Explicit computation of the height of a Gross-Schoen Cycle

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Explicit Computation of the Height of a Gross-Schoen Cycle

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Contents

Introduction

In this thesis, we study the Beilinson-Bloch heights of canonical Gross-Schoen cycles of genus 3 smooth curves.

Let X be a smooth projective curve defined over a field K (either a number field or the function field of a smooth curve over a field). We write Δ for a canonical Gross-Schoen cycle on X, an element in $\mathrm{CH}^2(X^3)_{\mathbb{Q}}$, and write $\langle\Delta, \Delta\rangle$ for its Beilinson-Bloch height. The height is well-defined although the cycle depends on a choice of a torsion point in $\mathrm{Pic}(X)$ (see Subsection 1.5.2 for further explanation).

There are several reasons for studying the height of this special cycle. According to some standard conjectures on cycles, the height should be non-negative (Conjecture 1.5.9). When the height $\langle\Delta, \Delta\rangle$ is positive, the cycle $\Delta \in \mathrm{CH}^2(X^3)_{\mathbb{Q}}$ is non-zero. Let M be the Chow motive corresponding to the kernel of the map

$$\begin{aligned}\bigwedge^3 H^1(X)(2) &\to H^1(X)(1)\\ a \wedge b \wedge c &\to a(b \cup c) + b(c \cup a) + c(a \cup b).\end{aligned}$$

In this situation, A. Beilinson and S. Bloch conjectured the following equality:

$$\mathrm{rank}(\mathrm{CH}(M)) = \mathrm{ord}_{s=1} L(M, s),$$

where $L(M, s)$ is the L-series associated to M. Then cycle Δ lies in the group $\mathrm{CH}(M)$. If $\langle\Delta, \Delta\rangle$ does not vanish, then the L-series vanishes at $s = 1$. Then the cycle is non-zero in $\mathrm{CH}^2(X^3)_{\mathbb{Q}}$. More details can be found in Section 1.5 in [69].

In Sections 1.1-1.3, we recall results on models of curves and Arakelov intersection theory. In Section 1.4, we study the pm-graphs associated to semistable curves. With these preparations, we state Zhang's main results on $\langle\Delta, \Delta\rangle$ (see Section 1.3 in [69]) in Section 1.5: a decomposition of $\langle\Delta, \Delta\rangle$ (Theorem 1.5.6) and a Northcott property of $\langle\Delta, \Delta\rangle$ (Theorem 1.5.11). Over number fields, we prove the following unboundedness property of the normalized heights in Theorem 1.5.17.

Theorem 0.0.1. *Let $g \geq 3$ be an integer. There exists an integer D_g and a family of genus g smooth curves $\{E_j\}_{j\in\mathbb{N}^+}$ defined over $\mathbb{Q}$ such that*

(1) For all $j \in \mathbb{N}^+$, the curve E_j has a semistable model over a number field k_j such that $[k_j : \mathbb{Q}] \leq D_g$,

(2) the normalized height of the canonical Gross-Schoen cycle on E_j, which is defined as $\frac{\langle\Delta_j,\Delta_j\rangle}{[k_j:\mathbb{Q}]}$, goes to infinity.

In Chapter 2, we study arithmetic and geometric properties of genus 3 curves. We first describe the moduli behaviour of semistable genus 3 curves. Then we introduce algebraic modular forms and bitangents of smooth plane quartics. We study the divisor of the modular form χ'_{18} on $\overline{\mathcal{M}}_3$ and Klein's formula for smooth plane quartics. These will be used in Chapter 3 and Chapter 4 for the decomposition of the height $\langle\Delta,\Delta\rangle$ (Theorem 3.3.2) and for a sanity check for our main computation (Section 4.9).

In Chapter 3, we turn to the study of the height $\langle\Delta,\Delta\rangle$ for smooth plane quartics. In Section 3.1, we show how to compute admissible invariants on dual graphs in genus 3. We give a proof of the contraction lemma (Proposition 1.4.33) for these invariants (Definition 1.4.20), which we have not been able to find in the literature. In Section 3.2, we recall K. Yamaki's result on genus 3 curves over function fields. A constant $h(\overline{\Gamma})$ introduced in Equation (3.1) will be used for computing $\mathrm{ord}_v(\chi'_{18})$ for finite places v in Subsection 4.4.2. In Section 3.3, we show the following property of $\langle\Delta,\Delta\rangle$, based on the work of R. de Jong in [13].

Theorem 0.0.2. *Let $\{L_m\}_{m\in\mathbb{N}^+}$ be a family of smooth non-hyperelliptic curves of genus 3 over $\mathbb{Q}$. If the following properties hold:*

(1) considering $\{L_m \otimes_{\mathbb{Q}} \mathbb{C}\}_{m\in\mathbb{N}^+}$ as a family of points in $\mathcal{M}_3(\mathbb{C})$, this family of points lies on a curve in $\overline{\mathcal{M}}_3(\mathbb{C})$ and converges to a point in $\overline{\mathcal{M}}_3(\mathbb{C})\backslash\mathcal{M}_3(\mathbb{C})$ which has a non-trivial dual graph satisfying Condition ($\mathfrak{H}$) (see Definition 3.3.6),

(2) the dual graphs of their stable models (which exist over finite extensions of the base field $\mathbb{Q}$, see Theorem 1.1.16) over finite places satisfy Condition ($\mathfrak{H}$),

then their heights of canonical Gross-Schoen cycles $\langle\Delta_m,\Delta_m\rangle$ go to infinity.

Basically, Condition ($\mathfrak{H}$) is to make sure that the contribution from finite places are non-negative and the contribution from infinite places approaches to infinity. In Subsection 3.3.4, we give an application of our criterion, and this is the main result of [13].

Theorem 0.0.1 holds for all genus $g \geq 3$ while Theorem 0.0.2 only works for $g = 3$.

Chapter 4 contains the main result of this thesis. We compute the height $\langle \Delta, \Delta \rangle$ for a specific plane quartic curve $\mathfrak{C}_{\mathbb{Q}}$ that has semistable reduction over $\mathbb{Q}$. As far as we know, this is the first attempt to numerically compute the height for a non-hyperelliptic curve of genus ≥ 3. Our computation can be summarized as follows.

Computation 0.0.3. *For the plane quartic curve* $\mathfrak{C} \to \mathrm{Spec}(\mathbb{Z})$ *defined by*

$$-X^3Y + X^2Y^2 - XY^2Z + Y^3Z + X^2Z^2 + XZ^3 = 0,$$

we have the following results:

(1) $\delta(\mathfrak{C}_{\mathbb{C}}) \approx -24.87$ *(Faltings delta invariant),*

(2) $\varphi(\mathfrak{C}_{\mathbb{C}}) \approx 1.17$ *(Faltings phi invariant),*

(3) $\deg \det f_* \overline{\omega}_{\mathfrak{C}} \approx -2.9190567336$ *(Faltings height),*

(4) $(\overline{\omega}, \overline{\omega})_{Ar} \approx 3.43$ *(self-intersection of the Arakelov dualising sheaf),*

(5) $(\hat{\omega}, \hat{\omega})_{ad} \approx 1.55$ *(self-intersection of the admissible dualising sheaf),*

(6) $\langle \Delta, \Delta \rangle \approx 0.60$ *(height of the canonical Gross-Schoen cycle),*

where notations can be found in Equation (2.8) (for $\deg \det f_* \overline{\omega}_{\mathfrak{C}}$*), Definition 1.2.14 (for* $(\overline{\omega}, \overline{\omega})_{Ar}$*), Theorem 1.5.3 (for* $(\hat{\omega}, \hat{\omega})_{ad}$*) and Theorem 1.5.6 (for* $\langle \Delta, \Delta \rangle$*).*

In Subsection 4.9, we discuss potential numerical issues in our computation, and explain why we are reasonably confident in their correctness. Code is attached in Appendices I-IX.

Throughout this thesis, a *curve* over a field k is a reduced, geometrically connected, projective k-scheme of pure dimension 1. All *Dedekind schemes* are assumed to have dimension 1. For a number field K, we write $K(\mathbb{C})$ for the set of complex embeddings of K. We write $\mathbb{N}^+$ for the set of positive integers.

Chapter 1

Arithmetic surfaces and intersection theory

This chapter is devoted to arithmetic surfaces and Arakelov theory. In Section 1.1, we discuss models of curves and general theory of arithmetic surfaces. Section 1.2 is about the foundation of Arakelov intersection theory. Section 1.3 is about G. Faltings' seminal paper [23] on Arakelov theory. In Section 1.4, we discuss dual graphs associated to semistable arithmetic surfaces and harmonic analysis on them. The heights of canonical Gross-Schoen cycles introduced in Section 1.5 are the main theme of this thesis.

We prove that Zhang's admissible invariants satisfy the contraction lemma (Proposition 1.4.33), which we have not found in literature. In Theorem 1.5.17, we prove an unboundedness property of the heights of canonical Gross-Schoen cycles for genus $g \geq 3$ smooth curves over number fields. To the best of the author's knowledge, this is a new result.

1.1 Models of curves

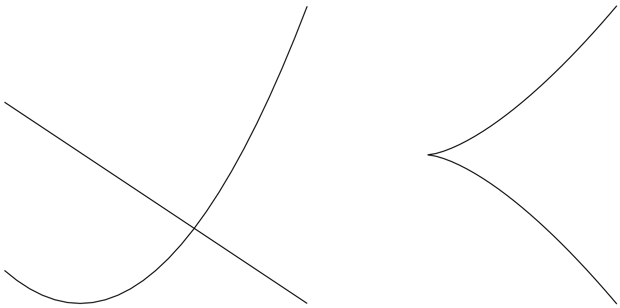
In Subsection 1.1.1, we define semistability and thickness. In Subsection 1.1.2, we define various kinds of models and state the semistable reduction theorem. In Subsection 1.1.3, we introduce the Deligne pairing on arithmetic surfaces. Proofs can be found in [48] and [52].

The definition of thickness appears in Proposition 1.1.8, which is essential for defining the dual graph of a semistable curve.

For simplicity, S is a Dedekind scheme (that is, a normal, irreducible Noetherian scheme of dimension 1) with function field $K(S)$ throughout this section. We write $k(p)$ for the residue field of a point p in a scheme.

1.1.1 Semistability

Semistable curves are curves with mildest possible singularities. By 'mildest', we mean intersections with two different tangent directions. In the graphs below, the singular point on the left one is considered as a 'mildest' singularity while the singular point on the right one is not.



(a) nodal point (b) cusp point

Figure 1.1.1: node and cusp

Definition 1.1.1. *Let C be a curve over an algebraically closed field k. A point p on C is called a nodal point or an ordinary double point if $\widehat{\mathcal{O}}_{C,p} \simeq \frac{k[[u,v]]}{(uv)}$.*

Definition 1.1.2. *A curve over an algebraically closed field k of arithmetic genus g is called semistable (resp. stable) if $g \geq 1$ (resp. $g \geq 2$), all of its singular points are nodal points and all of its components with arithmetic genus 0 meet other components in at least 2 (resp. 3) points.*

Remark 1.1.3. *We repeat here that all curves over fields are assumed to be geometrically connected.*

Example 1.1.4. *The curve $E_m : Y^2Z = X^3 + X^2Z$ in $\mathbb{P}^2_{\mathbb{C}}$ is semistable. It has only 1 nodal point at $(0:0:1)$.*

It is equivalent to define a stable curve as a curve having only nodal singularities and a finite automorphism group. The finiteness of the automorphism groups of stable curves can be compared with Hurwitz's automorphism theorem for Riemann surfaces which says that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is a finite group (containing at most $84(g-1)$ elements).

Remark 1.1.5. *From the definition, a semistable curve C over an algebraically closed field is a local complete intersection of codimension $n-1$ in $\mathbb{P}^n$, thus Serre duality can be applied and the dualizing sheaf of C is a line bundle (Theorem III.7.11 in [33]).*

Let C_0 be a semistable curve with 1 component and 1 node p_s. We describe the dualizing sheaf ω_{C_0} of C_0 as follows:

The normalization of C_0 is a smooth curve C with two specified points p and q (preimages of p_s under the normalization map). Let Ω be the dualising sheaf of C, which is isomorphic to the sheaf of differential forms on C. Then we have

$$\omega_{C_0} \simeq r_* \Omega(p+q),$$

where r_* is the pushforward along the normalization map r.

Definition 1.1.6. *For a general base scheme T, we define a curve over T to be a scheme X with a proper flat and finitely presented morphism $f : X \to T$ of pure relative dimension 1. We say f is a curve (resp. stable curve, resp. semistable curve) of genus g if $X_{\bar{t}}$ is a curve (resp. stable curve, resp. semistable curve) of genus g for all geometric points $\bar{t}$ of T.*

Example 1.1.7. *(Nice curves can be non-semistable) The curve C_F in $\mathbb{P}^2_{\mathbb{Z}}$ defined by $X^n + Y^n + Z^n = 0$ is not a semistable curve when $n \geq 2$. For any prime number $p|n$ we have $X^n + Y^n + Z^n = (X^{n/p} + Y^{n/p} + Z^{n/p})^p$ in $\overline{\mathbb{F}}_p[X, Y, Z]$ and thus the fiber of C_F at p is not even reduced. More generally, a plane curve defined by $G(X^p, Y^p, Z^p) = 0$ for some polynomial $G \in \mathbb{Z}[X, Y, Z]$ can not be semistable.*

By Definition 1.1.6, we can define a semistable curve over a non-algebraically closed field l. A point p_s on a semistable curve C over l is called a *split node* if $k(p_s) = l$ and $\widehat{\mathcal{O}}_{X,p_s} \simeq l[[u, v]]/(uv)$.

Proposition 1.1.8. *Recall that S is a Dedekind scheme. Let $X \to S$ be a semistable curve with smooth generic fiber X_η. For $s \in S$, let $x \in X_s$ be a singular point of X_s.*

(1) There exists a Dedekind scheme S', with a surjective and étale morphism $S' \to S$, such that any point $x' \in X' := X \times_S S'$ lying above x, belonging to a fiber $X'_{s'}$, is a split node of $X'_{s'} \to \operatorname{Spec} k(s')$.

(2) With the scheme S' obtained in (1), we have an isomorphism

$$\widehat{\mathcal{O}}_{X',x'} \simeq \widehat{\mathcal{O}}_{S',s'}[[u, v]]/(uv - c)$$

for some non-zero $c \in \mathfrak{m}_{s'}\mathcal{O}_{S',s'}$.

(3) Let e_x be the valuation of c for the normalized valuation of $\mathcal{O}_{S',s'}$. Then e_x is independent of the choice of S', s', and of x', and it is called the thickness of x in X.

Proof. See Corollary 10.3.22 in [48]. □

Example 1.1.9. *For a prime $p \geq 3$ and a positive integer n, the equation*

$$Y^2Z = X^3 + X^2Z + p^nZ^3$$

defines a semistable curve C in $\mathbb{P}^2_{\mathbb{Z}_p}$ with 1 nodal point at $p_s = (X, Y, p)$. It can be shown that $\widehat{\mathcal{O}}_{C,p_s} \simeq \mathbb{Z}_p[[u,v]]/(uv - p^n)$ and hence the thickness at p_s is n. More precisely, at the origin of the affine patch $y^2 - x^2(1+x) - p^n = 0$, we can construct $g(x) \in \mathbb{Z}_p[[x]]$ such that $g^2(x) = 1 + x$, and this gives $y^2 - (xg(x))^2 - p^n = 0$. Taking $u = y + xg(x)$ and $v = y - xg(x)$, we get $\widehat{\mathcal{O}}_{C,p_s} \simeq \frac{\mathbb{Z}_p[[u,v]]}{(uv-p^n)}$.

Remark 1.1.10. *Thickness can be considered as a measure of singularity in an arithmetic sense.*

1.1.2 Models

By a $fibered\ surface\ over\ S$, we mean an integral, projective, flat S-scheme $\pi : X \to S$ of dimension 2 (S is a Dedekind scheme). We say the fibered surface π is normal if X is normal.

Definition 1.1.11. *Let C be a smooth curve over $K(S)$. We call a normal fibered surface $X \to S$ together with an isomorphism $f : X_\eta \simeq C$ a model of C over S, where η is the generic point of S. If X is regular, we call it a regular model. For a model X of C, if every birational map $Y \dashrightarrow X$ of models can be extended to a morphism, we say X is a minimal model for C. Moreover, we say a model (X, f) of C has property P if the morphism $X \to S$ has the property P.*

Theorem 1.1.12. *For every excellent, reduced, Noetherian 2-dimensional scheme X, there exists a proper birational morphism $X' \to X$ where X' is a regular scheme.*

$Sketch\ of\ proof$: X' is attained by iteratively blowing up at the singular locus and taking normalization. J. Lipman proved that this procedure terminates in finitely many steps. See [47] for a complete proof. QED

Remark 1.1.13. *Theorem 1.1.12 can be considered as the desingularization of 2-dimensional schemes. For general dimensions, H. Hironaka proved that any variety over a field of characteristic* 0 *can be desingularized into a regular variety. In [11], A. J. de Jong introduced alteration and proved that a separated integral scheme of finite type over a complete discrete valuation ring (this includes fields of characteristic p) always has an alteration from a regular scheme (Theorem 6.5 in [11]).*

Let C be a smooth curve over $K(S)$. With Theorem 1.1.12, we can always get a regular model $X \to S$ of C. If we assume further that the genus $g > 0$, then $X \to S$ has a unique minimal regular model, up to a unique isomorphism (Theorem 9.3.21 in [48]).

Definition 1.1.14. *Let C be a smooth curve over $K(S)$. We say that C has good (resp. stable, resp. semistable) reduction at a closed point $s \in S$ if there exists a smooth (resp. stable, resp. semistable) model X of C over $\mathrm{Spec}(\mathcal{O}_{S,s})$. We say C has good (resp. stable, resp. semistable) reduction over S if it has good reduction at every closed point $s \in S$.*

Good reduction is easy to deal with since it has smooth special fiber, but it can happen that a curve does not have good reduction.

Example 1.1.15. *(A curve without good reduction) For a field k, set $k((\lambda))$ with the natural discrete valuation, that is $\mathrm{val}(\lambda) = 1$. Then $E : Y^2Z = X(X-Z)(X-\lambda Z)$ is an elliptic curve over $k((\lambda))$ with the j invariant*

$$j(E) = \frac{2^8(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

The curve E does not have a smooth model since its j invariant is not in the valuation ring of $k((\lambda))$ (Proposition 5.5 in [60]).

The following theorem, first proved by P. Deligne and D. Mumford, shows the existence of a stable model, after taking an adequate field extension.

Theorem 1.1.16. *(Stable reduction theorem) Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K of genus $g \geq 2$. Then there exists an extension of discrete valuation rings $R \subset R'$ inducing a finite separable extension of fraction fields K'/K and a stable curve $Y \to \mathrm{Spec}(R')$ of genus g with $Y_{K'} \simeq C_{K'}$ over K'.*

Proof. Tag 0E8C ◻

Remark 1.1.17. *The original version of this theorem in Stack Project also requires $H^0(C, \mathcal{O}_C) = K$. We omit this condition since we assume all curves to be geometrically connected in this thesis.*

1.1.3 Intersection theory on arithmetic surfaces

It is too much to require a curve over a general scheme to be smooth, and we will instead consider regular objects. There are several advantages for restricting our discussion to regular Noetherian schemes. First, the stalks of regular schemes are UFDs, thus there is a 1-1 correspondence between rational equivalence classes of Weil divisors and isomorphism classes of line bundles. Second, the Grothendieck groups of coherent sheaves and vector bundles on regular schemes coincide (Page 13 in [61]), thus the K-theory on regular schemes behaves better. Third, regularity is strong enough for having a moving lemma on schemes (Corollary 9.1.10 in [48]).

As in the last two subsections, we still write S for a Dedekind scheme with fraction field $K(S)$.

Definition 1.1.18. *We call a regular fibered surface $\mathcal{X} \to S$ an arithmetic surface when $\mathcal{X}_\eta$ is smooth for the generic point η of S.*

Definition 1.1.19. *Let $\pi : \mathcal{X} \to S$ be an arithmetic surface and let D be a prime Weil divisor on $\mathcal{X}$. We say D is horizontal if $\pi|_D : D \to S$ is surjective. We say D is vertical if its image is reduced to a point.*

Remark 1.1.20. *A prime horizontal divisor is just the Zariski closure of a L-point in $\mathcal{X}$ where L is a finite extension of $K(S)$. A prime vertical divisor is an irreducible component of the fiber $\mathcal{X}_p$ over some closed point p in S.*

Let D and E be two Weil divisors on an arithmetic surface $\pi : \mathcal{X} \to S$ with no common components. We define the *intersection multiplicity* of D and E at a closed point $x \in \mathcal{X}$ as

$$i_x(D,E) := \mathrm{length}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{X},x}/(f,g)), \tag{1.1}$$

where f and g are local equations of D and E at x. We obtain a 0-cycle on $\mathcal{X}$ that can be written as

$$I(D,E) := \sum_x i_x(D,E)x.$$

Then

$$\pi_* I(D,E) := \sum_x i_x(D,E)[k(x) : k(\pi(x))]\pi(x)$$

is a divisor on S.

The intersection theory above is less satisfying since we have not defined the intersection between divisors with common components. This can be done by applying the moving lemma (see Section 9.1 in [48]). The following theorem of P. Deligne generalizes the intersection theory above using the language of line bundles.

Theorem 1.1.21. *Let $\pi : \mathcal{X} \to S$ be an arithmetic surface. Let L and M be two line bundles on $\mathcal{X}$. We can associate a line bundle $\langle L, M\rangle$ on S such that the following properties are satisfied:*

(1) If $L' \simeq L$ and $M' \simeq M$ then $\langle L', M'\rangle \simeq \langle L, M\rangle$.

(2) *The pairing is symmetric and satisfies the following laws:*

$$\langle L, M\rangle \simeq \langle M, L\rangle$$

$$\langle L, M\otimes N\rangle \simeq \langle L, M\rangle \otimes \langle L, N\rangle.$$

(3) *Let l and m be two rational sections on L and M whose divisors have no common components. Then there exists a non-zero rational section $\langle l, m\rangle$ of $\langle L, M\rangle$ such that:*

(a) *Let f be a rational function on X such that fl and m have no common components. Then*

$$\langle fl, m\rangle = N_{(m)}(f)\langle l, m\rangle,$$

where the definition of N can be found in Page 19 *in [52].*

(b) *There is an isomorphism*

$$\langle L, M\rangle \simeq \mathcal{O}_S(N_{(l)}(m)),$$

which sends $\langle l, m\rangle$ to $1_{N_{(l)}(m)}$. In fact, we have

$$\operatorname{div}(\langle l, m\rangle) = N_{(l)}(m).$$

(4) *Let D be a horizontal divisor on X. Then for a line bundle L on X, we have a natural isomorphism $\langle L, \mathcal{O}_X(D)\rangle \simeq N_D(L)$ which sends $\langle l, 1_D\rangle$ to $N_D(l)$.*

(5) *Let $\rho : S' \to S$ a flat morphism between connected Dedekind schemes and let $X' := X\times_S S'$ be the base change of $X \to S$ by ρ with the following commutative diagram.*

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\rho} & S \end{array}$$

Then there is a natural isomorphism

$$\rho^*(\langle L, M\rangle) \simeq \langle \mu^*(L), \mu^*(M)\rangle.$$

Proof. See Theorem 4.7 in [52]. □

If S is a smooth curve over a field k or S is isomorphic to $\mathrm{Spec}(O_K)$ for some number field K, we can associate a degree map to the Deligne pairing in the following way.

When S is a smooth curve, we define $\deg\langle L, M\rangle$ as the degree of the line bundle $\langle L, M\rangle$ on the curve S. In the classical way, this is equal to

$$\sum_x i_x(D, E)[k(x) : k(\pi(x))] \deg(\pi(x)), \tag{1.2}$$

where D and E are divisors of some rational sections of L and M with no common components.

Remark 1.1.22. *For simplicity, we write* $(\cdot, \cdot)$ *for* $\deg\langle\cdot, \cdot\rangle$ *in this case.*

When $S = \mathrm{Spec}(O_K)$ for some number field K, we want to define $\deg\langle L, M\rangle$ as

$$\sum_x i_x(D, E)[k(x) : k(\pi(x))] \log(\# k(\pi(x))), \tag{1.3}$$

where D and E are divisors of some rational sections of L and M with no common components. However, this is not good, since the number given by Equation (1.3) really depends on the choice of rational sections. Instead, we will consider line bundles on $\mathcal{X}$ with Hermitian metrics. Given two metrized line bundles L and M on $\mathcal{X}$, we can endow $\langle L, M\rangle$ with a Hermitian metric (Section 3.3 in [52]). Then we define $\deg\langle L, M\rangle$ using the following definition.

Definition 1.1.23. *Let M be a Hermitian metrized line bundle on* $\mathrm{Spec}(O_K)$*. We define its degree by*

$$\deg(M) \coloneqq \log\#(M/O_K \cdot s) - \sum_{\sigma \in K(\mathbb{C})} \log\|s\|_\sigma$$

for an arbitrary non-zero element s of M. By the product formula, this degree does not depend on the choice of s.

The degree of the Deligne pairing has a close relation with the Arakelov intersection theory (Section 1.2). Actually, we have the following equality

$$(L, M)_{Ar} = \deg\langle L, M\rangle \tag{1.4}$$

for admissible line bundles L and M. See Section 6.3 in [18] for the construction of the metric on $\langle L, M\rangle$ and further discussion on Equation (1.4).

Remark 1.1.24. *For divisors D and E with no common components, we write* $(D, E)_{\mathrm{fin}}$ *for the number given by Equation (1.3).*

1.2 Arakelov's work

The main reference for this section is [1], in which Arakelov tried to establish an *arithmetic* intersection theory on arithmetic surfaces over number fields.

In Subsection 1.2.1, we introduce the Green's function on a Riemann surface which gives a metric on the line bundles on this Riemann surface. In Subsection 1.2.2, we explain Arakelov intersection theory and define the Arakelov dualising sheaf.

We will carry out some explicit computation for the Green's function in Section 4.6.

1.2.1 Green's functions on Riemann surfaces

In this subsection, X is a compact Riemann surface of genus $g \geq 1$, and we write Ω^1_X for its sheaf of holomorphic differential forms.

We can define a Hermitian inner product on $H^0(X, \Omega^1_X)$ as follows:

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \overline{\eta}. \tag{1.5}$$

With this inner product, we can choose an orthonormal basis $\{\omega_1, ..., \omega_g\}$, and we define the *volume form* or *canonical form* of X to be

$$\mu_{Ar} := \frac{i}{2g} \sum_{j=1}^{g} \omega_j \wedge \overline{\omega}_j. \tag{1.6}$$

The (1-1)-form μ_{Ar} on X does not depend on the choice of orthonormal basis.

Remark 1.2.1. *The word 'volume' comes from* $\int_X \mu_{Ar} = 1$.

Definition 1.2.2. *The canonical Arakelov-Green function of X is the unique function $G : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies:*

(1) Let P be any fixed point in an open set U with a local coordinate z on U. For $Q \in U$, we have $\log G(P, Q) = \log|z(Q)| + f(Q)$, where f is a C^∞ function on U.

(2) $G(\cdot, \cdot)^2$ is a C^∞ function on $X \times X$ and $\partial_Q \overline{\partial}_Q \log G(P, Q)^2 = 2\pi i \mu_{Ar}(Q)$ for $Q \neq P$.

(3) $\int_X \log G(P, Q) d\mu_{Ar}(Q) = 0$.

(4) G vanishes at the diagonal of $X \times X$.

Remark 1.2.3. *For simplicity, we will use the Green's function of X instead of the Arakelov-Green function of X in this thesis.*

Example 1.2.4. *Let $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$, and define a torus X by $X \simeq \mathbb{C}/\langle 1, \tau \rangle$. Let z be the coordinate of $\mathbb{C}$. Then the Green's function on X is*

$$\log G(z_1, z_2) = \log G(z_1 - z_2, 0) = \log\|\theta\| \left(\tau, z_1 - z_2 + \frac{1+\tau}{2}\right) - \log\|\eta\|(\tau),$$

where

$$\|\theta\|(a+bi, x+yi) = b^{1/4} e^{-\pi y^2/b} \cdot |\sum_{n\in\mathbb{Z}} e^{\pi i n^2 (a+bi)} e^{2\pi i n(x+yi)}|,$$

$$\|\eta\|(a+bi) = b^{1/4} \cdot |e^{\pi i(a+bi)/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i(a+bi)})|.$$

See Section 7 in [23] for details.

It can be proven by Green's formula that the Green's function is a symmetric function. The existence of the Green's function can be proven by partial differential equation tools and the uniqueness is trivial. However, it is still not easy to construct it from the definition. R. de Jong gave an explicit expression for the Green's function (Theorem 2.1.2 [12]).

Next we will assign a Hermitian metric on $\mathcal{O}(D)$ to each divisor D on X. The trivial line bundle is assigned with the constant function $\|1\|_{\mathcal{O}_X} = 1$. For a prime divisor $D = P$, we assign the smooth Hermitian metric

$$\|1\|_{\mathcal{O}(D)}(Q) \coloneqq G(P, Q) \tag{1.7}$$

on $\mathcal{O}(D)$. If $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are already assigned with metrics, we define

$$\|1\|_{\mathcal{O}(D_1+D_2)}(Q) \coloneqq \|1\|_{\mathcal{O}(D_1)}(Q) \cdot \|1\|_{\mathcal{O}(D_2)}(Q) \tag{1.8}$$

to be the metric on $\mathcal{O}(D_1 + D_2)$. These can give a Hermitian metric on every line bundle $\mathcal{O}(D)$ of X inductively.

Definition 1.2.5. *Let $\|\cdot\|$ be a smooth Hermitian metric on a line bundle $\mathcal{O}(D)$ of X. We say $\|\cdot\|$ is admissible if its curvature form is a multiple of μ_{Ar}.*

From Property (2) in Definition 1.2.2, we find that the curvature form of the metric $\|\cdot\|_{\mathcal{O}(D)}$ is $\deg(D)\mu_{Ar}$. This means that the metric we just defined on $\mathcal{O}(D)$ is admissible.

Remark 1.2.6. *There is an admissible metric on a line bundle $\mathcal{O}(D)$, unique up to a multiplicative scalar. This can be proven using Property* (2) *of the Green's function.*

We end this subsection by constructing an admissible metric on Ω_X.

(1) For any point $P \in X$, we already have a metric on $\mathcal{O}(P)$ given by the Green's function on X.

(2) The residue of rational sections of Ω_X^1 at P gives a natural isometry:

$$\Omega_X(P)|_P \simeq \mathbb{C},$$

where $\mathbb{C}$ has the standard Euclidean metric.

(3) We assign Ω_X^1 with the metric such that the following isomorphism gives an isometry at every point P:

$$\Omega_X(P) \simeq \mathcal{O}(P) \otimes \Omega_X.$$

Definition 1.2.7. *The metric defined above, denoted by $\|\cdot\|_{Ar}$, is called the Arakelov metric on Ω_X^1.*

Let $\Delta : X \to X \times X$ be the diagonal map. Then we know that $\Omega_X^1 \simeq \mathcal{O}_{X\times X}(-\Delta)|_\Delta$. If we assign a metric on $\mathcal{O}_{X\times X}(-\Delta)$, then its pullback along the diagonal map will induce a metric on Ω_X^1. The metric $\|\cdot\|_{Ar}$ is equal to the pullback of the metric $\|1\|_{\mathcal{O}_{X\times X}(-\Delta)}(P,Q) := G^{-1}(P,Q)$ along Δ.

Theorem 1.2.8. *The Arakelov metric $\|\cdot\|_{Ar}$ is admissible.*

Proof. See Section 4 in [1] or Section 4.5 in [52]. □

Remark 1.2.9. *In fact, we can associate a Green's function to any Kähler form on X (modifying Property (2) in Definition 1.2.2). The reason we choose the canonical form (Equation (1.6)) is that this is the only Kähler form, up to multiplicative scalar, that induces an admissible metric on Ω_X^1 by the construction above (Lemma 4.25 in [52]).*

1.2.2 Arakelov intersection theory

Now we are ready to show how Arakelov intersection theory is defined. In this subsection, K is a number field with integer ring O_K and $S = \mathrm{Spec}(O_K)$.

Let $\pi : \mathcal{X} \to S$ be an arithmetic surface of genus $g \geq 1$ with smooth generic fiber $\mathcal{X}_K$. A prime horizontal divisor D on $\mathcal{X}$ can be written in the form $\epsilon_*(\mathrm{Spec}(O_L))$, where L is a finite extension of K and $\mathrm{Spec}(O_L) \xrightarrow{\epsilon} \mathcal{X}$ is a section of π. Each embedding $\sigma : K \to \mathbb{C}$ corresponds to a compact Riemann surface $\mathcal{X}_\sigma$. By μ_σ, we mean the canonical form on $\mathcal{X}_\sigma$ defined in the last subsection.

In Arakelov intersection theory, the divisor group of $\mathcal{X}$ contains the divisors in the usual sense, which are called $finite\ divisors$, and also contains real linear combinations of $\mathcal{X}_\sigma$, which are called $infinite\ divisors$. The advantage of including infinite divisors is that we can make S into a 'compact' object by the product formula.

