Explicit computation of the height of a Gross-Schoen Cycle

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Chapter 1

Arithmetic surfaces and intersection theory

This chapter is devoted to arithmetic surfaces and Arakelov theory. In Section 1.1, we discuss models of curves and general theory of arithmetic surfaces. Section 1.2 is about the foundation of Arakelov intersection theory. Section 1.3 is about G. Faltings' seminal paper [23] on Arakelov theory. In Section 1.4, we discuss dual graphs associated to semistable arithmetic surfaces and harmonic analysis on them. The heights of canonical Gross-Schoen cycles introduced in Section 1.5 are the main theme of this thesis.

We prove that Zhang's admissible invariants satisfy the contraction lemma (Proposition 1.4.33), which we have not found in literature. In Theorem 1.5.17, we prove an unboundedness property of the heights of canonical Gross-Schoen cycles for genus $g \geq 3$ smooth curves over number fields. To the best of the author's knowledge, this is a new result.

1.1 Models of curves

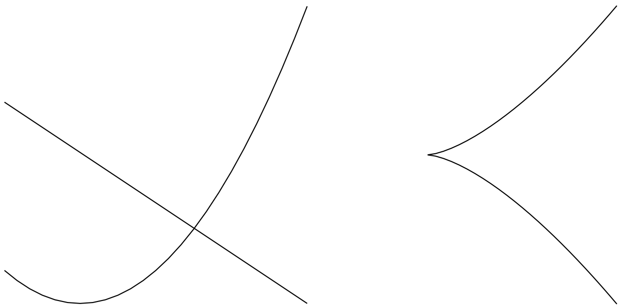
In Subsection 1.1.1, we define semistability and thickness. In Subsection 1.1.2, we define various kinds of models and state the semistable reduction theorem. In Subsection 1.1.3, we introduce the Deligne pairing on arithmetic surfaces. Proofs can be found in [48] and [52].

The definition of thickness appears in Proposition 1.1.8, which is essential for defining the dual graph of a semistable curve.

For simplicity, S is a Dedekind scheme (that is, a normal, irreducible Noetherian scheme of dimension 1) with function field $K(S)$ throughout this section. We write $k(p)$ for the residue field of a point p in a scheme.

1.1.1 Semistability

Semistable curves are curves with mildest possible singularities. By 'mildest', we mean intersections with two different tangent directions. In the graphs below, the singular point on the left one is considered as a 'mildest' singularity while the singular point on the right one is not.



(a) nodal point (b) cusp point

Figure 1.1.1: node and cusp

Definition 1.1.1. *Let C be a curve over an algebraically closed field k. A point p on C is called a nodal point or an ordinary double point if $\widehat{\mathcal{O}}_{C,p} \simeq \frac{k[[u,v]]}{(uv)}$.*

Definition 1.1.2. *A curve over an algebraically closed field k of arithmetic genus g is called semistable (resp. stable) if $g \geq 1$ (resp. $g \geq 2$), all of its singular points are nodal points and all of its components with arithmetic genus 0 meet other components in at least 2 (resp. 3) points.*

Remark 1.1.3. *We repeat here that all curves over fields are assumed to be geometrically connected.*

Example 1.1.4. *The curve $E_m : Y^2Z = X^3 + X^2Z$ in $\mathbb{P}^2_{\mathbb{C}}$ is semistable. It has only 1 nodal point at $(0 : 0 : 1)$.*

It is equivalent to define a stable curve as a curve having only nodal singularities and a finite automorphism group. The finiteness of the automorphism groups of stable curves can be compared with Hurwitz's automorphism theorem for Riemann surfaces which says that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is a finite group (containing at most $84(g-1)$ elements).

Remark 1.1.5. *From the definition, a semistable curve C over an algebraically closed field is a local complete intersection of codimension $n-1$ in $\mathbb{P}^n$, thus Serre duality can be applied and the dualizing sheaf of C is a line bundle (Theorem III.7.11 in [33]).*

Let C_0 be a semistable curve with 1 component and 1 node p_s. We describe the dualizing sheaf ω_{C_0} of C_0 as follows:

The normalization of C_0 is a smooth curve C with two specified points p and q (preimages of p_s under the normalization map). Let Ω be the dualising sheaf of C, which is isomorphic to the sheaf of differential forms on C. Then we have

$$\omega_{C_0} \simeq r_*\Omega(p+q),$$

where r_* is the pushforward along the normalization map r.

Definition 1.1.6. *For a general base scheme T, we define a curve over T to be a scheme X with a proper flat and finitely presented morphism $f : X \to T$ of pure relative dimension 1. We say f is a curve (resp. stable curve, resp. semistable curve) of genus g if $X_{\bar{t}}$ is a curve (resp. stable curve, resp. semistable curve) of genus g for all geometric points $\bar{t}$ of T.*

Example 1.1.7. *(Nice curves can be non-semistable) The curve C_F in $\mathbb{P}^2_{\mathbb{Z}}$ defined by $X^n + Y^n + Z^n = 0$ is not a semistable curve when $n \geq 2$. For any prime number $p|n$ we have $X^n + Y^n + Z^n = (X^{n/p} + Y^{n/p} + Z^{n/p})^p$ in $\overline{\mathbb{F}}_p[X, Y, Z]$ and thus the fiber of C_F at p is not even reduced. More generally, a plane curve defined by $G(X^p, Y^p, Z^p) = 0$ for some polynomial $G \in \mathbb{Z}[X, Y, Z]$ can not be semistable.*

By Definition 1.1.6, we can define a semistable curve over a non-algebraically closed field l. A point p_s on a semistable curve C over l is called a *split node* if $k(p_s) = l$ and $\widehat{\mathcal{O}}_{X,p_s} \simeq l[[u, v]]/(uv)$.

Proposition 1.1.8. *Recall that S is a Dedekind scheme. Let $X \to S$ be a semistable curve with smooth generic fiber X_η. For $s \in S$, let $x \in X_s$ be a singular point of X_s.*

(1) There exists a Dedekind scheme S', with a surjective and étale morphism $S' \to S$, such that any point $x' \in X' := X \times_S S'$ lying above x, belonging to a fiber $X'_{s'}$, is a split node of $X'_{s'} \to \operatorname{Spec} k(s')$.

(2) With the scheme S' obtained in (1), we have an isomorphism

$$\widehat{\mathcal{O}}_{X',x'} \simeq \widehat{\mathcal{O}}_{S',s'}[[u, v]]/(uv - c)$$

for some non-zero $c \in \mathfrak{m}_{s'}\mathcal{O}_{S',s'}$.

(3) Let e_x be the valuation of c for the normalized valuation of $\mathcal{O}_{S',s'}$. Then e_x is independent of the choice of S', s', and of x', and it is called the thickness of x in X.

Proof. See Corollary 10.3.22 in [48]. □

Example 1.1.9. *For a prime $p \geq 3$ and a positive integer n, the equation*

$$Y^2 Z = X^3 + X^2 Z + p^n Z^3$$

defines a semistable curve C in $\mathbb{P}^2_{\mathbb{Z}_p}$ with 1 nodal point at $p_s = (X, Y, p)$. It can be shown that $\widehat{\mathcal{O}}_{C,p_s} \simeq \mathbb{Z}_p[[u, v]]/(uv - p^n)$ and hence the thickness at p_s is n. More precisely, at the origin of the affine patch $y^2 - x^2(1 + x) - p^n = 0$, we can construct $g(x) \in \mathbb{Z}_p[[x]]$ such that $g^2(x) = 1 + x$, and this gives $y^2 - (xg(x))^2 - p^n = 0$. Taking $u = y + xg(x)$ and $v = y - xg(x)$, we get $\widehat{\mathcal{O}}_{C,p_s} \simeq \frac{\mathbb{Z}_p[[u,v]]}{(uv - p^n)}$.

Remark 1.1.10. *Thickness can be considered as a measure of singularity in an arithmetic sense.*

1.1.2 Models

By a *fibered surface over* S, we mean an integral, projective, flat S-scheme $\pi : X \to S$ of dimension 2 (S is a Dedekind scheme). We say the fibered surface π is normal if X is normal.

Definition 1.1.11. *Let C be a smooth curve over $K(S)$. We call a normal fibered surface $X \to S$ together with an isomorphism $f : X_\eta \simeq C$ a model of C over S, where η is the generic point of S. If X is regular, we call it a regular model. For a model X of C, if every birational map $Y \dashrightarrow X$ of models can be extended to a morphism, we say X is a minimal model for C. Moreover, we say a model (X, f) of C has property P if the morphism $X \to S$ has the property P.*

Theorem 1.1.12. *For every excellent, reduced, Noetherian 2-dimensional scheme X, there exists a proper birational morphism $X' \to X$ where X' is a regular scheme.*

Sketch of proof : X' is attained by iteratively blowing up at the singular locus and taking normalization. J. Lipman proved that this procedure terminates in finitely many steps. See [47] for a complete proof. QED

Remark 1.1.13. *Theorem 1.1.12 can be considered as the desingularization of 2-dimensional schemes. For general dimensions, H. Hironaka proved that any variety over a field of characteristic 0 can be desingularized into a regular variety. In [11], A. J. de Jong introduced alteration and proved that a separated integral scheme of finite type over a complete discrete valuation ring (this includes fields of characteristic p) always has an alteration from a regular scheme (Theorem 6.5 in [11]).*

Let C be a smooth curve over $K(S)$. With Theorem 1.1.12, we can always get a regular model $X \to S$ of C. If we assume further that the genus $g > 0$, then $X \to S$ has a unique minimal regular model, up to a unique isomorphism (Theorem 9.3.21 in [48]).

Definition 1.1.14. *Let C be a smooth curve over $K(S)$. We say that C has good (resp. stable, resp. semistable) reduction at a closed point $s \in S$ if there exists a smooth (resp. stable, resp. semistable) model X of C over* $\mathrm{Spec}(\mathcal{O}_{S,s})$*. We say C has good (resp. stable, resp. semistable) reduction over S if it has good reduction at every closed point $s \in S$.*

Good reduction is easy to deal with since it has smooth special fiber, but it can happen that a curve does not have good reduction.

Example 1.1.15. *(A curve without good reduction) For a field k, set $k((\lambda))$ with the natural discrete valuation, that is* $\mathrm{val}(\lambda) = 1$*. Then $E : Y^2Z = X(X-Z)(X-\lambda Z)$ is an elliptic curve over $k((\lambda))$ with the j invariant*

$$j(E) = \frac{2^8(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

The curve E does not have a smooth model since its j invariant is not in the valuation ring of $k((\lambda))$ (Proposition 5.5 in [60]).

The following theorem, first proved by P. Deligne and D. Mumford, shows the existence of a stable model, after taking an adequate field extension.

Theorem 1.1.16. *(Stable reduction theorem) Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K of genus $g \geq 2$. Then there exists an extension of discrete valuation rings $R \subset R'$ inducing a finite separable extension of fraction fields K'/K and a stable curve $Y \to$* $\mathrm{Spec}(R')$ *of genus g with $Y_{K'} \simeq C_{K'}$ over K'.*

Proof. Tag 0E8C □

Remark 1.1.17. *The original version of this theorem in Stack Project also requires $H^0(C, \mathcal{O}_C) = K$. We omit this condition since we assume all curves to be geometrically connected in this thesis.*

1.1.3 Intersection theory on arithmetic surfaces

It is too much to require a curve over a general scheme to be smooth, and we will instead consider regular objects. There are several advantages for restricting our discussion to regular Noetherian schemes. First, the stalks of regular schemes are UFDs, thus there is a 1-1 correspondence between rational equivalence classes of Weil divisors and isomorphism classes of line bundles. Second, the Grothendieck groups of coherent sheaves and vector bundles on regular schemes coincide (Page 13 in [61]), thus the K-theory on regular schemes behaves better. Third, regularity is strong enough for having a moving lemma on schemes (Corollary 9.1.10 in [48]).

As in the last two subsections, we still write S for a Dedekind scheme with fraction field $K(S)$.

