Explicit computation of the height of a Gross-Schoen Cycle

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Introduction

In this thesis, we study the Beilinson-Bloch heights of canonical Gross-Schoen cycles of genus 3 smooth curves.

Let X be a smooth projective curve defined over a field K (either a number field or the function field of a smooth curve over a field). We write Δ for a canonical Gross-Schoen cycle on X, an element in $\mathrm{CH}^2(X^3)_{\mathbb{Q}}$, and write $\langle \Delta, \Delta \rangle$ for its Beilinson-Bloch height. The height is well-defined although the cycle depends on a choice of a torsion point in $\mathrm{Pic}(X)$ (see Subsection 1.5.2 for further explanation).

There are several reasons for studying the height of this special cycle. According to some standard conjectures on cycles, the height should be non-negative (Conjecture 1.5.9). When the height $\langle \Delta, \Delta \rangle$ is positive, the cycle $\Delta \in \mathrm{CH}^2(X^3)_{\mathbb{Q}}$ is non-zero. Let M be the Chow motive corresponding to the kernel of the map

$$\bigwedge^3 H^1(X)(2) \to H^1(X)(1)$$
$$a \wedge b \wedge c \to a(b \cup c) + b(c \cup a) + c(a \cup b).$$

In this situation, A. Beilinson and S. Bloch conjectured the following equality:

$$\mathrm{rank}(\mathrm{CH}(M)) = \mathrm{ord}_{s=1} L(M, s),$$

where $L(M, s)$ is the L-series associated to M. Then cycle Δ lies in the group $\mathrm{CH}(M)$. If $\langle \Delta, \Delta \rangle$ does not vanish, then the L-series vanishes at $s = 1$. Then the cycle is non-zero in $\mathrm{CH}^2(X^3)_{\mathbb{Q}}$. More details can be found in Section 1.5 in [69].

In Sections 1.1-1.3, we recall results on models of curves and Arakelov intersection theory. In Section 1.4, we study the pm-graphs associated to semistable curves. With these preparations, we state Zhang's main results on $\langle \Delta, \Delta \rangle$ (see Section 1.3 in [69]) in Section 1.5: a decomposition of $\langle \Delta, \Delta \rangle$ (Theorem 1.5.6) and a Northcott property of $\langle \Delta, \Delta \rangle$ (Theorem 1.5.11). Over number fields, we prove the following unboundedness property of the normalized heights in Theorem 1.5.17.

Theorem 0.0.1. *Let $g \geq 3$ be an integer. There exists an integer D_g and a family of genus g smooth curves $\{E_j\}_{j\in\mathbb{N}^+}$ defined over $\mathbb{Q}$ such that*

(1) For all $j \in \mathbb{N}^+$, the curve E_j has a semistable model over a number field k_j such that $[k_j : \mathbb{Q}] \leq D_g$,

(2) the normalized height of the canonical Gross-Schoen cycle on E_j, which is defined as $\frac{\langle\Delta_j,\Delta_j\rangle}{[k_j:\mathbb{Q}]}$, goes to infinity.

In Chapter 2, we study arithmetic and geometric properties of genus 3 curves. We first describe the moduli behaviour of semistable genus 3 curves. Then we introduce algebraic modular forms and bitangents of smooth plane quartics. We study the divisor of the modular form χ'_{18} on $\overline{\mathcal{M}}_3$ and Klein's formula for smooth plane quartics. These will be used in Chapter 3 and Chapter 4 for the decomposition of the height $\langle\Delta,\Delta\rangle$ (Theorem 3.3.2) and for a sanity check for our main computation (Section 4.9).

In Chapter 3, we turn to the study of the height $\langle\Delta,\Delta\rangle$ for smooth plane quartics. In Section 3.1, we show how to compute admissible invariants on dual graphs in genus 3. We give a proof of the contraction lemma (Proposition 1.4.33) for these invariants (Definition 1.4.20), which we have not been able to find in the literature. In Section 3.2, we recall K. Yamaki's result on genus 3 curves over function fields. A constant $h(\overline{\Gamma})$ introduced in Equation (3.1) will be used for computing $\mathrm{ord}_v(\chi'_{18})$ for finite places v in Subsection 4.4.2. In Section 3.3, we show the following property of $\langle\Delta,\Delta\rangle$, based on the work of R. de Jong in [13].