Definition 1.2.10. *An Arakelov divisor on $\mathcal{X}$ is a formal sum $D_{\text{fin}} + D_{\text{inf}}$, where D_{fin} is a Weil divisor on $\mathcal{X}$ and $D_{\text{inf}} = \sum_{\sigma: K \to \mathbb{C}} c_\sigma \mathcal{X}_\sigma$ is a formal linear combination of infinite fibers $\mathcal{X}_\sigma$ over $\mathbb{R}$. We write $\mathrm{Div}_{Ar}(\mathcal{X})$ for the group generated by Arakelov divisors. We write $D_{\text{fin},\sigma}$ for the divisor on $\mathcal{X}_\sigma$ induced by D_{fin}.*

For a rational section f of $\mathcal{O}_\mathcal{X}$, we define a principal divisor associated to it as

$$\mathrm{div}_{Ar}(f) \coloneqq \mathrm{div}(f) + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma} \mathcal{X}_\sigma,$$

where $\mathrm{div}(f)$ is the divisor of f in the usual sense and $v_{\mathcal{X}_\sigma} \coloneqq -\int_{\mathcal{X}_\sigma} \log|f|_\sigma \mu_\sigma$. We say two Arakelov divisors D_1 and D_2 are *linearly equivalent* if $D_1 = \mathrm{div}_{Ar}(f) + D_2$ for some rational section f of $\mathcal{O}_\mathcal{X}$. We denote

$$\widehat{\mathrm{Cl}}(\mathcal{X}) \coloneqq \mathrm{Div}_{Ar}(\mathcal{X})/(\text{principal divisors}).$$

We now construct an intersection pairing $(\cdot,\cdot)_{Ar}$ on $\widehat{\mathrm{Cl}}(\mathcal{X})$ which does not depend on the choice of representatives.

(1) For an infinite prime divisor σ and a prime divisor D (finite or infinite), we define $(\sigma, D)_{Ar} \coloneqq d$ (resp. 0) if D is horizontal (resp. vertical or infinite), where d is the degree of D over the generic fiber.

(2) Let D_1 be a prime vertical divisor and let D_2 be a finite divisor. If D_1 and D_2 have no common components, then we define

$$(D_1, D_2)_{Ar} \coloneqq (D_1, D_2)_{\text{fin}},$$

where $(\cdot,\cdot)_{\text{fin}}$ is defined in Equation (1.3).

(3) Let $D_1 : \mathrm{Spec}(O_{L_1}) \to \mathcal{X}$ and $D_2 : \mathrm{Spec}(O_{L_2}) \to \mathcal{X}$ be distinct prime horizontal divisors of $\mathcal{X}$. Then $D_{1,\sigma}$ and $D_{2,\sigma}$ determine two sets of points $\{P^\sigma_{1,j}\}_{1 \le j \le [L_1:K]}$ and $\{P^\sigma_{2,k}\}_{1 \le k \le [L_2:K]}$ on $\mathcal{X}_\sigma$ for each embedding $\sigma : K \to \mathbb{C}$. We define

$$(D_1, D_2)_{Ar} \coloneqq (D_1, D_2)_{\text{fin}} + \sum_{\sigma \in K(\mathbb{C})} \sum_{1 \le j \le [L_1:K]} \sum_{1 \le k \le [L_2:K]} -\log G_\sigma(P^\sigma_{1,j}, P^\sigma_{2,k}).$$

We still need to define the intersection pairing when two Arakelov divisors have common finite components. The following theorem will be useful.

Theorem 1.2.11. *Let $\mathcal{X} \to \mathrm{Spec}(O_K)$ be an arithmetic surface. Let D be an Arakelov divisor and let f be a rational function on $\mathcal{X}$. If $\mathrm{div}_{Ar}(f)$ and D have no common finite components, then $(D, \mathrm{div}_{Ar}(f))_{Ar} = 0$.*

Sketch of proof : We only need to prove that $(\mathrm{div}_{Ar}(f), D)_{Ar} = 0$ for any prime Arakelov divisor D. This is trivial if D is an infinite divisor. When D is a finite divisor, we assume D is not in the support of $\mathrm{div}_{Ar}(f)$.

When D is a prime vertical divisor, this follows from Theorem 3.1 in [44]. It remains to prove this for a horizontal prime divisor. For simplicity, we only consider the case $D = \epsilon(\mathrm{Spec}(O_K))$ for some section $\epsilon : \mathrm{Spec}(O_K) \to \mathcal{X}$.

The divisor D corresponds to a point P_σ for each $\sigma : K \to \mathbb{C}$. A rational section f of $\mathcal{O}_{\mathcal{X}}$ gives a meromorphic function f_σ on $\mathcal{X}_\sigma$. Since f_σ is meromorphic, we have

$$\partial_\sigma \overline{\partial}_\sigma \log(|f_\sigma(x)|) = 0$$

outside $\mathrm{div}(f_\sigma)$. Since $\deg(f_\sigma) = 0$, we have

$$\partial_{P_\sigma} \overline{\partial}_{P_\sigma} \log G_\sigma(\mathrm{div}(f_\sigma), P_\sigma) = 0$$

by Property (2) in Definition 1.2.2. This means that there exists a real constant α such that

$$G(\mathrm{div}(f), P_\sigma) = e^\alpha \cdot |f|(P_\sigma).$$

According to Property (3) in Definition 1.2.2, we obtain $\alpha = -\int_{\mathcal{X}_\sigma} \log|f_\sigma|\mu_\sigma$. Now we can compute

$$\begin{aligned}
(D, (f))_{Ar} &= (D, \mathrm{div}(f) + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) \cdot \mathcal{X}_\sigma)_{Ar} \\
&= (D, \mathrm{div}(f))_{Ar} + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) \\
&= (D, \mathrm{div}(f))_{\mathrm{fin}} - \sum_{\sigma \in K(\mathbb{C})} \log|f_\sigma|(P_\sigma) - \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) + \sum_{\sigma \in K(\mathbb{C})} v_{\mathcal{X}_\sigma}(f) \\
&= (D, \mathrm{div}(f))_{\mathrm{fin}} - \sum_{\sigma \in K(\mathbb{C})} \log|f_\sigma|(P_\sigma) \\
&= 0,
\end{aligned}$$

where $(D, \mathrm{div}(f))_{\mathrm{fin}}$ is defined in Remark 1.1.24 and the last step is due to the product formula for number fields. QED

We return to the Arakelov intersection pairing. The Moving Lemma (Corollary 9.1.10 in [48]) says that for any two Arakelov divisors E and F, we can find a rational function h on $\mathcal{X}$ such that $E_{\mathrm{fin}} + \mathrm{div}(h)$ and F_{fin} have no common components. We define

$$(E, F)_{Ar} := (E + \mathrm{div}_{Ar}(h), F)_{Ar}.$$

Theorem 1.2.11 implies that the intersection number is well-defined and $(\cdot,\cdot)_{Ar}$ factors through Arakelov principal divisors, that is:

$$(\cdot,\cdot)_{Ar} : \widehat{\mathrm{Cl}}(\mathcal{X}) \times \widehat{\mathrm{Cl}}(\mathcal{X}) \to \mathbb{R}.$$

We next define a dualising object $\overline{\omega}$ for Arakelov divisors. The finite divisors of $\overline{\omega}$ should correspond to the usual dualising sheaf on $\mathcal{X}$, and thus it remains to figure out the infinite part. Before that, we introduce $\widehat{\mathrm{Pic}}(\mathcal{X})$.

Definition 1.2.12. *Let $\mathcal{X} \to \mathrm{Spec}(O_K)$ be an arithmetic surface. An admissible line bundle on $\mathcal{X}$ is the datum of a line bundle L on $\mathcal{X}$ and an admissible metric $\|\cdot\|_\sigma$ on the line bundle L_σ on $\mathcal{X}_\sigma$ for each $\sigma : K \to \mathbb{C}$. The set of isomorphism classes of admissible line bundles on $\mathcal{X}$ has a natural group structure, and we denote it by $\widehat{\mathrm{Pic}}(\mathcal{X})$.*

Theorem 1.2.13. *There is a canonical isomorphism of groups $\widehat{\mathrm{Cl}}(\mathcal{X}) \simeq \widehat{\mathrm{Pic}}(\mathcal{X})$.*

Sketch of proof : See Proposition 2.2 in [1] for details. We only give a description of the map. Let $D = D_{\mathrm{fin}} + \sum_\sigma c_\sigma \cdot \mathcal{X}_\sigma$ be an Arakelov divisor. Then D_{fin} gives a line bundle $\mathcal{O}(D_{\mathrm{fin}})$ on $\mathcal{X}$. For each $\sigma : K \to \mathbb{C}$, we associate the line bundle $\mathcal{O}(D_{\mathrm{fin},\sigma})$ on $\mathcal{X}_\sigma$ with the admissible metric $e^{-c_\sigma} \cdot \|\cdot\|_{\mathcal{O}(D_{\mathrm{fin},\sigma})}$, where $\|\cdot\|_{\mathcal{O}(D_{\mathrm{fin},\sigma})}$ is the metric induced from the Green's function on $\mathcal{X}_\sigma$. QED

Definition 1.2.14. *The Arakelov dualising sheaf $\overline{\omega}$ on $\mathcal{X} \to \mathrm{Spec}(O_K)$ consists of the following datum:*

(1) the usual dualising sheaf $\omega_{\mathcal{X}/S}$ on $\mathcal{X}$,

(2) the Arakelov metric $\|\cdot\|_{Ar,\sigma}$ on $\Omega_{\mathcal{X}_\sigma}$ for each $\sigma : K \to \mathbb{C}$.

According to Theorem 1.2.13, this dualising sheaf corresponds to a unique element in $\widehat{\mathrm{Cl}}(\mathcal{X})$.

We end this subsection with stating the adjunction formula in Arakelov intersection theory, although we do not really use it in this thesis.

Theorem 1.2.15. *The divisor D given by a section $\mathrm{Spec}(O_K) \to \mathcal{X}$ leads to the following equality*

$$-(D,D)_{Ar} = (D,\overline{\omega})_{Ar}.$$

Proof. See Lemma 4.26 in [52]. □

1.3 Faltings' work

In [23], G. Faltings established

(1) the Faltings-Riemann-Roch theorem,

(2) the arithmetic Noether's formula,

(3) the positivity of the relative dualizing sheaf $\omega_{\mathcal{X}}$ (Arakelov theoretic version),

(4) the Hodge index theorem (Arakelov theoretic version).

In this section, we give a brief review of these results except the last one. Subsection 1.3.1 is about the Faltings metric on the determinant of cohomology. Subsection 1.3.2 is about the Faltings-Riemann-Roch theorem and its corollaries.

Corollary 1.3.11 will be used to decompose $\langle \Delta, \Delta \rangle$ in Subsection 3.3.1. The Faltings δ invariant introduced in Theorem 1.3.9 will be computed in Section 4.8.

1.3.1 Determinant of cohomology and Faltings metric

Let V be a complex vector space of dimension d. We can define $\det V := \Lambda^d V$ as the top exterior power of V.

Example 1.3.1. *Let C be a compact Riemann surface of genus $g \geq 1$. For an arbitrary line bundle L on X, we have a 1-dimensional vector space over $\mathbb{C}$*

$$\lambda(L) := \det H^0(C, L) \otimes \left(\det H^1(C, L)\right)^{-1}.$$

By Serre duality, we have a canonical morphism

$$\left(\det H^1(C, L)\right)^{-1} \simeq \det H^0(X, \Omega_C \otimes L^{-1}).$$

In the above construction, we start from a line bundle on a Riemann surface and end with a 1-dimensional complex vector space. The following theorem is a generalization of this construction. We refer to Section 5 in [53] for the definition of Rp_*F.

Theorem 1.3.2. *More generally, let $p : Y \to T$ be a proper morphism of Noetherian schemes. Then for each coherent sheaf F on Y, flat over T, we can associate a line bundle $\det Rp_*F$ on T, called the determinant of cohomology of F, that satisfies the following properties.*

(1) $\det Rp_*F$ is functorial for isomorphisms of coherent sheaves on Y.

(2) $\det Rp_*F$ commutes with base change.

(3) *If*

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of coherent sheaves on Y flat over T, then there is an isomorphism

$$\det Rp_*F \xrightarrow{\sim} \det Rp_*F' \otimes \det Rp_*F''$$

which is functorial with respect to base changes and isomorphisms of exact sequences.

(4) *Let*

$$\mathcal{E} : 0 \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

be a complex of vector bundles of finite rank which gives a quasi-isomorphism

$$\mathcal{E} \xrightarrow{\sim} Rp_*F.$$

Then we have a canonical isomorphism

$$\det Rp_*F \xrightarrow{\sim} \bigotimes_{k=0}^{n} (\det E^k)^{(-1)^k},$$

which commutes with base changes. Here $\det E^k$ is the top exterior power of the vector bundle E^k.

(5) *If T is connected and F is fixed, then the function $\chi : s \to \chi(F_s)$ is a constant function on T. Let u be a global section of O_T^*. The multiplication by u on F induces an automorphism $\det(u) : \det Rp_*F \xrightarrow{\sim} \det Rp_*F$ according to (1), and we have*

$$\det(u) = u^{\chi}.$$

(6) *If M is a line bundle on T (assume connected again), then there is a canonical isomorphism*

$$\det Rp_*(F \otimes p^*M) \xrightarrow{\sim} (\det Rp_*F) \otimes M^{\otimes \chi}$$

Proof. See Section 1 in [50]. □

Remark 1.3.3. *By (4) in the above theorem, if the higher pushforward sheaves R^ip_*F ($i \geq 0$) are vector bundles, then there is a natural isomorphism*

$$\det Rp_*F \xrightarrow{\sim} \bigotimes_{i=1}^{n} (\det R^ip_*F)^{(-1)^i}.$$

Let $f : \mathcal{X} \to S$ be a semistable arithmetic surface. The dualising sheaf ω is coherent (even a line bundle) and the higher pushforwards $R^i f_*(\omega)$ are coherent. Since coherent sheaves on regular Noetherian schemes have finite free resolutions, we can apply (3) and (4) to construct $\det Rf_*\omega$. The following theorem shows the relation between the determinant of cohomology and the Deligne pairing in Theorem 1.1.21.

Proposition 1.3.4. *Let $p : \mathcal{X} \to S$ be an arithmetic surface with line bundles L and M on $\mathcal{X}$. We have a canonical isomorphism*

$$\langle L, M\rangle \xrightarrow{\sim} \det Rp_*(L \otimes M) \otimes (\det Rp_*L)^{-1} \otimes (\det Rp_*M)^{-1} \otimes \det p_*\omega_{\mathcal{X}/S}.$$

Proof. See Page 14 in [12]. ☐

A corollary of the proposition above is that we have a Riemann-Roch theorem for arithmetic surfaces.

Corollary 1.3.5. *Let $p : \mathcal{X} \to S$ be an arithmetic surface with line bundles L and M on $\mathcal{X}$. We have a canonical isomorphism*

$$(\det Rp_*L)^{\otimes 2} \xrightarrow{\sim} \langle L, L \otimes \omega_{\mathcal{X}/S}^{-1}\rangle \otimes (\det p_*\omega_{\mathcal{X}/S})^{\otimes 2}.$$

Proof. See Theorem 9.9 in [18]. ☐

Now we return to Riemann surfaces. Let $f : X \to \mathrm{Spec}(\mathbb{C})$ be a compact Riemann surface of genus $g > 0$. By Theorem 1.3.2, or the beginning of this subsection, we have a complex vector space $\lambda(L) := \det Rf_*(L)$ for any line bundle L on X. In [23], G. Faltings defined a metric on $\lambda(L)$ which is known as the Faltings metric.

Theorem 1.3.6. *There exists, for every line bundle L on X together with an admissible Hermitian metric on L, a Hermitian metric on $\lambda(L)$, such that the following properties hold:*

(1) An isometric isomorphism of line bundles induces an isometry on these $\lambda(\cdot)$.

(2) If the metric on L is changed by a factor $\alpha > 0$, the metric on $\lambda(L)$ is changed by multiplying $\alpha^{\chi(L)}$, where

$$\chi(L) = \dim H^0(X, L) - \dim H^1(X, L) = \deg(L) + 1 - g.$$

(3) For a divisor D on X and a point $P \in X$, $\mathcal{O}(D)$ and $\mathcal{O}(D - P)$ have canonical admissible metrics (constructed by Equation (1.7) and Equation (1.8)). We set

$\mathcal{O}(D)[P]$ with the metric given by the restriction of the metric on $\mathcal{O}(D)$ to the fibre over P. The exact sequence

$$0 \to \mathcal{O}(D-P) \to \mathcal{O}(D) \to \mathcal{O}(D)[P] \to 0$$

induces an isomorphism

$$\lambda(\mathcal{O}(D)) \simeq \lambda(\mathcal{O}(D-P)) \otimes_{\mathbb{C}} \mathcal{O}(D)[P]$$

which is also an isometry.

(4) The metrics on the $\lambda(\cdot)$ are unique up to a common scalar factor.

(5) If $L = \Omega_X$, the metric on $\lambda(L) = \det H^0(X, \Omega_X)$ is induced from the inner product in Equation (1.5).

Proof. See Theorem 1 in [23]. Points (3) and (4) make it possible for us to construct this metric inductively. The symmetry property of the Green's function guarantees that the order of points we choose in our construction does not matter. Point (1) is the most technical one. An alternative proof using Proposition 1.3.4 can be found on Page 15 of [12]. □

1.3.2 Faltings-Riemann-Roch theorem

In this subsection, we assume $p : \mathcal{X} \to S$ to be a semistable arithmetic surface with $S = \mathrm{Spec}(O_K)$ for some number field K.

If L is an admissible line bundle on $\mathcal{X}$, then $L \otimes \overline{\omega}^{-1}$ is also an admissible line bundle. According to Theorem 1.3.6, we can assign metrics to $\det Rp_*L$ and $\det Rp_*\omega$. There is a unique metric on $\langle L, L \otimes \overline{\omega}^{-1} \rangle$ such that Corollary 1.3.5 is an isometry with respect to these metrics. We have following Faltings-Riemann-Roch theorem.

Theorem 1.3.7. $\deg \det Rp_*L = \frac{1}{2}(L, L \otimes \overline{\omega}^{-1})_{Ar} + \deg \det p_*\omega_{\mathcal{X}/S}$.

Proof. See Theorem 3 in [23]. □

Let B be a smooth curve over a field l. If $p : \mathcal{Y} \to B$ is a semistable curve with smooth generic fiber, then we have

$$12 \cdot \deg(\det p_*\omega_{\mathcal{Y}}) = (\omega_{\mathcal{Y}}, \omega_{\mathcal{Y}}) + \delta,$$

where δ is the number of singular points, counted according to the degree of their residue field extensions and thicknesses. This is known as the classical Noether's formula.

Let ω be the universal dualising sheaf of the universal curve $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ (see Tag 0DMG for details). Over the complex points of $\mathcal{M}_g$, we can assign Arakelov metric (using the Arakelov-Green function) to ω fiberwise. Then we can assign a metric on $\langle \omega, \omega \rangle$ (see Section 9 in [14]).

Remark 1.3.8. *(More about the metric on $\langle \omega, \omega \rangle$) Let $\mathcal{Z}$ be a smooth complex variety and let $p : \mathcal{Y} \to \mathcal{Z}$ be a smooth proper curve of genus $g \geq 1$. For two metrized line bundles L (with a non-zero rational section l) and M (with a non-zero rational section m) on $\mathcal{Y}$, we can construct a line bundle $\langle L, M \rangle$ (with a non-zero rational section $\langle l, m \rangle$) on $\mathcal{Z}$. All these constructions are similar to the technique in Theorem 1.1.21. Then we can assign a metric on $\langle L, M \rangle$ given by*

$$\log\|\langle l, m \rangle\| = (\log\|m\|)[\mathrm{div} l] + \int_p \log\|l\| c_1(M)$$

Theorem/Definition 1.3.9. *There exists an isomorphism of line bundles*

$$\mu : (\det \pi_* \omega)^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega \rangle \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta)$$

on $\overline{\mathcal{M}}_g$, which is unique up to a sign. The Faltings delta invariant is defined to be the number $\delta(\cdot)$ such that $(2\pi)^{-4g}\exp(\delta(\cdot))$ is the norm of the above isomorphism on $\mathcal{M}_g(\mathbb{C})$.

Proof. See Theorem 2.1 in [51]. □

Remark 1.3.10. *In [23], G. Faltings gave an interpretation of the Faltings delta invariant using the theta divisor Θ associated to the corresponding compact Riemann surface. In [12], R. de Jong gave a more explicit method for numerically computing this invariant. We will apply this method in Section 4.7.*

The following corollary is known as the Noether's formula for an arithmetic surface over a ring of integers O_K. Recall that $\delta(X)$ is the Faltings delta invariant for the Riemann surface X, $\delta(\overline{\Gamma})$ is the total volume of the pm-graph $\overline{\Gamma}$ (see Definition 1.4.8 and the discussion after Remark 1.4.14), and $p : \mathcal{X} \to S$ is a semistable arithmetic surface.

Corollary 1.3.11. *We write $\det p_* \overline{\omega}$ for the line bundle $\det p_* \omega$ with the metric induced from Equation (1.5). Then we have*

$$12 \deg \det p_* \overline{\omega} = (\overline{\omega}, \overline{\omega})_{Ar} + \sum_s \delta(\overline{\Gamma}_s) \log(\# k(s)) + \sum_{\sigma \in K(\mathbb{C})} \delta(\mathcal{X}_\sigma) - 4g[K : \mathbb{Q}] \log(2\pi),$$

where the first (resp. second) sum goes through all closed points $s \in \mathrm{Spec}(O_K)$ (resp. complex embeddings of K) and $\delta(\mathcal{X}_\sigma)$ is the Faltings delta invariant (Theorem 1.3.9).

Proof. We first pull back the isomorphism in Theorem 1.3.9 along the classifying map $\mathrm{Spec}(O_K) \to \overline{\mathcal{M}}_g$. Then the assertion is proved by taking the degree of both sides in Theorem 1.3.9. □

At the end of this subsection, we state the non-negativity of the self-intersection of the Arakelov dualising sheaf, although we will not use it in an essential way.

Theorem 1.3.12. *Let D be an effective divisor on the semistable arithmetic surface $\mathcal{X} \to \mathrm{Spec}(O_K)$ of genus g. We have the following results:*

(1) $(\overline{\omega}, \overline{\omega})_{Ar} \geq 0$,

(2) $(\overline{\omega}, D)_{Ar} \geq \frac{(\overline{\omega}, \overline{\omega})_{Ar}}{4g(g-1)} \cdot \deg(D)$.

Proof. See Theorem 5 in [23]. □

1.4 Dual graph

In Subsection 1.4.1, we introduce pm-graphs and some basic notions. In Subsection 1.4.2, we discuss the Green's function on a pm-graph and the admissible invariants introduced by S. Zhang. In Subsection 1.4.3, we will introduce more invariants on pm-graphs and use them to prove the second half of Proposition 1.4.33. More details can be found in [69] and [70].

We will return to the explicit computation of admissible invariants in Section 3.1. And we will compute the admissible invariants of our main curve $\mathfrak{C}$ in Proposition 4.4.1.

All graphs are assumed to be finite.

1.4.1 Polarized metrized graph

Definition 1.4.1. *A star-shaped set of valence n and radius ϵ is a metric space that is isometric to*

$$S(n, \epsilon) = \{z \in \mathbb{C} : z = te^{\frac{2\pi i k}{n}} \text{for some } 0 \leq t < \epsilon \text{ and } k \in \mathbb{Z}\}.$$

Definition 1.4.2. *A metrized graph Γ is a compact connected metric space that either is a point or satisfies that for each point p of Γ there exists a neighbourhood U_p that is isometric to a star-shaped set of finite valence and radius $\epsilon > 0$. The valence is well-defined and we denote the valence of a point p by $v(p)$.*

We define the *canonical divisor* of Γ as

$$K_\Gamma := \sum_{x \in \Gamma} (v(x) - 2)x. \tag{1.9}$$

The canonical divisor is well-defined since all but finitely many points on Γ have valence 2. Let V_0 be the set containing exactly points $p \in \Gamma$ such that $v(p) \neq 2$. Since we only consider compact metrized graphs, V_0 is a finite set. A non-empty finite set $V' \subset \Gamma$ containing V_0 is called a *vertex set* of Γ. For a vertex set V', the complement $\Gamma \backslash V'$ is a union of finitely many connected components. Each component e° in $\Gamma \backslash V'$ is called an *edge* associated to V' and is isometric to an open interval. The closure of each edge e° in Γ is a closed segment (we call it an *ordinary edge*) or a circle (we call it a *self-loop*), denoted by e. We call $e \backslash e^\circ$ the *endpoints* of e° in Γ. We can associate a unique positive number $l(e^\circ)$ to each edge e° such that e° is isometric to the interval $(0, l(e^0))$. The real number $l(e^\circ)$ is called the *weight* of e°. For simplicity, we sometimes also say e is an edge of weight $l(e^\circ)$.

Remark 1.4.3. *Note that we require a vertex set to be non-empty, thus the empty set is not a vertex set for a 1-loop graph (containing* 1 *loop only).*

Example 1.4.4. *The following is an illustration of metrized graphs with specified vertex sets. We omit the weight information in the figure.*

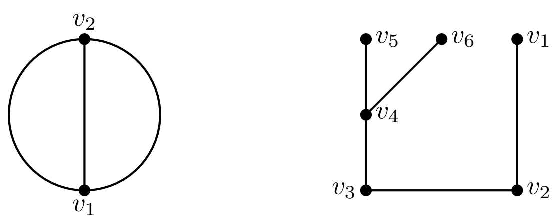


Figure 1.4.1: Metrized graphs with specified vertex sets

We can interpret metrized graphs in a graph-theoretic way. Let $G = (V, E)$ be an undirected graph with the vertex set V and the edge set E. Let $w : E \to \mathbb{R}_{>0}$ be a function on E, then we call the pair (G, w) a *weighted graph*.

There is a natural way to construct a metrized graph Γ with a specified vertex set from a connected weighted graph (G, w): V gives the specified vertex set ($V = V'$) and the vertices are connected by $e \in E$ with length $w(e)$. On the other hand, given a metrized graph Γ with a specified vertex set V', we can construct a connected weighted graph (G, w) by taking V to be V', taking elements in E to be the components in $\Gamma \backslash V$ and taking the induced weights from $\Gamma \backslash V$. Thus we have a correspondence

$$(\Gamma, V') \rightleftharpoons (G, w),$$

from the set of metrized graphs with specified vertex sets to the set of connected weighted graphs.

Definition 1.4.5. *Let* (G, w) *be a connected weighted graph. The* $first$ *Betti number of* (G, w) *is defined to be* $b_1(\Gamma) := \#E - \#V + 1$.

Remark 1.4.6. *According to the correspondence, we can define the first Betti number on a metrized graph* Γ *with a specified vertex set* V. *It is easy to show that this number only depends on the metric graph* Γ.

Definition 1.4.7. *Let the pair* (Γ, V) *be a metrized graph with a specified vertex set. A divisor* D *on* (Γ, V) *is an element in* $\mathbb{Z}^V$. *We define the degree of* D *(denoted by* $\deg(D)$*) to be the sum of all its values.*

Definition 1.4.8. *Fixing a map* $\mathfrak{q} : V \to \mathbb{Z}$ *for* (Γ, V), *we define the canonical divisor to be*

$$K_{\mathfrak{q}} := \sum_{p \in V} (v(p) - 2 + 2\mathfrak{q}(p))p.$$

We call the pair $\overline{\Gamma} = (\Gamma, \mathfrak{q})$ *a polarized metrized graph (or a pm-graph) if* $\mathfrak{q}$ *is non-negative and the associated canonical divisor* $K_{\mathfrak{q}}$ *is effective. The function* $\mathfrak{q}$ *is called a polarization of* (Γ, V).

Remark 1.4.9. *(Important) Throughout this thesis, we deal with pm-graphs in a flexible way. We can denote a pm-graph by* $\overline{\Gamma}$, $(G, w, \mathfrak{q})$, $(\Gamma, \mathfrak{q})$, $(V, E, w, \mathfrak{q})$ *and so on since they are equivalent. Notions can also be translated freely between graph-theoretic objects and metrized objects, for example, the first Betti number we already defined, the genus of a pm-graph which we will define and so on. Furthermore, notions can also be inherited, for example, pm-graphs inherit the notion the* $first$ *Betti number from metrized graphs.*

Definition 1.4.10. *Suppose* $\overline{\Gamma} = (\Gamma, \mathfrak{q})$ *is a pm-graph, the genus of* $\overline{\Gamma}$ *is defined to be*

$$g(\overline{\Gamma}) := \frac{1}{2}(\deg K_{\mathfrak{q}} + 2) = b_1(\Gamma) + \sum_{v \in V} \mathfrak{q}(v),$$

where $b_1(\Gamma)$ *is the first Betti number of* Γ.

Let e be an edge in a pm-graph $\overline{\Gamma} = (\Gamma, \mathfrak{q})$. We say it is of *type* 0 if we get a connected graph after removing the interior points of e from $\overline{\Gamma}$. For an integer i in $[1, g(\overline{\Gamma})/2]$, we say e is of type i if the removal of its interior points from $\overline{\Gamma}$ gives two disjoint pm-graphs of genus i and $g(\overline{\Gamma}) - i$. We write $\delta_i(\overline{\Gamma})$ for the total weight of edges of type i and write $\delta(\overline{\Gamma})$ for the total weight of $\overline{\Gamma}$. It follows from the definition that $\delta(\overline{\Gamma}) = \sum_{i=0}^{\lfloor \frac{g(\overline{\Gamma})}{2} \rfloor} \delta_i(\overline{\Gamma})$.

Definition 1.4.11. *Let* $\overline{\Gamma} = (\Gamma, \mathfrak{q})$ *be a pm-graph. We say the vertex* p *is eliminable if* $v(p) = 2$ *and* $\mathfrak{q}(p) = 0$.

Remark 1.4.12. *If v is an eliminable vertex on $\overline{\Gamma}$ and is the endpoint of edges e_1 and e_2, then we can get a new pm-graph by removing v from the vertex set and connecting e_1 and e_2 into one edge e with the weight $l(e_1) + l(e_2)$. We can also get new pm-graphs by adding eliminable vertices in an opposite way. This gives an equivalence relation for pm-graphs. Every pm-graph $\overline{\Gamma}$ of genus $g \neq 1$ is equivalent to a unique pm-graph with no eliminable points. The assumption $g \neq 1$ excludes the case when $\overline{\Gamma}$ is a genus 1 self-loop.*

Lemma 1.4.13. *A pm-graph $\overline{\Gamma}$ has no eliminable edges if and only if every coefficient $v(p) - 2 + 2\mathfrak{q}(p)$ in the canonical divisor (Definition 1.4.8) is positive.*

Proof. This follows from the definition. □

Remark 1.4.14. *Most invariants that we already defined or will define only depend on the equivalence class of the pm-graph.*

Let R be a discrete valuation ring. Let $f : X \to \mathrm{Spec}(R)$ be a semistable curve of genus g with smooth generic fiber. We can associate a pm-graph $\overline{\Gamma} = (V, E, w, \mathfrak{q})$ to f:

(1) Vertices in V correspond to irreducible components of its geometric special fiber $X_{\overline{s}}$.

(2) Edges in E correspond to nodal points and the endpoints correspond to the intersecting irreducible components. The weight $w(e)$ is the thickness of the nodal point corresponding to e.

(3) $\mathfrak{q}(v)$ is the geometric genus of the component corresponding to v.

The assumption that f is semistable guarantees that the canonical divisor $K_{\mathfrak{q}}$ is effective, thus this is a pm-graph. In general, we denote this graph by $\overline{\Gamma}_s$ and we call it the *dual graph* of f. We say the dual graph is *trivial* if it is a one-point graph.

Remark 1.4.15. *The arithmetic genus of the special fiber $X_{\overline{s}}$ is equal to $g(\overline{\Gamma}_s)$ (see the discussion at the end of Section 2 in [13]). For the dual graphs of stable curves, the pm-graphs have no eliminable vertices and every coefficient $v(p) - 2 + 2\mathfrak{q}(p)$ in the canonical divisor (Definition 1.4.8) is positive.*

For a general semistable curve over a Dedekind scheme $C \to S$ with smooth generic fiber, we have a dual graph for each closed point $s \in S$. Since we assume the curve to be generically smooth, the geometric special fiber $C_{\overline{s}}$ is non-smooth only for a 0-dimensional closed subset of S, which is finite. This means that we have trivial dual graphs for all but finitely many closed points in S.

Now we are going to introduce two operations on pm-graphs and metrized graphs which are the edge contraction and the wedge sum.

Let $\overline{\Gamma} = (E, V, w, \mathfrak{q})$ be a pm-graph of genus g and $e \in E$. We define a new pm-graph $\overline{\Gamma}_{\{e\}}$ as follows:

(1) If e is a self-loop with endpoint v_0, then we define

$$\overline{\Gamma}_{\{e\}} := (E\backslash\{e\}, V, w|_{E\backslash\{e\}}, \mathfrak{q}').$$

Here, $\mathfrak{q}'(v)$ is the same as $\mathfrak{q}(v)$ except $\mathfrak{q}'(v_0) = \mathfrak{q}(v_0) + 1$.

(2) If e is an ordinary edge with endpoints v_0 and v_1, then we define

$$\overline{\Gamma}_{\{e\}} := (E\backslash\{e\}, V', w|_{E\backslash\{e\}}, \mathfrak{q}'). \tag{1.10}$$

Here, V' is induced from V with v_0 and v_1 identified (denoted by v'), and $\mathfrak{q}'(v) = \mathfrak{q}(v)$ except $\mathfrak{q}'(v') = \mathfrak{q}(v_1) + \mathfrak{q}(v_2)$.

We call this pm-graph $\overline{\Gamma}_{\{}e\}$ the *contraction* of $\overline{\Gamma}$ at e.

Let $U = \{e_1, e_2, ..., e_n\}$ be a subset of E. We can get a new graph by taking the contraction of e_i one by one. The pm-graph we get in this way is denoted by $\overline{\Gamma}_U$. We write $\overline{\Gamma}^U$ for $\overline{\Gamma}_{E\backslash\{U\}}$. The contraction operation does not change the genus g.

Lemma 1.4.16. *$\overline{\Gamma}_U$ is well-defined, that is, taking contraction for edges in U in different orders gives the same pm-graph $\overline{\Gamma}_U$.*

Proof. We just need to consider the case $U = \{e_1, e_2\}$, which is easy to prove case-by-case. ☐

Lemma 1.4.17. *If $\overline{\Gamma}$ has no eliminable vertices, then neither do its contractions.*

Proof. We only need to check the endpoint(s) of the contracting edge e. We use Lemma 1.4.13 as the criterion for a pm-graph without eliminable edges.