Definition 1.1.18. *We call a regular fibered surface $\mathcal{X} \to S$ an arithmetic surface when $\mathcal{X}_\eta$ is smooth for the generic point η of S.*

Definition 1.1.19. *Let $\pi : \mathcal{X} \to S$ be an arithmetic surface and let D be a prime Weil divisor on $\mathcal{X}$. We say D is horizontal if $\pi|_D : D \to S$ is surjective. We say D is vertical if its image is reduced to a point.*

Remark 1.1.20. *A prime horizontal divisor is just the Zariski closure of a L-point in $\mathcal{X}$ where L is a finite extension of $K(S)$. A prime vertical divisor is an irreducible component of the fiber $\mathcal{X}_p$ over some closed point p in S.*

Let D and E be two Weil divisors on an arithmetic surface $\pi : \mathcal{X} \to S$ with no common components. We define the *intersection multiplicity* of D and E at a closed point $x \in \mathcal{X}$ as

$$i_x(D, E) := \mathrm{length}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{X},x}/(f, g)), \tag{1.1}$$

where f and g are local equations of D and E at x. We obtain a 0-cycle on $\mathcal{X}$ that can be written as

$$I(D, E) := \sum_x i_x(D, E)x.$$

Then

$$\pi_* I(D, E) := \sum_x i_x(D, E)[k(x) : k(\pi(x))]\pi(x)$$

is a divisor on S.

The intersection theory above is less satisfying since we have not defined the intersection between divisors with common components. This can be done by applying the moving lemma (see Section 9.1 in [48]). The following theorem of P. Deligne generalizes the intersection theory above using the language of line bundles.

Theorem 1.1.21. *Let $\pi : \mathcal{X} \to S$ be an arithmetic surface. Let L and M be two line bundles on $\mathcal{X}$. We can associate a line bundle $\langle L, M\rangle$ on S such that the following properties are satisfied:*

(1) If $L' \simeq L$ and $M' \simeq M$ then $\langle L', M'\rangle \simeq \langle L, M\rangle$.

(2) *The pairing is symmetric and satisfies the following laws:*

$$\langle L, M\rangle \simeq \langle M, L\rangle$$

$$\langle L, M \otimes N\rangle \simeq \langle L, M\rangle \otimes \langle L, N\rangle.$$

(3) *Let l and m be two rational sections on L and M whose divisors have no common components. Then there exists a non-zero rational section $\langle l, m\rangle$ of $\langle L, M\rangle$ such that:*

(a) *Let f be a rational function on X such that fl and m have no common components. Then*

$$\langle fl, m\rangle = N_{(m)}(f)\langle l, m\rangle,$$

where the definition of N can be found in Page 19 *in [52].*

(b) *There is an isomorphism*

$$\langle L, M\rangle \simeq \mathcal{O}_S(N_{(l)}(m)),$$

which sends $\langle l, m\rangle$ to $1_{N_{(l)}(m)}$. In fact, we have

$$\mathrm{div}(\langle l, m\rangle) = N_{(l)}(m).$$

(4) *Let D be a horizontal divisor on X. Then for a line bundle L on X, we have a natural isomorphism $\langle L, \mathcal{O}_X(D)\rangle \simeq N_D(L)$ which sends $\langle l, 1_D\rangle$ to $N_D(l)$.*

(5) *Let $\rho : S' \to S$ a flat morphism between connected Dedekind schemes and let $X' := X \times_S S'$ be the base change of $X \to S$ by ρ with the following commutative diagram.*

$$\begin{tikzcd} X' \arrow[r, "\mu"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "\rho"] & S \end{tikzcd}$$

Then there is a natural isomorphism

$$\rho^*(\langle L, M\rangle) \simeq \langle \mu^*(L), \mu^*(M)\rangle.$$

Proof. See Theorem 4.7 in [52]. □

If S is a smooth curve over a field k or S is isomorphic to $\mathrm{Spec}(O_K)$ for some number field K, we can associate a degree map to the Deligne pairing in the following way.

When S is a smooth curve, we define $\deg\langle L, M\rangle$ as the degree of the line bundle $\langle L, M\rangle$ on the curve S. In the classical way, this is equal to

$$\sum_x i_x(D,E)[k(x):k(\pi(x))]\deg(\pi(x)), \tag{1.2}$$

where D and E are divisors of some rational sections of L and M with no common components.

Remark 1.1.22. *For simplicity, we write* $(\cdot,\cdot)$ *for* $\deg\langle\cdot,\cdot\rangle$ *in this case.*

When $S = \mathrm{Spec}(O_K)$ for some number field K, we want to define $\deg\langle L, M\rangle$ as

$$\sum_x i_x(D,E)[k(x):k(\pi(x))]\log(\#k(\pi(x))), \tag{1.3}$$

where D and E are divisors of some rational sections of L and M with no common components. However, this is not good, since the number given by Equation (1.3) really depends on the choice of rational sections. Instead, we will consider line bundles on $\mathcal{X}$ with Hermitian metrics. Given two metrized line bundles L and M on $\mathcal{X}$, we can endow $\langle L, M\rangle$ with a Hermitian metric (Section 3.3 in [52]). Then we define $\deg\langle L, M\rangle$ using the following definition.

Definition 1.1.23. *Let M be a Hermitian metrized line bundle on* $\mathrm{Spec}(O_K)$*. We define its degree by*

$$\deg(M) := \log\#(M/O_K\cdot s) - \sum_{\sigma\in K(\mathbb{C})}\log\|s\|_\sigma$$

for an arbitrary non-zero element s of M. By the product formula, this degree does not depend on the choice of s.

The degree of the Deligne pairing has a close relation with the Arakelov intersection theory (Section 1.2). Actually, we have the following equality

$$(L,M)_{Ar} = \deg\langle L, M\rangle \tag{1.4}$$

for admissible line bundles L and M. See Section 6.3 in [18] for the construction of the metric on $\langle L, M\rangle$ and further discussion on Equation (1.4).

Remark 1.1.24. *For divisors D and E with no common components, we write* $(D,E)_{\mathrm{fin}}$ *for the number given by Equation (1.3).*

1.2 Arakelov's work

The main reference for this section is [1], in which Arakelov tried to establish an *arithmetic* intersection theory on arithmetic surfaces over number fields.

In Subsection 1.2.1, we introduce the Green's function on a Riemann surface which gives a metric on the line bundles on this Riemann surface. In Subsection 1.2.2, we explain Arakelov intersection theory and define the Arakelov dualising sheaf.

We will carry out some explicit computation for the Green's function in Section 4.6.

1.2.1 Green's functions on Riemann surfaces

In this subsection, X is a compact Riemann surface of genus $g \geq 1$, and we write Ω^1_X for its sheaf of holomorphic differential forms.

We can define a Hermitian inner product on $H^0(X, \Omega^1_X)$ as follows:

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \overline{\eta}. \tag{1.5}$$

With this inner product, we can choose an orthonormal basis $\{\omega_1, ..., \omega_g\}$, and we define the *volume form* or *canonical form* of X to be

$$\mu_{Ar} := \frac{i}{2g} \sum_{j=1}^{g} \omega_j \wedge \overline{\omega}_j. \tag{1.6}$$

The (1-1)-form μ_{Ar} on X does not depend on the choice of orthonormal basis.

Remark 1.2.1. *The word 'volume' comes from* $\int_X \mu_{Ar} = 1$.

Definition 1.2.2. *The canonical Arakelov-Green function of X is the unique function $G : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies:*

(1) Let P be any fixed point in an open set U with a local coordinate z on U. For $Q \in U$, we have $\log G(P,Q) = \log|z(Q)| + f(Q)$, where f is a C^∞ function on U.

(2) $G(\cdot,\cdot)^2$ is a C^∞ function on $X \times X$ and $\partial_Q \overline{\partial}_Q \log G(P,Q)^2 = 2\pi i \mu_{Ar}(Q)$ for $Q \neq P$.

(3) $\int_X \log G(P,Q) d\mu_{Ar}(Q) = 0$.

(4) G vanishes at the diagonal of $X \times X$.

Remark 1.2.3. *For simplicity, we will use the $Green's\ function$ of X instead of the Arakelov-Green function of X in this thesis.*

Example 1.2.4. *Let $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$, and define a torus X by $X \simeq \mathbb{C}/\langle 1, \tau\rangle$. Let z be the coordinate of $\mathbb{C}$. Then the Green's function on X is*

$$\log G(z_1, z_2) = \log G(z_1 - z_2, 0) = \log\|\theta\| \left(\tau, z_1 - z_2 + \frac{1+\tau}{2}\right) - \log\|\eta\|(\tau),$$

where

$$\|\theta\|(a+bi, x+yi) = b^{1/4} e^{-\pi y^2/b} \cdot |\sum_{n\in\mathbb{Z}} e^{\pi i n^2(a+bi)} e^{2\pi i n(x+yi)}|,$$

$$\|\eta\|(a+bi) = b^{1/4} \cdot |e^{\pi i(a+bi)/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i(a+bi)})|.$$

See Section 7 in [23] for details.

It can be proven by Green's formula that the Green's function is a symmetric function. The existence of the Green's function can be proven by partial differential equation tools and the uniqueness is trivial. However, it is still not easy to construct it from the definition. R. de Jong gave an explicit expression for the Green's function (Theorem 2.1.2 [12]).

Next we will assign a Hermitian metric on $\mathcal{O}(D)$ to each divisor D on X. The trivial line bundle is assigned with the constant function $\|1\|_{\mathcal{O}_X} = 1$. For a prime divisor $D = P$, we assign the smooth Hermitian metric

$$\|1\|_{\mathcal{O}(D)}(Q) \coloneqq G(P, Q) \tag{1.7}$$

on $\mathcal{O}(D)$. If $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are already assigned with metrics, we define

$$\|1\|_{\mathcal{O}(D_1+D_2)}(Q) \coloneqq \|1\|_{\mathcal{O}(D_1)}(Q) \cdot \|1\|_{\mathcal{O}(D_2)}(Q) \tag{1.8}$$

to be the metric on $\mathcal{O}(D_1 + D_2)$. These can give a Hermitian metric on every line bundle $\mathcal{O}(D)$ of X inductively.

Definition 1.2.5. *Let $\|\cdot\|$ be a smooth Hermitian metric on a line bundle $\mathcal{O}(D)$ of X. We say $\|\cdot\|$ is admissible if its curvature form is a multiple of μ_{Ar}.*

From Property (2) in Definition 1.2.2, we find that the curvature form of the metric $\|\cdot\|_{\mathcal{O}(D)}$ is $\deg(D)\mu_{Ar}$. This means that the metric we just defined on $\mathcal{O}(D)$ is admissible.

Remark 1.2.6. *There is an admissible metric on a line bundle $\mathcal{O}(D)$, unique up to a multiplicative scalar. This can be proven using Property* (2) *of the Green's function.*

We end this subsection by constructing an admissible metric on Ω_X.

(1) For any point $P \in X$, we already have a metric on $\mathcal{O}(P)$ given by the Green's function on X.

(2) The residue of rational sections of Ω^1_X at P gives a natural isometry:

$$\Omega_X(P)|_P \simeq \mathbb{C},$$

where $\mathbb{C}$ has the standard Euclidean metric.

(3) We assign Ω^1_X with the metric such that the following isomorphism gives an isometry at every point P:

$$\Omega_X(P) \simeq \mathcal{O}(P) \otimes \Omega_X.$$

Definition 1.2.7. *The metric defined above, denoted by $\|\cdot\|_{Ar}$, is called the Arakelov metric on Ω^1_X.*

Let $\Delta : X \to X \times X$ be the diagonal map. Then we know that $\Omega^1_X \simeq \mathcal{O}_{X\times X}(-\Delta)|_\Delta$. If we assign a metric on $\mathcal{O}_{X\times X}(-\Delta)$, then its pullback along the diagonal map will induce a metric on Ω^1_X. The metric $\|\cdot\|_{Ar}$ is equal to the pullback of the metric $\|1\|_{\mathcal{O}_{X\times X}(-\Delta)}(P,Q) := G^{-1}(P,Q)$ along Δ.

Theorem 1.2.8. *The Arakelov metric $\|\cdot\|_{Ar}$ is admissible.*

Proof. See Section 4 in [1] or Section 4.5 in [52]. □

Remark 1.2.9. *In fact, we can associate a Green's function to any Kähler form on X (modifying Property (2) in Definition 1.2.2). The reason we choose the canonical form (Equation (1.6)) is that this is the only Kähler form, up to multiplicative scalar, that induces an admissible metric on Ω^1_X by the construction above (Lemma 4.25 in [52]).*

1.2.2 Arakelov intersection theory

Now we are ready to show how Arakelov intersection theory is defined. In this subsection, K is a number field with integer ring O_K and $S = \mathrm{Spec}(O_K)$.