Theorem 0.0.2. *Let $\{L_m\}_{m\in\mathbb{N}^+}$ be a family of smooth non-hyperelliptic curves of genus 3 over $\mathbb{Q}$. If the following properties hold:*

(1) considering $\{L_m \otimes_{\mathbb{Q}} \mathbb{C}\}_{m\in\mathbb{N}^+}$ as a family of points in $\mathcal{M}_3(\mathbb{C})$, this family of points lies on a curve in $\overline{\mathcal{M}}_3(\mathbb{C})$ and converges to a point in $\overline{\mathcal{M}}_3(\mathbb{C})\backslash\mathcal{M}_3(\mathbb{C})$ which has a non-trivial dual graph satisfying Condition ($\mathfrak{H}$) (see Definition 3.3.6),

(2) the dual graphs of their stable models (which exist over finite extensions of the base field $\mathbb{Q}$, see Theorem 1.1.16) over finite places satisfy Condition ($\mathfrak{H}$),

then their heights of canonical Gross-Schoen cycles $\langle\Delta_m,\Delta_m\rangle$ go to infinity.

Basically, Condition ($\mathfrak{H}$) is to make sure that the contribution from finite places are non-negative and the contribution from infinite places approaches to infinity. In Subsection 3.3.4, we give an application of our criterion, and this is the main result of [13].

Theorem 0.0.1 holds for all genus $g \geq 3$ while Theorem 0.0.2 only works for $g = 3$.

Chapter 4 contains the main result of this thesis. We compute the height $\langle\Delta,\Delta\rangle$ for a specific plane quartic curve $\mathfrak{C}_{\mathbb{Q}}$ that has semistable reduction over $\mathbb{Q}$. As far as we know, this is the first attempt to numerically compute the height for a non-hyperelliptic curve of genus ≥ 3. Our computation can be summarized as follows.

Computation 0.0.3. *For the plane quartic curve* $\mathfrak{C} \to \operatorname{Spec}(\mathbb{Z})$ *defined by*

$$-X^3Y + X^2Y^2 - XY^2Z + Y^3Z + X^2Z^2 + XZ^3 = 0,$$

we have the following results:

(1) $\delta(\mathfrak{C}_{\mathbb{C}}) \approx -24.87$ *(Faltings delta invariant),*

(2) $\varphi(\mathfrak{C}_{\mathbb{C}}) \approx 1.17$ *(Faltings phi invariant),*

(3) $\deg\det f_*\overline{\omega}_{\mathfrak{C}} \approx -2.9190567336$ *(Faltings height),*

(4) $(\overline{\omega},\overline{\omega})_{Ar} \approx 3.43$ *(self-intersection of the Arakelov dualising sheaf),*

(5) $(\hat{\omega},\hat{\omega})_{ad} \approx 1.55$ *(self-intersection of the admissible dualising sheaf),*

(6) $\langle\Delta,\Delta\rangle \approx 0.60$ *(height of the canonical Gross-Schoen cycle),*

where notations can be found in Equation (2.8) (for $\deg\det f_*\overline{\omega}_{\mathfrak{C}}$*), Definition 1.2.14 (for* $(\overline{\omega},\overline{\omega})_{Ar}$*), Theorem 1.5.3 (for* $(\hat{\omega},\hat{\omega})_{ad}$*) and Theorem 1.5.6 (for* $\langle\Delta,\Delta\rangle$*).*

In Subsection 4.9, we discuss potential numerical issues in our computation, and explain why we are reasonably confident in their correctness. Code is attached in Appendices I-IX.

Throughout this thesis, a *curve* over a field k is a reduced, geometrically connected, projective k-scheme of pure dimension 1. All *Dedekind schemes* are assumed to have dimension 1. For a number field K, we write $K(\mathbb{C})$ for the set of complex embeddings of K. We write $\mathbb{N}^+$ for the set of positive integers.