Contracting a self-loop will make the endpoint have positive polarization. If e is an ordinary edge in $\overline{\Gamma}$, we only need to consider the case when endpoints of e are polarized by 0. The summation of their valences is at least 6, thus the point corresponding the contraction has valence at least 4. ☐

Definition 1.4.18. *A graph G is said to be reducible if there exist non-trivial subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a point. We say a graph is irreducible if it is not reducible.*

When G_1, G_2 are subgraphs of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ consists of one point, we say that G is the *wedge sum* of G_1 and G_2 (represented by $G = G_1 \vee G_2$). We define $G_1 \vee ... \vee G_n$ to be $(G_1 \vee ... \vee G_{n-1}) \vee G_n$. Every graph can be decomposed into a wedge sum of irreducible subgraphs, and the decomposition is unique up to order. These irreducible subgraphs are called *irreducible components* of G

Example 1.4.19. *In Figure 1.4.1, the first example is irreducible and the second one can be decomposed as a wedge sum of line segments.*

We end this subsection by introducing two important properties of functions on pm-graphs. We will mainly focus on functions satisfying these two properties.

Definition 1.4.20. *Let F be a real-valued function on the set of pm-graphs, and let $\overline{\Gamma} = (E, V, w, \mathfrak{q})$ be a pm-graph. For any $e \in E$ and $l > 0$, we can associate a pm-graph $\overline{\Gamma}_{e,l}$ by taking $w(e) = l$ with other parts unchanged. We say that F satisfies the contraction lemma on $\overline{\Gamma}$ if*

$$F(\overline{\Gamma}_{\{e\}}) = \lim_{l \to 0} F(\overline{\Gamma}_{e,l})$$

holds for all $e \in E$. We say the function F satisfies the contraction lemma if F satisfies the contraction lemma on every pm-graph.

It might look strange to introduce this property since all thicknesses of nodes of semistable curves are integers. A function satisfying the contraction lemma can be easier to compute. Let F' be a function on pm-graphs satisfying the contraction lemma. Let $\overline{\Gamma}$ be a pm-graph and e be an edge of $\overline{\Gamma}$. To compute $F'(\overline{\Gamma}_{\{e\}})$, we only need to compute a limit related to $F'(\overline{\Gamma})$. This is especially useful for genus 3 pm-graphs with only type 0 edges since we only have two maximal models in that case (Lemma 3.2.3).

Definition 1.4.21. *Let F be a real-valued function on the set of pm-graphs, and let $\overline{\Gamma} = (E, V, w, \mathfrak{q})$ be a pm-graph. Let $\{\overline{\Gamma}_i\}_{1 \leq i \leq k}$ be the irreducible components of $\overline{\Gamma}$ with induced polarizations from contractions. We say the function F is additive on $\overline{\Gamma}$ if*

$$F(\overline{\Gamma}) = \sum_{i=1}^{k} F(\overline{\Gamma}_i)$$

holds. We say the function F is additive if F is additive on every pm-graph.

Example 1.4.22. *It is easy to see that $\delta_i(\cdot)$ (defined before Definition 1.4.11) is additive.*

1.4.2 Green's functions and admissible invariants of $\overline{\Gamma}$

Let Γ be a metrized graph. Let $PS(\Gamma)$ be the space of real-valued continuous and piecewise smooth functions on Γ.

Remark 1.4.23. *By a piecewise smooth function, we mean a function f on Γ that there exists a finite subset of points $X_f \subset \Gamma$ such that $\Gamma \backslash X_f$ is a disjoint union of open intervals, and the restriction of f on each of these intervals is a smooth function.*

We write $PS(\Gamma)^\wedge$ for the vector space of linear maps from $PS(\Gamma)$ to $\mathbb{R}$. For a point u on Γ of valence n and $f \in PS(\Gamma)$, the function f has n directional derivatives at u. We define $\delta f(u) \in PS(\Gamma)^\wedge$ by $\delta f(u)(g) := g(u) \sum_{i=1}^{n} \lim_{x_i \to 0} f'_{\vec{i}}(x_i)$ for $g \in PS(\Gamma)$, where $f'_{\vec{i}}(\cdot)$ is the i-th outward directional derivative of f. It is easy to see that $\delta f(u)$ is zero for all but finitely many u, thus $\delta f := \sum_{u \in \Gamma} \delta f(u)$ is a well-defined element in $PS(\Gamma)^\wedge$.

For $f \in PS(\Gamma)$ and $x \in \Gamma$ of valence 2, we define $f''(x)$ as the second derivative of f at x in an arbitrary direction. The metric on Γ induces a uniform measure on Γ which we denote by μ_u (on each segment in Γ, the measure μ_u coincides with the Lebesgue measure). Then we can define the Laplacian on Γ as follows.

Remark 1.4.24. *f'' is well-defined at all but finitely many points on Γ. Assume f is smooth in a neighbourhood of x_0, we have two orientations for taking derivatives, denoted by $f'_+(x_0)$ and $f'_-(x_0)$. We have $f'_+(x_0) = -f'_-(x_0)$ and $f''_+(x_0) = (-1)^2 \cdot f''_-(x_0) = f''_-(x_0)$, thus the second derivatives are well-defined at all but finitely many points on Γ.*

Definition 1.4.25. *The Laplacian Δ on a metrized graph Γ is defined as the linear map from $PS(\Gamma)$ to $PS(\Gamma)^\wedge$ such that*

$$\Delta f(g) = -\int_\Gamma f'' g d\mu_u - \delta f(g)$$

for all $g \in PS(\Gamma)$.

Lemma 1.4.26. *Let Γ be a metrized graph. There is a unique continuous, symmetric, and piecewise smooth function $g(x,y)$ on $\Gamma \times \Gamma$ satisfying:*

$$\Delta_y g(x,y) = \delta_x - \frac{\mu_u}{\delta(\Gamma)},$$
$$\int_\Gamma g(x,y) d\mu_u(y) = 0,$$

for all $x \in \Gamma$, where δ_x is the Dirac measure at x.

Proof. See Appendix in [70]. □

This function is called the $Green's\ function$ on Γ associated to μ_u. Now, for a general measure μ on Γ with volume 1, we define:

$$g_\mu(x,y) := g(x,y) - \int_\Gamma g(x,y) d\mu(y) - \int_\Gamma g(x,y) d\mu(x) + \iint_{\Gamma \times \Gamma} g(x,y) d\mu(x) d\mu(y). \quad (1.11)$$

Then $g_\mu(x,y)$ is the unique function on G satisfying the following conditions:

$$\Delta_y G(x,y) = \delta_x - \mu,$$
$$\int_\Gamma G(x,y) d\mu(y) = 0.$$

Definition 1.4.27. *This function $g_\mu(x,y)$ is called the $Green's\ function$ on Γ associated to μ.*

Remark 1.4.28. *If we consider Γ as an electrical circuit such that the resistance is locally induced from the distance, then $g_{\delta_p}(q,q)$ is equal to the electrical resistance $r(p,q)$ between p and q (Page 179 in [70]).*

Lemma 1.4.29. *On a metrized graph Γ, we have $r(p,q) = g_\nu(q,q) - 2g_\nu(q,p) + g_\nu(p,p)$, where ν is any measure on Γ of volume 1.*

Proof. It is easy to check the following linear dependence

$$g_{\delta_p}(x,y) = g_\nu(x,y) - g_\nu(x,p) - g_\nu(p,y) + g_\nu(p,p).$$

By the remark above and the symmetry property of the Green's function, we prove the lemma by taking $x = y = q$. □

Theorem 1.4.30. *Let $D = \sum_{i=1}^{n} c_i \cdot x_i$ be a divisor on Γ with $\deg(D) \neq 2$. Then there is a unique measure μ_D on Γ of volume 1 and a unique constant c such that the following equality holds for any point x on Γ:*

$$c + g_{\mu_D}(D,x) + g_{\mu_D}(x,x) = 0, \tag{1.12}$$

where $g_{\mu_D}(D,x) := \sum_{i=1}^{n} c_i g_{\mu_D}(x_i,x)$.

Proof. See Theorem 3.2 in [70]. □

Remark 1.4.31. *The theorem above is only part of Theorem 3.2 in [70], and the remaining part of that theorem says that μ_D is positive if $D - K_\Gamma$ is effective, where K_Γ is defined in Equation (1.9).*

The function $g_{\mu_D}(x,y)$ is called the $admissible\ Green\ function$ of (Γ, D) and μ_D is called the $admissible\ measure$ of (Γ, D). Recall the definition of $K_\mathfrak{q}$ in Definition 1.4.8. For a pm-graph $\overline{\Gamma} = (G, w, \mathfrak{q})$, we denote by μ_{ad} the measure $\mu_{K_\mathfrak{q}}$ and by $g_{ad}(x,y)$ the Green's function $g_{K_\mathfrak{q}}(x,y)$. We are interested in the following three admissible invariants (see Section 4.1 in [69] for details):

$$\varphi(\overline{\Gamma}) := -\frac{1}{4}\delta(\overline{\Gamma}) + \frac{1}{4}\int_\Gamma g_{ad}(x,x)((10g(\overline{\Gamma}) + 2)d\mu_{ad}(x) - \delta_{K_\mathfrak{q}}(x)), \tag{1.13}$$

$$\epsilon(\overline{\Gamma}) := \int_\Gamma g_{ad}(x,x)((2g(\overline{\Gamma}) - 2)d\mu_{ad}(x) + \delta_{K_\mathfrak{q}}(x)), \tag{1.14}$$

$$\lambda(\overline{\Gamma}) := \frac{g(\overline{\Gamma}) - 1}{6(2g(\overline{\Gamma}) + 1)}\varphi(\overline{\Gamma}) + \frac{1}{12}(\epsilon(\overline{\Gamma}) + \delta(\overline{\Gamma})). \tag{1.15}$$

By integrating with respect to the second variable in Lemma 1.4.29, we have

$$g_{ad}(x,x) = \int_{\Gamma} r(x,y)\mu_{ad}(y) - \frac{1}{2}\int_{\Gamma} r(x,y)d\mu_{ad}(x)d\mu_{ad}(y).$$

Substituting the equation above to Equation (1.14), we get

$$\epsilon(\overline{\Gamma}) = \int_{\Gamma\times\Gamma} r(x,y)\delta_{K_{\mathrm{q}}}(x)d\mu_{ad}(y). \tag{1.16}$$

We can get similar expression of φ by r, $\delta_{K_{\mathrm{q}}}$ and μ_{ad}. In Section 3.1, we will use a more explicit way to compute these invariants for pm-graphs of genus 3.

Remark 1.4.32. *Many notions (including the Green function on metrized graphs we just defined) introduced in this subsection are motivated by and similar to the notions in Arakelov theory. The Green's function on a metrized graph is an analogue of the Arakelov-Green function on Riemann surfaces and so does these admissible invariants. In [70], S. Zhang used these invariants on metrized graphs to establish the admissible pairing theory.*

Proposition 1.4.33. *All the three invariants above satisfy the contraction lemma (Definition 1.4.20) and are additive (Definition 1.4.21) for pm-graphs of genus $g > 1$.*

Proof. For additivity, see Theorem 4.3.2 in [69]. For the contraction lemma, K. Yamaki proved the case of $g(\overline{\Gamma}) = 3$ in [66] Proposition 3.1. In Subsection 1.4.3, we will give a proof of general pm-graphs of genus $g > 1$, based on the work of Z. Cinkir, R. de Jong and F. Shokrieh. □

The following property was conjectured by S. Zhang in [69], and proved by Z. Cinkir (S. Zhang only conjectured the existence of the constant $c(g)$).

Theorem 1.4.34. *Let $\overline{\Gamma}$ be a pm-graph with genus $g > 1$. Then we have*

$$\varphi(\overline{\Gamma}) \geq c(g)\delta_0(\overline{\Gamma}) + \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \frac{2i(g-i)}{g}\delta_i(\overline{\Gamma})$$

where $c(2) = \frac{1}{27}$ and $c(g) = \frac{(g-1)^2}{2g(7g+5)}$ for $g \geq 3$. In particular, $c(3) = \frac{1}{39}$.

Proof. See Theorem 2.11 in [8]. □

Corollary 1.4.35. *Let $\overline{\Gamma}$ be a pm-graph of genus $g > 1$. Then we have $\varphi(\overline{\Gamma}) \geq c(g)\delta(\overline{\Gamma})$, where $c(2) = \frac{1}{27}$ and $c(g) = \frac{(g-1)^2}{2g(7g+5)}$ for $g \geq 3$.*

Proof. If i is an integer in $[1, \lfloor \frac{g}{2} \rfloor]$, then $\frac{2i(g-i)}{g} \geq 1$ and $c(g) \leq 1$. By Theorem 1.4.34, we get $\varphi(\overline{\Gamma}) \geq c(g)\delta(\overline{\Gamma})$. □

Corollary 1.4.36. *For any $\epsilon > 0$, there exists an integer g_ϵ such that $\varphi(\overline{\Gamma}) \geq (\frac{1}{14} - \epsilon)\delta(\overline{\Gamma})$ for all pm-graphs $\overline{\Gamma}$ with $g(\overline{\Gamma}) > \max\{g_\epsilon, 1\}$.*

Proof. This follows from the corollary above. □

1.4.3 Other invariants

When the genus g is fixed, there are only finitely many types of pm-graphs without eliminable edges of genus g, and these can be computed combinatorically. We still write Γ (resp. $\overline{\Gamma}$) for a metrized graph (resp. pm-graph).

For metrized graphs, M. Baker and R. Rumely defined the $\tau(\cdot)$ invariant (Section 14 in [4]) which has the following elementary interpretation by the electrical resistance function (Lemma 14.4 in [4]). We will use this interpretation to compute the $\tau(\cdot)$ invariant.

Definition 1.4.37. *Let $r(x, y)$ be the resistance function on a metrized graph Γ (Remark 1.4.28). For any point y in Γ, we have*

$$\tau(\Gamma) = \frac{1}{4}\int_\Gamma (r_x(x, y))^2 dx,$$

where r_x is the partial derivative of r with respect to the first variable x.

Remark 1.4.38. *The τ invariant is defined on metrized graphs, and thus we can extend it to pm-graphs. S. Zhang also defined a τ invariant on metrized graphs (Equation 4.1.2 in [69]), but that is different from our τ here.*

In [8], Z. Cinkir defined the following invariant of a pm-graph $\overline{\Gamma} = (V, E, w, \mathfrak{q})$:

$$\theta(\overline{\Gamma}) = \sum_{p,q\in V} (v(p) - 2 + 2\mathfrak{q}(p))(v(q) - 2 + 2\mathfrak{q}(q))r(p, q). \tag{1.17}$$

By Definition 1.4.8, all terms in $\theta(\overline{\Gamma})$ are non-negative, thus we have $\theta(\overline{\Gamma}) \geq 0$.

The reason we introduce $\tau(\overline{\Gamma})$ and $\theta(\overline{\Gamma})$ is that Zhang's admissible invariants can be written as linear combinations of $\delta(\overline{\Gamma})$, $\theta(\overline{\Gamma})$ and $\tau(\overline{\Gamma})$.

Theorem 1.4.39. *Let $\overline{\Gamma}$ be a pm-graph of genus $g > 1$. Then we have*

$$\epsilon(\overline{\Gamma}) = \frac{(4g-4)\tau(\overline{\Gamma})}{g} + \frac{\theta(\overline{\Gamma})}{2g},$$
$$\varphi(\overline{\Gamma}) = \frac{(5g-2)\tau(\overline{\Gamma})}{g} + \frac{\theta(\overline{\Gamma})}{4g} - \frac{\delta(\overline{\Gamma})}{4},$$
$$\lambda(\overline{\Gamma}) = \frac{(3g-3)\tau(\overline{\Gamma})}{4g+2} + \frac{\theta(\overline{\Gamma})}{16g+8} + \frac{(g+1)\delta(\overline{\Gamma})}{16g+8}.$$

Proof. See Propositions 4.6, 4.9 and Theorem 4.8 in [8]. □

Corollary 1.4.40. *If pm-graphs $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are equivalent (Remark 1.4.12), then all the six invariants are the same for the two pm-graphs.*

Proof. $\delta(\overline{\Gamma}_1) = \delta(\overline{\Gamma}_2)$ is trivial. From the definition of the θ invariant (Equation (1.17)), we find $v(p) - 2 + 2\mathfrak{q}(p) = 0$ when p is eliminable, thus $\theta(\overline{\Gamma}_1) = \theta(\overline{\Gamma}_2)$. Since the τ invariant is defined on metrized graphs, the polarization will not make any difference.

By Theorem 1.4.39, Zhang's invariants are determined by $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$, which completes the proof. □

Remark 1.4.41. *The transformation matrix in Theorem 1.4.39 is invertible. Thus computing $\varphi(\overline{\Gamma})$, $\lambda(\overline{\Gamma})$ and $\epsilon(\overline{\Gamma})$ is equivalent to computing $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$. As a corollary, the additivity of Zhang's admissible invariants is equivalent to the additivity of $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$.*

In the two lemmas below, we make the following assumptions. Let Γ be an irreducible electrical circuit (weighted graph). We assume that e is an ordinary edge (an edge that is not a self-loop) on Γ, and that it has endpoints p and q. Let Γ' be the contraction (Equation (1.10)) of Γ at e and let $\Gamma - e$ be the graph given by removing the interior points of e in Γ. We write $l(e)$ for the length of e and write u_{pq} for the point given by the contraction of e. We denote the electrical resistance between points x and y on Γ by $r(x, y; \Gamma)$.

Lemma 1.4.42. *If $\Gamma - e$ is connected, we have*

$$\tau(\Gamma') - \tau(\Gamma) = -\frac{l(e)}{12} + A_{p,q,\Gamma-e} \cdot \left(\frac{1}{R} - \frac{1}{l(e) + R}\right),$$

where $R = r(p, q; \Gamma - e)$, and $A_{p,q,\Gamma-e}$ only depends on p, q and $\Gamma - e$.

Proof. This is a direct result of Corollary 5.3 and Lemma 6.1 in [7]. □

Lemma 1.4.43. *Let x and y be two points on Γ but not in the interior of e. Then we have*

$$r(x, y; \Gamma') = r(x, y; \Gamma) - c(x, y, p, q; \Gamma),$$

where $c(x, y, p, q; \Gamma) \to 0$ as $l(e) \to 0$.

Proof. See Corollary 8.5 in [17]. □

We can now give a proof of the second half of Proposition 1.4.33 with the two lemmas above.

Proof of Proposition 1.4.33: By Theorem 1.4.39 and Remark 1.4.41, we only need to prove that θ, δ and τ satisfy the contraction lemma for irreducible graphs (Definition 1.4.18).

When $\overline{\Gamma}$ has only 1 edge, the assertion is trivial. Thus we assume that $\overline{\Gamma}$ is an irreducible pm-graph containing more than 1 edge. Let e be an edge on $\overline{\Gamma}$. By the irreducibility of $\overline{\Gamma}$, the weighted graph $\overline{\Gamma} - e$ is connected and e cannot be a self-loop. Thus we assume that e has endpoints p and q. It remains to show that the three invariants θ, τ and δ satisfy the contraction lemma for the edge e.

It is easy to see that δ satisfies the contraction lemma, since δ is just the sum of all lengths of the edges. Lemma 1.4.42 implies that τ also satisfies the contraction lemma. Thus we only need to check θ.

When x and y are vertices on $\overline{\Gamma}$ but not the endpoints of e, the contraction does not change the polarization of x and y, and $r(x, y; \overline{\Gamma}) \to r(x, y; \overline{\Gamma}')$ as $l(e) \to 0$ by Lemma 1.4.43. When $x = p$ and $y = q$, we have $r(x, y; \overline{\Gamma}) \to 0 = r(u_{pq}, u_{pq}; \overline{\Gamma}')$ as $l(e) \to 0$. When $x = y = p$ or $x = y = q$, the claim is trivial. It remains to consider the case $x = p$ but $y \neq p$ or q.

When $x = p$ but $y \neq p$ or q, we still have $r(s, y; \overline{\Gamma}) \to r(u_{pq}, y; \overline{\Gamma})$ as $l(e) \to 0$ for $s \in \{p, q\}$ by Lemma 1.4.43. We also have

$$\sum_{s \in \{p,q\}} (v(s) - 2 + 2\mathfrak{q}(s)) = v'(u_{pq}) - 2 + 2\mathfrak{q}(u_{pq})$$

by the construction of the contraction in Equation (1.10), where v' ($\mathfrak{q}$) is the valence (polarization) function on $\overline{\Gamma}'$. Now we can say that

$$\sum_{s \in \{p,q\}} (v(s) - 2 + 2\mathfrak{q}(s))(v(y) - 2 + 2\mathfrak{q}(y)) r(s, y; \overline{\Gamma})$$

converges to

$$(v'(u_{pq}) - 2 + 2\mathfrak{q}'(u_{pq}))(v'(y) - 2 + 2\mathfrak{q}'(y)) r(u_{pq}, y; \overline{\Gamma}')$$

as $l(e) \to 0$. Thus the contraction lemma holds for θ. QED

Remark 1.4.44. *Theorem 1.4.39 and Remark 1.4.41 reduce the computation of Zhang's admissible invariants on pm-graphs to the computation of $\theta(\overline{\Gamma})$, $\tau(\overline{\Gamma})$ and $\delta(\overline{\Gamma})$ on irreducible pm-graphs. This decomposition simplifies the computation even more, since $\delta(\overline{\Gamma})$ and $\theta(\overline{\Gamma})$ are finite sums and $\tau(\overline{\Gamma})$ is an integration of the derivative of r against the natural measure (compared with Zhang's admissible measure).*

1.5 Zhang's work

Subsection 1.5.1 is a rather sketchy description about the admissible pairing. Subsection 1.5.2 is about the decomposition and the Northcott property of $\langle \Delta, \Delta \rangle$. The main references for this section are [69] and [70].

Theorem 1.5.3 makes it possible to compute $(\hat{\omega}, \hat{\omega})_{ad}$. The whole of Chapter 4 is devoted to the computation of $\langle \Delta, \Delta \rangle$ for a specific curve $\mathfrak{C}_{\mathbb{Q}}$ by Theorem 1.5.6. The goal of Sections 4.5-4.7 is numerically computing the λ invariant defined in Equation (1.18).

1.5.1 Admissible pairing

Let B be either a smooth curve over a field or the spectrum of a ring of a number field. Let k be the fraction field of B. Let X be a smooth curve over k. Let $\mathcal{X}$ be an arithmetic surface over B whose generic fiber is isomorphic to X over k. We write $M(k)$ (resp. $M(k)_0$, resp. $M(k)_\infty$) for the set of places (resp. finite places, resp. infinite places) of k.

For a finite place v of k, we write $N(v)$ for $e^{\deg(v)}$ when B is a curve, for $\#k(v)$ (the cardinality of the residue field of B at v) when $B = \operatorname{Spec} O_k$. We also write $N(v)$ for e (resp. e^2) when v is a real (resp. complex) infinite place of k.

Remark 1.5.1. *A complex infinite place is a pair of conjugate complex embeddings.*

When B is a curve, we have a dualising sheaf ω on $\mathcal{X}$ that gives an adjunction formula in the usual intersection theory. When $B = \operatorname{Spec} O_k$, by assigning admissible metrics to the Archimedean places, there is an Arakelov dualising sheaf $\overline{\omega}$ that gives an adjunction formula (Theorem 1.2.15). G. Faltings proved a Hodge index theorem (Theorem 4 in [23]) for Arakelov intersection theory.

Inspired by the above results, S. Zhang established the *admissible* intersection theory $(\cdot, \cdot)_{ad}$ for smooth curves over a global field in [70]. This intersection theory is done by extending usual divisors on X to pairs (D, G), where D is a usual divisor on $\mathcal{X}$ (a model of X) and G includes the Arakelov-Green function and the Green's function on the dual graphs. In this intersection theory, there is a Hodge index theorem and a dualising sheaf $\hat{\omega}$ which gives an adjunction formula.

Remark 1.5.2. *We do not give the expression of $\hat{\omega}$ in this thesis. Instead, we use Theorem 1.5.3 to decompose it into the objects that we are more familiar with.*

At Archimedean places, this adelic Green's function contains information from the Arakelov-Green function, and at non-Archimedean places, this adelic Green's function contains information from the Green's function on the dual graphs we discussed in Section 1.4. Thus it makes sense to compare admissible intersection theory with two other intersection theories. The following theorem will be used repeatedly throughout this thesis (recall the definition of $\epsilon(\cdot)$ in Equation (1.14)).

Theorem 1.5.3. *We have the following equalities*

$$(\hat{\omega},\hat{\omega})_{ad} = (\overline{\omega},\overline{\omega}) - \sum_{v\in M(l)_0} \epsilon(\overline{\Gamma}_v)\log(N(v)),$$

where $(\overline{\omega},\overline{\omega})$ *is the self-intersection of the Arakelov dualising sheaf when* k *is a number field and is the self-intersection of the usual dualising sheaf when* k *is a function field.*

Proof. See Theorem 5.5 in [70]. □

Corollary 1.5.4. $(\hat{\omega},\hat{\omega})_{ad} \leq (\overline{\omega},\overline{\omega})$.

Proof. This comes from the fact that $\epsilon(\overline{\Gamma}) \geq 0$, which is proven in Theorem 4.4 in [70]. Alternatively, we can also get this from Theorem 1.4.39. □

1.5.2 Gross-Schoen cycle

Let X be a smooth curve over a field k. Let $\alpha = \sum_{i=1}^{t} a_i p_i$ be a divisor on X over k with rational coefficients and degree $\sum_{i=1}^{t} a_i \deg p_i = 1$. We define cycles of X^3 associated to α as follows:

$$\begin{aligned}
\Delta_{123} &:= \{(x,x,x) : x \in X\},\\
\Delta_{12} &:= \sum_{i=1}^{t} a_i\{(x,x,p_i) : x \in X\},\\
\Delta_{23} &:= \sum_{i=1}^{t} a_i\{(p_i,x,x) : x \in X\},\\
\Delta_{31} &:= \sum_{i=1}^{t} a_i\{(x,p_i,x) : x \in X\},\\
\Delta_{1} &:= \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j\{(x,p_i,p_j) : x \in X\},\\
\Delta_{2} &:= \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j\{(p_i,x,p_j) : x \in X\},\\
\Delta_{3} &:= \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j\{(p_i,p_j,x) : x \in X\}.
\end{aligned}$$

In [30], B. Gross and C. Schoen constructed an element $\Delta_\alpha \in \mathrm{CH}^2(X^3)_{\mathbb{Q}}$ associated to α as

$$\Delta_\alpha = \Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3 \in \mathrm{CH}^2(X^3)_{\mathbb{Q}}.$$

They also proved that this cycle is homologous to 0 (Proposition 3.1 in [30]) and is rationally equivalent to 0 if X is rational, or elliptic, or hyperelliptic and α is a Weierstrass point (Section 4 in [30]). Thus it is natural to ask whether $\Delta_\alpha \neq 0$ when X is non-hyperelliptic.

Now we assume that the base field k is a number field or the function field of a smooth curve over a field. B. Gross and C. Schoen defined a canonical height $\langle \Delta_\alpha, \Delta_\alpha \rangle$ for Δ_α, which is actually a special case of the Beilinson-Bloch height.

Remark 1.5.5. *For our goal, we will use the first formula in Theorem 1.5.6 as the expression of* $\langle \Delta_\alpha, \Delta_\alpha \rangle$.

From now, we assume $g \geq 2$. Let x_α be the divisor $\alpha - K_X/(2g-2)$ in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$, where K_X is the canonical divisor of X. Then we have the following theorem by S. Zhang.

Theorem 1.5.6. *Let X be a smooth curve of genus $g > 1$ over a field k which is either a number field or the function field of a smooth curve over a field. Assume that X has a semistable model $\mathcal{X}$ over k. Then*

$$\langle \Delta_\alpha, \Delta_\alpha \rangle = \frac{2g+1}{2g-2}(\hat{\omega}, \hat{\omega})_{ad} + 6(g-1)\langle x_\alpha, x_\alpha \rangle - \sum_{v \in M(k)} \varphi(X_v) \log(N(v)).$$

Here $\langle x_\alpha, x_\alpha \rangle$ is the Néron-Tate height of the class $\alpha - K_X/(2g-2)$ in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$, and the $\varphi(X_v)$ are defined as follows.

(1) If v is an Archimedean place, then

$$\varphi(X_v) := \sum_{\substack{l \in \mathbb{N} \\ 1 \leq m,n \leq g}} \frac{2}{\lambda_l} \left| \int_{X_v} \phi_l \omega_m(x) \overline{\omega}_n(x) \right|^2$$

where the ϕ_l are the normalized real eigenforms of the Arakelov Laplacian:

$$\frac{\partial\overline{\partial}}{\pi i}\phi_l = \lambda_l \cdot \phi_l \cdot d\mu_v, \qquad \int \phi_k \phi_l d\mu = \delta_{k,l},$$

and $\{\omega_i\}_{1 \leq i \leq g}$ is an orthonormal basis of $H^0(X_v, \Omega_{X_v})$ with respect to the inner product in Equation (1.5). The eigenvalues λ_l are non-negative (see Section 3 in [13]).

(2) *If v is a non-Archimedean place, then $\varphi(X_v) \coloneqq \varphi(\overline{\Gamma}_v)$ which we defined in Equation (1.13).*

Proof. See Theorem 1.3.1 in [69]. □

Remark 1.5.7. *The invariant φ at an Archimedean place is known as the Zhang-Kawazumi invariant. Let $\delta(C)$ be the Faltings delta invariant of the compact Riemann surface C (Theorem 1.3.9) and $\delta'(C) \coloneqq \delta(C) - 4g\log(2\pi)$, then we define*

$$\lambda(C) \coloneqq \frac{g-1}{6(2g+1)}\varphi(C) + \frac{1}{12}\delta'(C), \tag{1.18}$$

where the definition of $\varphi(C)$ can be found in Theorem 1.5.6.

After replacing k by a sufficiently large extension, we can assume $(2g-2)\xi = K_X$ for some $\xi \in \mathrm{Pic}(X)$. The height $\langle \Delta_\alpha, \Delta_\alpha \rangle$ reaches its minimal value precisely when α and ξ are equal up to a torsion divisor (according to the non-negativity of the Néron-Tate height). The cycle Δ_ξ is known as a *canonical Gross-Schoen cycle* of X. The image of Δ_ξ in $\mathrm{CH}^2(X^3)_{\mathbb{Q}}$ does not depend on the choice of ξ, thus the number $\langle \Delta_\xi, \Delta_\xi \rangle$ is well-defined.

Corollary 1.5.8. *When k is the function field of a smooth curve B, we can rewrite $\langle \Delta_\xi, \Delta_\xi \rangle$ in the following way*

$$\langle \Delta_\xi, \Delta_\xi \rangle = \frac{2g+1}{2g-2}\left((\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) - \sum_{s \in M(k)_0} \psi(\overline{\Gamma}_s)\log N(s) \right), \tag{1.19}$$

where s runs over all closed points of B and ψ is defined to be

$$\psi(\overline{\Gamma}) \coloneqq \epsilon(\overline{\Gamma}) + \frac{2g-2}{2g+1}\varphi(\overline{\Gamma}). \tag{1.20}$$

Proof. This comes from Theorem 1.5.3 and Theorem 1.5.6. □

When k is a function field of characteristic 0, the height $\langle \Delta_\xi, \Delta_\xi \rangle$ is non-negative by the Hodge index theorem. When k is a number field or a function field with positive characteristic, we have the following conjecture (Conjecture 1.4.1 in [69]).

Conjecture 1.5.9. *Let k be a number field or a function field with positive characteristic, then*

$$\langle \Delta_\xi, \Delta_\xi \rangle \geq 0$$

with equality if and only if Δ_ξ is rationally equivalent to 0.

B. Gross and C. Schoen's work shows that the height vanishes when the curve is of genus 0, 1 or hyperelliptic (including genus 2). This thesis is mainly about the height for genus 3 curves. Is it unbounded (Theorem 1.5.17, Section 3.3), can it be explicitly computed (Chapter 4)?

Remark 1.5.10. *In [69], S. Zhang asked when will the height be zero. The height of a canonical Gross-Schoen cycle on a hyperelliptic curve vanishes (Proposition 4.8 in [30]). In Section 3.2, we explain a result of K. Yamaki which partially answers the converse of this problem for genus 3 curves.*

Now we shift our attention to the finiteness property of $\langle \Delta_\xi, \Delta_\xi \rangle$. For a variety T over a field k and a geometric point $t : \mathrm{Spec}(\overline{k}) \to T$, there exists a minimal finite extension k_0 of k such that t factors through $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k_0)$. The integer $\deg(t) := [k_0 : k]$ is well-defined, and we have the following theorem by S. Zhang.

Theorem 1.5.11. *Let $Y \to T$ be a smooth and projective family of curves of genus $g \geq 3$ over a projective variety T over a number field k, or the function field of a curve over a finite field. If the classifying map $T \to M_g$ from T to the coarse moduli space of genus g smooth curves over k is finite, then we have a Northcott property: for any positive numbers D and H,*

$$\# \left\{ t \in T(\overline{k}) : \deg(t) \leq D, \frac{\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle}{\deg(t)} \leq H \right\} < \infty. \tag{1.21}$$

Proof. See Theorem 1.3.5 in [69]. □

Remark 1.5.12. *In the theorem above, we use a different convention from that of S. Zhang. In Theorem 1.3.5 in [69], Zhang denoted $\frac{\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle}{\deg(t)}$ simply by $\langle \Delta_\xi(Y_t), \Delta_\xi(Y_t) \rangle$.*

Remark 1.5.13. *For a stable curve $q : \mathcal{X} \to S$ of genus $g \geq 2$ where S is either a smooth curve over a field or the spectrum of a number ring, there is a height associated to the Ceresa cycle $c(\mathcal{X}/S)$. We have the following relation between the two heights*

$$c(\mathcal{X}/S) = \frac{2}{3} \langle \Delta_\xi, \Delta_\xi \rangle. \tag{1.22}$$

See Theorem 1.5.6 in [69] for more details.