Let $\pi : \mathcal{X} \to S$ be an arithmetic surface of genus $g \geq 1$ with smooth generic fiber $\mathcal{X}_K$. A prime horizontal divisor D on $\mathcal{X}$ can be written in the form $\epsilon_*(\mathrm{Spec}(O_L))$, where L is a finite extension of K and $\mathrm{Spec}(O_L) \xrightarrow{\epsilon} \mathcal{X}$ is a section of π. Each embedding $\sigma : K \to \mathbb{C}$ corresponds to a compact Riemann surface $\mathcal{X}_\sigma$. By μ_σ, we mean the canonical form on $\mathcal{X}_\sigma$ defined in the last subsection.

In Arakelov intersection theory, the divisor group of $\mathcal{X}$ contains the divisors in the usual sense, which are called $finite\ divisors$, and also contains real linear combinations of $\mathcal{X}_\sigma$, which are called $infinite\ divisors$. The advantage of including infinite divisors is that we can make S into a 'compact' object by the product formula.

Definition 1.2.10. *An Arakelov divisor on $\mathcal{X}$ is a formal sum $D_{\mathrm{fin}} + D_{\mathrm{inf}}$, where D_{fin} is a Weil divisor on $\mathcal{X}$ and $D_{\mathrm{inf}} = \sum_{\sigma: K \to \mathbb{C}} c_\sigma \mathcal{X}_\sigma$ is a formal linear combination of infinite fibers $\mathcal{X}_\sigma$ over $\mathbb{R}$. We write $\mathrm{Div}_{Ar}(\mathcal{X})$ for the group generated by Arakelov divisors. We write $D_{\mathrm{fin},\sigma}$ for the divisor on $\mathcal{X}_\sigma$ induced by D_{fin}.*

For a rational section f of $\mathcal{O}_{\mathcal{X}}$, we define a principal divisor associated to it as

$$\mathrm{div}_{Ar}(f) := \mathrm{div}(f) + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma} \mathcal{X}_\sigma,$$

where $\mathrm{div}(f)$ is the divisor of f in the usual sense and $v_{\mathcal{X}_\sigma} := -\int_{\mathcal{X}_\sigma} \log|f|_\sigma \mu_\sigma$. We say two Arakelov divisors D_1 and D_2 are *linearly equivalent* if $D_1 = \mathrm{div}_{Ar}(f) + D_2$ for some rational section f of $\mathcal{O}_{\mathcal{X}}$. We denote

$$\widehat{\mathrm{Cl}}(\mathcal{X}) := \mathrm{Div}_{Ar}(\mathcal{X})/(\text{principal divisors}).$$

We now construct an intersection pairing $(\cdot,\cdot)_{Ar}$ on $\widehat{\mathrm{Cl}}(\mathcal{X})$ which does not depend on the choice of representatives.

(1) For an infinite prime divisor σ and a prime divisor D (finite or infinite), we define $(\sigma, D)_{Ar} := d$ (resp. 0) if D is horizontal (resp. vertical or infinite), where d is the degree of D over the generic fiber.

(2) Let D_1 be a prime vertical divisor and let D_2 be a finite divisor. If D_1 and D_2 have no common components, then we define

$$(D_1, D_2)_{Ar} := (D_1, D_2)_{\mathrm{fin}},$$

where $(\cdot,\cdot)_{\mathrm{fin}}$ is defined in Equation (1.3).

(3) Let $D_1 : \mathrm{Spec}(O_{L_1}) \to \mathcal{X}$ and $D_2 : \mathrm{Spec}(O_{L_2}) \to \mathcal{X}$ be distinct prime horizontal divisors of $\mathcal{X}$. Then $D_{1,\sigma}$ and $D_{2,\sigma}$ determine two sets of points $\{P^\sigma_{1,j}\}_{1 \le j \le [L_1:K]}$ and $\{P^\sigma_{2,k}\}_{1 \le k \le [L_2:K]}$ on $\mathcal{X}_\sigma$ for each embedding $\sigma : K \to \mathbb{C}$. We define

$$(D_1, D_2)_{Ar} := (D_1, D_2)_{\mathrm{fin}} + \sum_{\sigma \in K(\mathbb{C})} \sum_{1 \le j \le [L_1:K]} \sum_{1 \le k \le [L_2:K]} -\log G_\sigma(P^\sigma_{1,j}, P^\sigma_{2,k}).$$

We still need to define the intersection pairing when two Arakelov divisors have common finite components. The following theorem will be useful.

Theorem 1.2.11. *Let $\mathcal{X} \to \mathrm{Spec}(O_K)$ be an arithmetic surface. Let D be an Arakelov divisor and let f be a rational function on $\mathcal{X}$. If $\mathrm{div}_{Ar}(f)$ and D have no common finite components, then $(D, \mathrm{div}_{Ar}(f))_{Ar} = 0$.*

Sketch of proof : We only need to prove that $(\mathrm{div}_{Ar}(f), D)_{Ar} = 0$ for any prime Arakelov divisor D. This is trivial if D is an infinite divisor. When D is a finite divisor, we assume D is not in the support of $\mathrm{div}_{Ar}(f)$.

When D is a prime vertical divisor, this follows from Theorem 3.1 in [44]. It remains to prove this for a horizontal prime divisor. For simplicity, we only consider the case $D = \epsilon(\mathrm{Spec}(O_K))$ for some section $\epsilon : \mathrm{Spec}(O_K) \to \mathcal{X}$.

The divisor D corresponds to a point P_σ for each $\sigma : K \to \mathbb{C}$. A rational section f of $\mathcal{O}_{\mathcal{X}}$ gives a meromorphic function f_σ on $\mathcal{X}_\sigma$. Since f_σ is meromorphic, we have

$$\partial_\sigma \overline{\partial}_\sigma \log(|f_\sigma(x)|) = 0$$

outside $\mathrm{div}(f_\sigma)$. Since $\deg(f_\sigma) = 0$, we have

$$\partial_{P_\sigma} \overline{\partial}_{P_\sigma} \log G_\sigma(\mathrm{div}(f_\sigma), P_\sigma) = 0$$

by Property (2) in Definition 1.2.2. This means that there exists a real constant α such that

$$G(\mathrm{div}(f), P_\sigma) = e^\alpha \cdot |f|(P_\sigma).$$

According to Property (3) in Definition 1.2.2, we obtain $\alpha = -\int_{\mathcal{X}_\sigma} \log|f_\sigma|\mu_\sigma$. Now we can compute

$$\begin{aligned}
(D, (f))_{Ar} &= (D, \mathrm{div}(f) + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) \cdot \mathcal{X}_\sigma)_{Ar} \\
&= (D, \mathrm{div}(f))_{Ar} + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) \\
&= (D, \mathrm{div}(f))_{\mathrm{fin}} - \sum_{\sigma \in K(\mathbb{C})} \log|f_\sigma|(P_\sigma) - \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) \\
&= (D, \mathrm{div}(f))_{\mathrm{fin}} - \sum_{\sigma \in K(\mathbb{C})} \log|f_\sigma|(P_\sigma) \\
&= 0,
\end{aligned}$$

where $(D, \mathrm{div}(f))_{\mathrm{fin}}$ is defined in Remark 1.1.24 and the last step is due to the product formula for number fields. QED

We return to the Arakelov intersection pairing. The Moving Lemma (Corollary 9.1.10 in [48]) says that for any two Arakelov divisors E and F, we can find a rational function h on $\mathcal{X}$ such that $E_{\mathrm{fin}} + \mathrm{div}(h)$ and F_{fin} have no common components. We define

$$(E, F)_{Ar} := (E + \mathrm{div}_{Ar}(h), F)_{Ar}.$$

Theorem 1.2.11 implies that the intersection number is well-defined and $(\cdot,\cdot)_{Ar}$ factors through Arakelov principal divisors, that is:

$$(\cdot,\cdot)_{Ar} : \widehat{\mathrm{Cl}}(\mathcal{X}) \times \widehat{\mathrm{Cl}}(\mathcal{X}) \to \mathbb{R}.$$

We next define a dualising object $\overline{\omega}$ for Arakelov divisors. The finite divisors of $\overline{\omega}$ should correspond to the usual dualising sheaf on $\mathcal{X}$, and thus it remains to figure out the infinite part. Before that, we introduce $\widehat{\mathrm{Pic}}(\mathcal{X})$.

Definition 1.2.12. *Let $\mathcal{X} \to \mathrm{Spec}(O_K)$ be an arithmetic surface. An admissible line bundle on $\mathcal{X}$ is the datum of a line bundle L on $\mathcal{X}$ and an admissible metric $\|\cdot\|_\sigma$ on the line bundle L_σ on $\mathcal{X}_\sigma$ for each $\sigma : K \to \mathbb{C}$. The set of isomorphism classes of admissible line bundles on $\mathcal{X}$ has a natural group structure, and we denote it by $\widehat{\mathrm{Pic}}(\mathcal{X})$.*

Theorem 1.2.13. *There is a canonical isomorphism of groups $\widehat{\mathrm{Cl}}(\mathcal{X}) \simeq \widehat{\mathrm{Pic}}(\mathcal{X})$.*

Sketch of proof : See Proposition 2.2 in [1] for details. We only give a description of the map. Let $D = D_{\text{fin}} + \sum_\sigma c_\sigma \cdot \mathcal{X}_\sigma$ be an Arakelov divisor. Then D_{fin} gives a line bundle $\mathcal{O}(D_{\text{fin}})$ on $\mathcal{X}$. For each $\sigma : K \to \mathbb{C}$, we associate the line bundle $\mathcal{O}(D_{\text{fin},\sigma})$ on $\mathcal{X}_\sigma$ with the admissible metric $e^{-c_\sigma} \cdot \|\cdot\|_{\mathcal{O}(D_{\text{fin},\sigma})}$, where $\|\cdot\|_{\mathcal{O}(D_{\text{fin},\sigma})}$ is the metric induced from the Green's function on $\mathcal{X}_\sigma$. QED

Definition 1.2.14. *The Arakelov dualising sheaf $\overline{\omega}$ on $\mathcal{X} \to \mathrm{Spec}(O_K)$ consists of the following datum:*

(1) the usual dualising sheaf $\omega_{\mathcal{X}/S}$ on $\mathcal{X}$,

(2) the Arakelov metric $\|\cdot\|_{Ar,\sigma}$ on $\Omega_{\mathcal{X}_\sigma}$ for each $\sigma : K \to \mathbb{C}$.

According to Theorem 1.2.13, this dualising sheaf corresponds to a unique element in $\widehat{\mathrm{Cl}}(\mathcal{X})$.

We end this subsection with stating the adjunction formula in Arakelov intersection theory, although we do not really use it in this thesis.

Theorem 1.2.15. *The divisor D given by a section $\mathrm{Spec}(O_K) \to \mathcal{X}$ leads to the following equality*

$$-(D,D)_{Ar} = (D,\overline{\omega})_{Ar}.$$

Proof. See Lemma 4.26 in [52]. □

1.3 Faltings' work

In [23], G. Faltings established

(1) the Faltings-Riemann-Roch theorem,

(2) the arithmetic Noether's formula,

(3) the positivity of the relative dualizing sheaf $\omega_{\mathcal{X}}$ (Arakelov theoretic version),

(4) the Hodge index theorem (Arakelov theoretic version).

In this section, we give a brief review of these results except the last one. Subsection 1.3.1 is about the Faltings metric on the determinant of cohomology. Subsection 1.3.2 is about the Faltings-Riemann-Roch theorem and its corollaries.

Corollary 1.3.11 will be used to decompose $\langle \Delta, \Delta \rangle$ in Subsection 3.3.1. The Faltings δ invariant introduced in Theorem 1.3.9 will be computed in Section 4.8.

1.3.1 Determinant of cohomology and Faltings metric

Let V be a complex vector space of dimension d. We can define $\det V := \Lambda^d V$ as the top exterior power of V.

Example 1.3.1. *Let C be a compact Riemann surface of genus $g \geq 1$. For an arbitrary line bundle L on X, we have a 1-dimensional vector space over $\mathbb{C}$*

$$\lambda(L) := \det H^0(C, L) \otimes \left(\det H^1(C, L)\right)^{-1}.$$

By Serre duality, we have a canonical morphism

$$\left(\det H^1(C, L)\right)^{-1} \simeq \det H^0(X, \Omega_C \otimes L^{-1}).$$

In the above construction, we start from a line bundle on a Riemann surface and end with a 1-dimensional complex vector space. The following theorem is a generalization of this construction. We refer to Section 5 in [53] for the definition of Rp_*F.

