Explicit computation of the height of a Gross-Schoen Cycle

Wang, R.

Chapter 2

Arithmetic and geometric properties of genus 3 curves

In this chapter, we study geometric and arithmetic properties of genus 3 curves. In Section 2.1, we recall general notions and results. In Section 2.2, we discuss properties of χ'_{18}, including C. Ritzenthaler's work on Klein's formula. We will freely use the moduli language.

2.1 General background

In Subsection 2.1.1, we explain the classification of stable curves of genus 3. In Subsection 2.1.2, we explain the relation between various kinds of modular forms and state the Torelli theorem. In Subsection 2.1.3, we recall some notions in invariant theory. In Subsection 2.1.4, we introduce bitangents of plane quartic curves, and explain their relation with semicanonical divisors and theta characteristics.

The modular form χ_h defined in Equation (2.3) will play an important role in Section 2.2 and Section 3.3. Corollary 2.1.20 will be used to evaluate $\|\theta\|_{g-1}$ in Section 4.5.

2.1.1 Classification and moduli

We begin with a simple classification of smooth curves of genus 3 over an algebraically closed field. Most statements in this subsection can be found in [19].

Proposition 2.1.1. *Let k be an algebraically closed field. A non-hyperelliptic smooth curve of genus 3 over k always has a plane quartic model in the projective plane $\mathbb{P}^2$.*

Proof. See Page 519 in [63]. □

We have the following models representing smooth genus 3 curves over an algebraically closed field k. When $\operatorname{char} k \neq 2$, hyperelliptic curves of genus 3 have the following affine model

$$C : y^2 = \prod_{i=1}^{7}(x - c_i), \quad \text{where } c_i \in k$$

while when $\operatorname{char} k = 2$ (Theorem 7.4.24 in [48]), they have the following affine model

$$C : y^2 + f(x)y = g(x)$$

with

$$7 \leq \max\{2 \deg f(x), \deg g(x)\} \leq 8.$$

Plane quartic curves over k can be expressed as

$$\sum_{l+m+n=4} c_{lmn} X^l Y^m Z^n = 0,$$

where $c_{lmn} \in k$.

Example 2.1.2. *(Klein quartic) The plane curve defined by $X^3Y + Y^3Z + Z^3X = 0$ is called the Klein quartic curve. As a compact Riemann surface, it has 168 automorphisms. As a curve over $\mathbb{Z}$, it has potentially good reduction at 7 (Page 81 in [20]).*

We write $\mathcal{M}_3$ (resp. M_3) for the moduli stack (resp. coarse moduli space) of smooth genus 3 curves. Similarly, we write $\overline{\mathcal{M}}_3$ (resp. $\overline{M}_3$) for the moduli stack (resp. coarse moduli space) of stable curves of genus 3.

According to Theorem 3.19 and Theorem 5.1 in [58], we have the following results. The moduli space $\overline{\mathcal{M}}_3$ is an algebraic stack over $\operatorname{Spec}(\mathbb{Z})$ of relative dimension 6, which contains $\mathcal{M}_3$ as an open substack.

Singular curves of genus 3 make up a divisor Δ in $\overline{\mathcal{M}}_3$, which can be decomposed as

$$\Delta = \Delta_0 \cup \Delta_1,$$

where Δ_0 denotes the closure of the irreducible singular curves of geometric genus 2 with exactly one nodal point, and Δ_1 denotes the closure of reducible curves with exactly two components of genus 1 and 2. Both Δ_0 and Δ_1 are prime divisors of $\overline{\mathcal{M}}_3$. General statements for higher genus g can be found in Page 411 in [23].

The hyperelliptic locus H in $\mathcal{M}_3$ is an irreducible algebraic stack of codimension 1 (Theorem 2.1 in [26]). Let $\overline{H}$ be the closure of H in $\overline{\mathcal{M}}_3$.

2.1.2 Modular forms and the Torelli theorem

The main references for this subsection are [13] and [43]. We assume the integer $g \geq 3$ in this subsection.

Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian schemes of relative dimension g and denote by $p : \mathcal{U}_g \to \mathcal{A}_g$ the universal abelian variety. Let $\Omega_{\mathcal{U}_g/\mathcal{A}_g}$ denote the sheaf of relative 1-forms of p. Then we get a rank g vector bundle $\mathcal{E} = p_*\Omega_{\mathcal{U}_g/\mathcal{A}_g}$ (known as the *Hodge bundle*), and its determinant $\mathcal{L} = \det p_*\Omega_{\mathcal{U}_g/\mathcal{A}_g}$ on $\mathcal{A}_g$.

Definition 2.1.3. *An algebraic Siegel modular form of genus g and weight $h \in \mathbb{Z}_{>0}$ over a commutative ring R is an element of the R-module*

$$S_{g,h}(R) = \Gamma(\mathcal{A}_g \otimes R, \mathcal{L}^{\otimes h}).$$

Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be the universal smooth curve of genus g. We have a vector bundle $\mathcal{E}_\pi = \pi_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ and an invertible bundle $\mathcal{L}_\pi = \det \pi_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ on $\mathcal{M}_g$ associated to π.

Definition 2.1.4. *A Teichmüller modular form of genus g and weight h over R is an element of the R-module*

$$T_{g,h}(R) = \Gamma(\mathcal{M}_g \otimes R, \mathcal{L}_\pi^{\otimes h}).$$

For a ring homomorphism $R_1 \to R_2$, elements in $S_{g,h}(R_1)$ (resp. $T_{g,h}(R_1)$) can be mapped to elements in $S_{g,h}(R_2)$ (resp. $T_{g,h}(R_2)$). Thus it makes sense to ask if a modular form in $S_{g,h}(R_2)$ (resp. $T_{g,h}(R_2)$) can be lifted to an element in $S_{g,h}(R_1)$ (resp. $T_{g,h}(R_1)$). In Lemma 2.1.9, we will find that the modular form $\chi_h(\tau)$ in $S_{g,h}(\mathbb{C})$ can be lifted to an element in $S_{g,h}(\mathbb{Z})$ (denoted by χ'_h) with respect to the ring homomorphism $\mathbb{Z} \to \mathbb{C}$.