In the remaining part of this subsection, we prove the unboundedness of the height $\langle \Delta_\xi, \Delta_\xi \rangle$ for genus $g \geq 3$ curves over number fields.

Lemma 1.5.14. *If $g > 2$, then there exists a finite morphism of schemes $C \to M_g$, where C is a smooth curve defined over k.*

Proof. We denote the coarse moduli space of principally polarized Abelian varieties by A_g. The Torelli map $M_g \to A_g$ is an immersion (Corollary 1.5 and the remark after it in [56]). We get a projective compactification $\tilde{M}_g$ for M_g in the Satake compactification A_g^{sc} of A_g by taking the closure of its image. The boundary $\tilde{M}_g \backslash M_g$ has codimension ≥ 2, since $A_g^{st} = A_g \sqcup A_{g-1} \cdots A_1 \sqcup A_0$ and $\dim(A_m) = \frac{m(m+1)}{2}$.

We can then get an irreducible curve T in M_g by cutting out sufficiently many hypersurfaces in general position (we might need to choose an irreducible component). The induced morphism $T \to M_g$ is a closed immersion by the construction, thus it is also finite. We write $C \to T$ for the normalization of T. Then we have a finite morphism $C \to M_g$ since it is the composition of finite morphisms $C \to T$ and $T \to M_g$. ☐

Remark 1.5.15. *The Satake compactification and the Torelli map can be defined over k (even over $\mathrm{Spec}(\mathbb{Z})$). See Page 179 in [10] and Page 150 in [24] for details. Explicit curves on M_g for $g > 2$ can be found in [28] and [68].*

Lemma 1.5.16. *Let Z be an irreducible smooth projective variety of positive dimension defined over a number field k. There exists a sufficiently large integer d such that there are infinitely many closed points on Z whose degree is less than d.*

Proof. We first fix a closed embedding $Z \to \mathbb{P}_k^n$. By Bertini's theorem, we can find $\dim(Z) - 1$ hyperplanes in $\mathbb{P}_k^n$ whose intersection with Z is a 1-dimensional smooth projective variety. We choose one irreducible component if there are more than one. Thus, we just need to prove the lemma when is Z a smooth curve.

Every non-zero rational function f on Z gives a morphism $Z \to \mathbb{P}_k^1$. We denote the degree of this morphism by d_f. The fiber of every k-point in $\mathbb{P}_k^1$ is an effective divisor of Z of degree d_f, thus every closed point in the divisor is of degree not bigger than d_f. Since there are infinitely many k-points in $\mathbb{P}_k^1$, we can obtain infinitely many closed points on Z whose degree is not bigger than d_f. ☐

Theorem 1.5.17. *Let $g \geq 3$ be an integer. There exists an integer D_g and a family of genus g smooth curves $\{E_j\}_{j \in \mathbb{N}^+}$ defined over $\overline{\mathbb{Q}}$ such that*

(1) For all $j \in \mathbb{N}^+$, the curve E_j has semistable model over a number field k_j such that $[k_j : \mathbb{Q}] \leq D_g$,

(2) the normalized height of the canonical Gross-Schoen cycle on E_j, which is defined as $\frac{\langle \Delta_j, \Delta_j \rangle}{[k_j : \mathbb{Q}]}$, goes to infinity.

Proof. We can obtain a finite morphism $C \to M_g$ by Lemma 1.5.14, where C is a smooth curve over k. According to Lemma 1.5.16, there exists an integer D_g such that there are infinitely many points on C whose degree is smaller than D_g. Now we can prove the assertion by applying the Northcott property in Theorem 1.5.11. ☐

So far, we know that $\langle \Delta_\xi, \Delta_\xi \rangle$ can be 0 and can be arbitrarily large. Nobody has yet numerically computed $\langle \Delta_\xi, \Delta_\xi \rangle$ for a non-hyperelliptic curve of genus $g \geq 3$. In Chapter 4, we will numerically compute $\langle \Delta_\xi, \Delta_\xi \rangle$ for a specific plane quartic curve over $\mathbb{Q}$.

For simplicity, we will mainly use $\langle \Delta, \Delta \rangle$ to denote the height of a canonical Gross-Schoen cycle from now on.

Chapter 2

Arithmetic and geometric properties of genus 3 curves

In this chapter, we study geometric and arithmetic properties of genus 3 curves. In Section 2.1, we recall general notions and results. In Section 2.2, we discuss properties of χ'_{18}, including C. Ritzenthaler's work on Klein's formula. We will freely use the moduli language.

2.1 General background

In Subsection 2.1.1, we explain the classification of stable curves of genus 3. In Subsection 2.1.2, we explain the relation between various kinds of modular forms and state the Torelli theorem. In Subsection 2.1.3, we recall some notions in invariant theory. In Subsection 2.1.4, we introduce bitangents of plane quartic curves, and explain their relation with semicanonical divisors and theta characteristics.

The modular form χ_h defined in Equation (2.3) will play an important role in Section 2.2 and Section 3.3. Corollary 2.1.20 will be used to evaluate $\|\theta\|_{g-1}$ in Section 4.5.

2.1.1 Classification and moduli

We begin with a simple classification of smooth curves of genus 3 over an algebraically closed field. Most statements in this subsection can be found in [19].

Proposition 2.1.1. *Let k be an algebraically closed field. A non-hyperelliptic smooth curve of genus 3 over k always has a plane quartic model in the projective plane $\mathbb{P}^2$.*

Proof. See Page 519 in [63]. $\square$

We have the following models representing smooth genus 3 curves over an algebraically closed field k. When $\operatorname{char} k \neq 2$, hyperelliptic curves of genus 3 have the following affine model

$$C : y^2 = \prod_{i=1}^{7}(x - c_i), \quad \text{where } c_i \in k$$

while when $\operatorname{char} k = 2$ (Theorem 7.4.24 in [48]), they have the following affine model

$$C : y^2 + f(x)y = g(x)$$

with

$$7 \leq \max\{2 \deg f(x), \deg g(x)\} \leq 8.$$

Plane quartic curves over k can be expressed as

$$\sum_{l+m+n=4} c_{lmn} X^l Y^m Z^n = 0,$$

where $c_{lmn} \in k$.

Example 2.1.2. *(Klein quartic) The plane curve defined by $X^3Y + Y^3Z + Z^3X = 0$ is called the Klein quartic curve. As a compact Riemann surface, it has 168 automorphisms. As a curve over $\mathbb{Z}$, it has potentially good reduction at 7 (Page 81 in [20]).*

We write $\mathcal{M}_3$ (resp. M_3) for the moduli stack (resp. coarse moduli space) of smooth genus 3 curves. Similarly, we write $\overline{\mathcal{M}}_3$ (resp. $\overline{M}_3$) for the moduli stack (resp. coarse moduli space) of stable curves of genus 3.

According to Theorem 3.19 and Theorem 5.1 in [58], we have the following results. The moduli space $\overline{\mathcal{M}}_3$ is an algebraic stack over $\operatorname{Spec}(\mathbb{Z})$ of relative dimension 6, which contains $\mathcal{M}_3$ as an open substack.

Singular curves of genus 3 make up a divisor Δ in $\overline{\mathcal{M}}_3$, which can be decomposed as

$$\Delta = \Delta_0 \cup \Delta_1,$$

where Δ_0 denotes the closure of the irreducible singular curves of geometric genus 2 with exactly one nodal point, and Δ_1 denotes the closure of reducible curves with exactly two components of genus 1 and 2. Both Δ_0 and Δ_1 are prime divisors of $\overline{\mathcal{M}}_3$. General statements for higher genus g can be found in Page 411 in [23].

The hyperelliptic locus H in $\mathcal{M}_3$ is an irreducible algebraic stack of codimension 1 (Theorem 2.1 in [26]). Let $\overline{H}$ be the closure of H in $\overline{\mathcal{M}}_3$.

2.1.2 Modular forms and the Torelli theorem

The main references for this subsection are [13] and [43]. We assume the integer $g \geq 3$ in this subsection.

Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian schemes of relative dimension g and denote by $p : \mathcal{U}_g \to \mathcal{A}_g$ the universal abelian variety. Let $\Omega_{\mathcal{U}_g/\mathcal{A}_g}$ denote the sheaf of relative 1-forms of p. Then we get a rank g vector bundle $\mathcal{E} = p_*\Omega_{\mathcal{U}_g/\mathcal{A}_g}$ (known as the *Hodge bundle*), and its determinant $\mathcal{L} = \det p_*\Omega_{\mathcal{U}_g/\mathcal{A}_g}$ on $\mathcal{A}_g$.

Definition 2.1.3. *An algebraic Siegel modular form of genus g and weight $h \in \mathbb{Z}_{>0}$ over a commutative ring R is an element of the R-module*

$$S_{g,h}(R) = \Gamma(\mathcal{A}_g \otimes R, \mathcal{L}^{\otimes h}).$$

Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be the universal smooth curve of genus g. We have a vector bundle $\mathcal{E}_\pi = \pi_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ and an invertible bundle $\mathcal{L}_\pi = \det \pi_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ on $\mathcal{M}_g$ associated to π.

Definition 2.1.4. *A Teichmüller modular form of genus g and weight h over R is an element of the R-module*

$$T_{g,h}(R) = \Gamma(\mathcal{M}_g \otimes R, \mathcal{L}_\pi^{\otimes h}).$$

For a ring homomorphism $R_1 \to R_2$, elements in $S_{g,h}(R_1)$ (resp. $T_{g,h}(R_1)$) can be mapped to elements in $S_{g,h}(R_2)$ (resp. $T_{g,h}(R_2)$). Thus it makes sense to ask if a modular form in $S_{g,h}(R_2)$ (resp. $T_{g,h}(R_2)$) can be lifted to an element in $S_{g,h}(R_1)$ (resp. $T_{g,h}(R_1)$). In Lemma 2.1.9, we will find that the modular form $\chi_h(\tau)$ in $S_{g,h}(\mathbb{C})$ can be lifted to an element in $S_{g,h}(\mathbb{Z})$ (denoted by χ'_h) with respect to the ring homomorphism $\mathbb{Z} \to \mathbb{C}$.

Now we take R to be a field k. For a principally polarized abelian variety $(A, a) \in \mathcal{A}_g(k)$ of dimension g over k (resp. a smooth curve C of genus g over k), we denote by

$$\omega_k[A] \coloneqq \bigwedge^g H^0(A, \Omega_{A/k}) \quad (\text{resp. } \lambda_k[C] \coloneqq \bigwedge^g H^0(C, \Omega_{C/k}))$$

the k-vector space of global sections of $\mathcal{L}$ (resp. $\mathcal{L}_\pi$) over (A, a) (resp. C). For $f \in S_{g,h}(k)$ (resp. $f \in T_{g,h}(k)$) and a basis ω of $\omega_k[A]$ (resp. a basis λ of $\lambda_k[C]$), we put

$$f((A, a), \omega) = f(A, a)/\omega^{\otimes h} \in k, \quad (\text{resp. } f(C, \lambda) = f(C)/\lambda^{\otimes h} \in k). \tag{2.1}$$

This sends a algebraic Siegel modular form (resp. Teichmüller modular form) to a k-valued function on $\mathcal{A}_g(k)$ (resp. $\mathcal{M}_g(k)$).

The map $\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g$ sending every smooth curve C of genus g to its Jacobian with the canonical polarization $(\mathrm{Jac}(C), j)$ is known as the Torelli map. This gives a translation from $S_{g,h}(k)$ to $T_{g,h}(k)$.

Lemma 2.1.5. *The Torelli map* $\mathfrak{t}$ *satisfies* $\mathfrak{t}^*\mathcal{L} = \mathcal{L}_\pi$ *and induces a linear map*

$$\mathfrak{t}^* : S_{g,h}(k) = \Gamma(\mathcal{A}_g \otimes k, \mathcal{L}^{\otimes h}) \to T_{g,h}(k) = \Gamma(\mathcal{M}_g \otimes k, \mathcal{L}_\pi^{\otimes h})$$

for any field k*.*

Proof. See Section 2.1 in [35]. □

On Page 89 in [42], we can find the following precise form of the Torelli theorem.

Theorem 2.1.6. *Let* (A, a) *be a principally polarized abelian variety of dimension* $g \geq 1$ *over a field* k*. We assume* (A, a) *is isomorphic over* $\overline{k}$ *to the Jacobian of a curve* X_0 *of genus* g *defined over* $\overline{k}$*. Then the following holds :*

(1) If X_0 *is hyperelliptic, then there is a curve* X/k *isomorphic to* X_0 *over* $\overline{k}$ *such that* (A, a) *is* k*-isomorphic to* $(\mathrm{Jac}X, j)$ *where* j *is the canonical polarization.*

(2) If X_0 *is not hyperelliptic, there is a curve* X/k *isomorphic to* X_0 *over* $\overline{k}$*, and a quadratic character*

$$\varepsilon : \mathrm{Gal}(k_{\mathrm{sep}}/k) \longrightarrow \{\pm 1\}$$

such that the twisted abelian variety $(A, a)_\varepsilon$ *(see X.5 in [60] for the explanation of 'twisted') is* k*-isomorphic to* $(\mathrm{Jac}X, j)$*. The character* ε *is trivial if and only if* (A, a) *is* k*-isomorphic to a Jacobian.*

Now we shift our attention to the case $k = \mathbb{C}$. Let $\mathbb{H}_g := \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) \mid {}^t\tau = \tau, \mathrm{Im}\tau > 0\}$ be the Siegel upper half space of genus g.

Definition 2.1.7. *An analytic Siegel modular form of genus* g *and weight* h *is a complex holomorphic function* $\phi(\cdot)$ *on* $\mathbb{H}_g$ *satisfying*

$$\phi(M\tau) = \det(c\tau + d)^h \cdot \phi(\tau),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *for matrices* a, b, c, $d \in \mathrm{Mat}(g \times g, \mathbb{Z})$*, and*

$$M\tau := (a\tau + b)(c\tau + d)^{-1}.$$

We denote the $\mathbb{C}$*-vector space of such functions by* $R_{g,h}$*.*

There is a complex torus over $\mathbb{H}_g$ given by

$$\mathbb{U}_g := \frac{\mathbb{H}_g \times \mathbb{C}^g}{((\tau_1, z) \sim (\tau_2, z_2) \text{ if and only if } \tau_1 = \tau_2 \text{ and } z_1 - z_2 \in \mathbb{Z}^g + \tau_1 \mathbb{Z}^g)}.$$

We have a map of complex manifolds $u : \mathbb{H}_g \to \mathcal{A}_g(\mathbb{C})$ and an isomorphism

$$\mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z}) \xrightarrow{\sim} \mathcal{A}_g(\mathbb{C}).$$

The map u induces an isomorphism between $\mathbb{U}_g$ and the pull-back of $\mathcal{U}_g(\mathbb{C})$ along u. The tangent space along the unit section of $\mathbb{U}_g \to \mathbb{H}_g$ is canonically identified with $\mathbb{C}^g$, giving a trivialization on the Hodge bundle $\tilde{\mathcal{E}} = q_*\Omega_{\mathbb{U}_g/\mathbb{H}_g}$ on $\mathbb{H}_g$ by the frame

$$(d\zeta_1/\zeta_1, \dots, d\zeta_g/\zeta_g) = (2\pi i dz_1, \dots, 2\pi i dz_g),$$

where $\zeta_i = \exp(2\pi i z_i)$. Then the line bundle $\tilde{\mathcal{L}} = \det \tilde{\mathcal{E}}$ is trivialized by the frame $\omega = \frac{d\zeta_1}{\zeta_1} \wedge \cdots \wedge \frac{d\zeta_g}{\zeta_g} = (2\pi i)^g (dz_1 \wedge \cdots \wedge dz_g)$. See Pages 141-142 in [24] for details.

Proposition 2.1.8. *We write (A_τ, a_τ) for a principally polarized complex abelian variety with the period matrix τ. Let $f \in S_{g,h}(\mathbb{C})$ and let $\tilde{f}$ be the following $\mathbb{C}$-valued function on $\mathbb{H}_g$*

$$\tilde{f}(\tau) \coloneqq (2\pi i)^{-gh} f(A_\tau, a_\tau)/(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h},$$

where $(z_1, \dots, z_g)$ is the canonical basis of $\mathbb{C}^g$. The map $f \to \tilde{f}$ induces an isomorphism $S_{g,h}(\mathbb{C}) \simeq R_{g,h}$.

Proof. See Page 141 in [24]. $\square$

We denote the subset of $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ containing exactly all elements $\epsilon = (a', a'')$ such that $4a' \cdot a'' \equiv 0 (\mathrm{mod}\, 2)$ by S_g. We take $h = \frac{\# S_g}{2}$ and define a holomorphic function on $\mathbb{H}_g$ by

$$\tilde{\chi}_h(\tau) \coloneqq \frac{(-1)^{gh/2}}{2^{2^{g-1}(2^g-1)}} \cdot \prod_{\epsilon \in S_g} \theta_\epsilon(0, \tau), \tag{2.2}$$

where

$$\theta_\epsilon(z, \tau) \coloneqq \sum_{n \in \mathbb{Z}^g} \exp(\pi i\, {}^t(n+a')\tau(n+a') + 2\pi i\, {}^t(n+a')(z+a'')), \quad z \in \mathbb{C}^g.$$

Under the assumption $g \geq 3$, we have $\tilde{\chi}_h \in R_{g,h}$ (Lemma 10 in [37]). By Proposition 2.1.8, this corresponds to a algebraic Siegel modular form

$$\chi_h(A_\tau) \coloneqq (2\pi i)^{gh} \cdot \tilde{\chi}_h(\tau)(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h} \in S_{g,h}(\mathbb{C}). \tag{2.3}$$

By Lemma 2.1.5, we can get a Teichmüller modular form in $T_{g,h}(\mathbb{C})$. Actually, we have the following result.

Lemma 2.1.9. *The algebraic Siegel modular form χ_h is a primitive (not congruent to 0 modulo p for any prime p) element in $S_{g,h}(\mathbb{Z})$. Moreover, there exists a Teichmüller modular form $\mu_{h/2} \in T_{g,h/2}(\mathbb{Z})$ such that*

$$\mathsf{t}^*(\chi_h) = (\mu_{h/2})^2.$$

Proof. See Proposition 3.4 in [35] and Proposition 4.5 in [36]. □

Remark 2.1.10. *To distinguish from the modular form χ_h in $S_{g,h}(\mathbb{C})$, we denote this modular form in $S_{g,h}(\mathbb{Z})$ by χ'_h.*

Let (A, a) be a principally polarized complex abelian variety with a fixed basis of differential 1-forms $\{\omega_i\}_{1\leq i\leq g}$ and a symplectic homology basis of $H_1(A, \mathbb{Z})$. We can obtain a period matrix $(\Omega_1|\Omega_2)$ by taking the integration of the differential forms along the homology basis. Using the notations in Equation (2.1) and Proposition 2.1.8, we have the following proposition.

Proposition 2.1.11. *Let f be an algebraic Siegel modular form in $S_{g,h}(k_0)$ for some subfield $k_0 \subset \mathbb{C}$. Let $\omega = \omega_1 \wedge \cdots \wedge \omega_g \in \omega_{k_0}[A]$. Then*

$$f((A, a), \omega) = (2\pi i)^{gh} \frac{\tilde{f}(\tau)}{\det \Omega_1^h}.$$

Proof. See Proposition 1.2.4 in [43]. □

2.1.3 Notions in invariant theory

Let d be a positive integer. Let L be an algebraically closed field such that d is invertible in L. Let V be an n-dimensional vector space over L. We have the following two interpretations of $X_d := \mathrm{Sym}^d(V^*)$ which we will use freely.

(1) Fixing a basis $v = (v_1, \ldots, v_n)$ of V, elements in $\mathrm{Sym}^d(V^*)$ can be considered as degree d homogeneous polynomials (or d-forms) in $k[x_1, \ldots, x_n]$ where $x_i(v_j) = \delta_{ij}$.

(2) We can also consider X_d as an affine scheme which is isomorphic to $\mathbb{A}^{\mathfrak{d}}$, where $\mathfrak{d} = \dim(\mathrm{Sym}^d(V^*))$.

We define an action of $GL(V)$ (resp. $SL(V)$) on X_d by

$$r(s) : F(x_1, \ldots, x_n) \to (s \cdot F)(x_1, \ldots, x_n) = F(s(x_1, \ldots, x_n))$$

for $s \in GL(V)$ (resp. $SL(V)$). This induces a natural action of $GL(V)$ (or $SL(V)$) on regular (or rational) functions on X_d.

Definition 2.1.12. *Let U be a Zariski open set of X_d that is stable under the action of $SL(V)$. An element Ψ of $\mathcal{O}(U)$ is called an invariant on U if $\Psi = s \cdot \Psi$ for all $s \in SL(V)$. We denote the subspace of $\mathcal{O}(U)$ consisting of homogeneous invariants of degree h by $\mathrm{Inv}_h(U)$.*

If $\Psi \in \mathcal{O}(U)$, and if w and h are integers such that $hd = nw$, then $\Psi \in \mathrm{Inv}_h(U)$ if and only if

$$s \cdot \Psi = (\det s)^w \Psi \quad \text{for every } s \in GL(V),$$

where w is called the *weight* of Ψ (Section 2.1 in [43]).

In the following part of this subsection, we assume $n = 3$. Let $\mathfrak{I}_d$ be the set of all non-negative integer tuples (c_1, c_2, c_3) such that $c_1 + c_2 + c_3 = d$. Let $\mathrm{Res}(\cdot)$ be the multivariate resultant (Theorem IX.3.5 in [45]). We write x for the tuple (x_1, x_2, x_3) and write $x^{(c_1,c_2,c_3)}$ for the monomial $x_1^{c_1} x_2^{c_2} x_3^{c_3}$. We call the polynomial $\mathfrak{P} := \sum_{I \in \mathfrak{I}_d} c_I x^I$ over $L[c_I]_{I \in \mathfrak{I}_d}$ the *universal ternary form* of degree d. The polynomial

$$\mathrm{Disc}_{\mathfrak{P}} := d^{-((d-1)^n-(-1)^n)/d} \mathrm{Res}\left(\frac{\partial \mathfrak{P}}{\partial x_1}, \frac{\partial \mathfrak{P}}{\partial x_2}, \frac{\partial \mathfrak{P}}{\partial x_3}\right) \tag{2.4}$$

in $L[c_I]_{I \in \mathfrak{I}_d}$ has the property that its zero locus classifies exactly all non-smooth plane curves of degree d (Section 2.2 in [42]). For a specific ternary form F of degree d, we write $\mathrm{Disc}(F)$ for the value of $\mathrm{Disc}_{\mathfrak{P}}$ at F.

By the *universal plane curve* of degree d over X_d, we mean the variety

$$\mathfrak{U}_d := \{(F, \mathfrak{x}) \in X_d \times \mathbb{P}^2 |\, F(\mathfrak{x}) = 0\}.$$

We write X_d^0 for the Zariski open set

$$X_d^0 := (X_d)_{\mathrm{Disc}_{\mathfrak{P}}} = \{F \in X_d |\, \mathrm{Disc}(F) \neq 0\}.$$

of X_d. We write $\mathfrak{U}_d^0$ for the universal curve over the non-singular locus X_d^0 with the smooth projection map

$$\mathfrak{U}_d^0 \to X_d^0.$$

Explicitly speaking, *invariants* for ternary quartic forms ($d = 4$, $n = 3$) are polynomials in 15 coefficient variables that are stable under the action of $SL_3(L)$ (this is compatible with Definition 2.1.12). The discriminant is an invariant of degree 27 (Section 7 in [21]).

2.1.4 Bitangents

A plane smooth quartic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ intersects a straight line $l \subset \mathbb{P}^2$ at 4 points, counted with multiplicity (Bézout's theorem). We say l is a *bitangent* of C if l is tangent to C at two distinct points. The following theorem was proven by J. Plücker in [57].

Theorem 2.1.13. *Every smooth plane quartic curve over $\mathbb{C}$ has precisely 28 bitangent lines.*

Remark 2.1.14. *Theorem 2.1.13 also holds for plane quartics over a separably closed field k with* $\operatorname{char} k \neq 2$.

The following result of D. Lehavi implies a close relation between plane quartics and their bitangents.

Theorem 2.1.15. *Every smooth plane quartic curve over $\mathbb{C}$ can be reconstructed from its bitangents.*

Proof. See Theorem 1.4 in [46]. □

Now we consider plane quartics over a general separably closed field k with $\operatorname{char} k \neq 2$. Let $f : C \hookrightarrow \mathbb{P}^2$ be a smooth plane quartic over k.

Lemma 2.1.16. *The effective canonical divisors on C are exactly the divisors $(C \cdot L)$, the intersection of C and L, for arbitrary lines $L \subset \mathbb{P}^2$.*

Proof. This comes from the fact that $\Omega_C \simeq \mathcal{O}_C(1) = f^*\mathcal{O}(1)$ for plane smooth quartics. □

Definition 2.1.17. *A theta characteristic on a smooth plane quartic curve C is a line bundle L on C such that $L \otimes L \simeq \Omega_C$. A theta characteristic is said to be odd (resp. even) if $h^0(C, L)$ is odd (resp. even). We denote the set of odd theta characteristics of C by $OT(C)$.*

We have the following well-known correspondence (see Page 289 in [31]).

Theorem 2.1.18. *There is a canonical bijection of bitangents of a smooth plane quartic C and odd theta characteristics of C given by*

$$L \to \frac{1}{2}(C \cdot L).$$

Proof. Let L be a bitangent of C, then the divisor $F := \frac{1}{2}(C \cdot L)$ is a theta characteristic by Lemma 2.1.16. Since F is effective, we have $h^0(C, \mathcal{O}(F)) \geq 1$. Since $F = \Omega_C(-F)$ and $\deg(F) = 2$, we get $h^0(C, \mathcal{O}(F)) \leq 1$ by Clifford's theorem (Theorem IV.5.4 in [33]). Thus $h^0(C, \mathcal{O}(F)) = 1$ and F is an odd characteristic. It remains to prove that this is a bijection.

Let D be an odd theta characteristic on C. Since we have $h^0(C, \mathcal{O}(D)) > 0$, the linear system $|D|$ is non-empty with an effective representative $E = P + Q$. Since $2E$ is canonical, we have $2E = (C \cdot L)$ for some line L by Lemma 2.1.16. This proves the surjectivity in the theorem.

Suppose bitangents L_1 and L_2 give the same theta characteristic, then we have $L_1 \cap C = 2(P + Q)$ and $L_2 \cap C = 2(R + S)$ for points P, Q, R and S on C such that $\{P, Q\} \neq \{R, S\}$. Thus $P + Q - R - S = \mathrm{div}(g)$ for some rational function g on C. This is impossible, otherwise g gives an hyperelliptic map $C \to \mathbb{P}^1$. This proves the injectivity in the theorem. $\square$

Now we shift our attention to $\mathrm{Jac}(C)$, an abelian variety of dimension 3. We denote the group of 2-torsion k-points of $\mathrm{Jac}(C)$ by $\mathrm{Jac}(C)[2]$. Since $\mathrm{char}\, k \neq 2$, $\mathrm{Jac}(C)[2]$ is isomorphic to $\mathbb{F}_2^{\oplus 6}$. We have the Weil pairing

$$\langle \cdot, \cdot \rangle_W : \mathrm{Jac}(C)[2] \times \mathrm{Jac}(C)[2] \to \mathbb{F}_2.$$

There exists a symplectic basis $\{g_1, g_2, g_3, h_1, h_2, h_3\}$ of $\mathrm{Jac}(C)[2]$ such that

$$\langle g_i, g_j \rangle_W = \langle h_i, h_j \rangle_W = 0$$

and

$$\langle g_i, h_j \rangle_W = \delta_{i,j}.$$

We call $Q : \mathrm{Jac}(C)[2] \to \mathbb{F}_2$ a *quadratic form with polar form* $\langle \cdot, \cdot \rangle_W$ if

$$Q(x + y) - Q(x) - Q(y) = \langle x, y \rangle_W \quad \text{for all } x, y \in \mathrm{Jac}(C)[2].$$

We denote the set of quadratic forms with polar form $\langle \cdot, \cdot \rangle_W$ by $\mathbf{T}_C$. Then the *Arf invariant* of an element $Q(\cdot)$ in $\mathbf{T}_C$ is

$$\mathrm{Arf}(Q) := \sum_{1 \leq i \leq 3} Q(g_i) Q(h_i) \in \mathbb{F}_2,$$

which is independent on the choice of the symplectic basis. The set of quadratic forms with polar form $\langle \cdot, \cdot \rangle_W$ forms a torsor over $\mathrm{Jac}(C)[2]$. This structure is defined by

$$(Q + \eta)(x) = Q(x) + \langle x, \eta \rangle_W = Q(x + \eta) + Q(\eta)$$

for $Q(\cdot) \in \mathbf{T}_C$ and $\eta \in \mathrm{Jac}(C)[2] \simeq \mathbb{F}_2^{\oplus 6}$.

We denote the subset of $\mathbf{T}_C$ consisting of quadratic forms of Arf invariant 0 (resp. 1) by $\mathbf{E}_C$ (resp. $\mathbf{O}_C$). The set $\mathbf{E}_C$ (resp. $\mathbf{O}_C$) contains 36 (resp. 28) elements. The symplectic group $\mathrm{Sp}_6(\mathbb{F}_2)$ gives a natural action on $\mathbf{O}_C$ and $\mathbf{E}_C$, which is also transitive.

Theorem 2.1.19. *There is a canonical bijection between the set* $\mathbf{O}_C$ *and the set of bitangents of* C*.*

Proof. See Proposition 6.2 in [39] and the end of Section 2 in [40]. $\square$

Corollary 2.1.20. *We have a* 1-1 *correspondence among the three sets*

$$\mathrm{Bitangents}(C) \leftrightarrow \mathbf{O}_C \leftrightarrow OT(C)$$

Proof. This is a trivial corollary from Theorem 2.1.18 and Theorem 2.1.19. □

Remark 2.1.21. *The set of theta characteristics for complex smooth curves of genus g has a bijection to the set $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$. The even characteristics correspond to elements $(a,b) \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ such that $4 \times (a \cdot b) \equiv 0 \bmod 2$, and the odd characteristics correspond to other elements in $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$. This interpretation appears in Riemann's theta function with characteristic $\theta_\epsilon(z,\tau)$, which we already used in the definition of S_g in Equation (2.2).*

We give two examples of plane quartics with special behaviour of their bitangents.

Example 2.1.22. *Let k be a field with $\operatorname{char} k \neq 2, 7$. Let ζ be a primitive 7-th root of unity in k^{sep}. We set $\epsilon_1 \coloneqq \zeta + \zeta^{-1}$, $\epsilon_2 = \zeta^2 + \zeta^{-2}$, $\epsilon_3 \coloneqq \zeta^4 + \zeta^{-4}$. Then the 28 bitangents of the Klein curve (Example 2.1.2) over k^{sep} are*

$$\begin{aligned} l_{0,j} &: Z = -\zeta^j Y - \zeta^{3j} X, \\ l_{1,j} &: Z = -\zeta^j \epsilon_1^2 Y - \zeta^{3j} \epsilon_3^{-2} X, \\ l_{2,j} &: Z = -\zeta^j \epsilon_2^2 Y - \zeta^{3j} \epsilon_1^{-2} X, \\ l_{3,j} &: Z = -\zeta^j \epsilon_3^2 Y - \zeta^{3j} \epsilon_2^{-2} X, \end{aligned}$$

where $j = 0, 1, \ldots, 6$.

Example 2.1.23. *The following plane quartic over $\mathbb{Q}$ has 28 bitangents over $\mathbb{Q}$.*

$$\begin{aligned} &3X^3Z + X(Y^3 - 11054979YZ^2 - 148224431 34Z^3) + 38Y^4 + 243542Y^3Z \\ &+631949994Y^2Z^2 + 822588784146YZ^3 + 4605878924287 44Z^4 = 0 \end{aligned}$$

Details can be found in (6.6) *in [59].*

We end this subsection with a short discussion of the case $\operatorname{char} k = 2$.

When $\operatorname{char} k = 2$ and $k = k^{\text{sep}}$, the dimension r of $\operatorname{Jac}(C)[2]$ over $\mathbb{F}_2$ and number l of bitangents satisfy the following condition: $r = \lfloor l/2 \rfloor$ with $l \in \{1, 2, 4, 7\}$. See Page 60 in [62] for details.

Example 2.1.24. *Let k be an algebraically closed field with $\operatorname{char} k = 2$. Then all smooth plane quartic curves over k with only 1 bitangent can be represented as*

$$(aX^2 + bY^2 + cZ^2 + dXY + eYZ + fZX)^2 = X(Y^3 + X^2Z),$$

where $c \in k^*$. See Proposition 2.1 in [54] for details.

2.2 χ'_{18} and Klein's formula

In Subsection 2.2.1, we show how χ'_{18} behaves on $\overline{\mathcal{M}}_3$, and define the Hodge metric on $\det q_*\Omega_{\mathbb{U}_g/\mathbb{H}_g}$. In Subsection 2.2.2, we talk about the Klein's formula for plane quartics. The main references for this section are [13] and [43].

We will use Proposition 2.2.4 and Equation (2.7) to compute $\mathrm{ord}_v(\chi'_{18})$ at finite places and $\|\chi'_{18}\|_{\mathrm{Hdg}}$ at the infinite place in Section 4.4.

2.2.1 Moduli property of χ'_{18}

In this section we assume $g \geq 2$ (we will specialise to $g = 3$ soon) and use the notation introduced in Subsection 2.1.2. Let $\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g$ be the Torelli map. For the universal stable curve $\overline{\pi} : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$, we can define vector bundles $\mathcal{E}_{\overline{\pi}}$ and $\mathcal{L}_{\overline{\pi}}$. For a stable curve $f : \mathcal{X} \to S$ over a scheme S, we denote the pullback of $\mathcal{E}_{\overline{\pi}}$ (resp. $\mathcal{L}_{\overline{\pi}}$) along the classifying map $J : S \to \overline{\mathcal{M}}_g$ by $\mathcal{E}_f$ (resp. $\mathcal{L}_f$).