Theorem 1.3.2. *More generally, let $p : Y \to T$ be a proper morphism of Noetherian schemes. Then for each coherent sheaf F on Y, flat over T, we can associate a line bundle $\det Rp_*F$ on T, called the determinant of cohomology of F, that satisfies the following properties.*

(1) $\det Rp_*F$ is functorial for isomorphisms of coherent sheaves on Y.

(2) $\det Rp_*F$ commutes with base change.

(3) *If*

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of coherent sheaves on Y flat over T, then there is an isomorphism

$$\det Rp_*F \xrightarrow{\sim} \det Rp_*F' \otimes \det Rp_*F''$$

which is functorial with respect to base changes and isomorphisms of exact sequences.

(4) *Let*

$$\mathcal{E} : 0 \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

be a complex of vector bundles of finite rank which gives a quasi-isomorphism

$$\mathcal{E} \xrightarrow{\sim} Rp_*F.$$

Then we have a canonical isomorphism

$$\det Rp_*F \xrightarrow{\sim} \bigotimes_{k=0}^{n} (\det E^k)^{(-1)^k},$$

which commutes with base changes. Here $\det E^k$ *is the top exterior power of the vector bundle E^k.*

(5) *If T is connected and F is fixed, then the function $\chi : s \to \chi(F_s)$ is a constant function on T. Let u be a global section of O_T^*. The multiplication by u on F induces an automorphism* $\det(u) : \det Rp_*F \xrightarrow{\sim} \det Rp_*F$ *according to* (1), *and we have*

$$\det(u) = u^{\chi}.$$

(6) *If M is a line bundle on T (assume connected again), then there is a canonical isomorphism*

$$\det Rp_*(F \otimes p^*M) \xrightarrow{\sim} (\det Rp_*F) \otimes M^{\otimes \chi}$$

Proof. See Section 1 in [50]. □

Remark 1.3.3. *By* (4) *in the above theorem, if the higher pushforward sheaves R^ip_*F $(i \geq 0)$ are vector bundles, then there is a natural isomorphism*

$$\det Rp_*F \xrightarrow{\sim} \bigotimes_{i=1}^{n} (\det R^ip_*F)^{(-1)^i}.$$

Let $f : \mathcal{X} \to S$ be a semistable arithmetic surface. The dualising sheaf ω is coherent (even a line bundle) and the higher pushforwards $R^i f_*(\omega)$ are coherent. Since coherent sheaves on regular Noetherian schemes have finite free resolutions, we can apply (3) and (4) to construct $\det Rf_*\omega$. The following theorem shows the relation between the determinant of cohomology and the Deligne pairing in Theorem 1.1.21.

Proposition 1.3.4. *Let $p : \mathcal{X} \to S$ be an arithmetic surface with line bundles L and M on $\mathcal{X}$. We have a canonical isomorphism*

$$\langle L, M\rangle \xrightarrow{\sim} \det Rp_*(L \otimes M) \otimes (\det Rp_*L)^{-1} \otimes (\det Rp_*M)^{-1} \otimes \det p_*\omega_{\mathcal{X}/S}.$$

Proof. See Page 14 in [12]. □

A corollary of the proposition above is that we have a Riemann-Roch theorem for arithmetic surfaces.

Corollary 1.3.5. *Let $p : \mathcal{X} \to S$ be an arithmetic surface with line bundles L and M on $\mathcal{X}$. We have a canonical isomorphism*

$$(\det Rp_*L)^{\otimes 2} \xrightarrow{\sim} \langle L, L \otimes \omega^{-1}_{\mathcal{X}/S}\rangle \otimes (\det p_*\omega_{\mathcal{X}/S})^{\otimes 2}.$$

Proof. See Theorem 9.9 in [18]. □

Now we return to Riemann surfaces. Let $f : X \to \mathrm{Spec}(\mathbb{C})$ be a compact Riemann surface of genus $g > 0$. By Theorem 1.3.2, or the beginning of this subsection, we have a complex vector space $\lambda(L) := \det Rf_*(L)$ for any line bundle L on X. In [23], G. Faltings defined a metric on $\lambda(L)$ which is known as the Faltings metric.

Theorem 1.3.6. *There exists, for every line bundle L on X together with an admissible Hermitian metric on L, a Hermitian metric on $\lambda(L)$, such that the following properties hold:*

(1) An isometric isomorphism of line bundles induces an isometry on these $\lambda(\cdot)$.

(2) If the metric on L is changed by a factor $\alpha > 0$, the metric on $\lambda(L)$ is changed by multiplying $\alpha^{\chi(L)}$, where

$$\chi(L) = \dim H^0(X, L) - \dim H^1(X, L) = \deg(L) + 1 - g.$$

(3) For a divisor D on X and a point $P \in X$, $\mathcal{O}(D)$ and $\mathcal{O}(D - P)$ have canonical admissible metrics (constructed by Equation (1.7) and Equation (1.8)). We set

$\mathcal{O}(D)[P]$ with the metric given by the restriction of the metric on $\mathcal{O}(D)$ to the fibre over P. The exact sequence

$$0 \to \mathcal{O}(D-P) \to \mathcal{O}(D) \to \mathcal{O}(D)[P] \to 0$$

induces an isomorphism

$$\lambda(\mathcal{O}(D)) \simeq \lambda(\mathcal{O}(D-P)) \otimes_{\mathbb{C}} \mathcal{O}(D)[P]$$

which is also an isometry.

(4) The metrics on the $\lambda(\cdot)$ are unique up to a common scalar factor.

(5) If $L = \Omega_X$, the metric on $\lambda(L) = \det H^0(X, \Omega_X)$ is induced from the inner product in Equation (1.5).

Proof. See Theorem 1 in [23]. Points (3) and (4) make it possible for us to construct this metric inductively. The symmetry property of the Green's function guarantees that the order of points we choose in our construction does not matter. Point (1) is the most technical one. An alternative proof using Proposition 1.3.4 can be found on Page 15 of [12]. □

1.3.2 Faltings-Riemann-Roch theorem

In this subsection, we assume $p : \mathcal{X} \to S$ to be a semistable arithmetic surface with $S = \operatorname{Spec}(O_K)$ for some number field K.

If L is an admissible line bundle on $\mathcal{X}$, then $L \otimes \overline{\omega}^{-1}$ is also an admissible line bundle. According to Theorem 1.3.6, we can assign metrics to $\det Rp_*L$ and $\det Rp_*\omega$. There is a unique metric on $\langle L, L \otimes \overline{\omega}^{-1} \rangle$ such that Corollary 1.3.5 is an isometry with respect to these metrics. We have following Faltings-Riemann-Roch theorem.

Theorem 1.3.7. $\deg \det Rp_*L = \frac{1}{2}(L, L \otimes \overline{\omega}^{-1})_{Ar} + \deg \det p_*\omega_{\mathcal{X}/S}$.

Proof. See Theorem 3 in [23]. □

Let B be a smooth curve over a field l. If $p : \mathcal{Y} \to B$ is a semistable curve with smooth generic fiber, then we have

$$12 \cdot \deg(\det p_*\omega_{\mathcal{Y}}) = (\omega_{\mathcal{Y}}, \omega_{\mathcal{Y}}) + \delta,$$

where δ is the number of singular points, counted according to the degree of their residue field extensions and thicknesses. This is known as the classical Noether's formula.

Let ω be the universal dualising sheaf of the universal curve $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ (see Tag 0DMG for details). Over the complex points of $\mathcal{M}_g$, we can assign Arakelov metric (using the Arakelov-Green function) to ω fiberwise. Then we can assign a metric on $\langle \omega, \omega \rangle$ (see Section 9 in [14]).

Remark 1.3.8. *(More about the metric on $\langle \omega, \omega \rangle$) Let $\mathcal{Z}$ be a smooth complex variety and let $p : \mathcal{Y} \to \mathcal{Z}$ be a smooth proper curve of genus $g \geq 1$. For two metrized line bundles L (with a non-zero rational section l) and M (with a non-zero rational section m) on $\mathcal{Y}$, we can construct a line bundle $\langle L, M \rangle$ (with a non-zero rational section $\langle l, m \rangle$) on $\mathcal{Z}$. All these constructions are similar to the technique in Theorem 1.1.21. Then we can assign a metric on $\langle L, M \rangle$ given by*

$$\log\|\langle l, m\rangle\| = (\log\|m\|)[\mathrm{div} l] + \int_p \log\|l\| c_1(M)$$

Theorem/Definition 1.3.9. *There exists an isomorphism of line bundles*

$$\mu : (\det \pi_* \omega)^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega \rangle \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta)$$

on $\overline{\mathcal{M}}_g$, which is unique up to a sign. The Faltings delta invariant is defined to be the number $\delta(\cdot)$ such that $(2\pi)^{-4g}\exp(\delta(\cdot))$ is the norm of the above isomorphism on $\mathcal{M}_g(\mathbb{C})$.

Proof. See Theorem 2.1 in [51]. □

Remark 1.3.10. *In [23], G. Faltings gave an interpretation of the Faltings delta invariant using the theta divisor Θ associated to the corresponding compact Riemann surface. In [12], R. de Jong gave a more explicit method for numerically computing this invariant. We will apply this method in Section 4.7.*

The following corollary is known as the Noether's formula for an arithmetic surface over a ring of integers O_K. Recall that $\delta(X)$ is the Faltings delta invariant for the Riemann surface X, $\delta(\overline{\Gamma})$ is the total volume of the pm-graph $\overline{\Gamma}$ (see Definition 1.4.8 and the discussion after Remark 1.4.14), and $p : \mathcal{X} \to S$ is a semistable arithmetic surface.

Corollary 1.3.11. *We write $\det p_* \overline{\omega}$ for the line bundle $\det p_* \omega$ with the metric induced from Equation (1.5). Then we have*

$$12 \deg \det p_* \overline{\omega} = (\overline{\omega}, \overline{\omega})_{Ar} + \sum_s \delta(\overline{\Gamma}_s) \log(\#k(s)) + \sum_{\sigma \in K(\mathbb{C})} \delta(\mathcal{X}_\sigma) - 4g[K : \mathbb{Q}]\log(2\pi),$$

where the first (resp. second) sum goes through all closed points $s \in \mathrm{Spec}(O_K)$ (resp. complex embeddings of K) and $\delta(\mathcal{X}_\sigma)$ is the Faltings delta invariant (Theorem 1.3.9).

Proof. We first pull back the isomorphism in Theorem 1.3.9 along the classifying map $\mathrm{Spec}(O_K) \to \overline{\mathcal{M}}_g$. Then the assertion is proved by taking the degree of both sides in Theorem 1.3.9. □

At the end of this subsection, we state the non-negativity of the self-intersection of the Arakelov dualising sheaf, although we will not use it in an essential way.

Theorem 1.3.12. *Let D be an effective divisor on the semistable arithmetic surface $\mathcal{X} \to \mathrm{Spec}(O_K)$ of genus g. We have the following results:*

(1) $(\overline{\omega}, \overline{\omega})_{Ar} \geq 0$,

(2) $(\overline{\omega}, D)_{Ar} \geq \frac{(\overline{\omega},\overline{\omega})_{Ar}}{4g(g-1)} \cdot \deg(D)$.

Proof. See Theorem 5 in [23]. □

1.4 Dual graph

In Subsection 1.4.1, we introduce pm-graphs and some basic notions. In Subsection 1.4.2, we discuss the Green's function on a pm-graph and the admissible invariants introduced by S. Zhang. In Subsection 1.4.3, we will introduce more invariants on pm-graphs and use them to prove the second half of Proposition 1.4.33. More details can be found in [69] and [70].

We will return to the explicit computation of admissible invariants in Section 3.1. And we will compute the admissible invariants of our main curve $\mathfrak{C}$ in Proposition 4.4.1.

All graphs are assumed to be finite.