Now we take R to be a field k. For a principally polarized abelian variety $(A, a) \in \mathcal{A}_g(k)$ of dimension g over k (resp. a smooth curve C of genus g over k), we denote by

$$\omega_k[A] \coloneqq \bigwedge^g H^0(A, \Omega_{A/k}) \quad (\text{resp. } \lambda_k[C] \coloneqq \bigwedge^g H^0(C, \Omega_{C/k}))$$

the k-vector space of global sections of $\mathcal{L}$ (resp. $\mathcal{L}_\pi$) over (A, a) (resp. C). For $f \in S_{g,h}(k)$ (resp. $f \in T_{g,h}(k)$) and a basis ω of $\omega_k[A]$ (resp. a basis λ of $\lambda_k[C]$), we put

$$f((A, a), \omega) = f(A, a)/\omega^{\otimes h} \in k, \quad (\text{resp. } f(C, \lambda) = f(C)/\lambda^{\otimes h} \in k). \tag{2.1}$$

This sends a algebraic Siegel modular form (resp. Teichmüller modular form) to a k-valued function on $\mathcal{A}_g(k)$ (resp. $\mathcal{M}_g(k)$).

The map $\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g$ sending every smooth curve C of genus g to its Jacobian with the canonical polarization $(\mathrm{Jac}(C), j)$ is known as the Torelli map. This gives a translation from $S_{g,h}(k)$ to $T_{g,h}(k)$.

Lemma 2.1.5. *The Torelli map* $\mathfrak{t}$ *satisfies* $\mathfrak{t}^*\mathcal{L} = \mathcal{L}_\pi$ *and induces a linear map*

$$\mathfrak{t}^* : S_{g,h}(k) = \Gamma(\mathcal{A}_g \otimes k, \mathcal{L}^{\otimes h}) \to T_{g,h}(k) = \Gamma(\mathcal{M}_g \otimes k, \mathcal{L}_\pi^{\otimes h})$$

for any field k.

Proof. See Section 2.1 in [35]. □

On Page 89 in [42], we can find the following precise form of the Torelli theorem.

Theorem 2.1.6. *Let* (A, a) *be a principally polarized abelian variety of dimension* $g \geq 1$ *over a field* k*. We assume* (A, a) *is isomorphic over* $\overline{k}$ *to the Jacobian of a curve* X_0 *of genus* g *defined over* $\overline{k}$*. Then the following holds :*

(1) If X_0 *is hyperelliptic, then there is a curve* X/k *isomorphic to* X_0 *over* $\overline{k}$ *such that* (A, a) *is* k*-isomorphic to* $(\mathrm{Jac}X, j)$ *where* j *is the canonical polarization.*

(2) If X_0 *is not hyperelliptic, there is a curve* X/k *isomorphic to* X_0 *over* $\overline{k}$*, and a quadratic character*

$$\varepsilon : \mathrm{Gal}(k_{\mathrm{sep}}/k) \longrightarrow \{\pm 1\}$$

such that the twisted abelian variety $(A, a)_\varepsilon$ *(see X.5 in [60] for the explanation of 'twisted') is* k*-isomorphic to* $(\mathrm{Jac}X, j)$*. The character* ε *is trivial if and only if* (A, a) *is* k*-isomorphic to a Jacobian.*

Now we shift our attention to the case $k = \mathbb{C}$. Let $\mathbb{H}_g := \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) \mid {}^t\tau = \tau, \mathrm{Im}\tau > 0\}$ be the Siegel upper half space of genus g.

Definition 2.1.7. *An analytic Siegel modular form of genus* g *and weight* h *is a complex holomorphic function* $\phi(\cdot)$ *on* $\mathbb{H}_g$ *satisfying*

$$\phi(M\tau) = \det(c\tau + d)^h \cdot \phi(\tau),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *for matrices* a*,* b*,* c*,* $d \in \mathrm{Mat}(g \times g, \mathbb{Z})$*, and*

$$M\tau := (a\tau + b)(c\tau + d)^{-1}.$$

We denote the $\mathbb{C}$*-vector space of such functions by* $R_{g,h}$*.*

There is a complex torus over $\mathbb{H}_g$ given by

$$\mathbb{U}_g := \frac{\mathbb{H}_g \times \mathbb{C}^g}{((\tau_1, z) \sim (\tau_2, z_2) \text{ if and only if } \tau_1 = \tau_2 \text{ and } z_1 - z_2 \in \mathbb{Z}^g + \tau_1\mathbb{Z}^g)}.$$

We have a map of complex manifolds $u : \mathbb{H}_g \to \mathcal{A}_g(\mathbb{C})$ and an isomorphism

$$\mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z}) \xrightarrow{\sim} \mathcal{A}_g(\mathbb{C}).$$

The map u induces an isomorphism between $\mathbb{U}_g$ and the pull-back of $\mathcal{U}_g(\mathbb{C})$ along u. The tangent space along the unit section of $\mathbb{U}_g \to \mathbb{H}_g$ is canonically identified with $\mathbb{C}^g$, giving a trivialization on the Hodge bundle $\tilde{\mathcal{E}} = q_*\Omega_{\mathbb{U}_g/\mathbb{H}_g}$ on $\mathbb{H}_g$ by the frame

$$(d\zeta_1/\zeta_1, \ldots, d\zeta_g/\zeta_g) = (2\pi i dz_1, \ldots, 2\pi i dz_g),$$

where $\zeta_i = \exp(2\pi i z_i)$. Then the line bundle $\tilde{\mathcal{L}} = \det \tilde{\mathcal{E}}$ is trivialized by the frame $\omega = \frac{d\zeta_1}{\zeta_1} \wedge \cdots \wedge \frac{d\zeta_g}{\zeta_g} = (2\pi i)^g (dz_1 \wedge \cdots \wedge dz_g)$. See Pages 141-142 in [24] for details.