By Lemma 2.1.5, there are natural isomorphisms $\mathcal{E}_\pi \simeq \mathfrak{t}^*\mathcal{E}$ and $\mathcal{L}_\pi \simeq \mathfrak{t}^*\mathcal{L}$. We can get a algebraic Siegel modular form $\chi'_{18} \in S_{3,18}(\mathbb{C})$ by taking $g = 3$ and $h = 18$ in Equation (2.3). By Lemma 2.1.9, this can be lifted to a modular form in $S_{3,18}(\mathbb{Z})$ which we denote by χ'_{18}. The pullback of χ'_{18} along the Torelli map gives a Teichmüller modular form in $T_{3,18}(\mathbb{Z})$ which we also denote by χ'_{18}. Now χ'_{18} can be considered as a global section of the line bundle $\mathcal{L}_\pi^{\otimes 18}$ on $\mathcal{M}_3$ and a rational section of the line bundle $\mathcal{L}_{\overline{\pi}}^{\otimes 18}$ on $\overline{\mathcal{M}}_3$.

Lemma 2.2.1. *The divisor of χ'_{18} on $\mathcal{M}_3$ equals $2H$, where H is the hyperelliptic locus.*

Proof. See Theorem 8.1 in [13]. □

In this paragraph, S is the spectrum of a discrete valuation ring. Let $f\colon \mathcal{X} \to S$ be a stable curve of genus 3 with smooth and non-hyperelliptic generic fiber. By the lemma above, we know χ'_{18} is a non-zero rational section of $\mathcal{L}_f^{\otimes 18}$ on S. Thus we can define $\mathrm{ord}_v(\chi'_{18})$, where v is the closed point of S.

Returning to the case where S is an arbitrary integral scheme. Then we have vector bundles $\mathcal{E}_f = f_*\omega_{\mathcal{X}/S}$ and $\mathcal{G}_f = f_*\omega_{\mathcal{X}/S}^{\otimes 2}$ on S.

Lemma 2.2.2. *Let $f\colon \mathcal{X} \to S$ be a stable curve of genus 3 with smooth and non-hyperelliptic generic fiber. Then both $\mathcal{E}_f$ and $\mathcal{G}_f$ are of rank 6.*

Proof. The ranks of these vector bundles can be computed over any point in S. Thus we only need to compute $\dim(\mathrm{Sym}^2(H^0(X, \Omega_X)))$ and $h^0(X, \Omega_X^{\otimes 2})$ for smooth curves X of genus 3.

By Riemann-Roch, we have $h^0(X, \Omega_X) = 3$ and $h^0(X, \Omega_X^{\otimes 2}) = 6$. This implies that

$$\dim(\mathrm{Sym}^2(H^0(X, \Omega_X))) = h^0(X, \Omega_X^{\otimes 2}) = 6.$$

□

We have a canonical map

$$\nu_f \colon \mathrm{Sym}^2 \mathcal{E}_f \to \mathcal{G}_f\,, \quad \eta_1 \cdot \eta_2 \mapsto \eta_1 \otimes \eta_2\,, \tag{2.5}$$

which is functorial in f. Both $\mathrm{Sym}^2\mathcal{E}_f$ and $\mathcal{G}_f$ are vector bundles of rank 6 and thus we have a natural map of invertible sheaves

$$\det \nu_f \colon \det \mathrm{Sym}^2 \mathcal{E}_f \to \det \mathcal{G}_f, \tag{2.6}$$

which is functorial in f. The map ν_f is surjective if f is smooth and nowhere hyperelliptic. We can view $\det \nu_f$ as a global section s_f of the invertible sheaf $(\det \mathrm{Sym}^2\mathcal{E}_f)^{\otimes -1} \otimes \det \mathcal{G}_f$ on S. Then the zero locus of s_f is contained in the hyperelliptic locus. Standard multi-linear algebra yields a canonical isomorphism

$$\det \mathrm{Sym}^2 \mathcal{E}_f \xrightarrow{\sim} \mathcal{L}_f^{\otimes 4}$$

of invertible sheaves on S, where $\mathcal{L}_f = \det \mathcal{E}_f$ as before, and this shows that we may as well view s_f as a global section of the invertible sheaf $\mathcal{L}_f^{\otimes -4} \otimes \det \mathcal{G}_f$ on S. Let $\pi : \mathcal{C}_3 \to \mathcal{M}_3$ be the universal smooth curve of genus 3, then we can associate a section s_π.

Lemma 2.2.3. *The section s_π is not identically equal to 0, and the divisor of s_π on $\mathcal{M}_3$ is equal to the reduced hyperelliptic divisor H.*

Proof. See Proposition 9.1 in [13]. □

Now we want to consider $\mathrm{div}(\chi'_{18})$ on $\overline{\mathcal{M}}_3$. We denote the divisor of $s_{\overline{\pi}}$ on $\overline{\mathcal{M}}_3$ by K, and denote the divisor of singular curves on $\overline{\mathcal{M}}_3$ by Δ.

Proposition 2.2.4. *If we take χ'_{18} as a rational section of the line bundle $\mathcal{L}_{\overline{\pi}}^{\otimes 18}$ on $\overline{\mathcal{M}}_3$, then we have the equality of effective divisors*

$$\mathrm{div}(\chi'_{18}) = 2K + 2\Delta$$

on $\overline{\mathcal{M}}_3$.

Sketch of proof : See Proposition 9.2 in [13] for a complete proof.

Let H be the hyperelliptic locus of $\mathcal{M}_3$ with closure $\overline{H}$ in $\overline{\mathcal{M}}_3$. By Lemma 2.2.3, we have $\operatorname{div} s_\pi = H$ on $\mathcal{M}_3$. By Lemma 2.2.1, the modular form χ'_{18} is a global section of $\mathcal{L}_\pi^{\otimes 18}$ with divisor $2H$. Thus $\chi'_{18} \otimes s_\pi^{\otimes -2}$ is a trivializing section of $\mathcal{L}_\pi^{\otimes 26} \otimes (\det \mathcal{G}_\pi)^{\otimes -2}$ over $\mathcal{M}_3$.

We have a canonical isomorphism of line bundles on $\mathcal{M}_3$

$$\mu : \det \mathcal{G}_\pi \simeq \mathcal{L}_\pi^{\otimes 13},$$

which comes from the Mumford's functorial Riemann-Roch (Theorem 2.1 and Equation 2.1.2 in [51]). Then $\mu^{\otimes 2}$ gives another trivialization section of $\mathcal{L}_\pi^{\otimes 26} \otimes (\det \mathcal{G}_\pi)^{\otimes -2}$, denoted by w. Since the only invertible regular functions on $\mathcal{M}_3$ are ± 1, this means that w and $\chi'_{18} \otimes s_\pi^{\otimes -2}$ are equal up to a sign.

Mumford's functorial Riemann-Roch on $\overline{\mathcal{M}}_3$ extends μ to an isomorphism

$$\det \mathcal{G}_{\overline{\pi}} \otimes \mathcal{O}(\Delta) \simeq \mathcal{L}_{\overline{\pi}}^{\otimes 13}$$

of line bundles on $\overline{\mathcal{M}}_3$. This extends $\chi'_{18} \otimes s_{\overline{\pi}}^{\otimes -2}$ on the trivial line bundle

$$\mathcal{L}_{\overline{\pi}}^{\otimes 26} \otimes (\det \mathcal{G}_{\overline{\pi}})^{\otimes -2} \otimes \mathcal{O}(-2\Delta).$$

The assertion is proven by taking the divisor of the trivial section on the line bundle above. QED

At the end of this subsection, we explain the relation between χ'_{18} and the Faltings height. Details can be found in Section 6 in [13].

For a metrized line bundle $(L, (\|\cdot\|_v)_{v \in M(k)_\infty})$ on a ring of integers O_k, its *arithmetic degree* is given by choosing a non-zero rational section s of L and setting

$$\deg(L, (\|\cdot\|_v)_{v \in M(k)_\infty}) := \sum_{v \in M(k)_0} \operatorname{ord}_v(s) \log(Nv) - \sum_{v \in M(k)_\infty} \log\|s\|_v,$$

where $M(k)_0$ (resp. $M(k)_\infty$) is the set of finite (resp. infinite) places of O_k.

Recall that $q : \mathbb{U}_g \to \mathbb{H}_g$ is the universal principally polarized complex abelian variety over the Siegel upper-half space. We write $\tilde{\mathcal{L}}$ for the line bundle $\det q_* \Omega_{\mathbb{U}_g/\mathbb{H}_g}$. The *Hodge metric* of $\tilde{\mathcal{L}}$ is given by

$$\|dz_1 \wedge \ldots dz_g\|_{\mathrm{Hdg}}(\tau) = \sqrt{\det \mathrm{Im}\tau} \tag{2.7}$$

for all $\tau \in \mathbb{H}_g$.

Let $f : \mathcal{X} \to \mathrm{Spec}(O_k)$ be a semistable arithmetic surface of genus 3 over a ring of integers with non-hyperelliptic smooth generic fiber. Let $\overline{\omega}$ be the Arakelov dualising sheaf. The *Faltings height* of f is given by

$$\deg \det f_*\overline{\omega}_{\mathcal{X}/S} = \frac{\sum_{v\in M(k)_0} \mathrm{ord}_v(\chi'_{18})\log(Nv)}{18} - \frac{\sum_{v\in M(k)_\infty} \log\|\chi'_{18}\|_{\mathrm{Hdg},v}}{18}. \tag{2.8}$$

2.2.2 Klein formula

Recall that S_g is the set of even theta characteristics of genus g. By Page 851 in [37], the function

$$\widetilde{\Sigma}_{140} := \prod_{\substack{\epsilon\in S_3\\ \epsilon\neq 0}} \theta_\epsilon(0,\tau)^8$$

is an analytic Siegel modular form of weight 140.

Theorem 2.2.5. *Let (A,a) be a principally polarized abelian variety of dimension 3 defined over $k \subset \mathbb{C}$. Let $\omega_1,\omega_2,\omega_3$ be a basis of $H^0(A,\Omega^1_{A/k})$ and $\gamma_1,\dots\gamma_6$ a symplectic basis of $H_1(A,\mathbb{Z})$. Then we can associate the period matrix $\Omega = [\Omega_1,\Omega_2]$ of (A,a). Put $\tau = \Omega_1^{-1}\Omega_2 \in \mathbb{H}_3$.*

(1) If $\widetilde{\Sigma}_{140}(\tau) = 0$ and $\tilde{\chi}_{18}(\tau) = 0$, then (A,a) is decomposable over $\overline{k}$. In particular it is not a Jacobian.

(2) If $\widetilde{\Sigma}_{140}(\tau) \neq 0$ and $\tilde{\chi}_{18}(\tau) = 0$, then there exists a hyperelliptic curve X/k such that $(\mathrm{Jac}X, j) \simeq (A,a)$.

(3) If $\tilde{\chi}_{18}(\tau) \neq 0$ then (A,a) is isomorphic to a non-hyperelliptic Jacobian if and only if

$$\chi_{18}((A,a),\omega) = (2\pi i)^{54}\frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}$$

is a square in k, where $\omega := \omega_1\wedge\omega_2\wedge\omega_3$.

Sketch of proof : The first two are proven in Lemma 10, Lemma 11 in [37]. By Proposition 2.1.11, we have the equality in (3). We give a sketch for the remaining part of (3), and a complete proof can be found in Theorem 1.3.3 in [43].

We assume (A,a) to be isomorphic to the Jacobian of a non-hyperelliptic genus 3 curve C/k. By Lemma 2.1.5 and Lemma 2.1.9, we have

$$\chi_{18}((A,a),\omega) = \mathbf{t}^*(\chi_{18})(C,\lambda) = \mu_9(C,\lambda)^2 \in k^{\times 2},$$

with $\lambda = \mathbf{t}^*\omega$.

Now we assume (A, a) is not isomorphic to the Jacobian of a non-hyperelliptic genus 3 curve C/k. By (3) in Theorem 2.1.6, we know (A, a) is a quadratic twist of a Jacobian (A', a'). Then it can be shown that

$$\chi_{18}((A,a),\omega) \equiv c^9 \chi_{18}((A',a'),\omega') (\operatorname{mod} k^{\times 18})$$

for some non-square element $c \in k^*$ (Corollary 1.2.3 in [43]). This implies that $\chi_{18}((A,a),\omega)$ is not a square, which completes the proof. QED

With this theorem, we can show the following formula of Klein in [41] which links the discriminant of a plane quartic and the analytic Siegel modular form $\tilde{\chi}_{18}$.

We fix a smooth plane curve C_F defined by a homogeneous degree d polynomial $F(X,Y,Z) = 0$. We write f for $F(x,y,1)$ and write $k[x,y]_{<d}$ for the subspace of $k[x,y]$ containing polynomials of degree less than d. By a *classical basis* of Ω_{C_F}, we mean a basis of Ω_{C_F} in the form $\{\frac{g_i dx}{\frac{\partial f}{\partial y}}\}_{1\leq i \leq \frac{(d-1)(d-2)}{2}}$ where $\{g_i\}_{1\leq i\leq \frac{(d-1)(d-2)}{2}}$ is a basis of $k[x,y]_{<d}$ (see Theorem 4.6.10).

Theorem 2.2.6. *Let C_F be a smooth plane quartic curve over $\mathbb{C}$ defined by $F(X,Y,Z) = 0$. Let $\Omega = (\Omega_1|\Omega_2)$ be the period matrix of C_F with respect to a classical basis of differential forms and a symplectic homology basis. We denote $\Omega_1^{-1}\Omega_2$ by τ. Then we have*

$$\operatorname{Disc}(F)^2 = (2\pi)^{54} \frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}.$$

Sketch of proof : See Theorem 2.2.3 in [43] for a complete proof.

We define a function I on X_4^0 as

$$I(F) \coloneqq (2\pi)^{54} \frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}.$$

It can be shown that I is an invariant of degree 54 in the sense of Definition 2.1.12 (Corollary 2.2.2 in [43]). This means that $I(F)$ is a degree 54 homogeneous polynomial of the coefficients of F.

By Theorem 2.2.5, we have $I(F) \neq 0$ for all $F \in X_4^0$. Recall that the discriminant is a multiple of the resultant (Equation (2.4)) which is an irreducible polynomial of degree 27 (Page 113 Section 7 in [21]). By Hilbert's Nullstellensatz, we have $I = c\operatorname{Disc}_{\mathfrak{U}}^n$ for some constant $c \in \mathbb{C}^{\times}$.

The exponent n can be computed by the degree counting $n = 54/27 = 2$. The constant c can be computed for Ciani curves, which is equal to 1 (Corollary 4.2 in [42]). QED

Remark 2.2.7. *The Ciani curves are plane curves defined by*

$$X^4 + Y^4 + Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2 = 0,$$

for a, b and $c \in k$.

Chapter 3

Arakelov geometry in genus 3

In this chapter, we discuss Arakelov geometry with an emphasis on genus 3 curves. In Section 3.1, based on the work of Z. Cinkir, we show a classification of pm-graphs of genus 3 and compute admissible invariants associated to it. In Section 3.2, based on the work of K. Yamaki, we show that for a genus 3 graphically hyperelliptic curve over a function field, the height $\langle \Delta, \Delta \rangle$ vanishes if and only if the curve is hyperelliptic. In Section 3.3, based on the work of R. de Jong, we show a unboundedness result of $\langle \Delta, \Delta \rangle$ for genus 3 curves over number fields.

We get a result on hyperelliptic graphs and apply it to genus 3 polarized graphs (Proposition 3.2.20). In Theorem 3.3.12, we give a criterion for the unboundedness of the heights of a family of curves over $\mathbb{Q}$. To the best of the author's knowledge, these are new results.

3.1 Admissible invariants for genus 3 curves

Subsection 3.1.1 is about the explicit computation for genus 3 pm-graphs. We refer to Section 1.4 for terminology on pm-graphs. Subsection 3.1.2 contains two tables for the invariants on genus 3 pm-graphs with the first Betti number $b_1 = 0$ or 1.

We will use results in this section to compute the admissible invariants of our main curve $\mathfrak{C}_{\mathbb{Q}}$ in Theorem 4.4.1.

3.1.1 Computation for genus 3 curves

In this subsection, we explain how to explicitly compute the six invariants discussed in Theorem 1.4.39. To begin with, we specialize Theorem 1.4.39 to the case $g = 3$.

Proposition 3.1.1. *Let $\overline{\Gamma}$ be a pm-graph of genus* 3*. Then we have*

$$\varphi(\overline{\Gamma}) = \frac{13}{3}\tau(\overline{\Gamma}) + \frac{\theta(\overline{\Gamma})}{12} - \frac{\delta(\overline{\Gamma})}{4},$$
$$\lambda(\overline{\Gamma}) = \frac{3}{7}\tau(\overline{\Gamma}) + \frac{\theta(\overline{\Gamma})}{56} + \frac{\delta(\overline{\Gamma})}{14},$$
$$\epsilon(\overline{\Gamma}) = \frac{8}{3}\tau(\overline{\Gamma}) + \frac{\theta(\overline{\Gamma})}{6}.$$

Proof. Substitute $g = 3$ to Theorem 1.4.39. □

Now we show how to compute the six invariants of $\overline{\Gamma}_{ex}$ (Figure 3.1.1), a genus 3 pm-graph with no eliminable points. This pm-graph is non-irreducible and contains 1 cycle. We would like to use this example to show that it is possible to compute the invariants by techniques described. This method is also used by Z. Cinkir in [9]. Letters are the lengths of edges and integers are the polarization.

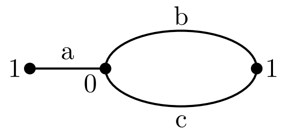


Figure 3.1.1: $\overline{\Gamma}_{ex}$

Proposition 3.1.2. *For the pm-graph $\overline{\Gamma}_{ex}$, we have*

$$\delta(\overline{\Gamma}_{ex}) = a + b + c,$$
$$\tau(\overline{\Gamma}_{ex}) = \frac{\delta(\overline{\Gamma}_{ex})}{12} + \frac{a}{6},$$
$$\theta(\overline{\Gamma}_{ex}) = 6a + \frac{8bc}{b+c},$$
$$\varphi(\overline{\Gamma}_{ex}) = \frac{\delta(\overline{\Gamma}_{ex})}{9} + \frac{6bc + 11a(b+c)}{9(b+c)},$$
$$\lambda(\overline{\Gamma}_{ex}) = \frac{3\delta(\overline{\Gamma}_{ex})}{28} + \frac{4bc + 5a(b+c)}{28(b+c)},$$
$$\epsilon(\overline{\Gamma}_{ex}) = \frac{2\delta(\overline{\Gamma}_{ex})}{9} + \frac{12bc + 13a(b+c)}{9(b+c)}.$$

Proof. As a metrized graph, Γ_{ex} can be written as the wedge sum of two irreducible components $\Gamma_a \vee \Gamma_{bc}$, where Γ_a is obtained by contracting the edges of length b and c and Γ_{bc} is obtained by contracting the edge of length a. By the additivity of the six

Figure 3.1.2: Irreducible components

invariants (Remark 1.4.41), we just need to compute the invariants on $\overline{\Gamma}_a$ and $\overline{\Gamma}_{bc}$, where the polarization is induced from that on $\overline{\Gamma}_{ex}$. Figure 3.1.2 is an illustration for this.

For δ, it is trivial that $\delta(\overline{\Gamma}_a) = a$ and $\delta(\overline{\Gamma}_{bc}) = b + c$, thus

$$\delta(\overline{\Gamma}_{ex}) = a + b + c.$$

For θ, by Equation (1.17), we have

$$\begin{aligned}
\theta(\overline{\Gamma}_a) &= 2 \times (1 - 2 + 2) \times (1 - 2 + 4) \times a = 6a, \\
\theta(\overline{\Gamma}_{bc}) &= 2 \times (2 - 2 + 2) \times (2 - 2 + 2) \times \frac{bc}{b+c} = \frac{8bc}{b+c},
\end{aligned}$$

thus

$$\theta(\overline{\Gamma}_{ex}) = 6a + \frac{8bc}{b+c}.$$

Recall the interpretation of τ in Definition 1.4.37. For $\overline{\Gamma}_a$, we take y to be a vertex p, and then we get $r(x,p) = d(x,p)$, where $d(\cdot,\cdot)$ is the path distance function. Thus we have

$$\tau(\overline{\Gamma}_a) = \frac{1}{4}\int_{\Gamma_a} r_x(x,p)^2 dx = \frac{1}{4}\int_0^a dx = \frac{a}{4}.$$

For $\overline{\Gamma}_{bc}$, by the formula of electrical resistance in a parallel connection, we get

$$r(x,y) = \frac{d(x,y)(b + c - d(x,y))}{b+c}.$$

Taking y to be a vertex p, we have

$$\tau(\overline{\Gamma}_{bc}) = \frac{1}{4}\int_{\Gamma_{bc}} r_x(x,p)^2 dx = \frac{1}{4}\int_0^{b+c} \left(\frac{b+c-2x}{b+c}\right)^2 dx = \frac{b+c}{12}.$$

By the additivity of τ, we get

$$\tau(\overline{\Gamma}_{ex}) = \frac{\delta(\overline{\Gamma}_{ex})}{12} + \frac{a}{6}.$$

According to Proposition 3.1.1, we get

$$\begin{aligned}
\varphi(\overline{\Gamma}_{ex}) &= \frac{a+b+c}{9} + \frac{6bc+11a(b+c)}{9(b+c)}, \\
\lambda(\overline{\Gamma}_{ex}) &= \frac{3(a+b+c)}{28} + \frac{4bc+5a(b+c)}{28(b+c)}, \\
\epsilon(\overline{\Gamma}_{ex}) &= \frac{2(a+b+c)}{9} + \frac{12bc+13a(b+c)}{9(b+c)}.
\end{aligned}$$

□

The whole list of genus 3 pm-graphs without eliminable points and their invariants can be found in [9]. In this thesis, we copy part of this list (containing those pm-graphs with the first Betti number $b_1 = 0$ or 1) in Table 3.1 and Table 3.2.

We can find from Table 3.2 that $\lambda(\overline{\Gamma}) \geq \frac{3\delta(\overline{\Gamma})}{28}$ and $\epsilon(\overline{\Gamma}) \geq \frac{2\delta(\overline{\Gamma})}{9}$. These two bounds actually hold for all pm-graphs of genus 3.

The invariant $\varphi(\overline{\Gamma})$ is more complicated. When $b_1 \leq 1$, we can find from Table 3.2 that $\varphi(\overline{\Gamma}) \geq \frac{1}{9}\delta(\overline{\Gamma})$. This bound does not hold for a general genus 3 pm-graph. By a technical analysis of inequalities, Z. Cinkir proved the following proposition which is conjectured by X. Faber in Remark 5.1 in [22].

Proposition 3.1.3. *For a pm-graph $\overline{\Gamma}$ of genus 3, we have $\varphi(\overline{\Gamma}) \geq \frac{17\delta(\overline{\Gamma})}{288}$.*

Proof. See the proof of Claim on Page 332 in [9]. □

Remark 3.1.4. *Proposition 3.1.3 is not a corollary of Theorem 1.4.34 since $c(3) \leq \frac{17}{288}$.*

3.1.2 Tables for genus 3 pm-graphs

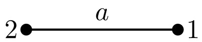
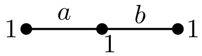
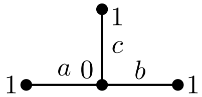
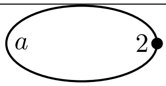
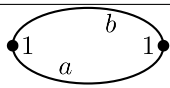
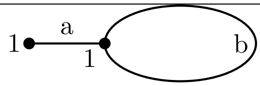
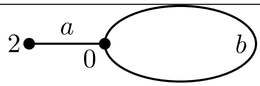
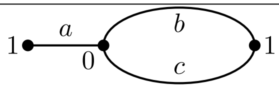
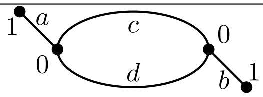
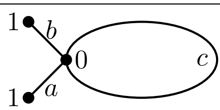
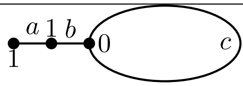
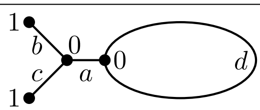
	$\overline{\Gamma}$	$\delta(\overline{\Gamma})$	$\tau(\overline{\Gamma})$	$\theta(\overline{\Gamma})$
$0I$		0	0	0
$0II$		a	$\frac{\delta(\overline{\Gamma})}{4}$	$6\delta(\overline{\Gamma})$
$0III$		$a+b$	$\frac{\delta(\overline{\Gamma})}{4}$	$6\delta(\overline{\Gamma})$
$0IV$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{4}$	$6\delta(\overline{\Gamma})$
$1I$		a	$\frac{\delta(\overline{\Gamma})}{12}$	0
$1II$		$a+b$	$\frac{\delta(\overline{\Gamma})}{12}$	$\frac{8ab}{a+b}$
$1III$		$a+b$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a}{6}$	$6a$
$1IV$		$a+b$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a}{6}$	$6a$
$1V$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a}{6}$	$6a+\frac{8bc}{b+c}$
$1VI$		$a+b+c+d$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b}{6}$	$6(a+b)+\frac{8cd}{c+d}$
$1VII$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b}{6}$	$6(a+b)$
$1VIII$		$a+b+c$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b}{6}$	$6(a+b)$
$1IX$		$a+b+c+d$	$\frac{\delta(\overline{\Gamma})}{12}+\frac{a+b+c}{6}$	$6(a+b+c)$

Table 3.1: Table of $\overline{\Gamma}$, $\delta(\overline{\Gamma})$, $\theta(\overline{\Gamma})$ and $\tau(\overline{\Gamma})$

	$\varphi(\overline{\Gamma})$	$\lambda(\overline{\Gamma})$	$\epsilon(\overline{\Gamma})$
$0I$	0	0	0
$0II$	$\frac{4\delta(\overline{\Gamma})}{3}$	$\frac{2\delta(\overline{\Gamma})}{7}$	$\frac{5\delta(\overline{\Gamma})}{3}$
$0III$	$\frac{4\delta(\overline{\Gamma})}{3}$	$\frac{2\delta(\overline{\Gamma})}{7}$	$\frac{5\delta(\overline{\Gamma})}{3}$
$0IV$	$\frac{4\delta(\overline{\Gamma})}{3}$	$\frac{2\delta(\overline{\Gamma})}{7}$	$\frac{5\delta(\overline{\Gamma})}{3}$
$1I$	$\frac{\delta(\overline{\Gamma})}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}$
$1II$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{2ab}{3(a+b)}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{ab}{7(a+b)}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{4ab}{3(a+b)}$
$1III$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{11a}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{5a}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{13a}{9}$
$1IV$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{11a}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{5a}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{13a}{9}$
$1V$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{6bc+11a(b+c)}{9(b+c)}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{4bc+5a(b+c)}{28(b+c)}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{12bc+13a(b+c)}{9(b+c)}$
$1VI$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{6cd+11(a+b)(c+d)}{9(c+d)}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{4cd+5(a+b)(c+d)}{28(c+d)}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{12cd+13(a+b)(c+d)}{9(c+d)}$
$1VII$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{11(a+b)}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{5(a+b)}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{13(a+b)}{9}$
$1VIII$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{11(a+b)}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{5(a+b)}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{13(a+b)}{9}$
$1IX$	$\frac{\delta(\overline{\Gamma})}{9}+\frac{11(a+b+c)}{9}$	$\frac{3\delta(\overline{\Gamma})}{28}+\frac{5(a+b+c)}{28}$	$\frac{2\delta(\overline{\Gamma})}{9}+\frac{13(a+b+c)}{9}$

Table 3.2: Table of $\varphi(\overline{\Gamma})$, $\lambda(\overline{\Gamma})$ and $\epsilon(\overline{\Gamma})$

3.2 Graphically hyperelliptic curves over function fields

In this section, B is a smooth curve over an algebraically closed field k with function field K. Subsections 3.2.1-3.2.2 are still about pm-graphs of genus 3 (with an application to the height $\langle\Delta,\Delta\rangle$). In Subsection 3.2.3, we show that the height $\langle\Delta,\Delta\rangle$ of a graphically hyperelliptic genus 3 curve over K vanishes if and only the curve is hyperelliptic. We refer to Section 1.4 for the terminology on pm-graphs and Subsection 1.5.2 for the theory of Gross-Schoen cycles.

The number $h(\overline{\Gamma})$ introduced in Equation (3.1) will be used in Theorem 3.3.12 and Proposition 4.4.5.

3.2.1 An inequality for $\langle\Delta,\Delta\rangle$

By *a polarized graph*, we mean a pm-graph without the metric, in other words, it is a pair $\overline{G} = (G, \mathfrak{q})$ where $G = (V, E)$ is a graph and $\mathfrak{q}$ is a polarization making the canonical divisor (Definition 1.4.8) effective.

The polarized graphs $\overline{\mathbf{H}} = (\mathbf{H}, 0)$ and $\overline{\mathbf{N}} = (\mathbf{N}, 0)$ in Figure 3.2.1 are two irreducible polarized graphs (the polarization is the constant function 0) without eliminable vertices, and we call the two graphs *maximal models*.

Definition 3.2.1. *We say $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ is a model for a polarized graph $\overline{G}$ if $\overline{G}$ is equivalent to a contraction of $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ with the induced polarization.*

Remark 3.2.2. *For simplicity, we use the same notations for pm-graphs and polarized graphs (like the contraction $\overline{G}_S$ and $\overline{G}^S$). We also use Table 3.1 for the types of genus 3 polarized graphs when $b_1 = 0$ or 1.*

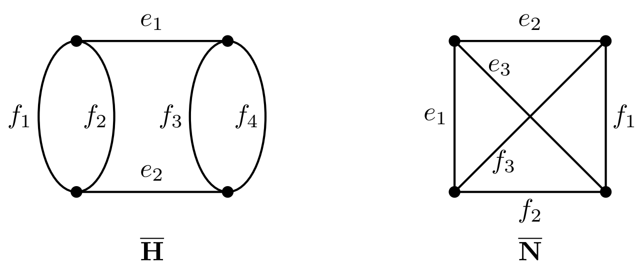


Figure 3.2.1: Maximal models

Lemma 3.2.3. *Every polarized graph $\overline{G}$ of genus 3 with only edges of type 0 is equivalent to a polarized graph having $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ as a model. If we assume further that $\overline{G}$ is not equivalent to $\overline{\mathbf{N}}$, then it has $\overline{\mathbf{H}}$ as a model.*

Proof. This can be proven by a combinatorial checking. □

Definition 3.2.4. *Let $\overline{G} = (G, \mathfrak{q})$ be a polarized graph of genus 3 with no eliminable vertices. We say a pair of edges $\{e, e'\}$ of $\overline{G}$ is of h-type if $\overline{G}^{\{e,e'\}}$ is of type $1II$ in Table 3.1.*

Example 3.2.5. *In Figure 3.2.1, $\{e_1, e_2\}$ is the only pair of h-type edges in $\overline{\mathbf{H}}$ while $\overline{\mathbf{N}}$ has no edges of h-type.*

Lemma 3.2.6. *A polarized graph $\overline{G}$ of genus 3 without eliminable vertices has at most one pair of edges of h-type.*

Proof. If $\{e_1, e_2\}$ is a pair of edges of h-type, then e_1 and e_2 sit in the same irreducible component otherwise $\overline{G}^{\{e_1,e_2\}}$ is reducible.

Let $\{e_3, e_4\}$ be another pair of edges of h-type. The two pairs lie on the same irreducible component of $\{e_1, e_2\}$, otherwise $\overline{G}^{\{e_1,e_2,e_3,e_4\}}$ can not be a graph without eliminable vertices, which contradicts Lemma 1.4.17. We denote this irreducible component with induced polarization by $\overline{G}_1$. By Lemma 3.2.3, the polarized graph $\overline{G}_1$ is equivalent to a certain contraction of $\overline{\mathbf{H}}$ or $\overline{\mathbf{N}}$ with the induced polarization. Since $\overline{\mathbf{H}}$ and $\overline{\mathbf{N}}$ have at most 1 pair of edges of h-type (Example 3.2.5), so does their contraction. Thus $\{e_1, e_2\} = \{e_3, e_4\}$. □

Since pm-graphs are polarized graphs with metrics, our discussion so far can be extended to pm-graphs easily.

Let $\overline{\Gamma} = (G, w, \mathfrak{q})$ be a genus 3 pm-graph with no eliminable vertices. If there exists a pair of edges of h-type $\{e_1, e_2\}$ on $\overline{\Gamma}$, we define

$$h(\overline{\Gamma}) \coloneqq \min\{w(e_1), w(e_2)\}, \tag{3.1}$$

otherwise we set $h(\overline{\Gamma}) = 0$. For a general pm-graph $\overline{\Gamma}$ which is equivalent to $\overline{\Gamma}_0$ with no eliminable vertices, we define

$$h(\overline{\Gamma}) \coloneqq h(\overline{\Gamma}_0).$$

Lemma 3.2.7. *$h(\cdot)$ is additive on pm-graphs of genus 3.*

Proof. This is trivial from the definition of $h(\cdot)$. □

Recall the definition of $\psi(\overline{\Gamma})$ in Corollary 1.5.8. For a genus 3 pm-graph $\overline{\Gamma}$ with only edges of type 0, we define

$$\Phi(\overline{\Gamma}) \coloneqq \frac{1}{3}\delta_0(\overline{\Gamma}) + \frac{4}{3}h(\overline{\Gamma}) - \psi(\overline{\Gamma}). \tag{3.2}$$

Lemma 3.2.8. *The invariant Φ is additive for pm-graphs with only type 0 edges.*

Proof. The function ψ is additive since it is a linear combination of admissible invariants (Corollary 1.5.8). On pm-graphs with only edges of type 0, the invariant δ_0 is additive since $\delta_0 = \delta$. The additivity of h is trivial according to Lemma 3.2.6. $\square$

Lemma 3.2.9. *For a tree pm-graph $\overline{\Gamma}$ of genus g, we have*

$$\psi(\overline{\Gamma}) = \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \left(\frac{12i(g-i)}{2g+1} - 1 \right) \delta_i(\overline{\Gamma}).$$

Proof. In this case, we have $\delta_0(\overline{\Gamma}) = 0$ and $h(\overline{\Gamma}) = 0$, thus ψ is a linear combination of ϵ and φ (Equation (1.20)). Theorem 1.4.39 implies that we can reduce the problem to the computation of τ, θ and δ. Since $\overline{\Gamma}$ is a tree, the underlying graph Γ is the wedge sum of segments. By the additivity of these invariants, we only need to compute them for the pm-graph with one segment and two endpoints polarized by i and $g-i$ for $0 < i \leq \lfloor \frac{g}{2} \rfloor$. $\square$

Now we give a lower bound of $\langle \Delta, \Delta \rangle$ for non-hyperelliptic curves by Lemma 3.2.11. For a semistable curve $f : \mathcal{X} \to B$, we denote the dual graph at a closed point $s \in B$ by $\overline{\Gamma}_s$. If the genus of f is 3, all edges in $\overline{\Gamma}_s$ are of type 0 or 1. We denote the contraction of all type 1 (resp. 0) edges in $\overline{\Gamma}_s$ with the induced polarization by $\overline{\Gamma}_s^{\circ}$ (resp. $\overline{\Gamma}_s^{+}$).