1.4.1 Polarized metrized graph

Definition 1.4.1. *A star-shaped set of valence n and radius ϵ is a metric space that is isometric to*

$$S(n, \epsilon) = \{z \in \mathbb{C} : z = te^{\frac{2\pi i k}{n}} \text{for some } 0 \leq t < \epsilon \text{ and } k \in \mathbb{Z}\}.$$

Definition 1.4.2. *A metrized graph Γ is a compact connected metric space that either is a point or satisfies that for each point p of Γ there exists a neighbourhood U_p that is isometric to a star-shaped set of finite valence and radius $\epsilon > 0$. The valence is well-defined and we denote the valence of a point p by $v(p)$.*

We define the *canonical divisor* of Γ as

$$K_\Gamma := \sum_{x \in \Gamma} (v(x) - 2)x. \tag{1.9}$$

The canonical divisor is well-defined since all but finitely many points on Γ have valence 2. Let V_0 be the set containing exactly points $p \in \Gamma$ such that $v(p) \neq 2$. Since we only consider compact metrized graphs, V_0 is a finite set. A non-empty finite set $V' \subset \Gamma$ containing V_0 is called a *vertex set* of Γ. For a vertex set V', the complement $\Gamma \backslash V'$ is a union of finitely many connected components. Each component e° in $\Gamma \backslash V'$ is called an *edge* associated to V' and is isometric to an open interval. The closure of each edge e° in Γ is a closed segment (we call it an *ordinary edge*) or a circle (we call it a *self-loop*), denoted by e. We call $e \backslash e^\circ$ the *endpoints* of e° in Γ. We can associate a unique positive number $l(e^\circ)$ to each edge e° such that e° is isometric to the interval $(0, l(e^0))$. The real number $l(e^\circ)$ is called the *weight* of e°. For simplicity, we sometimes also say e is an edge of weight $l(e^\circ)$.

Remark 1.4.3. *Note that we require a vertex set to be non-empty, thus the empty set is not a vertex set for a 1-loop graph (containing* 1 *loop only).*

Example 1.4.4. *The following is an illustration of metrized graphs with specified vertex sets. We omit the weight information in the figure.*

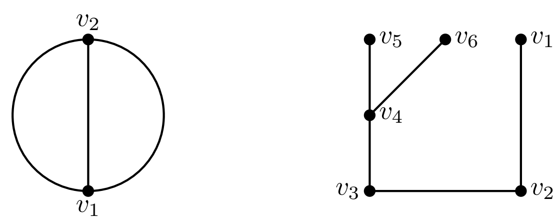


Figure 1.4.1: Metrized graphs with specified vertex sets

We can interpret metrized graphs in a graph-theoretic way. Let $G = (V, E)$ be an undirected graph with the vertex set V and the edge set E. Let $w : E \to \mathbb{R}_{>0}$ be a function on E, then we call the pair (G, w) a *weighted graph*.

There is a natural way to construct a metrized graph Γ with a specified vertex set from a connected weighted graph (G, w): V gives the specified vertex set ($V = V'$) and the vertices are connected by $e \in E$ with length $w(e)$. On the other hand, given a metrized graph Γ with a specified vertex set V', we can construct a connected weighted graph (G, w) by taking V to be V', taking elements in E to be the components in $\Gamma \backslash V$ and taking the induced weights from $\Gamma \backslash V$. Thus we have a correspondence

$$(\Gamma, V') \rightleftharpoons (G, w),$$

from the set of metrized graphs with specified vertex sets to the set of connected weighted graphs.

Definition 1.4.5. *Let (G, w) be a connected weighted graph. The first Betti number of (G, w) is defined to be $b_1(\Gamma) := \#E - \#V + 1$.*

Remark 1.4.6. *According to the correspondence, we can define the first Betti number on a metrized graph Γ with a specified vertex set V. It is easy to show that this number only depends on the metric graph Γ.*

Definition 1.4.7. *Let the pair (Γ, V) be a metrized graph with a specified vertex set. A divisor D on (Γ, V) is an element in $\mathbb{Z}^V$. We define the degree of D (denoted by* $\deg(D)$*) to be the sum of all its values.*

Definition 1.4.8. *Fixing a map $\mathfrak{q} : V \to \mathbb{Z}$ for (Γ, V), we define the canonical divisor to be*

$$K_{\mathfrak{q}} := \sum_{p \in V} (v(p) - 2 + 2\mathfrak{q}(p))p.$$

We call the pair $\overline{\Gamma} = (\Gamma, \mathfrak{q})$ a polarized metrized graph (or a pm-graph) if $\mathfrak{q}$ is non-negative and the associated canonical divisor $K_{\mathfrak{q}}$ is effective. The function $\mathfrak{q}$ is called a polarization of (Γ, V).

Remark 1.4.9. *(Important) Throughout this thesis, we deal with pm-graphs in a flexible way. We can denote a pm-graph by $\overline{\Gamma}$, $(G, w, \mathfrak{q})$, $(\Gamma, \mathfrak{q})$, $(V, E, w, \mathfrak{q})$ and so on since they are equivalent. Notions can also be translated freely between graph-theoretic objects and metrized objects, for example, the first Betti number we already defined, the genus of a pm-graph which we will define and so on. Furthermore, notions can also be inherited, for example, pm-graphs inherit the notion the first Betti number from metrized graphs.*

Definition 1.4.10. *Suppose $\overline{\Gamma} = (\Gamma, \mathfrak{q})$ is a pm-graph, the genus of $\overline{\Gamma}$ is defined to be*

$$g(\overline{\Gamma}) := \frac{1}{2}(\deg K_{\mathfrak{q}} + 2) = b_1(\Gamma) + \sum_{v \in V} \mathfrak{q}(v),$$

where $b_1(\Gamma)$ is the first Betti number of Γ.

Let e be an edge in a pm-graph $\overline{\Gamma} = (\Gamma, \mathfrak{q})$. We say it is of *type* 0 if we get a connected graph after removing the interior points of e from $\overline{\Gamma}$. For an integer i in $[1, g(\overline{\Gamma})/2]$, we say e is of type i if the removal of its interior points from $\overline{\Gamma}$ gives two disjoint pm-graphs of genus i and $g(\overline{\Gamma}) - i$. We write $\delta_i(\overline{\Gamma})$ for the total weight of edges of type i and write $\delta(\overline{\Gamma})$ for the total weight of $\overline{\Gamma}$. It follows from the definition that $\delta(\overline{\Gamma}) = \sum_{i=0}^{\lfloor \frac{g(\overline{\Gamma})}{2} \rfloor} \delta_i(\overline{\Gamma})$.

Definition 1.4.11. *Let $\overline{\Gamma} = (\Gamma, \mathfrak{q})$ be a pm-graph. We say the vertex p is eliminable if $v(p) = 2$ and $\mathfrak{q}(p) = 0$.*

Remark 1.4.12. *If v is an eliminable vertex on $\overline{\Gamma}$ and is the endpoint of edges e_1 and e_2, then we can get a new pm-graph by removing v from the vertex set and connecting e_1 and e_2 into one edge e with the weight $l(e_1) + l(e_2)$. We can also get new pm-graphs by adding eliminable vertices in an opposite way. This gives an equivalence relation for pm-graphs. Every pm-graph $\overline{\Gamma}$ of genus $g \neq 1$ is equivalent to a unique pm-graph with no eliminable points. The assumption $g \neq 1$ excludes the case when $\overline{\Gamma}$ is a genus 1 self-loop.*

Lemma 1.4.13. *A pm-graph $\overline{\Gamma}$ has no eliminable edges if and only if every coefficient $v(p) - 2 + 2\mathfrak{q}(p)$ in the canonical divisor (Definition 1.4.8) is positive.*

Proof. This follows from the definition. ☐

Remark 1.4.14. *Most invariants that we already defined or will define only depend on the equivalence class of the pm-graph.*

Let R be a discrete valuation ring. Let $f : X \to \mathrm{Spec}(R)$ be a semistable curve of genus g with smooth generic fiber. We can associate a pm-graph $\overline{\Gamma} = (V, E, w, \mathfrak{q})$ to f:

(1) Vertices in V correspond to irreducible components of its geometric special fiber $X_{\overline{s}}$.

(2) Edges in E correspond to nodal points and the endpoints correspond to the intersecting irreducible components. The weight $w(e)$ is the thickness of the nodal point corresponding to e.

(3) $\mathfrak{q}(v)$ is the geometric genus of the component corresponding to v.

The assumption that f is semistable guarantees that the canonical divisor $K_{\mathfrak{q}}$ is effective, thus this is a pm-graph. In general, we denote this graph by $\overline{\Gamma}_s$ and we call it the *dual graph* of f. We say the dual graph is *trivial* if it is a one-point graph.

Remark 1.4.15. *The arithmetic genus of the special fiber $X_{\overline{s}}$ is equal to $g(\overline{\Gamma}_s)$ (see the discussion at the end of Section 2 in [13]). For the dual graphs of stable curves, the pm-graphs have no eliminable vertices and every coefficient $v(p) - 2 + 2\mathfrak{q}(p)$ in the canonical divisor (Definition 1.4.8) is positive.*

For a general semistable curve over a Dedekind scheme $C \to S$ with smooth generic fiber, we have a dual graph for each closed point $s \in S$. Since we assume the curve to be generically smooth, the geometric special fiber $C_{\overline{s}}$ is non-smooth only for a 0-dimensional closed subset of S, which is finite. This means that we have trivial dual graphs for all but finitely many closed points in S.

Now we are going to introduce two operations on pm-graphs and metrized graphs which are the edge contraction and the wedge sum.

Let $\overline{\Gamma} = (E, V, w, \mathfrak{q})$ be a pm-graph of genus g and $e \in E$. We define a new pm-graph $\overline{\Gamma}_{\{e\}}$ as follows:

(1) If e is a self-loop with endpoint v_0, then we define

$$\overline{\Gamma}_{\{e\}} := (E\backslash\{e\}, V, w|_{E\backslash\{e\}}, \mathfrak{q}').$$

Here, $\mathfrak{q}'(v)$ is the same as $\mathfrak{q}(v)$ except $\mathfrak{q}'(v_0) = \mathfrak{q}(v_0) + 1$.

(2) If e is an ordinary edge with endpoints v_0 and v_1, then we define

$$\overline{\Gamma}_{\{e\}} := (E\backslash\{e\}, V', w|_{E\backslash\{e\}}, \mathfrak{q}'). \tag{1.10}$$

Here, V' is induced from V with v_0 and v_1 identified (denoted by v'), and $\mathfrak{q}'(v) = \mathfrak{q}(v)$ except $\mathfrak{q}'(v') = \mathfrak{q}(v_1) + \mathfrak{q}(v_2)$.

We call this pm-graph $\overline{\Gamma}_{\{}e\}$ the *contraction* of $\overline{\Gamma}$ at e.

Let $U = \{e_1, e_2, ..., e_n\}$ be a subset of E. We can get a new graph by taking the contraction of e_i one by one. The pm-graph we get in this way is denoted by $\overline{\Gamma}_U$. We write $\overline{\Gamma}^U$ for $\overline{\Gamma}_{E\backslash\{U\}}$. The contraction operation does not change the genus g.

Lemma 1.4.16. *$\overline{\Gamma}_U$ is well-defined, that is, taking contraction for edges in U in different orders gives the same pm-graph $\overline{\Gamma}_U$.*

Proof. We just need to consider the case $U = \{e_1, e_2\}$, which is easy to prove case-by-case. □

Lemma 1.4.17. *If $\overline{\Gamma}$ has no eliminable vertices, then neither do its contractions.*

Proof. We only need to check the endpoint(s) of the contracting edge e. We use Lemma 1.4.13 as the criterion for a pm-graph without eliminable edges.

Contracting a self-loop will make the endpoint have positive polarization. If e is an ordinary edge in $\overline{\Gamma}$, we only need to consider the case when endpoints of e are polarized by 0. The summation of their valences is at least 6, thus the point corresponding the contraction has valence at least 4. □

Definition 1.4.18. *A graph G is said to be reducible if there exist non-trivial subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a point. We say a graph is irreducible if it is not reducible.*

When G_1, G_2 are subgraphs of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ consists of one point, we say that G is the *wedge sum* of G_1 and G_2 (represented by $G = G_1 \vee G_2$). We define $G_1 \vee ... \vee G_n$ to be $(G_1 \vee ... \vee G_{n-1}) \vee G_n$. Every graph can be decomposed into a wedge sum of irreducible subgraphs, and the decomposition is unique up to order. These irreducible subgraphs are called *irreducible components* of G

Example 1.4.19. *In Figure 1.4.1, the first example is irreducible and the second one can be decomposed as a wedge sum of line segments.*

We end this subsection by introducing two important properties of functions on pm-graphs. We will mainly focus on functions satisfying these two properties.