Proposition 2.1.8. *We write (A_τ, a_τ) for a principally polarized complex abelian variety with the period matrix τ. Let $f \in S_{g,h}(\mathbb{C})$ and let $\tilde{f}$ be the following $\mathbb{C}$-valued function on $\mathbb{H}_g$*

$$\tilde{f}(\tau) \coloneqq (2\pi i)^{-gh} f(A_\tau, a_\tau)/(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h},$$

where $(z_1, \ldots, z_g)$ is the canonical basis of $\mathbb{C}^g$. The map $f \to \tilde{f}$ induces an isomorphism $S_{g,h}(\mathbb{C}) \simeq R_{g,h}$.

Proof. See Page 141 in [24]. □

We denote the subset of $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ containing exactly all elements $\epsilon = (a', a'')$ such that $4a' \cdot a'' \equiv 0 (\mathrm{mod}\, 2)$ by S_g. We take $h = \frac{\# S_g}{2}$ and define a holomorphic function on $\mathbb{H}_g$ by

$$\tilde{\chi}_h(\tau) \coloneqq \frac{(-1)^{gh/2}}{2^{2^{g-1}(2^g-1)}} \cdot \prod_{\epsilon \in S_g} \theta_\epsilon(0, \tau), \tag{2.2}$$

where

$$\theta_\epsilon(z,\tau) \coloneqq \sum_{n \in \mathbb{Z}^g} \exp(\pi i \, {}^t(n+a')\tau(n+a') + 2\pi i \, {}^t(n+a')(z+a'')), \quad z \in \mathbb{C}^g.$$

Under the assumption $g \geq 3$, we have $\tilde{\chi}_h \in R_{g,h}$ (Lemma 10 in [37]). By Proposition 2.1.8, this corresponds to a algebraic Siegel modular form

$$\chi_h(A_\tau) \coloneqq (2\pi i)^{gh} \cdot \tilde{\chi}_h(\tau)(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h} \in S_{g,h}(\mathbb{C}). \tag{2.3}$$

By Lemma 2.1.5, we can get a Teichmüller modular form in $T_{g,h}(\mathbb{C})$. Actually, we have the following result.

Lemma 2.1.9. *The algebraic Siegel modular form χ_h is a primitive (not congruent to 0 modulo p for any prime p) element in $S_{g,h}(\mathbb{Z})$. Moreover, there exists a Teichmüller modular form $\mu_{h/2} \in T_{g,h/2}(\mathbb{Z})$ such that*

$$\mathsf{t}^*(\chi_h) = (\mu_{h/2})^2.$$

Proof. See Proposition 3.4 in [35] and Proposition 4.5 in [36]. □

Remark 2.1.10. *To distinguish from the modular form χ_h in $S_{g,h}(\mathbb{C})$, we denote this modular form in $S_{g,h}(\mathbb{Z})$ by χ'_h.*

Let (A, a) be a principally polarized complex abelian variety with a fixed basis of differential 1-forms $\{\omega_i\}_{1 \leq i \leq g}$ and a symplectic homology basis of $H_1(A, \mathbb{Z})$. We can obtain a period matrix $(\Omega_1|\Omega_2)$ by taking the integration of the differential forms along the homology basis. Using the notations in Equation (2.1) and Proposition 2.1.8, we have the following proposition.

Proposition 2.1.11. *Let f be an algebraic Siegel modular form in $S_{g,h}(k_0)$ for some subfield $k_0 \subset \mathbb{C}$. Let $\omega = \omega_1 \wedge \cdots \wedge \omega_g \in \omega_{k_0}[A]$. Then*

$$f((A, a), \omega) = (2\pi i)^{gh} \frac{\tilde{f}(\tau)}{\det \Omega_1^h}.$$

Proof. See Proposition 1.2.4 in [43]. □

2.1.3 Notions in invariant theory

Let d be a positive integer. Let L be an algebraically closed field such that d is invertible in L. Let V be an n-dimensional vector space over L. We have the following two interpretations of $X_d := \mathrm{Sym}^d(V^*)$ which we will use freely.

(1) Fixing a basis $v = (v_1, \ldots, v_n)$ of V, elements in $\mathrm{Sym}^d(V^*)$ can be considered as degree d homogeneous polynomials (or d-forms) in $k[x_1, \ldots, x_n]$ where $x_i(v_j) = \delta_{ij}$.

(2) We can also consider X_d as an affine scheme which is isomorphic to $\mathbb{A}^{\mathfrak{d}}$, where $\mathfrak{d} = \dim(\mathrm{Sym}^d(V^*))$.

We define an action of $GL(V)$ (resp. $SL(V)$) on X_d by

$$r(s) : F(x_1, \ldots, x_n) \to (s \cdot F)(x_1, \ldots, x_n) = F(s(x_1, \ldots, x_n))$$

for $s \in GL(V)$ (resp. $SL(V)$). This induces a natural action of $GL(V)$ (or $SL(V)$) on regular (or rational) functions on X_d.