Remark 3.2.10. *The pm-graph $\overline{\Gamma}_s^{\circ}$ is the wedge sum of irreducible components in $\overline{\Gamma}_s$ which are not isomorphic to segments. And $\overline{\Gamma}_s^{+}$ is a wedge sum of segment components in $\overline{\Gamma}_s$. Every edge in $\overline{\Gamma}_s$ corresponds to an edge in either $\overline{\Gamma}_s^{\circ}$ or $\overline{\Gamma}_s^{+}$. If F is an additive function on pm-graphs, then $F(\overline{\Gamma}_s) = F(\overline{\Gamma}_s^{\circ}) + F(\overline{\Gamma}_s^{+})$.*

Lemma 3.2.11. *Let $f : \mathcal{X} \to B$ be a semistable curve of genus 3 with smooth non-hyperelliptic generic fiber. We have*

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq \sum_{s \in B} \left(\frac{\delta_0(\overline{\Gamma}_s)}{3} + 3\delta_1(\overline{\Gamma}_s) + \frac{4h(\overline{\Gamma}_s)}{3} \right),$$

where $\overline{\Gamma}_s$ is the dual graph over $s \in B$.

Proof. See Corollary 3.8 in [66]. $\square$

Proposition 3.2.12. *Let $f : \mathcal{X} \to B$ be a semistable curve of genus 3 with smooth non-hyperelliptic generic fiber C. Then we have*

$$\langle \Delta, \Delta \rangle \geq \frac{7}{4} \sum_s \Phi(\overline{\Gamma}_s^\circ) + \delta_1(C),$$

where $\delta_1(C) := \sum_s \delta_1(\overline{\Gamma}_s)$.

Proof. Every irreducible component of $\overline{\Gamma}_s$ is an irreducible component of $\overline{\Gamma}_s^\circ$ or $\overline{\Gamma}_s^+$ and vice versa (Remark 3.2.10). Thus by the additivity of ψ, we have

$$\psi(\overline{\Gamma}_s) = \psi(\overline{\Gamma}_s^\circ) + \psi(\overline{\Gamma}_s^+).$$

From Corollary 1.5.8 and Lemma 3.2.11, we get

$$\begin{aligned}\langle \Delta, \Delta \rangle =& \frac{7}{4}\left((\omega_{X/B}, \omega_{X/B}) - \sum_{s\in B} \psi(\overline{\Gamma}_s)\right)\\ \geq& \frac{7}{4}\left(\sum_{s\in B} \Phi(\overline{\Gamma}_s^\circ) + \sum_{y\in B}\left(3\delta_1(\overline{\Gamma}_s^+) - \psi(\overline{\Gamma}_s^+)\right)\right).\end{aligned}$$

Since $\overline{\Gamma}_s^+$ is a tree, by Lemma 3.2.9, we have $\psi(\overline{\Gamma}_s^+) = \frac{17}{7}\delta_1(\overline{\Gamma}_s^+)$. Thus

$$3\delta_1(\overline{\Gamma}_s^+) - \psi(\overline{\Gamma}_s^+) = \frac{4\delta_1(\overline{\Gamma}_s^+)}{7} = \frac{4\delta_1(\overline{\Gamma}_s)}{7}. \tag{3.3}$$

We get the result by substituting Equation (3.3) into the inequality above. □

Proposition 3.2.12 reduces the positivity of $\langle \Delta, \Delta \rangle$ to the computation for Φ and δ_1 at special fibers. In [67], K. Yamaki shows the following result for Φ.

Theorem 3.2.13. *Let $\overline{\Gamma} = (G, w, \mathfrak{q})$ be a pm-graph of genus 3 without eliminable vertices. Suppose that $\overline{\mathbf{H}}$ is a model of $\overline{\Gamma}$, then we have $\Phi(\overline{\Gamma}) \geq 0$. Moreover, $\Phi(\overline{\Gamma}) = 0$ if and only if one of the following cases occurs:*

(1) $\overline{\Gamma}$ *is the trivial pm-graph.*

(2) $\overline{\Gamma}$ *is isomorphic to* $\mathbf{E}_1$ *in Figure 3.2.2 with the weight condition*

$$w(f_1) = w(f_2), \quad w(f_3) = w(f_4), \quad w(e) = w(f_1) + w(f_3).$$

(3) $\overline{\Gamma}$ *is isomorphic to* $\mathbf{E}_2$ *in Figure 3.2.2 with the weight condition*

$$w(e_1) = w(e_2) = w(e_3).$$

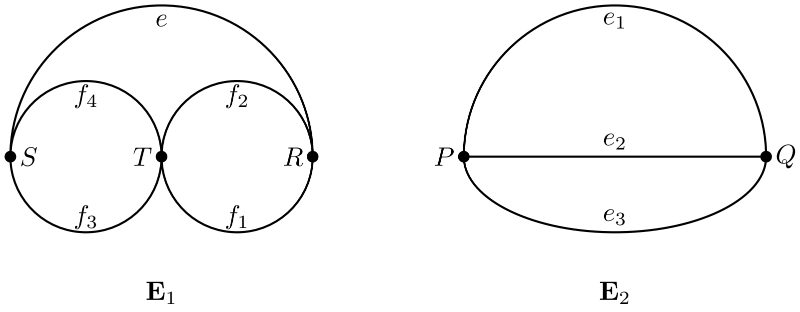


Figure 3.2.2: Two polarized graphs with model $\overline{\mathbf{H}}$

Proof. See Theorem 2.7 in [67]. □

Corollary 3.2.14. *Let $f : \mathcal{X} \to B$ be a semistable curve of genus 3 with smooth non-hyperelliptic generic fiber X. If $\Phi(\overline{\Gamma}_s) \geq 0$ for all s that $\overline{\Gamma}_s$ is equivalent to $\overline{\mathbf{N}}$, then $\langle \Delta, \Delta \rangle \geq 0$. In addition, if there exists s such that $\overline{\Gamma}_s$ is not equivalent to one of the pm-graphs in Theorem 3.2.13, then we have $\langle \Delta, \Delta \rangle > 0$.*

Proof. This is a consequence of Lemma 3.2.3 and Theorem 3.2.13. □

3.2.2 Hyperelliptic polarized graph

Definition 3.2.15. *A hyperelliptic graph $G = (V, E)$ is either the one-point graph, or a graph with an order 2 automorphism ι on G satisfying the following properties:*

(1) G has no self-loops.

(2) $\iota(e) \neq e$ for any $e \in E$.

(3) The quotient graph $G/\langle \iota \rangle$ is a tree.

(4) If a vertex $n \in V$ is not fixed by ι, then the valence satisfies $v(n) \geq 3$.

Lemma 3.2.16. *Let (G, ι) be a non-trivial hyperelliptic graph with $\iota(e_1) = e_2$. The graph $G_{\{e_1,e_2\}}$ given by contracting edges e_1 and e_2 is either a one-point graph or a non-trivial hyperelliptic graph with the induced automorphism ι_0 of order 2.*

Proof. We assume that $G_{\{e_1,e_2\}}$ is not a one-point graph, thus it has the induced automorphism ι_0 of order 2.

Condition (2) in Definition 3.2.15 is trivial.

The quotient graph $(G_{\{e_1,e_2\}}, \iota_0)/\langle\iota_0\rangle$ is given by contracting an edge from the tree $G/\langle\iota\rangle$, thus is also a tree. So Condition (3) in Definition 3.2.15 is verified.

If c is a self-loop in $G_{\{e_1,e_2\}}$, then Condition (2) says that $\iota(c)$ is a different self-loop. Thus the quotient $(G_{\{e_1,e_2\}}, \iota_0)/\langle\iota_0\rangle$ must have at least 1 self-loop, which contradicts Condition (3) we just proved. So Condition (1) in Definition 3.2.15 is verified.

We assume p to be a vertex in $G_{\{e_1,e_2\}}$ that is not fixed by ι_0. If p is the contraction point of $e_1 \in E$, then by the assumption, neither of e_1's endpoints are fixed by ι. It can be checked that e_1 and e_2 can not share the endpoints, otherwise $G_{\{e_1,e_2\}}$ contains a self-loop. Then we obtain $v(p) \geq 3+3-2 = 4$ by Condition (4) in Definition 3.2.15.

If p does not belong to the endpoints of contracting edges, then its valence is the same as that of the original graph (we write p' for this point in G). Since p is not fixed by ι_0, p' cannot be fixed by ι and thus $v(p) \geq 3$. So Condition (4) in Definition 3.2.15 is verified.

In conclusion, $(G_{\{e_1,e_2\}}, \iota_0)$ is a hyperelliptic graph. □

This lemma says that the hyperelliptic graph behaves well under the quotient map.

Proposition 3.2.17. *A hyperelliptic graph G does not have vertices with valence 1.*

Proof. Let p be a vertex of G with valence 1. By Condition (4) in Definition 3.2.15, it is fixed by ι. Thus the only edge related to it is fixed by ι, which contradicts Condition (2). □

Lemma 3.2.18. *Let G be a non-trivial graph. If ι and ι' are two order 2 automorphisms of G that make G a hyperelliptic graph, then $\iota = \iota'$.*

Proof. See Lemma 3.1 in [67]. □

Definition 3.2.19. *A polarized graph $\overline{G} = (V, E, \mathfrak{q})$ is called a hyperelliptic polarized graph if G is a one-point graph or the following are satisfied:*

(1) G is a non-trivial hyperelliptic graph with the order 2 automorphism ι.

(2) ι preserves the polarization $\mathfrak{q}$.

(3) $\mathfrak{q}(n) = 0$ for any $n \in V$ with $\iota(n) \neq n$.

If $w : E \to \mathbb{R}_{>0}$ is a weight function on the edges of $\overline{G}$ with the property $w(e) = w(\iota(e))$ for all $e \in E$, then we call $(\overline{G}, w)$ a hyperelliptic weighted polarized graph or a hyperelliptic pm-graph.

Proposition 3.2.20. *(1) Let e be an edge on a non-trivial hyperelliptic graph G (with the order 2 automorphism ι). Then $\{e, \iota(e)\}$ is a pair of edges of h-type.*

(2) *All hyperelliptic polarized graphs of genus* 3 *without eliminable points are of type* $1II$ *in Table 3.1.*

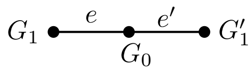
Proof. By applying Lemma 3.2.16 repeatedly, we find $G^{\{e,\iota(e)\}}$ is a hyperelliptic graph. Thus $\{e, \iota(e)\}$ has to be a pair of edges of h-type.

By Lemma 3.2.6, a polarized graph $\overline{G}$ of genus 3 without eliminable edges has at most 1 pair of edges of h-type. By the first assertion, the polarized graph $\overline{G}$ is equivalent to the type $1II$ graph in Table 3.1. ☐

Question 3.2.21. *Can we have a clearer description of hyperelliptic polarized graphs? What can we say for higher genus?*

Proposition 3.2.22. *A hyperelliptic polarized graph* $\overline{G}$ *only has edges of type* 0.

Proof. If there is an edge e of positive type, then $e' := \iota(e)$ is also an edge of positive type. Thus $\overline{G}$ looks like the following figure, where G_0, G_1 and G_1' are subgraphs instead of vertices.



The automorphism ι induces an isomorphism between G_1 and G_1', and an automorphism of G_0. By Condition (3) in Definition 3.2.15, the quotient graph $\overline{G}/\langle\iota\rangle$ is a tree, thus G_1 and G_1' are non-trivial trees or the one-point graph. Both of the two cases will lead to a vertex with valence 1, while this can not be true by Proposition 3.2.17. ☐

3.2.3 Graphically hyperelliptic curves

In this subsection, we take X to be a smooth curve over K of genus $g > 1$ with a semistable model $f : \mathcal{X} \to B$. Similarly to in Section 1.4.1, we denote $\overline{\Gamma}_s = (G_s, w_s, \mathfrak{q}_s)$ the dual graph of $\mathcal{X}$ at a closed point $s \in B$. By $\overline{\Gamma}_s^\circ$, we mean the induced pm-graph given by contracting edges of positive type in $\overline{\Gamma}_s$.

Definition 3.2.23. *If* $\overline{\Gamma}_s^\circ$ *is equivalent to a hyperelliptic pm-graph for all closed points* $s \in B$, *we call* X *or* f *a graphically hyperelliptic curve.*

Theorem 3.2.24. *Let* X *be a graphically hyperelliptic smooth genus 3 curve over* K *with a semistable model* $\mathcal{X} \to B$. *If* $\langle \Delta, \Delta \rangle = 0$ *and there is at least one closed point* $s \in B$ *such that* $\overline{\Gamma}_s^\circ$ *is non-trivial, then* X *is a hyperelliptic curve.*

Proof. Since $\overline{\Gamma}_s^\circ$ is a hyperelliptic graph, it cannot be of the form $\mathbf{N}$, $\mathbf{E}_1$ or $\mathbf{E}_2$. By Corollary 3.2.14, the curve X cannot be non-hyperelliptic, otherwise we have $\langle \Delta, \Delta \rangle > 0$. ☐

3.3 Non-hyperelliptic curves over number fields

In Subsection 3.3.1, we decompose $\langle\Delta,\Delta\rangle$ into the sum of contributions from (in)finite places (Theorem 3.3.2). In Subsection 3.3.2, we give a lower bound for $\mathrm{ord}_v(\chi'_{18})$ (Proposition 3.3.4). In Subsection 3.3.3, we prove an unboundedness result of $\langle\Delta,\Delta\rangle$. We refer to Section 1.5, Section 2.2 and Section 3.1 for terminology and theorems.

Theorem 3.3.2 will be the main tool for our computation of $\mathfrak{C}_{\mathbb{Q}}$ in Chapter 4. The Horikawa index will be used for the computation of $\mathrm{ord}_v(\chi'_{18})$ at a finite place v.

3.3.1 $\langle\Delta,\Delta\rangle$ for non-hyperelliptic curves of genus 3

Let k be a number field with $M(k)_0$ (resp. $M(k)_\infty$) its finite (resp. infinite) places and let $M(k)$ be the union $M(k)_0\cup M(k)_\infty$. We denote $\mathrm{Spec}(O_k)$ by S. Let X be a smooth curve of genus $g\geq 2$ over k which also has semistable reduction over k.

By Theorem 1.5.6, the height of a canonical Gross-Schoen cycle of X is

$$\langle\Delta,\Delta\rangle=\frac{2g+1}{2g-2}(\hat{\omega},\hat{\omega})_{ad}-\sum_{v\in M(k)}\varphi(X)\mathrm{log}Nv. \tag{3.4}$$

Let $f:\mathcal{X}\to S$ be a stable model of X and let $\omega_{\mathcal{X}/S}$ be the relative dualizing sheaf on $\mathcal{X}$. We endow the line bundle $\det f_*\omega_{\mathcal{X}/S}$ with the metric induced by Equation (1.6) at infinite places of k, and denote the metrized line bundle by $\det f_*\overline{\omega}_{\mathcal{X}/S}$.

By Corollary 1.3.11 and Theorem 1.5.3, we get

$$\begin{aligned}(\hat{\omega},\hat{\omega})_{ad}=&12\deg\det f_*\overline{\omega}_{\mathcal{X}/S}-\sum_{v\in M(k)_0}\delta(\overline{\Gamma}_v)\log(Nv)+\sum_{\sigma\in k(\mathbb{C})}\delta(\mathcal{X}_\sigma)\\&+\sum_{v\in M(k)_0}\epsilon(\overline{\Gamma}_s)\log(Nv)+4g[k:\mathbb{Q}]\log(2\pi).\end{aligned}$$

Substituting the equation above, Equation (1.15) and Equation (1.18) to Equation (3.4), we get the following proposition (Corollary 4.2 in [13]).

Proposition 3.3.1. *Let X be a smooth curve of genus $g\geq 2$ defined over the number field k and also has semistable reduction over k. Let $\Delta\in\mathrm{CH}^2(X^3)_{\mathbb{Q}}$ be a canonical Gross-Schoen cycle on X^3. Then the equality*

$$\langle\Delta,\Delta\rangle=\frac{6(2g+1)}{g-1}\left(\deg\det f_*\overline{\omega}_{\mathcal{X}/S}-\sum_{v\in M(k)}\lambda(X_v)\mathrm{log}Nv\right)$$

holds.

Let $\overline{\pi} : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal stable curve of genus $g \geq 2$ and let $\Omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ be the universal relative dualizing sheaf. For the stable curve $\mathcal{X} \to S$, the pull-back of $\mathcal{L}_{\overline{\pi}} := \det \overline{\pi}_* \Omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ along the classifying map $S \to \overline{\mathcal{M}}_g$ gives the metrized line bundle $\det f_* \overline{\omega}_{\mathcal{X}/S}$ on S.

Recall that we introduced a geometric Siegel modular form $\chi'_{18} \in S_{3,18}(\mathbb{Z})$ in Subsection 2.2.1, which corresponds to an element in $T_{3,18}(\mathbb{Z})$ (also denoted by χ'_{18}). Thus χ'_{18} can be considered as a rational section of $\mathcal{L}_{\overline{\pi}}^{\otimes 18}$. Now we assume that the generic fiber of $\mathcal{X} \to S$ is non-hyperelliptic and also of genus 3. Then the pull-back of χ'_{18} along the classifying map $S \to \overline{\mathcal{M}_3}$ gives a non-zero rational section of $\mathcal{L}_f^{\otimes 18}$ (Lemma 2.2.1). Over $\mathbb{C}$, the pullback of the Hodge metric (Equation (2.7)) on $\mathcal{L}_\pi$ coincides with the metric derived from Equation (1.5). Thus we have the following formula for $\deg \det f_* \overline{\omega}_{\mathcal{X}/S}$:

$$18 \deg \det f_* \overline{\omega}_{\mathcal{X}/S} = \sum_{v \in M(k)_0} \mathrm{ord}_v(\chi'_{18}) \log Nv - \sum_{v \in M(k)_\infty} \log \|\chi'_{18}\|_{\mathrm{Hdg},v}. \tag{3.5}$$

Applying this to Proposition 3.3.1, we get the following result (Theorem 8.2 in [13]).

Theorem 3.3.2. *Let X be a smooth non-hyperelliptic curve of genus 3 defined over the number field k which has semistable reduction over k. Let $f : \mathcal{X} \to \mathrm{Spec}(O_k)$ be the stable model of X over O_k and consider χ'_{18} as a rational section of the line bundle $\mathcal{L}_f^{\otimes 18}$. Then the height of a canonical Gross-Schoen cycle Δ on X^3 satisfies*

$$\begin{aligned} \frac{\langle \Delta, \Delta \rangle}{21} = & \sum_{v \in M(k)_0} \left(\frac{1}{18} \mathrm{ord}_v(\chi'_{18}) - \lambda(X_v) \right) \log Nv \\ & + \sum_{v \in M(k)_\infty} \left(-\frac{1}{18} \log \|\chi'_{18}\|_{\mathrm{Hdg},v} - \lambda(X_v) \right). \end{aligned}$$

3.3.2 The Horikawa index

In this subsection, S is the spectrum of a discrete valuation ring R with the closed point s and fraction field $K(S)$. Let $f : \mathcal{X} \to S$ be a stable curve of genus 3 with smooth non-hyperelliptic generic fiber.

Using the notation defined in the beginning of Subsection 2.2.1, the bundles $\mathcal{E}_f$ and $\mathcal{G}_f$ are locally free and the morphism $\nu_f : \mathrm{Sym}^2 \mathcal{E}_f \to \mathcal{G}_f$ given by $\eta_1 \cdot \eta_2 \to \eta_1 \otimes \eta_2$ is generically surjective (both are $K(S)$-linear spaces of dimension 6 at the generic fiber of S). Since R is a discrete valuation ring, we know $\mathrm{Sym}^2 \mathcal{E}_f$ and $\mathcal{G}_f$ can be viewed as free R-modules of rank 6. The generic surjectivity of ν_f also guarantees its global injectivity. This induces a short exact sequence:

$$0 \to \mathrm{Sym}^2 \mathcal{E}_f \to \mathcal{G}_f \to \mathcal{Q}_f \to 0. \tag{3.6}$$

Since $\mathrm{Sym}^2\mathcal{E}_f$ and $\mathcal{G}_f$ are isomorphic to $R^{\oplus 6}$, we find that $\mathcal{Q}_f$ is of finite length over R. We define $\mathrm{length}_{O_S}\mathcal{Q}_f$ as the *Horikawa index* of f at s, denoted by $\mathrm{Ind}_s(f)$.

Proposition 3.3.3. *If we consider χ'_{18} as a rational section of the line bundle $\mathcal{L}_f^{\otimes 18}$ on S, then we have the equality*

$$\mathrm{ord}_s(\chi'_{18}) = 2\mathrm{Ind}_s(f) + 2\delta(\overline{\Gamma}_s).$$

In particular, χ'_{18} is a global section of $\mathcal{L}_f^{\otimes 18}$.

Proof. See Proposition 9.3 in [13]. □

Proposition 3.3.4. *With the notation above, the inequality*

$$\mathrm{ord}_s(\chi'_{18}) \geq 2h(\overline{\Gamma}_s) + 2\delta_0(\overline{\Gamma}_s) + 6\delta_1(\overline{\Gamma}_s)$$

holds.

Proof. By Proposition 3.7 in [66], we have

$$\mathrm{Ind}_s(f) \geq h(\overline{\Gamma}_s) + 2\delta_1(\overline{\Gamma}_s),$$

where $h(\cdot)$ is defined in Equation (3.1). We prove the assertion by combining this with Proposition 3.3.3. □

Let $\overline{H}$ be the closure of the hyperelliptic locus of $\mathcal{M}_3$ in $\overline{\mathcal{M}}_3$. Pulling back the line bundle $\mathcal{O}_{\mathcal{M}_3}(\overline{H})$ and its canonical section along the classifying map $S \to \overline{\mathcal{M}}_3$, we can define the multiplicity $\mathrm{mult}_s\overline{H}$.

Proposition 3.3.5. *With the notation above, then we have*

$$\mathrm{Ind}_s(f) = \mathrm{mult}_s\overline{H} + 2\delta_1(\overline{\Gamma}_s).$$

Proof. See Proposition 9.6 in [13]. □

3.3.3 An unboundedness property of $\langle\Delta, \Delta\rangle$

In this subsection, we still write S for the spectrum of a discrete valuation ring. We denote the closed point of S by s. Recall Definition 1.4.18 and the paragraph after it for some graph-theoretic terminology.

Definition 3.3.6. *We say a genus 3 pm-graph $\overline{\Gamma} = (G, w, \mathfrak{q})$ satisfies Condition ($\mathfrak{H}$) if $\overline{\Gamma}$ is equivalent to a pm-graph $\overline{\Gamma}' = (G', w', \mathfrak{q}')$ such that*

(1) $\overline{\Gamma}'$ has no eliminable vertices,

(2) G' is the wedge sum of trees, type $1I$ or type $1II$ graphs in Table 3.1.

Proposition 3.3.7. *Let $C \to S$ be a genus 3 stable curve with smooth non-hyperelliptic generic fiber. If the dual graph $\overline{\Gamma}_s$ satisfies Condition ($\mathfrak{H}$), then we have*

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq 0,$$

where strict positivity holds if $\overline{\Gamma}_s$ is not trivial.

Proof. We mainly use the inequality in Proposition 3.3.4. Since the functions $h(\cdot)$, $\delta_0(\cdot)$ and $\delta_1(\cdot)$ are additive (Example 1.4.22 and Lemma 3.2.7), it remains to prove the assertion for trees, type $1I$ graphs and type $1II$ graphs in Table 3.1.

Claim 3.3.8. *If $\overline{\Gamma}_s$ is a tree, then $\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{1}{21}\delta(\overline{\Gamma}_s)$.*

Proof of claim: Table 3.1 contains all possible tree pm-graph of genus 3, then we have $\lambda(\overline{\Gamma}_s) = \frac{2}{7}\delta(\Gamma_s)$. A tree graph has no edges of type 0 thus $\delta_1(\overline{\Gamma}_s) = \delta(\overline{\Gamma}_s)$, so we obtain

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{6\delta(\overline{\Gamma}_s)}{18} - \frac{2}{7}\delta(\overline{\Gamma}_s) = \frac{1}{21}\delta(\overline{\Gamma}_s)$$

by Proposition 3.3.4. CLAIM PROVEN

Claim 3.3.9. *If $\overline{\Gamma}_s$ is of type $1I$ or $1II$ in Table 3.1, then $\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) > 0$.*

Proof of claim: For type $1I$ in Table 3.1, it is easy to see

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{a}{9} - \frac{3a}{28} = \frac{a}{252} > 0.$$

Now we consider $1II$ in Table 3.1. If we write $m_1, m_2 \in \mathbb{Z}_{>0}$ for the thicknesses of the two nodal points in C_s, then we get

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq \frac{m_1+m_2}{252} + \frac{\min\{m_1, m_2\}}{9} - \frac{m_1 m_2}{7(m_1+m_2)}$$

by Proposition 3.3.4 and Table 3.2. We assume $m_1 \geq m_2$. If we denote $\frac{m_1}{m_2}$ by m_3, then the right side of the inequality above becomes

$$\begin{aligned} & m_2 \cdot \frac{m_3^2 + 2m_3 + 1 + 28(1+m_3) - 36m_3}{252(1+m_3)} \\ =& m_2 \cdot \frac{m_3^2 - 6m_3 + 29}{252(1+m_3)} \\ =& m_2 \cdot \frac{(m_3-3)^3 + 20}{252(1+m_3)}, \end{aligned}$$

which proves the positivity. CLAIM PROVEN

Thus we have proved the proposition. □

The main tool we used in the proof of the last proposition is Proposition 3.3.4. This lower bound is not enough for our purposes if the dual graph contains more than 1 cycle (for example, the type $2III$ in [9]). However, even for type $2III$, we can prove

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq 0$$

where the equality holds when all edges are of the same length. Thus we would like to believe that the positivity holds in general. However, the inequality in Proposition 3.3.4 is not enough for this goal in the general case.

Conjecture 3.3.10. *Let $C \to S$ be a genus 3 stable curve whose generic fiber is non-hyperelliptic and smooth. We conjecture*

$$\frac{1}{18}\mathrm{ord}_s(\chi'_{18}) - \lambda(\overline{\Gamma}_s) \geq 0,$$

where strict positivity holds if $\overline{\Gamma}_s$ is not a one-point graph.

The theory of stable curves over Dedekind schemes can be extended to a complex manifold analogue. Let $\mathbb{D}$ be the complex unit disk. For a family of complex curves $g_{\mathbb{C}} : Y \to \mathbb{D}$ which is smooth over $\mathbb{D}^*$, there is a ramified map $\mathrm{j} : \mathbb{D} \to \mathbb{D}$ such that the pullback of $g_{\mathbb{C}}$ along j has a stable model over $\mathbb{D}$. See Proposition 7.2 in [38] and Page 173 in [49].

Lemma 3.3.11. *Let $f : \mathcal{Y} \to \mathbb{D}$ be a generically non-hyperelliptic stable curve of genus 3 that is smooth over $\mathbb{D}^*$. We consider χ'_{18} as a rational section of the line bundle $\mathcal{L}_f^{\otimes 18}$ on $\mathbb{D}$. Then the following asymptotics*

$$-\frac{1}{18}\mathrm{log}\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathcal{Y}_t) - \lambda(\mathcal{Y}_t) \sim -\left(\frac{1}{18}\mathrm{ord}_0(\chi'_{18}) - \lambda(\overline{\Gamma}_0)\right)\mathrm{log}|t|$$

holds as $t \to 0$, where the $\|\cdot\|_{\mathrm{Hdg}}$ is defined in Equation (2.7). The symbol $\sim$ here means that the difference of both sides can be extended to a continuous function on $\mathbb{D}^*$.

Proof. By Proposition 7.4 in [13], this is equivalent to

$$\lambda(\mathcal{Y}_t) \sim -\lambda(\overline{\Gamma}_0)\mathrm{log}|t| - \frac{1}{2}\mathrm{logdet}\,\mathrm{Im}\,\Omega(t),$$

as $t \to 0$. This asymptotic formula for λ was proven by R. de Jong and F. Shokrieh as Theorem C in [15]. □

Let $\{p_m\}_{m\in\mathbb{N}^+}$ be a family of points in the orbifold $\overline{\mathcal{M}}_3(\mathbb{C})$. Let $f : U \to \overline{\mathcal{M}}_3(\mathbb{C})$ be an étale map such that $\{p_{m+n_0}\}_{m\in\mathbb{N}^+} \subset f(U)$ for some positive integer n_0. Each point p_{m+n_0} can have several preimages on U along f. If there is a preimage p'_{m+n_0} of p_{m+n_0}

for each $m \in \mathbb{N}^+$ such that $\{p'_{m+n_0}\}_{m\in\mathbb{N}^+}$ converges to a point p'_s on U in the Euclidean topology, we say the family of points $\{p_m\}_{m\in\mathbb{N}^+}$ *converges* to the point $f(p'_s)$ on $\overline{\mathcal{M}}_3(\mathbb{C})$. If a family of points on $\overline{\mathcal{M}}_3(\mathbb{C})$ converges, then the converging point is well-defined (does not depend on the choices of f and the family of preimages).

Theorem 3.3.12. *Let $\{L_m\}_{m\in\mathbb{N}^+}$ be a family of smooth non-hyperelliptic curves of genus 3 over $\mathbb{Q}$. If the following properties hold:*

(1) considering $\{L_m \otimes_{\mathbb{Q}} \mathbb{C}\}_{m\in\mathbb{N}^+}$ as a family of points in $\mathcal{M}_3(\mathbb{C})$, this family of points lies on a curve in $\overline{\mathcal{M}}_3(\mathbb{C})$ and converges to a point in $\overline{\mathcal{M}}_3(\mathbb{C})\backslash\mathcal{M}_3(\mathbb{C})$ which has a non-trivial dual graph satisfying Condition ($\mathfrak{H}$),

(2) the dual graphs of their stable models (which exist over finite extensions of the base field $\mathbb{Q}$, see Theorem 1.1.16) over finite places satisfy Condition ($\mathfrak{H}$),

then their heights of canonical Gross-Schoen cycles $\langle\Delta_m, \Delta_m\rangle$ go to infinity.

Proof. We assume that L_m has semistable reduction over k_m with $[k_m : \mathbb{Q}] < +\infty$ for all $m \in \mathbb{N}^+$. Then we can decompose the height $\langle\Delta_m, \Delta_m\rangle$ with the formula in Theorem 3.3.2:

$$\begin{aligned}\langle\Delta_m, \Delta_m\rangle =& \frac{21}{[k_m : \mathbb{Q}]}\left(\sum_{v\in M(k_m)_0}\left(\frac{1}{18}\mathrm{ord}_v(\chi'_{18}) - \lambda(L_{m,v})\right)\log Nv\right)\\ &- \frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(L_m) - \lambda(L_m).\end{aligned}$$

Condition (2) implies that the contribution from finite places is non-negative (Proposition 3.3.7). It remains to show that the contribution from the infinite place $\mathbb{Q} \to \mathbb{C}$ goes to infinity as $m \to \infty$.

By Condition (1), after discarding finitely many curves in $\{L_m\}$, we can assume that there is a family of complex genus 3 curves $\mathfrak{f} : \mathcal{X} \to \mathbb{D}$ such that:

(1) $\mathfrak{f}$ is smooth over $\mathbb{D}^*$, and is singular at the centre of $\mathbb{D}$,

(2) there exists a series of points $\{t_m\}_{m\in\mathbb{N}^+}$ on $\mathbb{D}^*$ approaching to the centre as $m \to \infty$ such that $L_m \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathcal{X}_{t_m}$.

Taking a suitable ramified map $[n] : \mathbb{D} \to \mathbb{D}$ defined by $t \to t^n$, we can pass to a stable model $\mathfrak{f}' : \mathcal{X}' \to \mathbb{D}$ of $\mathfrak{f}$ and a family of points t'_m such that $[n](t'_m) = t_m$. Condition (1) implies that the fiber of $\mathfrak{f}'$ at the origin 0 is a singular stable curve satisfying Condition ($\mathfrak{H}$). By Lemma 3.3.11, we have

$$-\frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathcal{X}'_{t'_m}) - \lambda(\mathcal{X}'_{t'_m}) \sim -\left(\frac{1}{18}\mathrm{ord}_0(\chi'_{18}) - \lambda(\overline{\Gamma}_0)\right)\log|t'_m|.$$

Thus we get

$$-\frac{1}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathcal{X}_{t_m}) - \lambda(\mathcal{X}_{t_m}) \sim -\frac{1}{n}\cdot\left(\frac{1}{18}\mathrm{ord}_0(\chi'_{18}) - \lambda(\overline{\Gamma}_0)\right)\log|t_m|.$$

According to Proposition 3.3.7, we can say $\langle\Delta_m, \Delta_m\rangle \to \infty$ as $m \to \infty$. □

Remark 3.3.13. *If we can prove Conjecture 3.3.10, then we can discard mentioning the Condition ($\mathfrak{H}$) in (1) and remove the condition (2) in Theorem 3.3.12.*

3.3.4 An application of Theorem 3.3.12

In this subsection, we give an explicit family of curves that satisfies the conditions in Theorem 3.3.12.

