over $\mathbb{D}$ (parametrized by κ) is smooth over $\mathbb{D}^*$ and singular at $\kappa=0$ (the tacnodal curve $y^4=x^2(x-1)$).

Lemma 3.3.14. *Let $\mathfrak{D}_\kappa$ be the stable reduction of $D_\kappa\to\mathbb{D}$. Then $\mathfrak{D}_0$ is the union of two copies of the elliptic curve E given by the equation $y^2=x^3-x$, joined at two points.*

Proof. See Proposition 8 in [34]. □

By the lemma above, Condition (1) in Theorem 3.3.12 is also satisfied. Thus the heights of canonical Gross-Schoen cycles of $\{C_n\}_{n\in\mathfrak{N}}$ go to infinity as $n\to+\infty$.

Remark 3.3.15. *The unboundedness of $\langle\Delta,\Delta\rangle$ for $\{C_n\}_{n\in\mathfrak{N}}$ was first proved by R. de Jong in [13]. When the paper was written, the equality in Lemma 3.3.11 was only established when the dual graph of $\mathcal{Y}_0$ is of type $1II$.*