Definition 2.1.12. *Let U be a Zariski open set of X_d that is stable under the action of $SL(V)$. An element Ψ of $\mathcal{O}(U)$ is called an invariant on U if $\Psi = s \cdot \Psi$ for all $s \in SL(V)$. We denote the subspace of $\mathcal{O}(U)$ consisting of homogeneous invariants of degree h by $\mathrm{Inv}_h(U)$.*

If $\Psi \in \mathcal{O}(U)$, and if w and h are integers such that $hd = nw$, then $\Psi \in \mathrm{Inv}_h(U)$ if and only if

$$s \cdot \Psi = (\det s)^w \Psi \quad \text{for every } s \in GL(V),$$

where w is called the *weight* of Ψ (Section 2.1 in [43]).

In the following part of this subsection, we assume $n = 3$. Let $\mathfrak{I}_d$ be the set of all non-negative integer tuples (c_1, c_2, c_3) such that $c_1 + c_2 + c_3 = d$. Let $\mathrm{Res}(\cdot)$ be the multivariate resultant (Theorem IX.3.5 in [45]). We write x for the tuple (x_1, x_2, x_3) and write $x^{(c_1,c_2,c_3)}$ for the monomial $x_1^{c_1} x_2^{c_2} x_3^{c_3}$. We call the polynomial $\mathfrak{P} := \sum_{I \in \mathfrak{I}_d} c_I x^I$ over $L[c_I]_{I \in \mathfrak{I}_d}$ the *universal ternary form* of degree d. The polynomial

$$\mathrm{Disc}_{\mathfrak{P}} := d^{-((d-1)^n - (-1)^n)/d} \mathrm{Res}\left(\frac{\partial \mathfrak{P}}{\partial x_1}, \frac{\partial \mathfrak{P}}{\partial x_2}, \frac{\partial \mathfrak{P}}{\partial x_3}\right) \tag{2.4}$$

in $L[c_I]_{I \in \mathfrak{I}_d}$ has the property that its zero locus classifies exactly all non-smooth plane curves of degree d (Section 2.2 in [42]). For a specific ternary form F of degree d, we write $\mathrm{Disc}(F)$ for the value of $\mathrm{Disc}_{\mathfrak{P}}$ at F.

By the *universal plane curve* of degree d over X_d, we mean the variety

$$\mathfrak{U}_d := \{(F, \mathfrak{x}) \in X_d \times \mathbb{P}^2 |\, F(\mathfrak{x}) = 0\}.$$

We write X_d^0 for the Zariski open set

$$X_d^0 := (X_d)_{\mathrm{Disc}_{\mathfrak{P}}} = \{F \in X_d |\, \mathrm{Disc}(F) \neq 0\}.$$

of X_d. We write $\mathfrak{U}_d^0$ for the universal curve over the non-singular locus X_d^0 with the smooth projection map

$$\mathfrak{U}_d^0 \to X_d^0.$$

Explicitly speaking, *invariants* for ternary quartic forms ($d = 4$, $n = 3$) are polynomials in 15 coefficient variables that are stable under the action of $SL_3(L)$ (this is compatible with Definition 2.1.12). The discriminant is an invariant of degree 27 (Section 7 in [21]).

2.1.4 Bitangents

A plane smooth quartic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ intersects a straight line $l \subset \mathbb{P}^2$ at 4 points, counted with multiplicity (Bézout's theorem). We say l is a *bitangent* of C if l is tangent to C at two distinct points. The following theorem was proven by J. Plücker in [57].

Theorem 2.1.13. *Every smooth plane quartic curve over $\mathbb{C}$ has precisely 28 bitangent lines.*

Remark 2.1.14. *Theorem 2.1.13 also holds for plane quartics over a separably closed field k with* $\operatorname{char} k \neq 2$.

The following result of D. Lehavi implies a close relation between plane quartics and their bitangents.

Theorem 2.1.15. *Every smooth plane quartic curve over $\mathbb{C}$ can be reconstructed from its bitangents.*

Proof. See Theorem 1.4 in [46]. □

Now we consider plane quartics over a general separably closed field k with $\operatorname{char} k \neq 2$. Let $f : C \hookrightarrow \mathbb{P}^2$ be a smooth plane quartic over k.

Lemma 2.1.16. *The effective canonical divisors on C are exactly the divisors $(C \cdot L)$, the intersection of C and L, for arbitrary lines $L \subset \mathbb{P}^2$.*

Proof. This comes from the fact that $\Omega_C \simeq \mathcal{O}_C(1) = f^*\mathcal{O}(1)$ for plane smooth quartics. □

Definition 2.1.17. *A theta characteristic on a smooth plane quartic curve C is a line bundle L on C such that $L \otimes L \simeq \Omega_C$. A theta characteristic is said to be odd (resp. even) if $h^0(C, L)$ is odd (resp. even). We denote the set of odd theta characteristics of C by $OT(C)$.*

We have the following well-known correspondence (see Page 289 in [31]).

Theorem 2.1.18. *There is a canonical bijection of bitangents of a smooth plane quartic C and odd theta characteristics of C given by*

$$L \to \frac{1}{2}(C \cdot L).$$

Proof. Let L be a bitangent of C, then the divisor $F := \frac{1}{2}(C \cdot L)$ is a theta characteristic by Lemma 2.1.16. Since F is effective, we have $h^0(C, \mathcal{O}(F)) \geq 1$. Since $F = \Omega_C(-F)$ and $\deg(F) = 2$, we get $h^0(C, \mathcal{O}(F)) \leq 1$ by Clifford's theorem (Theorem IV.5.4 in [33]). Thus $h^0(C, \mathcal{O}(F)) = 1$ and F is an odd characteristic. It remains to prove that this is a bijection.

Let D be an odd theta characteristic on C. Since we have $h^0(C, \mathcal{O}(D)) > 0$, the linear system $|D|$ is non-empty with an effective representative $E = P + Q$. Since $2E$ is canonical, we have $2E = (C \cdot L)$ for some line L by Lemma 2.1.16. This proves the surjectivity in the theorem.