We define a family of plane curves by

$$\{C_n : y^4 = x^4 - (4n-2)x^2 + 1\}_{n\in\mathfrak{N}},$$

where $\mathfrak{N} = \{n \in \mathbb{N}^+ | n \equiv 2 (\mathrm{mod}\, 3), n \not\equiv 0, 1 (\mathrm{mod}\, 2^5)\}$.

J. Guàrdia proved that the dual graphs of the stable models of these curves over K_n (Notation 3.2 in [32]) are in Table 3.1 (all pm-graphs in Table 3.1 satisfy Condition ($\mathfrak{H}$)) for all finite places. This means that Condition (2) in Theorem 3.3.12 is satisfied.

As a compact Riemann surface, the curve C_n is isomorphic to $D_{1/n} : y^4 = x(x-1)(x-\frac{1}{n})$. The family of curves $D_\kappa : y^4 = x(x-1)(x-\kappa)$ over $\mathbb{D}$ (parametrized by κ) is smooth over $\mathbb{D}^*$ and singular at $\kappa = 0$ (the tacnodal curve $y^4 = x^2(x-1)$).

Lemma 3.3.14. *Let $\mathfrak{D}_\kappa$ be the stable reduction of $D_\kappa \to \mathbb{D}$. Then $\mathfrak{D}_0$ is the union of two copies of the elliptic curve E given by the equation $y^2 = x^3 - x$, joined at two points.*

Proof. See Proposition 8 in [34]. □

By the lemma above, Condition (1) in Theorem 3.3.12 is also satisfied. Thus the heights of canonical Gross-Schoen cycles of $\{C_n\}_{n\in\mathfrak{N}}$ go to infinity as $n \to +\infty$.

Remark 3.3.15. *The unboundedness of $\langle\Delta, \Delta\rangle$ for $\{C_n\}_{n\in\mathfrak{N}}$ was first proved by R. de Jong in [13]. When the paper was written, the equality in Lemma 3.3.11 was only established when the dual graph of $\mathcal{Y}_0$ is of type $1II$.*

Chapter 4

Explicit Computations

This chapter is devoted to explicit computations for the height $\langle\Delta, \Delta\rangle$ of the following plane curve over $\mathbb{Q}$, using SageMath and Magma. As far as we know, this is the first attempt to numerically compute $\langle\Delta, \Delta\rangle$ for a non-hyperelliptic curve of genus $g > 2$.

$$\boxed{\text{Main Curve : } -X^3Y + X^2Y^2 - XY^2Z + Y^3Z + X^2Z^2 + XZ^3 = 0}$$

We denote this plane curve over $\mathbb{Z}$ by $\mathfrak{C}$ and we use subscripts to distinguish its base changes, for example $\mathfrak{C}_{\mathbb{Z}_p}$, $\mathfrak{C}_{\mathbb{Q}}$ and so on. We denote the affine patch $Z = 1$ of $\mathfrak{C}$ by $U_{\mathfrak{C}}$. We denote the polynomial on the left hand side by $\mathfrak{F}$ and we write $\mathfrak{f}$ for $\mathfrak{F}(x, y, 1)$.

There are several reasons for choosing this curve. First, the curve $\mathfrak{C}$ is a stable model of $\mathfrak{C}_{\mathbb{Q}}$ over $\mathbb{Z}$. Second, all its residue fields at singular points are in the type $\mathbb{F}_p$ for some prime p (instead of the type $\mathbb{F}_{p^m}$ for some integer $m > 1$), which makes it easy to compute its thicknesses (Subsection 4.3). Third, it has no bad hyperelliptic reduction, thus we do not need to compute the hyperelliptic multiplicity in Corollary 4.4.4. The thicknesses and the hyperelliptic multiplicity are the main restrictions of our computation method. Other parts of our computation (like these numerical approximations in Sections 4.5-4.7) can be used for a general curve.

We sketch our plan of the computation as follows.

In Section 4.1, 4.2 and 4.3, we prove that $\mathfrak{C}_{\mathbb{Q}}$ has semistable reduction over $\mathbb{Q}$ and that $\mathfrak{C}$ is a regular stable model for it. Thus we can apply Theorem 3.3.2.

The reduction types of $\mathfrak{C}$ are summarized in Proposition 4.2.4 and Corollary 4.3.9. In Section 4.4, we show that all invariants except the infinite $\lambda(\mathfrak{C}_{\mathbb{C}})$ in Theorem 3.3.2

are computable. By Remark 1.5.7, to compute $\lambda(\mathfrak{C}_{\mathbb{C}})$ we only need to compute $\varphi(\mathfrak{C}_{\mathbb{C}})$ and $\delta(\mathfrak{C}_{\mathbb{C}})$.

In Section 4.5, we show how to evaluate the theta function $\|\theta\|_{g-1}$ on $\mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}})$. Using $\|\theta\|_{g-1}$, we numerically compute $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in Section 4.6. In Subsection 4.7.1, we compute another invariant $T(\mathfrak{C}_{\mathbb{C}})$. With $\log(S(\mathfrak{C}_{\mathbb{C}}))$ and $T(\mathfrak{C}_{\mathbb{C}})$, the invariant $\delta(\mathfrak{C}_{\mathbb{C}})$ can be computed by Theorem 4.7.3. Now, it remains to compute $\varphi(\mathfrak{C}_{\mathbb{C}})$.

By Theorem 4.7.7, we reduce the problem to the computation of $H(\mathfrak{C}_{\mathbb{C}})$. In the second half of Subsection 4.7.2, we explain the strategy for computing $H(\mathfrak{C}_{\mathbb{C}})$.

The results of our computation are summarized in Section 4.8. In Section 4.9, we explain the reliability of our results.

Longer sections of the code in this chapter can be found in Appendix I-IX.

4.1 Smoothness and bad reduction of $\mathfrak{C}$

In Subsection 4.1.1, we will show that $\mathfrak{C}_{\mathbb{Q}}$ is a smooth curve over $\mathbb{Q}$. In Subsection 4.1.2, we show $\mathfrak{C}_{\mathbb{Z}_p}$ has bad reduction at $p = 29, 163$ and good reduction at other primes.

4.1.1 Smoothness at the infinite place

By the Jacobian criterion for smoothness, we need to show that:

$$\sqrt{(\mathfrak{F}, \mathfrak{F}_X, \mathfrak{F}_Y, \mathfrak{F}_Z)} = (X, Y, Z).$$

This can be checked in SageMath by the following lines, thus $\mathfrak{C}_{\mathbb{Q}}$ is a smooth curve over $\mathbb{Q}$.

```
R.<x,y,z>=PolynomialRing(QQ)
f=-x^3*y + x^2*y^2 - x*y^2*z + y^3*z + x^2*z^2 + x*z^3
I=R.ideal(f,derivative(f,x),derivative(f,y),derivative(f,z))
I.radical()
Ideal (z, y, x) of Multivariate Polynomial Ring in x, y, z over Rational
    Field
```

4.1.2 Bad reduction at finite places

We first consider the reduction of the affine patch $U_{\mathfrak{C}}$. Since $U_{\mathfrak{C}} \simeq \mathrm{Spec}(\frac{\mathbb{Z}[x,y]}{(\mathfrak{f})})$ has smooth generic fiber, by the Jacobian criterion, the ideal $I_{\mathbb{Q}} = (\mathfrak{f}, \frac{\partial \mathfrak{f}}{\partial x}, \frac{\partial \mathfrak{f}}{\partial y})$ is the unit ideal in $\mathbb{Q}[x, y]$. This means that if we consider $I = (\mathfrak{f}, \frac{\partial \mathfrak{f}}{\partial x}, \frac{\partial \mathfrak{f}}{\partial y})$ as an ideal in $\mathbb{Z}[x, y]$, then $I \cap \mathbb{Z} = (n)$ for some positive integer n. Let p be a prime, then we can see that $p \nmid n$ if and only if the reduction of I to $\mathbb{F}_p[x, y]$ contains a unit $\overline{n} \in \mathbb{F}_p^*$ which is equivalent to saying that $U_{\mathfrak{C}}$ has good reduction at p. Thus $U_{\mathfrak{C}}$ has bad reduction exactly at the prime divisors of n.

By the above discussion, we have positive integers n_1, n_2 and n_3 for three affine patches. The curve $\mathfrak{C}$ has bad reduction exactly at the prime divisors of $n_1 n_2 n_3$. The following SageMath code can be used for computing the primes of bad reduction.

```
sage:PP.<x,y,z> = ProjectiveSpace(QQ,2)
sage:C = Curve(-x^3*y+x^2*y^2-x*y^2*z+y^3*z+x^2*z^2+x*z^3, PP)
sage:def MyBadPrimes(C):     #finding bad reduction primes
sage:  f = C.defining_polynomial()
sage:  RZ.<xZ,yZ,zZ> = PolynomialRing(ZZ, 3)
sage:  coeffs = f.coefficients()
sage:  dens = [c.denominator() for c in coeffs]
sage:  den = lcm(dens)
sage:  F = RZ(f*den)
sage:  Fx = F.derivative(xZ)
sage:  Fy = F.derivative(yZ)
sage:  Fz = F.derivative(zZ)
sage:  NaiveDisc = 1
sage:  for P in [[xZ,yZ,1],[xZ,1,zZ],[1,yZ,zZ]]:
sage:    I = ideal([g(P) for g in [F,Fx,Fy,Fz]])
sage:    G = I.groebner_basis()
sage:    n = G[len(G)-1]
sage:    NaiveDisc = lcm(n, NaiveDisc)
sage:  return [a[0] for a in factor(NaiveDisc)]
```

Bad reduction

Remark 4.1.1. *There exists an explicit formula for the discriminant of a plane quartic curve (Page 9 in [55]), and we can also find out the primes of bad reduction by factoring it. This computation is implemented in Magma (http://magma.maths.usyd.edu.au/magma/handbook/text/1547#17791).*

With the code above, we can obtain the following proposition.

Proposition 4.1.2. *The main curve $\mathfrak{C}$ has bad reduction at 29 and 163, and good reduction at every other prime.*

4.2 Semistability of $\mathfrak{C}$

In Subsection 4.2.1, we develop an algorithm for checking whether a singular point on a generically smooth plane curve over $\mathbb{Z}_p$ is a nodal singularity. In Subsection 4.2.2, we prove that $\mathfrak{C}_{\mathbb{Z}_p}$ is semistable over $p = 29$ or 163. Notions can be found in Section 1.1.

4.2.1 Algorithm for checking nodal singularities

By Definition 1.1.14, a generically smooth curve of genus $g \geq 1$ over $\mathbb{Z}_p$ is semistable if its geometric fiber at $\overline{\mathbb{F}}_p$ has only nodal singularities and all components of arithmetic genus 0 intersect other components in at least 2 points.

Let C be a generically smooth plane curve over $\mathbb{Z}$. Similar to Subsection 4.1.2, we check the nodal singularities on one affine patch at one time. We assume the curve is defined by $f(x, y) = 0$ on the affine patch $Z = 1$ of $\operatorname{Proj} \mathbb{Z}[X, Y, Z]$, and we denote the reduction of C and f at p by $C_{\mathbb{F}_p}$ and f_p. We sketch our strategy of checking nodal singularities of $C_{\mathbb{F}_p}$ as follows:

(1) We first check that the singular locus of $C_{\mathbb{F}_p}$ is 0-dimensional. It is possible that a curve over $\mathbb{Z}_p$ is smooth over $\mathbb{Q}_p$ but totally singular over $\mathbb{F}_p$, for example, the plane curve defined by $X^2 + pXY + pY^2 = 0$.

(2) If (1) is true, then we find out the coordinates of singular points in $C_{\overline{\mathbb{F}}_p}$. Since we start from base field $\mathbb{F}_p$, we will extend it to a field $\mathbb{F}_{p^D}$ such that all singular points have coordinates in $\mathbb{F}_{p^D}$.

(3) Fixing an arbitrary singular point of $C_{\overline{\mathbb{F}}_p}$, for example $p_s = (x_0, y_0)$, we translate p_s to the origin of the affine patch $Z = 1$. This induces a new polynomial $g(x, y) = f(x + x_0, y + y_0)$.

(4) After (3), we will check the singularity type of $g(x, y)$ at $O = (0, 0)$ in $\mathbb{A}^2$ by its non-zero homogeneous part of lowest degree.

For step (1), computing the height of an ideal is implemented in SageMath.

For step (2), computing associated primes is implemented in SageMath and the following lemma implies that the associated prime ideals of $\frac{\mathbb{F}_{p^D}[x,y]}{(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})}$ are exactly the singular points of $C_{\overline{\mathbb{F}}_p}|_{Z=1}$.

Lemma 4.2.1. *Let A be a Noetherian ring. Then the minimal prime ideals of A belong to $Ass(A)$.*

Proof. See Corollary 7.1.3 in [48]. $\square$

For step (4), we first point out $g \in \mathbb{F}_{p^D}[x, y]$ is square-free since otherwise the singular locus is of dimension 1 (contradicting (1)). Its singularity type at O is determined by its non-zero homogeneous part of lowest degree g_{min}. To be more precise, O is a smooth point if $\deg(g_{min}) = 1$, and O is a nodal (resp. cusp) point if $\deg(g_{min}) = 2$ and g_{min} can be factored into a product of two different (resp. the same) straight lines ([27] Page 66). This is also where we use the condition that C is a plane curve.

Example 4.2.2. *Let k be a field with* $\operatorname{char} k \neq 2, 3$. *The two plane curves $E_n : y^2 - x^3 - x^2 = 0$ and $E_c : y^2 - x^3 = 0$ have nodal and cusp points at the origin respectively. This can be observed by their homogeneous parts of lowest degree, which are $(y - x)(y + x)$ and y^2.*

The following is the pseudocode for our algorithm. The SageMath code can be found in Appendix IV.

Algorithm 1 (1) Checking the singular dimension

Input: f: a polynomial in $\mathbb{F}_p[x, y]$
Output: d: the dimension of singular locus of $f = 0$

```
Taking partial derivative f_x and f_y of f.
I = (f, f_x, f_y)
d = dimension of I = 2 − ht(I)
return d
```

Algorithm 1 (2) Finding out singular points

Input: $I = (f, f_x, f_y)$: an ideal in $\mathbb{F}_p[x, y]$ of height 2
Output: LST: list of singular points

```
Fieldext = 1
find = False
while find = False do
    primeideals = associated prime ideals of I
    for P in primeideals do
        if elements in Gröbner basis of P are not of degree 1 then
            break the for iteration
        end if
        find = True
        D = Fieldext
    end for
    Fieldext = Fieldext + 1
end while
change base field to F_{p^D}
LST = list of associated prime ideals of I in F_{p^D}[x, y]
```

return LST

Algorithm 1 (3) Checking the singularity type

Input: $\mathfrak{m} = (x-a, y-b)$: an element in LST with a and b in $\mathbb{F}_{p^D}$
Output: local behaviour of f at the point $\mathfrak{m}$

```
G(x,y) = F(x+a, y+b)
take H to be the non-zero homogeneous part of G in lowest degree
if degree(H) > 2 then
   result=Higher singularity
else
   result=the factorization of H over F_{p^{2D}}
end if
return result
```

From the output, we check the factorization of H manually for its singularity type.

Remark 4.2.3. *We can also check if a plane curve is a nodal curve by Magma (*$http : //magma.maths.usyd.edu.au/magma/handbook/text/1411\#15882$*).*

4.2.2 Semistability of $\mathfrak{C}$

With the algorithm in Subsection 4.2.1, we can get the following result of $\mathfrak{C}$.

Proposition 4.2.4. $\mathfrak{C}_{\mathbb{Z}_{29}}$ *has exactly one singular point at* $(X+3Z, Y-2Z, 29)$ *which is a nodal point. And* $\mathfrak{C}_{\mathbb{Z}_{163}}$ *has exactly one singular point at* $(X-49Z, Y-36Z, 163)$ *which is a nodal point. All other fibers are smooth.*

Remark 4.2.5. *The degree 2 parts of* $\mathfrak{C}$ *at these two points (*H *in the last algorithm) are* $-6x^2+3xy-11y^2$ *(for 29) and* $80x^2-56xy+15y^2$ *(for 163) respectively.*

Corollary 4.2.6. $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ *is stable and geometrically irreducible for every prime* p*. The geometric genus of* $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ *at* $p = 29, 163$ *is* 2*. Thus* $\mathfrak{C}$ *is a stable curve over* $\mathbb{Z}$*.*

Proof. When $p \neq 29$ or 163, the curve $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is a smooth plane quartic curve and thus stable and geometrically irreducible.

For $p = 29$ or 163, if $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ has multiple irreducible components, then each component corresponds to a polynomial $\mathfrak{F}_i$ such that $\mathfrak{F} = \prod \mathfrak{F}_i$. The polynomials $\mathfrak{F}_i$ are different otherwise $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ can not have only nodal singularities. If all $\mathfrak{F}_i$ are of degree 1, then the curve $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is the union of 4 straight lines in $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$, which can not be nodal and have exactly 1 singular point at the same time. When $\deg(\mathfrak{F}_{i_0}) > 1$ for some $\mathfrak{F}_{i_0}$, Bézout's theorem shows that there are more than 1 singular points which contradicts the fact that $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ has only 1 nodal point at $p = 29$ and 163. This shows that $\mathfrak{C}_{\overline{\mathbb{F}}_{29}}$ and $\mathfrak{C}_{\overline{\mathbb{F}}_{163}}$ are geometrically irreducible. Furthermore, their normalizations are curves of genus 2 which means that $\mathfrak{C}_{\overline{\mathbb{F}}_{29}}$ and $\mathfrak{C}_{\overline{\mathbb{F}}_{163}}$ are stable curves. $\square$

4.3 Thickness of $\mathfrak{C}$ at nodal points

In this section, we show that the thickness of $\mathfrak{C}$ at the two singular points is 1. This implies that $\mathfrak{C}$ is a regular stable model for $\mathfrak{C}_{\mathbb{Q}}$. We introduce the Fitting ideal in Subsection 4.3.1 and compute the thickness of $\mathfrak{C}$ in Subsection 4.3.2. The content of Subsection 4.3.3 is not used in our computation for $\mathfrak{C}$ but might be helpful for other curves.

4.3.1 Fitting ideal

In this section, we introduce the Fitting ideal and state its relation to thickness. Details can be found in Tag 0C3C and Subsections 2.2-2.4 of [5].

Definition 4.3.1. *If R is a commutative ring with 1 and M is a finitely generated R-module, then we have a free resolution of M*

$$\bigoplus_{l\in\mathfrak{L}} R \xrightarrow{\phi} \bigoplus_{j\in\mathfrak{J}} R \longrightarrow M \to 0 \tag{4.1}$$

where $\mathfrak{J}$ is a finite index set and $\mathfrak{L}$ can be infinite. The map ϕ corresponds to a $\#\mathfrak{J}\times\#\mathfrak{L}$ matrix N (might be an infinite matrix) and we define the k-th Fitting ideal $Fit_k^R(M)\subset R$ to be the ideal generated by all the $(\#\mathfrak{J}-k)\times(\#\mathfrak{J}-k)$ minors of N.

Remark 4.3.2. *The Fitting ideals are independent of the choice of the resolution (see Tag 07Z8).*

There is also a scheme version for the Fitting ideal. Lemma 4.3.3 shows that the Fitting ideal behaves well under localization and gluing.

Lemma 4.3.3. *Let X be a scheme. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type, then for each non-negative integer i, there exists a unique quasi-coherent sheaf of ideals $Fit_i^X(\mathcal{F})$ such that on each affine $U=\mathrm{Spec}(A)$ étale over X, we have*

$$Fit_i^X(\mathcal{F})(U) = Fit_i^A(\mathcal{F}(U)) \subset A.$$

Proof. Tag 0CZ3. $\square$

Lemma 4.3.4. *If $X\to S$ is a scheme morphism of finite type and $S'\to S$ is an affine morphism, then for any non-negative integer i, we have*

$$Fit_i^X(\Omega_{X/S})\otimes_S S' \simeq Fit_i^{X_{S'}}(\Omega_{X_{S'}/S'})$$

as $O_{S'}$-modules.

Proof. According to Lemma 4.3.3, we only need to consider the affine case. Let's assume that B is a finitely generated A-algebra, then we have a resolution as in Definition 4.3.1

$$\oplus_{i\in\mathfrak{L}}B \xrightarrow{\phi} \oplus_{j\in\mathfrak{J}}B \longrightarrow \Omega_{B/A} \to 0.$$

For a ring homomorphism $A \to A'$, since $\otimes_A A'$ is a right exact functor and the cotangent bundle is stable under base change, we have

$$\oplus_{i\in\mathfrak{L}}B_{A'} \xrightarrow{\phi'} \oplus_{j\in\mathfrak{J}}B_{A'} \longrightarrow \Omega_{B\otimes_A A'/A'} \to 0.$$

Since the matrix defining $Fit_i^B(\Omega_{B/A})$ is not changed after applying the functor $\otimes_A A'$, we have

$$Fit_i^B(\Omega_{B/A}) \otimes_A A' \simeq Fit_i^{B\otimes_A A'}(\Omega_{(B\otimes_A A')/A'})$$

as A'-modules. □

For a semistable curve X over S, the first Fitting ideal of the cotangent bundle $\Omega_{X/S}$ cuts out a closed subscheme of X which we denote by $\mathrm{Sing}(X/S)$. The complement of $\mathrm{Sing}(X/S)$ is exactly the smooth locus of $X \to S$. Remark 2.14 in [5] shows the following relation between the thicknesses of singular points on X and $\mathrm{Sing}(X/S)$.

Lemma 4.3.5. *Let S be the spectrum of a strict Henselian discrete valuation ring A with a uniformizer t. If $X \to S$ is a semistable curve with smooth generic fiber, then*

$$\mathrm{Sing}(X/S) \simeq \mathrm{Spec}\left(\prod_{e\in\mathfrak{N}} A/(t^{\alpha(e)})\right)$$

where $\mathfrak{N}$ is the set of nodal points on $X \otimes_S \mathrm{Spec}(A/(t))$ and $\alpha(e)$ is the thickness at e.

Example 4.3.6. *Let $\mathbb{Z}_p^{un}$ be the unramified closure of $\mathbb{Z}_p$. For the semistable elliptic curve*

$$C : Y^2Z - X^3 - aX^2Z - cZ^3 = 0$$

over $\mathbb{Z}_p^{un}$, where $p > 3$, $a \in (\mathbb{Z}_p^{un})^*$ and $c \in p\mathbb{Z}_p^{un}\backslash\{0\}$, we have

$$\mathrm{Sing}(C/S) \simeq \mathrm{Spec}(\mathbb{Z}_p^{un}/p^{\mathrm{ord}_p(c)}\mathbb{Z}_p^{un}).$$

Thus we conclude that C has a nodal singularity of thickness $\mathrm{ord}_p(c)$ at (x, y, p) on the affine patch $Z = 1$.

Proof. It is easy to show that there is only one nodal point (x, y, p) on the affine patch $Z = 1$. We denote $\frac{\mathbb{Z}_p^{un}[x,y]}{y^2-x^3-ax^2-c}$ by R and denote $y^2 - x^3 - ax^2 - c$ by f. By Definition 4.3.1 and Lemma 4.3.3, we make the following resolution of Ω_C on $Z = 1$:

$$R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} \Omega_{R/\mathbb{Z}^{un}} \longrightarrow 0$$

where ϕ sends 1 to $f_x \oplus f_y$ and ψ sends $u \oplus v$ to $udx + vdy$. This resolution just comes from the construction of the cotangent bundle, and thus is exact. By Lemma 4.3.3, the first Fitting ideal of this curve on $Z = 1$ is given by the ideal $I = (f_x, f_y)$ in R. Then we have

$$R/I \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{(y^2 - x^3 - ax^2 - c, 2y, -3x^2 - 2ax)} \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{(x,y,c)} \simeq \mathbb{Z}_p^{un}/(c).$$

By Lemma 4.3.5, we have

$$\mathrm{Sing}(C/\mathbb{Z}_p^{un}) \simeq \mathrm{Spec}(\mathbb{Z}_p^{un}/p^{\mathrm{ord}_p(c)}\mathbb{Z}_p^{un}),$$

and this shows the result. ☐

4.3.2 Thickness of $\mathfrak{C}$

Recall that in Section 4.2, we showed the following result for $\mathfrak{C}$:

- $\mathfrak{C}_{\mathbb{Z}_{29}}$ has exactly one nodal point at $(X+3, Y-2, 29)$ on the affine patch $Z = 1$ with residue field $\mathbb{F}_{29}$.
- $\mathfrak{C}_{\mathbb{Z}_{163}}$ has exactly one nodal point at $(X-49, Y-36, 163)$ on the affine patch $Z = 1$ with residue field $\mathbb{F}_{163}$.
- $\mathfrak{C}_{\mathbb{Z}}$ has no other singular points.

We start our computation on $\mathfrak{C}$ by the observation that all nodal points are on $U_{\mathfrak{C}}$. By Lemma 4.3.3, we can compute the Fitting ideal $Fit_1^{\mathfrak{C}}(\Omega_{\mathfrak{C}/\mathbb{Z}})(U_{\mathfrak{C}})$ on $U_{\mathfrak{C}}$ by the following resolution:

$$R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} \Omega_{\mathfrak{C}/\mathbb{Z}}(U_{\mathfrak{C}}) \longrightarrow 0 \tag{4.2}$$

where $R = \frac{\mathbb{Z}[x,y]}{(\mathfrak{f})}$, the map ϕ sends 1 to $\mathfrak{f}_x \oplus \mathfrak{f}_y$ and ψ sends $u \oplus v$ to $udx + vdy$.

This computation of the ideal $I = (\mathfrak{f}, \mathfrak{f}_x, \mathfrak{f}_y)$ can be carried out in SageMath by the following code:

```
R.<x,y>=PolynomialRing(ZZ)
f=-x^3*y+x^2*y^2-x*y^2+y^3+x^2+x
fx=derivative(f,x)
fy=derivative(f,y)
I=R.ideal([f,fx,fy])
B=I.groebner_basis()
B
[x + 3048, y + 2898, 4727]
```

```
factor(4727)
29 * 163
```

Thickness

Thus we have

$$\frac{R}{Fit_1^R(\Omega_{R/\mathbb{Z}})} \simeq \frac{\mathbb{Z}[x,y]}{(\mathfrak{f},\mathfrak{f}_x,\mathfrak{f}_y)} \simeq \frac{\mathbb{Z}[x,y]}{(x+3048, y+2898, 29\times 163)}. \tag{4.3}$$

By Lemma 4.3.4, we can tensor Equation (4.3) with $\otimes_{\mathbb{Z}}\mathbb{Z}_p^{un}$ for $p=29$ or 163 and get

$$\mathrm{Sing}(\mathfrak{C}_{\mathbb{Z}_{29}^{un}}) \simeq \mathbb{Z}_{29}^{un}/(29\cdot\mathbb{Z}_{29}^{un}),$$
$$\mathrm{Sing}(\mathfrak{C}_{\mathbb{Z}_{163}^{un}}) \simeq \mathbb{Z}_{163}^{un}/(163\cdot\mathbb{Z}_{163}^{un}).$$

Since $\mathbb{Z}_p^{un}$ is a strict Henselian discrete valuation ring, we conclude that the thickness of $\mathfrak{C}$ at these two points are both 1 by Lemma 4.3.5. We can summarize our computation into the following proposition.

Proposition 4.3.7. *$\mathfrak{C}_{\mathbb{Z}_{29}}$ has thickness 1 at the only nodal point, and $\mathfrak{C}_{\mathbb{Z}_{163}}$ has thickness 1 at the only nodal point.*

Corollary 4.3.8. *$\mathfrak{C}$ is the regular stable model of $\mathfrak{C}_{\mathbb{Q}}$ over $\mathbb{Z}$.*

Proof. By Corollary 4.2.6, the curve $\mathfrak{C}$ is stable. By Proposition 4.3.7, all singular points on $\mathfrak{C}$ have thickness 1, which means that $\mathfrak{C}$ is regular. □

Corollary 4.3.9. *The dual graphs of $\mathfrak{C}$ at 29 or 163 are of type $1I$ in Table 3.1 with the edge weighted by 1.*

Proof. An application of Corollary 4.2.6 and Proposition 4.3.7. □

4.3.3 Further discussion of thickness

For a polynomial f, we write $f^{\deg\leq i}$ (resp. $f^{\deg>i}$) for the polynomial containing monomials of f in degree not bigger (resp. bigger) than i. For example, if $f = x^4+x^3y^2+5x^2+xy+y^3$, then $f^{\deg\leq 3} = 5x^2+xy+y^3$.

Proposition 4.3.10. *For an odd prime p, we choose $U \simeq \mathrm{Spec}(A)$ to be an affine open subscheme of a semistable curve C over $\mathbb{Z}_p^{un}$ where $A = \mathbb{Z}_p^{un}[x,y]/(f)$. If U has only 1 nodal point $O_A = (x,y,p)$ and*

$$f^{\deg\leq 2} = ax^2+bxy+cy^2+d$$

for a, b, c in $\mathbb{Z}_p^{un}$ and d in $p\mathbb{Z}_p^{un}\backslash\{0\}$, then the thickness of C at O_A is equal to the thickness of $V = \mathrm{Spec}(B)$ at $O_B = (x,y,p)$ where $B = \frac{\mathbb{Z}_p[x,y]}{(f^{\deg\leq 2}(x,y))}$.

Proof. By the criterion of singularity type for plane curves (Page 66 in [27]), point O_A is a nodal point on U if and only if O_B is a nodal point on V. The geometric fiber $V_{\overline{\mathbb{F}}_p}$ is the union of two straight lines l_1 and l_2 on the affine plane. The two straight lines are not parallel otherwise the origin can not be a nodal point. This means that O_B is the only singular point on V.

We denote the partial derivative of $f^{\deg>2}$ with respect to x and y by $(f^{\deg>2})_x$ and $(f^{\deg>2})_y$. By Lemma 4.3.5, we have

$$\frac{\mathbb{Z}_p^{un}}{(p^{\alpha(O_A)})} \simeq \frac{A}{(Fit_1^A(\Omega_{A/\mathbb{Z}_p^{un}}))} \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{I_A} =: R_U \tag{4.4}$$

$$\frac{\mathbb{Z}_p^{un}}{(p^{\alpha(O_B)})} \simeq \frac{B}{(Fit_1^B(\Omega_{B/\mathbb{Z}_p^{un}}))} \simeq \frac{\mathbb{Z}_p^{un}[x,y]}{I_B} =: R_V \tag{4.5}$$

where

$$I_A = (f, 2ax + by + (f^{\deg>2})_x, bx + 2cy + (f^{\deg>2})_y),$$
$$I_B = (f^{\deg\leq 2}, 2ax + by, bx + 2cy).$$

These isomorphisms shows that R_U and R_V are local rings.

By the definition of nodal singularity, the image of $b^2 - 4ac$ in $\frac{\mathbb{Z}_p^{un}}{p\mathbb{Z}_p^{un}} \simeq \overline{\mathbb{F}}_p$ does not vanish, which means that $b^2 - 4ac \in (\mathbb{Z}_p^{un})^*$. We can simplify I_A and I_B to be

$$I_A = (f, x + l(x,y), y + m(x,y)),$$
$$I_B = (f^{\deg\leq 2}, x, y) = (x, y, d),$$

where

$$l(x,y) = \frac{b(f^{\deg>2})_y - 2c(f^{\deg>2})_x}{b^2 - 4ac},$$

$$m(x,y) = \frac{b(f^{\deg>2})_x - 2a(f^{\deg>2})_y}{b^2 - 4ac},$$

are polynomials in $\mathbb{Z}_p^{un}[x,y]$.

According to isomorphisms in Equation (4.4) and (4.5), in order to show $\alpha(O_A) = \alpha(O_B)$, we just need to show $R_U \simeq R_V$. Since the completion of $\frac{\mathbb{Z}_p^{un}}{(p^t)}$ with respect to the maximal ideal is still itself, we just need to show $\widehat{R_U} \simeq \widehat{R_V}$. By Equation (4.4) and (4.5), we get

$$\widehat{R_U} = \lim_n \frac{\mathbb{Z}_p^{un}[[x,y]]}{I_A + (x,y,p)^n},$$
$$\widehat{R_V} = \lim_n \frac{\mathbb{Z}_p^{un}[[x,y]]}{I_B + (x,y,p)^n}.$$

Claim 4.3.11. *We have the following equality*

$$I_A + (x, y, p)^n = I_B + (x, y, p)^n \tag{4.6}$$

for every positive integer n.

Proof of claim: When $n = 1$, we can see $I_A \subset (x, y, p)$ and $I_B \subset (x, y, p)$ and thus the claim is trivial.

For $n > 1$, we first note that

$$I_B + (x, y, p)^n = (x, y, d, p^n) = (x, y, p^{\min(n, \mathrm{ord}_p d)}),$$

and Equation (4.6) is equivalent to $(x, y) \subset I_A + (x, y, p)^n$.