Definition 1.4.20. *Let F be a real-valued function on the set of pm-graphs, and let $\overline{\Gamma} = (E, V, w, \mathfrak{q})$ be a pm-graph. For any $e \in E$ and $l > 0$, we can associate a pm-graph $\overline{\Gamma}_{e,l}$ by taking $w(e) = l$ with other parts unchanged. We say that F satisfies the contraction lemma on $\overline{\Gamma}$ if*

$$F(\overline{\Gamma}_{\{e\}}) = \lim_{l \to 0} F(\overline{\Gamma}_{e,l})$$

holds for all $e \in E$. We say the function F satisfies the contraction lemma if F satisfies the contraction lemma on every pm-graph.

It might look strange to introduce this property since all thicknesses of nodes of semistable curves are integers. A function satisfying the contraction lemma can be easier to compute. Let F' be a function on pm-graphs satisfying the contraction lemma. Let $\overline{\Gamma}$ be a pm-graph and e be an edge of $\overline{\Gamma}$. To compute $F'(\overline{\Gamma}_{\{e\}})$, we only need to compute a limit related to $F'(\overline{\Gamma})$. This is especially useful for genus 3 pm-graphs with only type 0 edges since we only have two maximal models in that case (Lemma 3.2.3).

Definition 1.4.21. *Let F be a real-valued function on the set of pm-graphs, and let $\overline{\Gamma} = (E, V, w, \mathfrak{q})$ be a pm-graph. Let $\{\overline{\Gamma}_i\}_{1 \le i \le k}$ be the irreducible components of $\overline{\Gamma}$ with induced polarizations from contractions. We say the function F is additive on $\overline{\Gamma}$ if*

$$F(\overline{\Gamma}) = \sum_{i=1}^{k} F(\overline{\Gamma}_i)$$

holds. We say the function F is additive if F is additive on every pm-graph.

Example 1.4.22. *It is easy to see that $\delta_i(\cdot)$ (defined before Definition 1.4.11) is additive.*

1.4.2 Green's functions and admissible invariants of $\overline{\Gamma}$

Let Γ be a metrized graph. Let $PS(\Gamma)$ be the space of real-valued continuous and piecewise smooth functions on Γ.

Remark 1.4.23. *By a piecewise smooth function, we mean a function f on Γ that there exists a finite subset of points $X_f \subset \Gamma$ such that $\Gamma \backslash X_f$ is a disjoint union of open intervals, and the restriction of f on each of these intervals is a smooth function.*

We write $PS(\Gamma)^\wedge$ for the vector space of linear maps from $PS(\Gamma)$ to $\mathbb{R}$. For a point u on Γ of valence n and $f \in PS(\Gamma)$, the function f has n directional derivatives at u. We define $\delta f(u) \in PS(\Gamma)^\wedge$ by $\delta f(u)(g) := g(u) \sum_{i=1}^{n} \lim_{x_i \to 0} f_{\vec{i}}'(x_i)$ for $g \in PS(\Gamma)$, where $f_{\vec{i}}'(\cdot)$ is the i-th outward directional derivative of f. It is easy to see that $\delta f(u)$ is zero for all but finitely many u, thus $\delta f := \sum_{u \in \Gamma} \delta f(u)$ is a well-defined element in $PS(\Gamma)^\wedge$.

For $f \in PS(\Gamma)$ and $x \in \Gamma$ of valence 2, we define $f''(x)$ as the second derivative of f at x in an arbitrary direction. The metric on Γ induces a uniform measure on Γ which we denote by μ_u (on each segment in Γ, the measure μ_u coincides with the Lebesgue measure). Then we can define the Laplacian on Γ as follows.

Remark 1.4.24. *f'' is well-defined at all but finitely many points on Γ. Assume f is smooth in a neighbourhood of x_0, we have two orientations for taking derivatives, denoted by $f_+'(x_0)$ and $f_-'(x_0)$. We have $f_+'(x_0) = -f_-'(x_0)$ and $f_+''(x_0) = (-1)^2 \cdot f_-''(x_0) = f_-''(x_0)$, thus the second derivatives are well-defined at all but finitely many points on Γ.*

Definition 1.4.25. *The Laplacian Δ on a metrized graph Γ is defined as the linear map from $PS(\Gamma)$ to $PS(\Gamma)^\wedge$ such that*

$$\Delta f(g) = -\int_\Gamma f'' g d\mu_u - \delta f(g)$$

for all $g \in PS(\Gamma)$.

Lemma 1.4.26. *Let Γ be a metrized graph. There is a unique continuous, symmetric, and piecewise smooth function $g(x, y)$ on $\Gamma \times \Gamma$ satisfying:*

$$\Delta_y g(x, y) = \delta_x - \frac{\mu_u}{\delta(\Gamma)},$$
$$\int_\Gamma g(x, y) d\mu_u(y) = 0,$$

for all $x \in \Gamma$, where δ_x is the Dirac measure at x.

Proof. See Appendix in [70]. □

This function is called the $Green's\ function$ on Γ associated to μ_u. Now, for a general measure μ on Γ with volume 1, we define:

$$g_\mu(x, y) := g(x, y) - \int_\Gamma g(x, y) d\mu(y) - \int_\Gamma g(x, y) d\mu(x) + \iint_{\Gamma \times \Gamma} g(x, y) d\mu(x) d\mu(y). \quad (1.11)$$

Then $g_\mu(x, y)$ is the unique function on G satisfying the following conditions:

$$\Delta_y G(x, y) = \delta_x - \mu,$$
$$\int_\Gamma G(x, y) d\mu(y) = 0.$$

Definition 1.4.27. *This function $g_\mu(x, y)$ is called the $Green's\ function$ on Γ associated to μ.*

Remark 1.4.28. *If we consider Γ as an electrical circuit such that the resistance is locally induced from the distance, then $g_{\delta_p}(q, q)$ is equal to the electrical resistance $r(p, q)$ between p and q (Page 179 in [70]).*

Lemma 1.4.29. *On a metrized graph Γ, we have $r(p, q) = g_\nu(q, q) - 2g_\nu(q, p) + g_\nu(p, p)$, where ν is any measure on Γ of volume 1.*

Proof. It is easy to check the following linear dependence

$$g_{\delta_p}(x, y) = g_\nu(x, y) - g_\nu(x, p) - g_\nu(p, y) + g_\nu(p, p).$$

By the remark above and the symmetry property of the Green's function, we prove the lemma by taking $x = y = q$. ☐

Theorem 1.4.30. *Let $D = \sum_{i=1}^{n} c_i \cdot x_i$ be a divisor on Γ with $\deg(D) \neq 2$. Then there is a unique measure μ_D on Γ of volume 1 and a unique constant c such that the following equality holds for any point x on Γ:*

$$c + g_{\mu_D}(D, x) + g_{\mu_D}(x, x) = 0, \tag{1.12}$$

where $g_{\mu_D}(D, x) := \sum_{i=1}^{n} c_i g_{\mu_D}(x_i, x)$.

Proof. See Theorem 3.2 in [70]. ☐

Remark 1.4.31. *The theorem above is only part of Theorem 3.2 in [70], and the remaining part of that theorem says that μ_D is positive if $D - K_\Gamma$ is effective, where K_Γ is defined in Equation (1.9).*

The function $g_{\mu_D}(x, y)$ is called the $admissible\ Green\ function$ of (Γ, D) and μ_D is called the $admissible\ measure$ of (Γ, D). Recall the definition of $K_\mathfrak{q}$ in Definition 1.4.8. For a pm-graph $\overline{\Gamma} = (G, w, \mathfrak{q})$, we denote by μ_{ad} the measure $\mu_{K_\mathfrak{q}}$ and by $g_{ad}(x, y)$ the Green's function $g_{K_\mathfrak{q}}(x, y)$. We are interested in the following three admissible invariants (see Section 4.1 in [69] for details):

$$\varphi(\overline{\Gamma}) := -\frac{1}{4}\delta(\overline{\Gamma}) + \frac{1}{4}\int_\Gamma g_{ad}(x, x)((10g(\overline{\Gamma}) + 2)d\mu_{ad}(x) - \delta_{K_\mathfrak{q}}(x)), \tag{1.13}$$

$$\epsilon(\overline{\Gamma}) := \int_\Gamma g_{ad}(x, x)((2g(\overline{\Gamma}) - 2)d\mu_{ad}(x) + \delta_{K_\mathfrak{q}}(x)), \tag{1.14}$$

$$\lambda(\overline{\Gamma}) := \frac{g(\overline{\Gamma}) - 1}{6(2g(\overline{\Gamma}) + 1)}\varphi(\overline{\Gamma}) + \frac{1}{12}(\epsilon(\overline{\Gamma}) + \delta(\overline{\Gamma})). \tag{1.15}$$

By integrating with respect to the second variable in Lemma 1.4.29, we have

$$g_{ad}(x,x) = \int_\Gamma r(x,y)\mu_{ad}(y) - \frac{1}{2}\int_\Gamma r(x,y)d\mu_{ad}(x)d\mu_{ad}(y).$$

Substituting the equation above to Equation (1.14), we get

$$\epsilon(\overline{\Gamma}) = \int_{\Gamma\times\Gamma} r(x,y)\delta_{K_{\mathrm{q}}}(x)d\mu_{ad}(y). \tag{1.16}$$

We can get similar expression of φ by r, $\delta_{K_{\mathrm{q}}}$ and μ_{ad}. In Section 3.1, we will use a more explicit way to compute these invariants for pm-graphs of genus 3.

Remark 1.4.32. *Many notions (including the Green function on metrized graphs we just defined) introduced in this subsection are motivated by and similar to the notions in Arakelov theory. The Green's function on a metrized graph is an analogue of the Arakelov-Green function on Riemann surfaces and so does these admissible invariants. In [70], S. Zhang used these invariants on metrized graphs to establish the admissible pairing theory.*

Proposition 1.4.33. *All the three invariants above satisfy the contraction lemma (Definition 1.4.20) and are additive (Definition 1.4.21) for pm-graphs of genus $g > 1$.*

Proof. For additivity, see Theorem 4.3.2 in [69]. For the contraction lemma, K. Yamaki proved the case of $g(\overline{\Gamma}) = 3$ in [66] Proposition 3.1. In Subsection 1.4.3, we will give a proof of general pm-graphs of genus $g > 1$, based on the work of Z. Cinkir, R. de Jong and F. Shokrieh. □

The following property was conjectured by S. Zhang in [69], and proved by Z. Cinkir (S. Zhang only conjectured the existence of the constant $c(g)$).

Theorem 1.4.34. *Let $\overline{\Gamma}$ be a pm-graph with genus $g > 1$. Then we have*

$$\varphi(\overline{\Gamma}) \geq c(g)\delta_0(\overline{\Gamma}) + \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \frac{2i(g-i)}{g}\delta_i(\overline{\Gamma})$$

where $c(2) = \frac{1}{27}$ and $c(g) = \frac{(g-1)^2}{2g(7g+5)}$ for $g \geq 3$. In particular, $c(3) = \frac{1}{39}$.

Proof. See Theorem 2.11 in [8]. □

Corollary 1.4.35. *Let $\overline{\Gamma}$ be a pm-graph of genus $g > 1$. Then we have $\varphi(\overline{\Gamma}) \geq c(g)\delta(\overline{\Gamma})$, where $c(2) = \frac{1}{27}$ and $c(g) = \frac{(g-1)^2}{2g(7g+5)}$ for $g \geq 3$.*

Proof. If i is an integer in $[1, \lfloor \frac{g}{2} \rfloor]$, then $\frac{2i(g-i)}{g} \geq 1$ and $c(g) \leq 1$. By Theorem 1.4.34, we get $\varphi(\overline{\Gamma}) \geq c(g)\delta(\overline{\Gamma})$. □

Corollary 1.4.36. *For any $\epsilon > 0$, there exists an integer g_ϵ such that $\varphi(\overline{\Gamma}) \geq (\frac{1}{14} - \epsilon)\delta(\overline{\Gamma})$ for all pm-graphs $\overline{\Gamma}$ with $g(\overline{\Gamma}) > \max\{g_\epsilon, 1\}$.*

Proof. This follows from the corollary above. □

1.4.3 Other invariants

When the genus g is fixed, there are only finitely many types of pm-graphs without eliminable edges of genus g, and these can be computed combinatorically. We still write Γ (resp. $\overline{\Gamma}$) for a metrized graph (resp. pm-graph).