Suppose bitangents L_1 and L_2 give the same theta characteristic, then we have $L_1 \cap C = 2(P + Q)$ and $L_2 \cap C = 2(R + S)$ for points P, Q, R and S on C such that $\{P, Q\} \neq \{R, S\}$. Thus $P + Q - R - S = \mathrm{div}(g)$ for some rational function g on C. This is impossible, otherwise g gives an hyperelliptic map $C \to \mathbb{P}^1$. This proves the injectivity in the theorem. $\square$

Now we shift our attention to $\mathrm{Jac}(C)$, an abelian variety of dimension 3. We denote the group of 2-torsion k-points of $\mathrm{Jac}(C)$ by $\mathrm{Jac}(C)[2]$. Since $\mathrm{char}\, k \neq 2$, $\mathrm{Jac}(C)[2]$ is isomorphic to $\mathbb{F}_2^{\oplus 6}$. We have the Weil pairing

$$\langle \cdot, \cdot \rangle_W : \mathrm{Jac}(C)[2] \times \mathrm{Jac}(C)[2] \to \mathbb{F}_2.$$

There exists a symplectic basis $\{g_1, g_2, g_3, h_1, h_2, h_3\}$ of $\mathrm{Jac}(C)[2]$ such that

$$\langle g_i, g_j \rangle_W = \langle h_i, h_j \rangle_W = 0$$

and

$$\langle g_i, h_j \rangle_W = \delta_{i,j}.$$

We call $Q : \mathrm{Jac}(C)[2] \to \mathbb{F}_2$ a *quadratic form with polar form* $\langle \cdot, \cdot \rangle_W$ if

$$Q(x + y) - Q(x) - Q(y) = \langle x, y \rangle_W \quad \text{for all } x, y \in \mathrm{Jac}(C)[2].$$

We denote the set of quadratic forms with polar form $\langle \cdot, \cdot \rangle_W$ by $\mathbf{T}_C$. Then the *Arf invariant* of an element $Q(\cdot)$ in $\mathbf{T}_C$ is

$$\mathrm{Arf}(Q) := \sum_{1 \leq i \leq 3} Q(g_i) Q(h_i) \in \mathbb{F}_2,$$

which is independent on the choice of the symplectic basis. The set of quadratic forms with polar form $\langle \cdot, \cdot \rangle_W$ forms a torsor over $\mathrm{Jac}(C)[2]$. This structure is defined by

$$(Q + \eta)(x) = Q(x) + \langle x, \eta \rangle_W = Q(x + \eta) + Q(\eta)$$

for $Q(\cdot) \in \mathbf{T}_C$ and $\eta \in \mathrm{Jac}(C)[2] \simeq \mathbb{F}_2^{\oplus 6}$.

We denote the subset of $\mathbf{T}_C$ consisting of quadratic forms of Arf invariant 0 (resp. 1) by $\mathbf{E}_C$ (resp. $\mathbf{O}_C$). The set $\mathbf{E}_C$ (resp. $\mathbf{O}_C$) contains 36 (resp. 28) elements. The symplectic group $\mathrm{Sp}_6(\mathbb{F}_2)$ gives a natural action on $\mathbf{O}_C$ and $\mathbf{E}_C$, which is also transitive.

Theorem 2.1.19. *There is a canonical bijection between the set* $\mathbf{O}_C$ *and the set of bitangents of* C.

Proof. See Proposition 6.2 in [39] and the end of Section 2 in [40]. $\square$

Corollary 2.1.20. *We have a* 1-1 *correspondence among the three sets*

$$\mathrm{Bitangents}(C) \leftrightarrow \mathbf{O}_C \leftrightarrow OT(C)$$

Proof. This is a trivial corollary from Theorem 2.1.18 and Theorem 2.1.19. □

Remark 2.1.21. *The set of theta characteristics for complex smooth curves of genus g has a bijection to the set $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$. The even characteristics correspond to elements $(a,b) \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ such that $4 \times (a \cdot b) \equiv 0 \bmod 2$, and the odd characteristics correspond to other elements in $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$. This interpretation appears in Riemann's theta function with characteristic $\theta_\epsilon(z,\tau)$, which we already used in the definition of S_g in Equation (2.2).*

We give two examples of plane quartics with special behaviour of their bitangents.

Example 2.1.22. *Let k be a field with $\operatorname{char} k \neq 2, 7$. Let ζ be a primitive 7-th root of unity in k^{sep}. We set $\epsilon_1 := \zeta + \zeta^{-1}$, $\epsilon_2 = \zeta^2 + \zeta^{-2}$, $\epsilon_3 := \zeta^4 + \zeta^{-4}$. Then the 28 bitangents of the Klein curve (Example 2.1.2) over k^{sep} are*

$$\begin{aligned}
l_{0,j} &: Z = -\zeta^j Y - \zeta^{3j} X,\\
l_{1,j} &: Z = -\zeta^j \epsilon_1^2 Y - \zeta^{3j} \epsilon_3^{-2} X,\\
l_{2,j} &: Z = -\zeta^j \epsilon_2^2 Y - \zeta^{3j} \epsilon_1^{-2} X,\\
l_{3,j} &: Z = -\zeta^j \epsilon_3^2 Y - \zeta^{3j} \epsilon_2^{-2} X,
\end{aligned}$$

where $j = 0, 1, \ldots, 6$.

Example 2.1.23. *The following plane quartic over $\mathbb{Q}$ has 28 bitangents over $\mathbb{Q}$.*

$$\begin{aligned}
&3X^3Z + X(Y^3 - 11054979YZ^2 - 14822443134Z^3) + 38Y^4 + 243542Y^3Z\\
&+631949994Y^2Z^2 + 822588784146YZ^3 + 4605878924287 44Z^4 = 0
\end{aligned}$$

Details can be found in (6.6) *in [59].*

We end this subsection with a short discussion of the case $\operatorname{char} k = 2$.

When $\operatorname{char} k = 2$ and $k = k^{\mathrm{sep}}$, the dimension r of $\mathrm{Jac}(C)[2]$ over $\mathbb{F}_2$ and number l of bitangents satisfy the following condition: $r = \lfloor l/2 \rfloor$ with $l \in \{1, 2, 4, 7\}$. See Page 60 in [62] for details.

Example 2.1.24. *Let k be an algebraically closed field with $\operatorname{char} k = 2$. Then all smooth plane quartic curves over k with only 1 bitangent can be represented as*

$$(aX^2 + bY^2 + cZ^2 + dXY + eYZ + fZX)^2 = X(Y^3 + X^2Z),$$

where $c \in k^*$. See Proposition 2.1 in [54] for details.

2.2 χ'_{18} and Klein's formula

In Subsection 2.2.1, we show how χ'_{18} behaves on $\overline{\mathcal{M}}_3$, and define the Hodge metric on $\det q_*\Omega_{\mathbb{U}_g/\mathbb{H}_g}$. In Subsection 2.2.2, we talk about the Klein's formula for plane quartics. The main references for this section are [13] and [43].

We will use Proposition 2.2.4 and Equation (2.7) to compute $\mathrm{ord}_v(\chi'_{18})$ at finite places and $\|\chi'_{18}\|_{\mathrm{Hdg}}$ at the infinite place in Section 4.4.

2.2.1 Moduli property of χ'_{18}

In this section we assume $g \geq 2$ (we will specialise to $g = 3$ soon) and use the notation introduced in Subsection 2.1.2. Let $\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g$ be the Torelli map. For the universal stable curve $\overline{\pi} : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$, we can define vector bundles $\mathcal{E}_{\overline{\pi}}$ and $\mathcal{L}_{\overline{\pi}}$. For a stable curve $f : \mathcal{X} \to S$ over a scheme S, we denote the pullback of $\mathcal{E}_{\overline{\pi}}$ (resp. $\mathcal{L}_{\overline{\pi}}$) along the classifying map $J : S \to \overline{\mathcal{M}}_g$ by $\mathcal{E}_f$ (resp. $\mathcal{L}_f$).

By Lemma 2.1.5, there are natural isomorphisms $\mathcal{E}_\pi \simeq \mathfrak{t}^*\mathcal{E}$ and $\mathcal{L}_\pi \simeq \mathfrak{t}^*\mathcal{L}$. We can get a algebraic Siegel modular form $\chi'_{18} \in S_{3,18}(\mathbb{C})$ by taking $g = 3$ and $h = 18$ in Equation (2.3). By Lemma 2.1.9, this can be lifted to a modular form in $S_{3,18}(\mathbb{Z})$ which we denote by χ'_{18}. The pullback of χ'_{18} along the Torelli map gives a Teichmüller modular form in $T_{3,18}(\mathbb{Z})$ which we also denote by χ'_{18}. Now χ'_{18} can be considered as a global section of the line bundle $\mathcal{L}_\pi^{\otimes 18}$ on $\mathcal{M}_3$ and a rational section of the line bundle $\mathcal{L}_{\overline{\pi}}^{\otimes 18}$ on $\overline{\mathcal{M}}_3$.

Lemma 2.2.1. *The divisor of χ'_{18} on $\mathcal{M}_3$ equals $2H$, where H is the hyperelliptic locus.*

Proof. See Theorem 8.1 in [13]. □

In this paragraph, S is the spectrum of a discrete valuation ring. Let $f : \mathcal{X} \to S$ be a stable curve of genus 3 with smooth and non-hyperelliptic generic fiber. By the lemma above, we know χ'_{18} is a non-zero rational section of $\mathcal{L}_f^{\otimes 18}$ on S. Thus we can define $\mathrm{ord}_v(\chi'_{18})$, where v is the closed point of S.

Returning to the case where S is an arbitrary integral scheme. Then we have vector bundles $\mathcal{E}_f = f_*\omega_{\mathcal{X}/S}$ and $\mathcal{G}_f = f_*\omega_{\mathcal{X}/S}^{\otimes 2}$ on S.

Lemma 2.2.2. *Let $f : \mathcal{X} \to S$ be a stable curve of genus 3 with smooth and non-hyperelliptic generic fiber. Then both $\mathcal{E}_f$ and $\mathcal{G}_f$ are of rank 6.*

Proof. The ranks of these vector bundles can be computed over any point in S. Thus we only need to compute $\dim(\mathrm{Sym}^2(H^0(X, \Omega_X)))$ and $h^0(X, \Omega_X^{\otimes 2})$ for smooth curves X of genus 3.

By Riemann-Roch, we have $h^0(X, \Omega_X) = 3$ and $h^0(X, \Omega_X^{\otimes 2}) = 6$. This implies that

$$\dim(\mathrm{Sym}^2(H^0(X, \Omega_X))) = h^0(X, \Omega_X^{\otimes 2}) = 6.$$

□

We have a canonical map

$$\nu_f \colon \mathrm{Sym}^2 \mathcal{E}_f \to \mathcal{G}_f\,, \quad \eta_1 \cdot \eta_2 \mapsto \eta_1 \otimes \eta_2\,, \tag{2.5}$$

which is functorial in f. Both $\mathrm{Sym}^2\mathcal{E}_f$ and $\mathcal{G}_f$ are vector bundles of rank 6 and thus we have a natural map of invertible sheaves

$$\det \nu_f \colon \det \mathrm{Sym}^2 \mathcal{E}_f \to \det \mathcal{G}_f, \tag{2.6}$$

which is functorial in f. The map ν_f is surjective if f is smooth and nowhere hyperelliptic. We can view $\det \nu_f$ as a global section s_f of the invertible sheaf $(\det \mathrm{Sym}^2\mathcal{E}_f)^{\otimes -1} \otimes \det \mathcal{G}_f$ on S. Then the zero locus of s_f is contained in the hyperelliptic locus. Standard multilinear algebra yields a canonical isomorphism

$$\det \mathrm{Sym}^2 \mathcal{E}_f \xrightarrow{\sim} \mathcal{L}_f^{\otimes 4}$$

of invertible sheaves on S, where $\mathcal{L}_f = \det \mathcal{E}_f$ as before, and this shows that we may as well view s_f as a global section of the invertible sheaf $\mathcal{L}_f^{\otimes -4} \otimes \det \mathcal{G}_f$ on S. Let $\pi : \mathcal{C}_3 \to \mathcal{M}_3$ be the universal smooth curve of genus 3, then we can associate a section s_π.