For metrized graphs, M. Baker and R. Rumely defined the $\tau(\cdot)$ invariant (Section 14 in [4]) which has the following elementary interpretation by the electrical resistance function (Lemma 14.4 in [4]). We will use this interpretation to compute the $\tau(\cdot)$ invariant.

Definition 1.4.37. *Let $r(x, y)$ be the resistance function on a metrized graph Γ (Remark 1.4.28). For any point y in Γ, we have*

$$\tau(\Gamma) = \frac{1}{4}\int_\Gamma (r_x(x, y))^2 dx,$$

where r_x is the partial derivative of r with respect to the first variable x.

Remark 1.4.38. *The τ invariant is defined on metrized graphs, and thus we can extend it to pm-graphs. S. Zhang also defined a τ invariant on metrized graphs (Equation 4.1.2 in [69]), but that is different from our τ here.*

In [8], Z. Cinkir defined the following invariant of a pm-graph $\overline{\Gamma} = (V, E, w, \mathfrak{q})$:

$$\theta(\overline{\Gamma}) = \sum_{p,q \in V} (v(p) - 2 + 2\mathfrak{q}(p))(v(q) - 2 + 2\mathfrak{q}(q))r(p, q). \tag{1.17}$$

By Definition 1.4.8, all terms in $\theta(\overline{\Gamma})$ are non-negative, thus we have $\theta(\overline{\Gamma}) \geq 0$.

The reason we introduce $\tau(\overline{\Gamma})$ and $\theta(\overline{\Gamma})$ is that Zhang's admissible invariants can be written as linear combinations of $\delta(\overline{\Gamma})$, $\theta(\overline{\Gamma})$ and $\tau(\overline{\Gamma})$.

Theorem 1.4.39. *Let $\overline{\Gamma}$ be a pm-graph of genus $g > 1$. Then we have*

$$\begin{aligned}
\epsilon(\overline{\Gamma}) &= \frac{(4g-4)\tau(\overline{\Gamma})}{g} + \frac{\theta(\overline{\Gamma})}{2g}, \\
\varphi(\overline{\Gamma}) &= \frac{(5g-2)\tau(\overline{\Gamma})}{g} + \frac{\theta(\overline{\Gamma})}{4g} - \frac{\delta(\overline{\Gamma})}{4}, \\
\lambda(\overline{\Gamma}) &= \frac{(3g-3)\tau(\overline{\Gamma})}{4g+2} + \frac{\theta(\overline{\Gamma})}{16g+8} + \frac{(g+1)\delta(\overline{\Gamma})}{16g+8}.
\end{aligned}$$

Proof. See Propositions 4.6, 4.9 and Theorem 4.8 in [8]. □

Corollary 1.4.40. *If pm-graphs $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are equivalent (Remark 1.4.12), then all the six invariants are the same for the two pm-graphs.*

Proof. $\delta(\overline{\Gamma}_1) = \delta(\overline{\Gamma}_2)$ is trivial. From the definition of the θ invariant (Equation (1.17)), we find $v(p) - 2 + 2\mathfrak{q}(p) = 0$ when p is eliminable, thus $\theta(\overline{\Gamma}_1) = \theta(\overline{\Gamma}_2)$. Since the τ invariant is defined on metrized graphs, the polarization will not make any difference.

By Theorem 1.4.39, Zhang's invariants are determined by $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$, which completes the proof. □

Remark 1.4.41. *The transformation matrix in Theorem 1.4.39 is invertible. Thus computing $\varphi(\overline{\Gamma})$, $\lambda(\overline{\Gamma})$ and $\epsilon(\overline{\Gamma})$ is equivalent to computing $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$. As a corollary, the additivity of Zhang's admissible invariants is equivalent to the additivity of $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$.*

In the two lemmas below, we make the following assumptions. Let Γ be an irreducible electrical circuit (weighted graph). We assume that e is an ordinary edge (an edge that is not a self-loop) on Γ, and that it has endpoints p and q. Let Γ' be the contraction (Equation (1.10)) of Γ at e and let $\Gamma - e$ be the graph given by removing the interior points of e in Γ. We write $l(e)$ for the length of e and write u_{pq} for the point given by the contraction of e. We denote the electrical resistance between points x and y on Γ by $r(x, y; \Gamma)$.

Lemma 1.4.42. *If $\Gamma - e$ is connected, we have*

$$\tau(\Gamma') - \tau(\Gamma) = -\frac{l(e)}{12} + A_{p,q,\Gamma-e} \cdot \left(\frac{1}{R} - \frac{1}{l(e) + R}\right),$$

where $R = r(p, q; \Gamma - e)$, and $A_{p,q,\Gamma-e}$ only depends on p, q and $\Gamma - e$.

Proof. This is a direct result of Corollary 5.3 and Lemma 6.1 in [7]. □

Lemma 1.4.43. *Let x and y be two points on Γ but not in the interior of e. Then we have*

$$r(x, y; \Gamma') = r(x, y; \Gamma) - c(x, y, p, q; \Gamma),$$

where $c(x, y, p, q; \Gamma) \to 0$ as $l(e) \to 0$.

Proof. See Corollary 8.5 in [17]. □

We can now give a proof of the second half of Proposition 1.4.33 with the two lemmas above.

Proof of Proposition 1.4.33: By Theorem 1.4.39 and Remark 1.4.41, we only need to prove that θ, δ and τ satisfy the contraction lemma for irreducible graphs (Definition 1.4.18).

When $\overline{\Gamma}$ has only 1 edge, the assertion is trivial. Thus we assume that $\overline{\Gamma}$ is an irreducible pm-graph containing more than 1 edge. Let e be an edge on $\overline{\Gamma}$. By the irreducibility of $\overline{\Gamma}$, the weighted graph $\overline{\Gamma} - e$ is connected and e cannot be a self-loop. Thus we assume that e has endpoints p and q. It remains to show that the three invariants θ, τ and δ satisfy the contraction lemma for the edge e.

It is easy to see that δ satisfies the contraction lemma, since δ is just the sum of all lengths of the edges. Lemma 1.4.42 implies that τ also satisfies the contraction lemma. Thus we only need to check θ.

When x and y are vertices on $\overline{\Gamma}$ but not the endpoints of e, the contraction does not change the polarization of x and y, and $r(x, y; \overline{\Gamma}) \to r(x, y; \overline{\Gamma}')$ as $l(e) \to 0$ by Lemma 1.4.43. When $x = p$ and $y = q$, we have $r(x, y; \overline{\Gamma}) \to 0 = r(u_{pq}, u_{pq}; \overline{\Gamma}')$ as $l(e) \to 0$. When $x = y = p$ or $x = y = q$, the claim is trivial. It remains to consider the case $x = p$ but $y \neq p$ or q.

When $x = p$ but $y \neq p$ or q, we still have $r(s, y; \overline{\Gamma}) \to r(u_{pq}, y; \overline{\Gamma})$ as $l(e) \to 0$ for $s \in \{p, q\}$ by Lemma 1.4.43. We also have

$$\sum_{s \in \{p,q\}} (v(s) - 2 + 2\mathfrak{q}(s)) = v'(u_{pq}) - 2 + 2\mathfrak{q}(u_{pq})$$

by the construction of the contraction in Equation (1.10), where v' ($\mathfrak{q}$) is the valence (polarization) function on $\overline{\Gamma}'$. Now we can say that

$$\sum_{s \in \{p,q\}} (v(s) - 2 + 2\mathfrak{q}(s))(v(y) - 2 + 2\mathfrak{q}(y)) r(s, y; \overline{\Gamma})$$

converges to

$$(v'(u_{pq}) - 2 + 2\mathfrak{q}'(u_{pq}))(v'(y) - 2 + 2\mathfrak{q}'(y)) r(u_{pq}, y; \overline{\Gamma}')$$

as $l(e) \to 0$. Thus the contraction lemma holds for θ. QED

Remark 1.4.44. *Theorem 1.4.39 and Remark 1.4.41 reduce the computation of Zhang's admissible invariants on pm-graphs to the computation of $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$ on irreducible pm-graphs. This decomposition simplifies the computation even more, since $\delta(\overline{\Gamma})$ and $\theta(\overline{\Gamma})$ are finite sums and $\tau(\overline{\Gamma})$ is an integration of the derivative of r against the natural measure (compared with Zhang's admissible measure).*

1.5 Zhang's work

Subsection 1.5.1 is a rather sketchy description about the admissible pairing. Subsection 1.5.2 is about the decomposition and the Northcott property of $\langle\Delta,\Delta\rangle$. The main references for this section are [69] and [70].

Theorem 1.5.3 makes it possible to compute $(\hat{\omega},\hat{\omega})_{ad}$. The whole of Chapter 4 is devoted to the computation of $\langle\Delta,\Delta\rangle$ for a specific curve $\mathfrak{C}_{\mathbb{Q}}$ by Theorem 1.5.6. The goal of Sections 4.5-4.7 is numerically computing the λ invariant defined in Equation (1.18).

1.5.1 Admissible pairing

Let B be either a smooth curve over a field or the spectrum of a ring of a number field. Let k be the fraction field of B. Let X be a smooth curve over k. Let $\mathcal{X}$ be an arithmetic surface over B whose generic fiber is isomorphic to X over k. We write $M(k)$ (resp. $M(k)_0$, resp. $M(k)_\infty$) for the set of places (resp. finite places, resp. infinite places) of k.

For a finite place v of k, we write $N(v)$ for $e^{\deg(v)}$ when B is a curve, for $\#k(v)$ (the cardinality of the residue field of B at v) when $B = \operatorname{Spec} O_k$. We also write $N(v)$ for e (resp. e^2) when v is a real (resp. complex) infinite place of k.

Remark 1.5.1. *A complex infinite place is a pair of conjugate complex embeddings.*

When B is a curve, we have a dualising sheaf ω on $\mathcal{X}$ that gives an adjunction formula in the usual intersection theory. When $B = \operatorname{Spec} O_k$, by assigning admissible metrics to the Archimedean places, there is an Arakelov dualising sheaf $\overline{\omega}$ that gives an adjunction formula (Theorem 1.2.15). G. Faltings proved a Hodge index theorem (Theorem 4 in [23]) for Arakelov intersection theory.

Inspired by the above results, S. Zhang established the *admissible* intersection theory $(\cdot,\cdot)_{ad}$ for smooth curves over a global field in [70]. This intersection theory is done by extending usual divisors on X to pairs (D, G), where D is a usual divisor on $\mathcal{X}$ (a model of X) and G includes the Arakelov-Green function and the Green's function on the dual graphs. In this intersection theory, there is a Hodge index theorem and a dualising sheaf $\hat{\omega}$ which gives an adjunction formula.

Remark 1.5.2. *We do not give the expression of $\hat{\omega}$ in this thesis. Instead, we use Theorem 1.5.3 to decompose it into the objects that we are more familiar with.*

At Archimedean places, this adelic Green's function contains information from the Arakelov-Green function, and at non-Archimedean places, this adelic Green's function contains information from the Green's function on the dual graphs we discussed in Section 1.4. Thus it makes sense to compare admissible intersection theory with two other intersection theories. The following theorem will be used repeatedly throughout this thesis (recall the definition of $\epsilon(\cdot)$ in Equation (1.14)).

Theorem 1.5.3. *We have the following equalities*

$$(\hat{\omega},\hat{\omega})_{ad} = (\overline{\omega},\overline{\omega}) - \sum_{v\in M(l)_0} \epsilon(\overline{\Gamma}_v)\log(N(v)),$$

where $(\overline{\omega},\overline{\omega})$ *is the self-intersection of the Arakelov dualising sheaf when* k *is a number field and is the self-intersection of the usual dualising sheaf when* k *is a function field.*

Proof. See Theorem 5.5 in [70]. □

Corollary 1.5.4. $(\hat{\omega},\hat{\omega})_{ad} \leq (\overline{\omega},\overline{\omega})$.