Lemma 2.2.3. *The section s_π is not identically equal to 0, and the divisor of s_π on $\mathcal{M}_3$ is equal to the reduced hyperelliptic divisor H.*

Proof. See Proposition 9.1 in [13]. □

Now we want to consider $\mathrm{div}(\chi'_{18})$ on $\overline{\mathcal{M}}_3$. We denote the divisor of $s_{\overline{\pi}}$ on $\overline{\mathcal{M}}_3$ by K, and denote the divisor of singular curves on $\overline{\mathcal{M}}_3$ by Δ.

Proposition 2.2.4. *If we take χ'_{18} as a rational section of the line bundle $\mathcal{L}_{\overline{\pi}}^{\otimes 18}$ on $\overline{\mathcal{M}}_3$, then we have the equality of effective divisors*

$$\mathrm{div}(\chi'_{18}) = 2K + 2\Delta$$

on $\overline{\mathcal{M}}_3$.

Sketch of proof : See Proposition 9.2 in [13] for a complete proof.

Let H be the hyperelliptic locus of $\mathcal{M}_3$ with closure $\overline{H}$ in $\overline{\mathcal{M}}_3$. By Lemma 2.2.3, we have $\operatorname{div} s_\pi = H$ on $\mathcal{M}_3$. By Lemma 2.2.1, the modular form χ'_{18} is a global section of $\mathcal{L}_\pi^{\otimes 18}$ with divisor $2H$. Thus $\chi'_{18} \otimes s_\pi^{\otimes -2}$ is a trivializing section of $\mathcal{L}_\pi^{\otimes 26} \otimes (\det \mathcal{G}_\pi)^{\otimes -2}$ over $\mathcal{M}_3$.

We have a canonical isomorphism of line bundles on $\mathcal{M}_3$

$$\mu : \det \mathcal{G}_\pi \simeq \mathcal{L}_\pi^{\otimes 13},$$

which comes from the Mumford's functorial Riemann-Roch (Theorem 2.1 and Equation 2.1.2 in [51]). Then $\mu^{\otimes 2}$ gives another trivialization section of $\mathcal{L}_\pi^{\otimes 26} \otimes (\det \mathcal{G}_\pi)^{\otimes -2}$, denoted by w. Since the only invertible regular functions on $\mathcal{M}_3$ are ± 1, this means that w and $\chi'_{18} \otimes s_\pi^{\otimes -2}$ are equal up to a sign.

Mumford's functorial Riemann-Roch on $\overline{\mathcal{M}}_3$ extends μ to an isomorphism

$$\det \mathcal{G}_{\overline{\pi}} \otimes \mathcal{O}(\Delta) \simeq \mathcal{L}_{\overline{\pi}}^{\otimes 13}$$

of line bundles on $\overline{\mathcal{M}}_3$. This extends $\chi'_{18} \otimes s_{\overline{\pi}}^{\otimes -2}$ on the trivial line bundle

$$\mathcal{L}_{\overline{\pi}}^{\otimes 26} \otimes (\det \mathcal{G}_{\overline{\pi}})^{\otimes -2} \otimes \mathcal{O}(-2\Delta).$$

The assertion is proven by taking the divisor of the trivial section on the line bundle above. QED

At the end of this subsection, we explain the relation between χ'_{18} and the Faltings height. Details can be found in Section 6 in [13].

For a metrized line bundle $(L, (\|\cdot\|_v)_{v \in M(k)_\infty})$ on a ring of integers O_k, its *arithmetic degree* is given by choosing a non-zero rational section s of L and setting

$$\deg(L, (\|\cdot\|_v)_{v \in M(k)_\infty}) := \sum_{v \in M(k)_0} \operatorname{ord}_v(s) \log(Nv) - \sum_{v \in M(k)_\infty} \log \|s\|_v,$$

where $M(k)_0$ (resp. $M(k)_\infty$) is the set of finite (resp. infinite) places of O_k.

Recall that $q : \mathbb{U}_g \to \mathbb{H}_g$ is the universal principally polarized complex abelian variety over the Siegel upper-half space. We write $\tilde{\mathcal{L}}$ for the line bundle $\det q_* \Omega_{\mathbb{U}_g/\mathbb{H}_g}$. The *Hodge metric* of $\tilde{\mathcal{L}}$ is given by

$$\|dz_1 \wedge \ldots dz_g\|_{\mathrm{Hdg}}(\tau) = \sqrt{\det \operatorname{Im} \tau} \tag{2.7}$$

for all $\tau \in \mathbb{H}_g$.

Let $f : \mathcal{X} \to \mathrm{Spec}(O_k)$ be a semistable arithmetic surface of genus 3 over a ring of integers with non-hyperelliptic smooth generic fiber. Let $\overline{\omega}$ be the Arakelov dualising sheaf. The *Faltings height* of f is given by

$$\deg \det f_* \overline{\omega}_{\mathcal{X}/S} = \frac{\sum\limits_{v \in M(k)_0} \mathrm{ord}_v(\chi'_{18}) \log(Nv)}{18} - \frac{\sum\limits_{v \in M(k)_\infty} \log \|\chi'_{18}\|_{\mathrm{Hdg},v}}{18}. \tag{2.8}$$

2.2.2 Klein formula

Recall that S_g is the set of even theta characteristics of genus g. By Page 851 in [37], the function

$$\widetilde{\Sigma}_{140} := \prod_{\substack{\epsilon \in S_3 \\ \epsilon \neq 0}} \theta_\epsilon(0,\tau)^8$$

is an analytic Siegel modular form of weight 140.

Theorem 2.2.5. *Let (A, a) be a principally polarized abelian variety of dimension 3 defined over $k \subset \mathbb{C}$. Let $\omega_1, \omega_2, \omega_3$ be a basis of $H^0(A, \Omega^1_{A/k})$ and $\gamma_1, \dots \gamma_6$ a symplectic basis of $H_1(A, \mathbb{Z})$. Then we can associate the period matrix $\Omega = [\Omega_1, \Omega_2]$ of (A, a). Put $\tau = \Omega_1^{-1}\Omega_2 \in \mathbb{H}_3$.*

(1) If $\widetilde{\Sigma}_{140}(\tau) = 0$ and $\tilde{\chi}_{18}(\tau) = 0$, then (A, a) is decomposable over $\overline{k}$. In particular it is not a Jacobian.

(2) If $\widetilde{\Sigma}_{140}(\tau) \neq 0$ and $\tilde{\chi}_{18}(\tau) = 0$, then there exists a hyperelliptic curve X/k such that $(\mathrm{Jac}X, j) \simeq (A, a)$.

(3) If $\tilde{\chi}_{18}(\tau) \neq 0$ then (A, a) is isomorphic to a non-hyperelliptic Jacobian if and only if

$$\chi_{18}((A, a), \omega) = (2\pi i)^{54} \frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}$$

is a square in k, where $\omega := \omega_1 \wedge \omega_2 \wedge \omega_3$.

Sketch of proof : The first two are proven in Lemma 10, Lemma 11 in [37]. By Proposition 2.1.11, we have the equality in (3). We give a sketch for the remaining part of (3), and a complete proof can be found in Theorem 1.3.3 in [43].

We assume (A, a) to be isomorphic to the Jacobian of a non-hyperelliptic genus 3 curve C/k. By Lemma 2.1.5 and Lemma 2.1.9, we have

$$\chi_{18}((A, a), \omega) = \mathbf{t}^*(\chi_{18})(C, \lambda) = \mu_9(C, \lambda)^2 \in k^{\times 2},$$

with $\lambda = \mathbf{t}^*\omega$.

Now we assume (A, a) is not isomorphic to the Jacobian of a non-hyperelliptic genus 3 curve C/k. By (3) in Theorem 2.1.6, we know (A, a) is a quadratic twist of a Jacobian (A', a'). Then it can be shown that

$$\chi_{18}((A, a), \omega) \equiv c^9 \chi_{18}((A', a'), \omega')(\mathrm{mod}\, k^{\times 18})$$

for some non-square element $c \in k^*$ (Corollary 1.2.3 in [43]). This implies that $\chi_{18}((A, a), \omega)$ is not a square, which completes the proof. QED

With this theorem, we can show the following formula of Klein in [41] which links the discriminant of a plane quartic and the analytic Siegel modular form $\tilde{\chi}_{18}$.

We fix a smooth plane curve C_F defined by a homogeneous degree d polynomial $F(X, Y, Z) = 0$. We write f for $F(x, y, 1)$ and write $k[x, y]_{<d}$ for the subspace of $k[x, y]$ containing polynomials of degree less than d. By a *classical basis* of Ω_{C_F}, we mean a basis of Ω_{C_F} in the form $\{\frac{g_i dx}{\frac{\partial f}{\partial y}}\}_{1 \le i \le \frac{(d-1)(d-2)}{2}}$ where $\{g_i\}_{1 \le i \le \frac{(d-1)(d-2)}{2}}$ is a basis of $k[x, y]_{<d}$ (see Theorem 4.6.10).

Theorem 2.2.6. *Let C_F be a smooth plane quartic curve over $\mathbb{C}$ defined by $F(X, Y, Z) = 0$. Let $\Omega = (\Omega_1 | \Omega_2)$ be the period matrix of C_F with respect to a classical basis of differential forms and a symplectic homology basis. We denote $\Omega_1^{-1}\Omega_2$ by τ. Then we have*

$$\mathrm{Disc}(F)^2 = (2\pi)^{54} \frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}.$$

Sketch of proof : See Theorem 2.2.3 in [43] for a complete proof.

We define a function I on X_4^0 as

$$I(F) := (2\pi)^{54} \frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}.$$

It can be shown that I is an invariant of degree 54 in the sense of Definition 2.1.12 (Corollary 2.2.2 in [43]). This means that $I(F)$ is a degree 54 homogeneous polynomial of the coefficients of F.

By Theorem 2.2.5, we have $I(F) \neq 0$ for all $F \in X_4^0$. Recall that the discriminant is a multiple of the resultant (Equation (2.4)) which is an irreducible polynomial of degree 27 (Page 113 Section 7 in [21]). By Hilbert's Nullstellensatz, we have $I = c\mathrm{Disc}_{\mathfrak{U}}^n$ for some constant $c \in \mathbb{C}^\times$.

The exponent n can be computed by the degree counting $n = 54/27 = 2$. The constant c can be computed for Ciani curves, which is equal to 1 (Corollary 4.2 in [42]). QED

Remark 2.2.7. *The Ciani curves are plane curves defined by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2 = 0,$$

for a, b and $c \in k$.