Proof. This comes from the fact that $\epsilon(\overline{\Gamma}) \geq 0$, which is proven in Theorem 4.4 in [70]. Alternatively, we can also get this from Theorem 1.4.39. □

1.5.2 Gross-Schoen cycle

Let X be a smooth curve over a field k. Let $\alpha = \sum_{i=1}^{t} a_i p_i$ be a divisor on X over k with rational coefficients and degree $\sum_{i=1}^{t} a_i \deg p_i = 1$. We define cycles of X^3 associated to α as follows:

$$\begin{aligned}
\Delta_{123} &:= \{(x,x,x) : x \in X\},\\
\Delta_{12} &:= \sum_{i=1}^{t} a_i\{(x,x,p_i) : x \in X\},\\
\Delta_{23} &:= \sum_{i=1}^{t} a_i\{(p_i,x,x) : x \in X\},\\
\Delta_{31} &:= \sum_{i=1}^{t} a_i\{(x,p_i,x) : x \in X\},\\
\Delta_{1} &:= \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j\{(x,p_i,p_j) : x \in X\},\\
\Delta_{2} &:= \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j\{(p_i,x,p_j) : x \in X\},\\
\Delta_{3} &:= \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j\{(p_i,p_j,x) : x \in X\}.
\end{aligned}$$

In [30], B. Gross and C. Schoen constructed an element $\Delta_\alpha \in \mathrm{CH}^2(X^3)_{\mathbb{Q}}$ associated to α as

$$\Delta_\alpha = \Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3 \in \mathrm{CH}^2(X^3)_{\mathbb{Q}}.$$

They also proved that this cycle is homologous to 0 (Proposition 3.1 in [30]) and is rationally equivalent to 0 if X is rational, or elliptic, or hyperelliptic and α is a Weierstrass point (Section 4 in [30]). Thus it is natural to ask whether $\Delta_\alpha \neq 0$ when X is non-hyperelliptic.

Now we assume that the base field k is a number field or the function field of a smooth curve over a field. B. Gross and C. Schoen defined a canonical height $\langle \Delta_\alpha, \Delta_\alpha \rangle$ for Δ_α, which is actually a special case of the Beilinson-Bloch height.

Remark 1.5.5. *For our goal, we will use the first formula in Theorem 1.5.6 as the expression of* $\langle \Delta_\alpha, \Delta_\alpha \rangle$.

From now, we assume $g \geq 2$. Let x_α be the divisor $\alpha - K_X/(2g-2)$ in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$, where K_X is the canonical divisor of X. Then we have the following theorem by S. Zhang.

Theorem 1.5.6. *Let X be a smooth curve of genus $g > 1$ over a field k which is either a number field or the function field of a smooth curve over a field. Assume that X has a semistable model $\mathcal{X}$ over k. Then*

$$\langle \Delta_\alpha, \Delta_\alpha \rangle = \frac{2g+1}{2g-2}(\hat{\omega}, \hat{\omega})_{ad} + 6(g-1)\langle x_\alpha, x_\alpha \rangle - \sum_{v \in M(k)} \varphi(X_v) \log(N(v)).$$

Here $\langle x_\alpha, x_\alpha \rangle$ is the Néron-Tate height of the class $\alpha - K_X/(2g-2)$ in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$, and the $\varphi(X_v)$ are defined as follows.

(1) If v is an Archimedean place, then

$$\varphi(X_v) := \sum_{\substack{l \in \mathbb{N} \\ 1 \leq m,n \leq g}} \frac{2}{\lambda_l} \left| \int_{X_v} \phi_l \omega_m(x) \overline{\omega}_n(x) \right|^2$$

where the ϕ_l are the normalized real eigenforms of the Arakelov Laplacian:

$$\frac{\partial\overline{\partial}}{\pi i}\phi_l = \lambda_l \cdot \phi_l \cdot d\mu_v, \qquad \int \phi_k \phi_l d\mu = \delta_{k,l},$$

and $\{\omega_i\}_{1 \leq i \leq g}$ is an orthonormal basis of $H^0(X_v, \Omega_{X_v})$ with respect to the inner product in Equation (1.5). The eigenvalues λ_l are non-negative (see Section 3 in [13]).

(2) *If v is a non-Archimedean place, then $\varphi(X_v) := \varphi(\overline{\Gamma}_v)$ which we defined in Equation (1.13).*

Proof. See Theorem 1.3.1 in [69]. □

Remark 1.5.7. *The invariant φ at an Archimedean place is known as the Zhang-Kawazumi invariant. Let $\delta(C)$ be the Faltings delta invariant of the compact Riemann surface C (Theorem 1.3.9) and $\delta'(C) := \delta(C) - 4g\log(2\pi)$, then we define*

$$\lambda(C) := \frac{g-1}{6(2g+1)}\varphi(C) + \frac{1}{12}\delta'(C), \tag{1.18}$$

where the definition of $\varphi(C)$ can be found in Theorem 1.5.6.

After replacing k by a sufficiently large extension, we can assume $(2g-2)\xi = K_X$ for some $\xi \in \mathrm{Pic}(X)$. The height $\langle \Delta_\alpha, \Delta_\alpha \rangle$ reaches its minimal value precisely when α and ξ are equal up to a torsion divisor (according to the non-negativity of the Néron-Tate height). The cycle Δ_ξ is known as a *canonical Gross-Schoen cycle* of X. The image of Δ_ξ in $\mathrm{CH}^2(X^3)_{\mathbb{Q}}$ does not depend on the choice of ξ, thus the number $\langle \Delta_\xi, \Delta_\xi \rangle$ is well-defined.

Corollary 1.5.8. *When k is the function field of a smooth curve B, we can rewrite $\langle \Delta_\xi, \Delta_\xi \rangle$ in the following way*

$$\langle \Delta_\xi, \Delta_\xi \rangle = \frac{2g+1}{2g-2}\left((\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) - \sum_{s \in M(k)_0} \psi(\overline{\Gamma}_s)\log N(s) \right), \tag{1.19}$$

where s runs over all closed points of B and ψ is defined to be

$$\psi(\overline{\Gamma}) := \epsilon(\overline{\Gamma}) + \frac{2g-2}{2g+1}\varphi(\overline{\Gamma}). \tag{1.20}$$

Proof. This comes from Theorem 1.5.3 and Theorem 1.5.6. □

When k is a function field of characteristic 0, the height $\langle \Delta_\xi, \Delta_\xi \rangle$ is non-negative by the Hodge index theorem. When k is a number field or a function field with positive characteristic, we have the following conjecture (Conjecture 1.4.1 in [69]).

Conjecture 1.5.9. *Let k be a number field or a function field with positive characteristic, then*

$$\langle \Delta_\xi, \Delta_\xi \rangle \geq 0$$

with equality if and only if Δ_ξ is rationally equivalent to 0.

B. Gross and C. Schoen's work shows that the height vanishes when the curve is of genus 0, 1 or hyperelliptic (including genus 2). This thesis is mainly about the height for genus 3 curves. Is it unbounded (Theorem 1.5.17, Section 3.3), can it be explicitly computed (Chapter 4)?

Remark 1.5.10. *In [69], S. Zhang asked when will the height be zero. The height of a canonical Gross-Schoen cycle on a hyperelliptic curve vanishes (Proposition 4.8 in [30]). In Section 3.2, we explain a result of K. Yamaki which partially answers the converse of this problem for genus 3 curves.*

Now we shift our attention to the finiteness property of $\langle \Delta_\xi, \Delta_\xi \rangle$. For a variety T over a field k and a geometric point $t : \mathrm{Spec}(\overline{k}) \to T$, there exists a minimal finite extension k_0 of k such that t factors through $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k_0)$. The integer $\deg(t) := [k_0 : k]$ is well-defined, and we have the following theorem by S. Zhang.

Theorem 1.5.11. *Let $Y \to T$ be a smooth and projective family of curves of genus $g \geq 3$ over a projective variety T over a number field k, or the function field of a curve over a finite field. If the classifying map $T \to M_g$ from T to the coarse moduli space of genus g smooth curves over k is finite, then we have a Northcott property: for any positive numbers D and H,*

$$\# \left\{ t \in T(\overline{k}) : \deg(t) \leq D, \frac{\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle}{\deg(t)} \leq H \right\} < \infty. \tag{1.21}$$

Proof. See Theorem 1.3.5 in [69]. □

Remark 1.5.12. *In the theorem above, we use a different convention from that of S. Zhang. In Theorem 1.3.5 in [69], Zhang denoted $\frac{\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle}{\deg(t)}$ simply by $\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle$.*

Remark 1.5.13. *For a stable curve $q : \mathcal{X} \to S$ of genus $g \geq 2$ where S is either a smooth curve over a field or the spectrum of a number ring, there is a height associated to the Ceresa cycle $c(\mathcal{X}/S)$. We have the following relation between the two heights*

$$c(\mathcal{X}/S) = \frac{2}{3} \langle \Delta_\xi, \Delta_\xi \rangle. \tag{1.22}$$

See Theorem 1.5.6 in [69] for more details.

In the remaining part of this subsection, we prove the unboundedness of the height $\langle \Delta_\xi, \Delta_\xi \rangle$ for genus $g \geq 3$ curves over number fields.

Lemma 1.5.14. *If $g > 2$, then there exists a finite morphism of schemes $C \to M_g$, where C is a smooth curve defined over k.*

Proof. We denote the coarse moduli space of principally polarized Abelian varieties by A_g. The Torelli map $M_g \to A_g$ is an immersion (Corollary 1.5 and the remark after it in [56]). We get a projective compactification $\tilde{M}_g$ for M_g in the Satake compactification A_g^{sc} of A_g by taking the closure of its image. The boundary $\tilde{M}_g \backslash M_g$ has codimension ≥ 2, since $A_g^{st} = A_g \sqcup A_{g-1} \cdots A_1 \sqcup A_0$ and $\dim(A_m) = \frac{m(m+1)}{2}$.

We can then get an irreducible curve T in M_g by cutting out sufficiently many hypersurfaces in general position (we might need to choose an irreducible component). The induced morphism $T \to M_g$ is a closed immersion by the construction, thus it is also finite. We write $C \to T$ for the normalization of T. Then we have a finite morphism $C \to M_g$ since it is the composition of finite morphisms $C \to T$ and $T \to M_g$. □

Remark 1.5.15. *The Satake compactification and the Torelli map can be defined over k (even over* $\mathrm{Spec}(\mathbb{Z})$*). See Page 179 in [10] and Page 150 in [24] for details. Explicit curves on M_g for $g > 2$ can be found in [28] and [68].*

Lemma 1.5.16. *Let Z be an irreducible smooth projective variety of positive dimension defined over a number field k. There exists a sufficiently large integer d such that there are infinitely many closed points on Z whose degree is less than d.*

Proof. We first fix a closed embedding $Z \to \mathbb{P}_k^n$. By Bertini's theorem, we can find $\dim(Z) - 1$ hyperplanes in $\mathbb{P}_k^n$ whose intersection with Z is a 1-dimensional smooth projective variety. We choose one irreducible component if there are more than one. Thus, we just need to prove the lemma when is Z a smooth curve.

Every non-zero rational function f on Z gives a morphism $Z \to \mathbb{P}_k^1$. We denote the degree of this morphism by d_f. The fiber of every k-point in $\mathbb{P}_k^1$ is an effective divisor of Z of degree d_f, thus every closed point in the divisor is of degree not bigger than d_f. Since there are infinitely many k-points in $\mathbb{P}_k^1$, we can obtain infinitely many closed points on Z whose degree is not bigger than d_f. □

Theorem 1.5.17. *Let $g \geq 3$ be an integer. There exists an integer D_g and a family of genus g smooth curves $\{E_j\}_{j \in \mathbb{N}^+}$ defined over $\overline{\mathbb{Q}}$ such that*

(1) For all $j \in \mathbb{N}^+$, the curve E_j has semistable model over a number field k_j such that $[k_j : \mathbb{Q}] \leq D_g$,

(2) the normalized height of the canonical Gross-Schoen cycle on E_j, which is defined as $\frac{\langle \Delta_j, \Delta_j \rangle}{[k_j:\mathbb{Q}]}$, goes to infinity.

Proof. We can obtain a finite morphism $C \to M_g$ by Lemma 1.5.14, where C is a smooth curve over k. According to Lemma 1.5.16, there exists an integer D_g such that there are infinitely many points on C whose degree is smaller than D_g. Now we can prove the assertion by applying the Northcott property in Theorem 1.5.11. □

So far, we know that $\langle \Delta_\xi, \Delta_\xi \rangle$ can be 0 and can be arbitrarily large. Nobody has yet numerically computed $\langle \Delta_\xi, \Delta_\xi \rangle$ for a non-hyperelliptic curve of genus $g \geq 3$. In Chapter 4, we will numerically compute $\langle \Delta_\xi, \Delta_\xi \rangle$ for a specific plane quartic curve over $\mathbb{Q}$.

For simplicity, we will mainly use $\langle \Delta, \Delta \rangle$ to denote the height of a canonical Gross-Schoen cycle from now on.