Now we show that $(x, y)^n \subset I_A + (x, y, p)^n$ implies $(x, y)^{n-1} \subset I_A + (x, y, p)^n$. We will show that $x^i y^{n-i-1} \in I_A + (x, y, p)^n$ for every integer i in $[0, n-1]$. Since either x's or y's exponent is positive, without loss of generality, we can assume $i \geq 1$. Then

$$x^i y^{n-i-1} = (x + l(x, y))x^{i-1}y^{n-i-1} - l(x, y) \cdot x^{i-1}y^{n-i-1},$$

where the degree of l is either equal to 0 or strictly bigger than 1. Since $x + l(x, y) \in I_A$ and $l(x, y) \cdot x^{i-1}y^{n-i-1} \in (x, y)^n \subset (x, y, p)^n$, we have

$$x^i y^{(n-i-1)} \in I_A + (x, y, p)^n$$

thus

$$(x, y)^{n-1} \subset I_A + (x, y, p)^n.$$

This procedure does not use the powers of p in the ideal. Repeating this procedure, we can finally show that (x, y) is contained in both sides in Equation (4.6), which implies

$$I_A + (x, y, p)^n = (x, y, p^{\min(n, \mathrm{ord}_p d)}) = I_B + (x, y, p)^n.$$

CLAIM PROVEN

By the claim, we have $\widehat{R_U} \simeq \widehat{R_V}$ which implies $\alpha(O_A) = \alpha(O_B)$. □

Corollary 4.3.12. *If $p^e \| d$ in $\mathbb{Z}_p^{un}$, the thickness of the curve C in Proposition 4.3.10 at the point $O_A = (x, y, p)$ is e.*

Proof. By Proposition 4.3.10, we only need to compute the thickness of $f^{\deg \leq 2}$ at the point $O_B = (x, y, p)$. By Lemma 4.3.5, we have the following on V

$$\frac{\mathbb{Z}_p^{un}}{(p^{\alpha(O_A)})} \simeq \frac{\mathbb{Z}_p^{un}[x, y]}{(f^{\deg \leq 2}, (f^{\deg \leq 2})_x, (f^{\deg \leq 2})_y)} \tag{4.7}$$

$$\simeq \frac{\mathbb{Z}_p^{un}[x, y]}{(ax^2 + bxy + cy^2 + d, 2ax + by, bx + 2cy)}. \tag{4.8}$$

By semistability of C, we have $b^2 - 4ac \in (\mathbb{Z}_p^{un})^*$, thus $(2ax + by, bx + 2cy) = (x, y)$. Substituting $(2ax + by, bx + 2cy) = (x, y)$ to Equation (4.8), we get

$$\frac{\mathbb{Z}_p^{un}[x,y]}{(ax^2 + bxy + cy^2 + d, 2ax + by, bx + 2cy)} \simeq \frac{\mathbb{Z}_p^{un}}{(d)},$$

which gives $\alpha(O_A) = e$. $\square$

Example 4.3.13. *In Proposition 4.3.10, f has no linear terms, and now we show that this requirement is essential. We assume that $U \simeq \operatorname{Spec}\left(\frac{\mathbb{Z}_p^{un}[x,y]}{(f)}\right)$ is an open subscheme of C where*

$$f = x^d + xy + p^m x - p^n y + p^l \tag{4.9}$$

for integers $m > 0$, $l > 0$, $n > 0$ and $d > 2$. Then

$$f^{\deg\leq 2} = xy + p^m x - p^n y + p^l. \tag{4.10}$$

We will compute the thickness of f and $f^{\deg\leq 2}$ at the origin (x, y, p).

(1) by substituting

$$\begin{aligned} x &\to x' + p^n \\ y &\to y' - dp^{n(d-1)} - p^m, \end{aligned}$$

into Equation (4.9), we get

$$\begin{aligned} f_1(x', y') &= f(x' + p^n, y' - dp^{n(d-1)} - p^m) \\ &= f_1^{\deg>2} + \frac{d(d-1)p^{n(d-2)}}{2} x'^2 + x'y' + p^{nd} + p^{m+n} + p^l. \end{aligned}$$

(2) by substituting

$$\begin{aligned} x &\to x'' + p^n \\ y &\to y'' - p^m, \end{aligned}$$

into Equation (4.10), we get

$$\begin{aligned} f_2(x'', y'') &= f^{\deg\leq 2}(x'' + p^n, y'' - p^m) \\ &= x''y'' + p^{m+n} + p^l. \end{aligned}$$

Now we can apply Corollary 4.3.12 for computing the thickness at the origin. Taking $n = 2$, $d = 3$ and $m = l = 10$, we get the thickness is 6 in (1) and is 10 in (2).

4.4 $\langle \Delta, \Delta \rangle$ for $\mathfrak{C}_{\mathbb{Q}}$

Recall that in Theorem 3.3.2, we already decomposed $\langle \Delta, \Delta \rangle$ into a sum of contributions from Archimedean places and non-Archimedean places. In this section, we will show that all terms but the infinite λ invariants in Theorem 3.3.2 are computable now.

1 For a finite place v, by Proposition 4.2.4, Corollary 4.2.6 and Proposition 4.3.7, the dual graph of $\mathfrak{C}$ at v is known. We can get its admissible invariants (including $\lambda(\mathfrak{C}_v)$) from Table 3.1 and Table 3.2. There are only finitely many primes with bad reduction, and invariants at primes with good reduction contribute 0 to the height.

2 For a finite place v, by Proposition 3.3.3 and Proposition 3.3.5, the number $\mathrm{ord}_v(\chi'_{18})$ can be computed if we know the dual graph and the location of $\mathfrak{C}_v$ in the moduli space $\overline{\mathcal{M}}_3$.

3 For the infinite place $v : \mathbb{Q} \to \mathbb{C}$, in Equation (2.3) we have an explicit expression for χ'_{18} for v, and the Hodge metric is determined by the period matrix of $\mathfrak{C}_{\mathbb{C}}$ (Equation (2.7)).

4 For the infinite place $v : \mathbb{Q} \to \mathbb{C}$, the invariant $\lambda(\mathfrak{C}_v)$ is the most difficult one, we will show how to compute it in later sections. In fact, all the remaining sections are necessary for the computation of $\lambda(\mathfrak{C}_v)$.

4.4.1 λ at finite places

By Proposition 4.1.2 and Corollary 4.3.9, the only non-trivial dual graphs of $\mathfrak{C}$ come from 29 and 163, which are the type $1I$ graph in Table 3.1. We can get the admissible invariants of $\overline{\Gamma}_v$ for $v = 29$ and 163 from Table 3.1 and Table 3.2, which can be summarized as follows.

Proposition 4.4.1. *For $v = 29$ or 163, we have $\delta_0(\mathfrak{C}_v) = 1$, $\delta_1(\mathfrak{C}_v) = 0$, $\tau(\mathfrak{C}_v) = \frac{1}{12}$, $\theta(\mathfrak{C}_v) = 0$, $\varphi(\mathfrak{C}_v) = \frac{1}{9}$, $\lambda(\mathfrak{C}_v) = \frac{3}{28}$ and $\epsilon(\mathfrak{C}_v) = \frac{2}{9}$.*

Corollary 4.4.2. *The λ invariants from non-Archimedean places contribute*

$$\begin{aligned} -21 \sum_{v \in M(\mathbb{Q})_0} \lambda(\mathfrak{C}_v) N(v) = & -\frac{21 \times 3}{28}(\log 29 + \log 163) \\ \approx & -19.0373535692 \end{aligned}$$

to the height $\langle \Delta, \Delta \rangle$ of $\mathfrak{C}_{\mathbb{Q}}$.

Proof. Substitute Proposition 4.4.1 into Theorem 3.3.2. □

4.4.2 χ'_{18} at finite places

The closure of the hyperelliptic locus H (denoted by $\overline{H}$) in $\overline{\mathcal{M}_3}$ is a divisor. Proposition 3.3.3 and Proposition 3.3.5 relate $\overline{H}$ and $\mathrm{ord}_v(\chi'_{18})$. Thus we need to study the integer $\mathrm{mult}_v(\overline{H})$ for our curve $\mathfrak{C}$. Note that $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is geometrically irreducible and of geometric genus 2 (Corollary 4.2.6) at $p = 29$ and 163.

Let D be a smooth curve of genus 2 over an algebraically closed field. It is well-known that D is hyperelliptic and has a unique hyperelliptic involution, i.e. a non-trivial element $\sigma \in \mathrm{Aut}(D)$ such that $\sigma^2 = Id_D$ and $D/ < \sigma > \simeq \mathbb{P}^1$. We say points $p \neq q$ on D are *conjugate* if $\sigma(p) = q$.

Proposition 4.4.3. *Let H_0 be a smooth curve of genus 2 over an algebraically closed field K. Let p and q be conjugate points on H_0. The curve C given by identifying p and q (glue them into a nodal singularity) on H_0 is not a plane quartic curve.*

Proof. We assume that C is a plane quartic defined by $f(x, y) = 0$ on certain affine patch U_{xy}. Without losing generality, we can assume the nodal point is $(0, 0)$, then the equation becomes

$$f(x, y) = f^{\deg \geq 3}(x, y) + f^{\deg = 2}(x, y), \tag{4.11}$$

where $f^{\deg=2}(x, y)$ is a non-degenerate quadratic form.

Now we blow up the curve C at $(0, 0)$ by substituting $y = xt$, then we get an affine open set U_{xt} of H_0 given by

$$f^{\deg \geq 3}(x, xt)/x^2 + f^{\deg=2}(x, xt)/x^2 = 0. \tag{4.12}$$

By the non-degeneracy of $f^{\deg=2}(x, y)$, we know that $f^{\deg=2}(x, xt)/x^2$ is a polynomial in t with distinct roots t_1 and t_2. After the blow up, we get the original smooth curve H_0 and the nodal point $(0, 0)$ on C is resolved into two distinct points $(0, t_1)$ and $(0, t_2)$ on U_{xt}. These two points are exactly p and q.

In Equation (4.12), we have $f^{\deg \geq 3}(x, xt)/x^2 = xf_1(t) + x^2 f_2(t)$. At least, we know that f_2 is non-zero (we assumed that C is a plane quartic), and $f^{\deg=2}(x, xt)/x^2$ is a polynomial in t. In other words, t gives a 2-1 map from H_0 to $\mathbb{P}^1_K$. Since H_0 is hyperelliptic, it has a natural 2-1 map Quo to $\mathbb{P}^1_K$. By the uniqueness of the hyperelliptic 2-1 map for a hyperelliptic curve, we have a unique automorphism η of $\mathbb{P}^1_K$ that makes the following diagram commute:

$$\begin{tikzcd} H_0 \arrow[r, "Quo"] \arrow[d, "t"] & \mathbb{P}^1 \arrow[dl, "\eta"] \\ \mathbb{P}^1 & \end{tikzcd}$$

where conjugate points are mapped to the same point by Quo.

The images of p and q are 2 different points along t since they correspond to $(0, t_1)$ and $(0, t_2)$. This is impossible since they are conjugate and are mapped to the same point by Quo. Thus the curve C can not be a plane quartic. $\square$

Corollary 4.4.4. *When $p = 29$ or 163, we have $\mathrm{mult}_p(\overline{H}) = 0$ for $\mathfrak{C}_{\mathbb{Z}_p}$.*

Proof. According to Proposition 4.2.6, the reduction of $\mathfrak{C}$ at 29 (and 163) is an irreducible plane quartic with exactly 1 singular point. By Proposition 4.4.3, the reduction of $\mathfrak{C}$ at $p = 29$ and 163 are not obtained by gluing conjugate points on a genus 2 curve.

The singular curve $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ lies on the closure of the hyperelliptic locus if and only if $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ has an involution ι and the quotient map $\mathfrak{C}_{\overline{\mathbb{F}}_p}/\langle\iota\rangle$ is a tree of $\mathbb{P}^1$ connected by nodal points (Page 101 in [3]). The normalization of $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is a genus 2 curve (denoted by $\widetilde{\mathfrak{C}}_{\overline{\mathbb{F}}_p}$) thus is hyperelliptic. Suppose $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is hyperelliptic, then $\mathfrak{C}_{\overline{\mathbb{F}}_p}/\langle\iota\rangle \simeq \mathbb{P}^1$ since we already know that $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is irreducible (Proposition 4.2.6).

We write q_ι for the quotient map induced by ι and write n for the normalization map of $\mathfrak{C}_{\overline{\mathbb{F}}_p}$, that is

$$\widetilde{\mathfrak{C}}_{\overline{\mathbb{F}}_p} \xrightarrow{n} \mathfrak{C}_{\overline{\mathbb{F}}_p} \xrightarrow{q_\iota} \mathbb{P}^1.$$

According to the first paragraph, we know that n identifies two non-conjugate points. The composition of $q_\iota \circ n$ is a 2-1 map from $\widetilde{\mathfrak{C}}_{\overline{\mathbb{F}}_p}$ to $\mathbb{P}^1$. This is impossible since we already know that n identifies non-conjugate points. Thus $\mathfrak{C}_{\overline{\mathbb{F}}_p}$ is non-hyperelliptic. The multiplicity of $\mathfrak{C}_{\mathbb{Z}_p}$ at the hyperelliptic locus in $\overline{\mathcal{M}}_3$ is 0. $\square$

Proposition 4.4.5. *For $v = 29$ or 163, we have $\mathrm{ord}_v(\chi'_{18})(\mathfrak{C}_{\mathbb{Z}_v}) = 2$. And*

$$\mathrm{ord}_v(\chi'_{18})(\mathfrak{C}_{\mathbb{Z}_v}) = 0$$

for all other finite places v.

Proof. Combining Proposition 3.3.3 and Proposition 3.3.5 we have

$$\mathrm{ord}_v(\chi'_{18}) = 2\mathrm{mult}_v(\overline{H}) + 6\delta_1(\overline{\Gamma}_v) + 2\delta_0(\overline{\Gamma}_v).$$

Then we get the result by combining Proposition 4.2.4, Proposition 4.4.1 and Corollary 4.4.4. $\square$

Remark 4.4.6. $\mathrm{ord}_v(\chi'_{18})$ *vanishes at finite places of good reduction.*

Corollary 4.4.7. *For finite places, χ'_{18} contributes*

$$\frac{21}{18}(2\log 29 + 2\log 163) \approx 19.7424407385$$

to $\langle\Delta, \Delta\rangle$.

Proof. Substituting Proposition 4.4.5 into Theorem 3.3.2. $\square$

4.4.3 χ'_{18} at the infinite place

Recall notations introduced in Equation (2.2), Equation (2.3) and Remark 2.1.10. Using the metric given by Equation (2.7), we get that the contribution of $\log\|\chi'_{18}\|_{\mathrm{Hdg}}$ in Theorem 3.3.2 is:

$$\begin{aligned}
&-\frac{21}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\tau)\\
=&-\frac{21}{18}\log(\|2^{-28}(2\pi i)^{54}\tilde{\chi}_{18}(\tau)(dz_1\wedge dz_2\wedge dz_3)^{\otimes 18}(\tau)\|_{\mathrm{Hdg}})\\
=&-\frac{21}{18}\log|2^{-28}(2\pi)^{54}\prod_{\epsilon\in S_3}\theta_\epsilon(0,\tau)(\det\mathrm{Im}\tau)^9|.
\end{aligned}$$

All components except the list of even characteristics are implemented in Magma, while the list of even theta characteristics for dimension 3 is easy to compute by hand. Magma code for this computation can be found in Appendix VI. With our calculation, we get the following proposition.

Proposition 4.4.8. *At the infinite place, the χ'_{18} modular form contributes*

$$-\frac{21}{18}\log\|\chi'_{18}\|_{\mathrm{Hdg}}\approx -81.0426321447$$

to $\langle\Delta,\Delta\rangle$.

4.5 Evaluation of $\|\theta\|_{g-1}$

In this section, we will define and show how to evaluate $\|\theta\|_{g-1}$ at points in $\mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}})$. At the end of Subsection 4.5.1, we summarize our strategy. Subsection 4.5.2 is about the computation of a canonical divisor of $\mathfrak{C}_{\mathbb{C}}$. We can evaluate $\|\theta\|_{g-1}$ with Proposition 4.5.11 in Subsection 4.5.3.

To avoid confusion, in this section, we still use g in some notations even though we know $g=3$ for $\mathfrak{C}_{\mathbb{C}}$, for example we use $\|\theta\|_{g-1}$ rather than $\|\theta\|_2$.

4.5.1 Strategy

Fixing a base point P_{bs}, a basis of holomorphic forms $\{\omega_i\}_{1\leq i\leq g}$ and a symplectic homology basis $\{\eta_i\}$ of the genus g Riemann surface C, we have a period matrix $\Omega=(\Omega_1,\Omega_2)$ associated to these datum. Then we have an element $\tau=\Omega_1^{-1}\Omega_2$ in the Siegel upper half-space $\mathbb{H}_g$. Taking $\{\eta_i\}_{1\leq i\leq g}=\{\omega_i\}_{1\leq i\leq g}\cdot {}^t\Omega_1^{-1}$, we have the following map

$$\mathrm{Div}^{g-1}(C)\to\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g,\quad \sum_n n_kP_k\to\sum_n n_k\int_{P_{bs}}^{P_k}(\eta_1,\dots,\eta_g),$$

which induces a bijective map:

$$u : \mathrm{Pic}^{g-1}(C) \to \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g. \tag{4.13}$$

Remark 4.5.1. *The 'Abel-Jacobi' map above is well-defined for a chosen base point P_{bs}. We will write AJ for the Abel-Jacobi map from $\mathrm{Div}^0(C)$ to $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, which we can define without a base point.*

The zero locus of Riemann's theta function

$$\theta(z;\tau) := \sum_{n\in\mathbb{Z}^g} \exp(\pi i^t n\tau n + 2\pi i^t nz) \tag{4.14}$$

defines a divisor Θ_0 on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Recall that the theta divisor Θ in $\mathrm{Pic}^{g-1}(C)$ corresponds to classes of line bundles admitting a global section. The following theorem of Riemann (Theorem 1.4.2 in [12]) links Θ_0 and Θ.

Theorem 4.5.2. *We denote t_κ to be the translation map of the tori with respect to $\kappa \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, that is, an endomorphism of $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ sending $x \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ to $x + \kappa \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. There is a unique element $\kappa = \kappa(P_{bs})$ in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ such that $(t_\kappa \circ u)^*\Theta = \Theta_0$ which also induces a canonical isomorphism of line bundles $(t_\kappa \circ u)^*\mathcal{O}(\Theta_0) \xrightarrow{\sim} \mathcal{O}(\Theta)$ on $\mathrm{Pic}^{g-1}(C)$. Furthermore, we have $(t_\kappa\circ u)(K_C - D) = -(t_\kappa\circ u)(D)$ for any divisor D of degree $g-1$.*

By a *semi-canonical divisor* on C, we mean a divisor s on C of degree $g-1$ such that $2s \sim \Omega_C$. For a compact Riemann surface C of genus $g > 0$, there are 2^{2g} semi-canonical elements in $\mathrm{Pic}^{g-1}(C)$. These semi-canonical divisors are equal up to a 2-torsion point of $\mathrm{Jac}(C)$.

Corollary 4.5.3. *The map $t_\kappa \circ u$ identifies the set of classes of semi-canonical divisor on C with the set of 2-torsion points on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$.*

By Riemann's theorem, we can translate a metric on $\mathcal{O}(\Theta_0)$ to $\mathcal{O}(\Theta)$ along the map $t_\kappa \circ u$. The following paragraph shows how we choose the metric on $\mathcal{O}(\Theta_0)$.

We write s for the canonical section of $\mathcal{O}(\Theta_0)$ and fix a $(1,1)$-form on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ by

$$\nu := \frac{i}{2} \sum_{1\le k,l\le g} (\mathrm{Im}\tau)^{-1}_{k,l} dz_k \wedge d\bar{z}_l. \tag{4.15}$$

The $2g$-form $\frac{1}{g!}\nu^g$ gives the Haar measure on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. We choose $\|\cdot\|_{\Theta_0}$ to be the metric on $\mathcal{O}(\Theta_0)$ uniquely determined by:

(i) the curvature form of $\|\cdot\|_{\Theta_0}$ is equal to ν,

(ii) $\frac{1}{g!}\int_{\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g}\|s\|^2_{\Theta_0}\nu^g = 2^{-g/2}$.

For simplicity, we write $\|\theta\|$ for $\|(t_\kappa \cdot u)^* s\|_\Theta$ or $\|s\|_{\Theta_0}$. Then we have the following expression of $\|\theta\|$.

Proposition 4.5.4. *Let $z \in \mathbb{C}^g$ and $\tau \in \mathbb{H}_g$. Then the formula*

$$\|\theta\|(z;\tau) = (\det \mathrm{Im}\tau)^{1/4}\exp(-\pi {}^t y \cdot (\mathrm{Im}\tau)^{-1} \cdot y) \cdot |\theta(z;\tau)|$$

holds, where $y = \mathrm{Im} z$ and θ is defined in Equation (4.14).

Proof. See Page 413 in [23]. $\square$

Notation 4.5.5. *We write $\|\theta\|$ for the metric of the canonical section of $\mathcal{O}(\Theta_0)$ on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, and write $\|\theta\|_{g-1}$ for the metric of the canonical section of $\mathcal{O}(\Theta)$ on $\mathrm{Pic}^{g-1}(C)$ induced by $\|\theta\|$.*

By Theorem 4.5.2, there exist a unique $\Delta' \in \mathrm{Pic}^{g-1}(C)$ such that for all $D \in \mathrm{Pic}^{g-1}(C)$, we have:

$$2\Delta' = K_C, \tag{4.16}$$
$$\|\theta\|_{g-1}(D) = \|\theta\|(AJ(D - \Delta')), \tag{4.17}$$

where AJ is the Abel-Jacobi map from $\mathrm{Pic}^0(C)$ to $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$.

Now we explain our strategy for evaluating $\|\theta\|_{g-1}(D)$ where $D \in \mathrm{Pic}^{g-1}(C)$. Recall that we fixed an isomorphism $\mathrm{Pic}^{g-1}(C) \xrightarrow{\sim} \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\tau$ in Equation (4.13). By Proposition 4.5.4 and Equation (4.17), we reduce the problem to computing $AJ(D - \Delta')$. By the equality

$$AJ(D - \Delta') = AJ(D - (g-1)P_{bs}) - AJ(\Delta' - (g-1)P_{bs}), \tag{4.18}$$

we only need to compute $AJ(\Delta' - (g-1)P_{bs})$. Since Pic^{g-1} is a 3-dimensional abelian variety, there are 64 elements in $\mathrm{Pic}^{g-1}(C)$ satisfying the Equation (4.16). These elements give 64 possibilities for $AJ(\Delta' - (g-1)P_{bs})$. In Subsection 4.5.2, we compute the canonical divisor K_C. In Subsection 4.5.3, we explain our algorithm for finding the correct $AJ(\Delta' - (g-1)P_{bs})$ among the 64 possibilities.

Remark 4.5.6. *θ and AJ are implemented in Magma. Using Proposition 4.5.4, we can evaluate $\|\theta\|$.*

4.5.2 Canonical divisor of $\mathfrak{C}_{\mathbb{C}}$

In this subsection, we compute a canonical divisor of $\mathfrak{C}_{\mathbb{C}}$. Recall that a canonical divisor of a Riemann surface is the divisor of a non-zero meromorphic differential form.

We write the equation of $\mathfrak{C}_{\mathbb{C}}$ on the affine patch $U = \mathfrak{C}_{\mathbb{C}}|_{X=1}$ as

$$f_0 = -y_X + y_X^2 - y_X^2 z_X + y_X^3 z_X + z_X^2 + z_X^3,$$

then the differential form on U

$$\omega_0 = \frac{z_X dz_X}{(f_0)_{y_X}} = \frac{z_X dz_X}{3y_X^2 z_X - 2z_X y_X + 2y_X - 1} \tag{4.19}$$

can be extended to a global holomorphic form ω (Theorem 4.6.10). Thus we just need to compute $\operatorname{div}(\omega)$. Since ω is a holomorphic form, we only need to consider zeroes of ω.

The locally defined function z_X is a local parameter for all but finite points on U. We write U_1 for the open subset of U where z_X is a local parameter. The numerator z_X of ω_0 on U vanishes to order 1 at the points

$$P_1 = (1:0:0),$$
$$P_2 = (1:1:0),$$

while the denominator does not. Thus z_X is a local parameter near P_1 and P_2, and ω_0 has simple zeroes at P_1 and P_2. So we obtain

$$\operatorname{div}(\omega)|_{U_1} = [P_1] + [P_2].$$

It can be shown that points in $U \backslash U_1$ are not in the support of $\operatorname{div}(\omega)|_U$. Thus we have

$$\operatorname{div}(\omega)_U = [P_1] + [P_2].$$

There are two points of $\mathfrak{C}_{\mathbb{C}}$ not lying on U:

$$P_3 = (0:1:0),$$
$$P_4 = (0:0:1).$$

$P_3 = (0:1:0)$ lies on the affine patch $V = \mathfrak{C}_{\mathbb{C}}|_{Y=1}$. Substituting

$$z_X \to \frac{z_Y}{x_Y}$$
$$y_X \to \frac{1}{x_Y}$$

into Equation (4.19) and the defining polynomial of $\mathfrak{C}_{\mathbb{C}}$, we get

$$\omega|_V = \frac{z_Y x_Y dz_Y - z_Y^2 dx_Y}{3z_Y - 2z_Y x_Y + 2x_Y^2 - x_Y^3}, \tag{4.20}$$

and

$$z_Y + x_Y^2 = x_Y^3 + z_Y x_Y - z_Y^3 x_Y - z_Y^2 x_Y^2. \tag{4.21}$$

The coordinate of P_3 in this affine patch is $(0,0)_V$, and either x_Y or z_Y is the local parameter for P_3. From Equation (4.21), we can see x_Y is a local parameter at P_3 ($\mathrm{ord}_{P_3}(x_Y) = 1$) and thus $\mathrm{ord}_{P_3}(z_Y) = 2$. Then the right hand side of Equation (4.21) has order strictly bigger than 2 (that is, $\mathrm{ord}_{P_3}(z_Y + x_Y^2) > 2$). Substituting

$$z_Y = -x_Y^2 + \text{higher degree terms}$$

into Equation (4.20), we get $\mathrm{ord}_{P_3}(\omega) = 2$.

It remains to compute the order of ω at P_4. Substituting the order of ω at P_1, P_2 and P_3 into the equations below

$$4 = \deg(K_{\mathfrak{C}_{\mathbb{C}}}) = \mathrm{ord}_{P_1}(\omega) + \mathrm{ord}_{P_2}(\omega) + \mathrm{ord}_{P_3}(\omega) + \mathrm{ord}_{P_4}(\omega),$$

we get $\mathrm{ord}_{P_4}(\omega) = 0$. In conclusion, we get the following proposition.

Proposition 4.5.7.

$$\begin{aligned} K_{\mathfrak{C}_{\mathbb{C}}} = \mathrm{div}(\omega) &= [P_1] + [P_2] + 2[P_3] \\ &= [(1:0:0)] + [(1:1:0)] + 2[(0:1:0)]. \end{aligned}$$

Remark 4.5.8. *We can also compute the canonical divisor in Magma.*

4.5.3 2-translation

Given the canonical divisor K_C of C, we get 64 possibilities for $AJ(\Delta' - (g-1)P_{bs})$. In this subsection, we explain how to find the correct one among the 64.

We use the base point P_{bs} fixed in Subsection 4.5.1, and write T_C for the torus $\mathrm{Jac}(C) = \mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\tau$. We use the isomorphism $u : \mathrm{Pic}^{g-1}(C) \xrightarrow{\sim} T_C$ given in Equation (4.13). According to the last paragraph in Subsection 4.5.1, there is a subset V of T_C, containing 64 elements, such that each element v_Δ in V satisfies

$$2v_\Delta = AJ(K_C - 2(g-1)P_{bs}).$$

We want to find the one that makes Equation (4.17) hold.

The difference of any two elements in V is a 2-torsion point of T_C. If we fix a $v_{\Delta'}$ in V, then we just need to figure out the correct translation by a 2-torsion point η in T_C that makes the equation

$$\|\theta\|_{g-1}(D) = \|\theta\|(AJ(D - (g-1)P_{bs}) - (v_{\Delta'} + \eta)) \tag{4.22}$$

hold.

Lemma 4.5.9. $\|\theta\|_{g-1}$ *vanishes at points in* $\mathrm{Pic}^{g-1}(C)$ *that have effective representative divisors.*

Proof. This follows from Theorem 4.5.2 and the definition of Θ (the paragraph before Theorem 4.5.2). □

We use the lemma above to compute the 2-torsion point η. For example, $\|\theta\|_{g-1}$ vanishes at $(g-1)P_{bs}$. We sketch the algorithm as follows. Magma code and the period matrix τ chosen by Magma can be found in Appendix V.

Algorithm 2 Computation of the correct $v_{\Delta'} + \eta$

Input: C: a plane quartic over $\mathbb{C}$
P_{bs}: a default base point of C
Output: $v_{\Delta'} + \eta$ in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$

```
Compute the small period matrix τ of C.
Generate a set S consisting all vectors of the form ∑ c_i · v_i where v_i are column
   vectors of (1|τ) and c_i ∈ {0, 1/2}
Compute v_{K_C} = AJ(K_C − 2[P_bs])
v_{Δ'} = v_{K_C}/2
for η in S do
   if ‖θ‖(AJ(2[P_bs]) − v_{Δ'} − η) ≤ 0.00000001 then
      return v_{Δ'} + η
   end if
end for
```

Remark 4.5.10. *(1) Since we only check the vanishing of* $\|\theta\|_{g-1}$ *at one specific effective divisor* $2P_{bs}$*, it can happen that more than one* 2*-torsion point of* T_C *makes this specific theta value vanish. For our curve* $\mathfrak{C}_{\mathbb{C}}$*, the computation result of our code shows that this does not happen (only one the 64 choices makes the function vanish).*

(2) The correctness of the 2-translation can be checked by using a different effective divisor of degree $g-1$*, since they should give the same answer.*

Proposition 4.5.11. *Using the default base point* P_{bs} *and the (co)homology basis chosen by Magma, the point* $AJ(\Delta' - (g-1)P_{bs})$ *in* $T_{\mathfrak{C}_{\mathbb{C}}}$ *that makes Equation (4.17) hold is given by:*

$$z_1 = 0.47925054265168018676 - 0.00334176833187451614 * I$$
$$z_2 = 0.69868487750843232229 + 0.19949572388256356310 * I$$
$$z_3 = 0.00722266620787249385 - 0.04301020693432081496 * I.$$

With this proposition, we can evaluate $\|\theta\|_{g-1}$ by Equation (4.22). Note that the Abel-Jacobi map AJ, the modified theta function $\|\theta\|$ and the addition of divisors on $\mathfrak{C}_{\mathbb{C}}$ can be implemented in Magma.

Remark 4.5.12. *The point P_{bs} chosen by Magma is*

$$(-2.000000000 : -4.214319743 : 1).$$

4.6 Computation of the Green's function

In this section, we compute the Green's function on $\mathfrak{C}_{\mathbb{C}}$. The invariant $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in the Green's function will be used in the computation of $\lambda(\mathfrak{C}_{\mathbb{C}})$. In Subsection 4.6.1, we compute the Weierstrass points of $\mathfrak{C}_{\mathbb{C}}$. In Subsection 4.6.2, we compute the volume form of $\mathfrak{C}_{\mathbb{C}}$. In Subsection 4.6.3, we explain our algorithm for computing $\log(S(\mathfrak{C}_{\mathbb{C}}))$. We refer to Subsection 1.2.1 and [12] for definitions and theorems.

Instead of constructing $G(x, y)$ from Definition 1.2.2, we give an explicit formula of the Green's function discovered by R. de Jong in [12].

Following Proposition 2.2.6 in [12], we write $\mathfrak{W}$ for the set of Weierstrass points counted with weights and define the invariant $S(X)$ of a compact Riemann surface X of genus g as

$$\begin{aligned}\log(S(X)) = &-g^2 \cdot \int_X \log\|\theta\|_{g-1}(gP - Q) \cdot \mu(Q) \\ &+ \frac{1}{g} \cdot \sum_{W \in \mathfrak{W}} \log\|\theta\|_{g-1}(gP - W).\end{aligned} \tag{4.23}$$

Theorem 4.6.1. *If P and Q are distinct points on a compact Riemann surface X of genus $g > 1$ and P is not a Weierstrass point, then we have*

$$G(P, Q)^g = S(X)^{1/g^2} \cdot \frac{\|\theta\|_{g-1}(gP - Q)}{\prod_{W \in \mathfrak{W}} \|\theta\|_{g-1}(gP - W)^{1/g^3}}.$$

Proof. See the proof of Theorem 2.1.2 and Proposition 2.2.6 in [12]. □

By the computation in the last subsection, we are now able to evaluate

$$\|\theta\|_{g-1} : \mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}}) \to \mathbb{R}$$

with $\|\theta\|$ (Proposition 4.5.4). With this explicit formula for the Green's function, our goal is reduced to the computation of the Weierstrass points and the invariant $S(\mathfrak{C}_{\mathbb{C}})$.

4.6.1 Weierstrass points of plane quartic curves

We will first recall definitions related to Weierstrass points and some fundamental properties, further results can be found in [2], Page 41.

Definition 4.6.2. *Let X be a compact Riemann surface of genus $g > 1$ with canonical divisor K. An effective divisor D on X is called special if $h^0(\mathcal{O}(K-D)) > 0$. A point P is called a Weierstrass point if gP is a special divisor.*

Definition 4.6.3. *If we write $Gap(P)$ for*

$$Gap(P) \coloneqq \{n \in \mathbb{Z}_{>0} : h^0(\mathcal{O}(nP)) = h^0(\mathcal{O}((n-1)P))\},$$

the weight of a point P is defined as $w(P) \coloneqq \sum_{n \in \mathrm{Gap}(P)} n - g(g-1)/2$.

Example 4.6.4. *For a hyperelliptic curve of genus g, the Weierstrass points are exactly the $2g+2$ ramification points of a hyperelliptic 2-1 map with equal weight $\frac{g(g-1)}{2}$.*

We know that $w(P) = 0$ for all but finitely many points, thus the divisor

$$W_X \coloneqq \sum_{P \in X} w(P)P$$

is well-defined. It is well-known (Proposition 1.12 in [64]) that this effective divisor is of degree $g(g-1)(g+1)$. Thus for a plane quartic curve C, we have $\deg(W_C)$ =24.

Let C be a smooth plane curve. For $x \in C$, we write T_x for the tangent line of C at x. A point $p \in C$ is called a $flex\ point$ if p is a smooth point and $I(p, T_p \cap C) \geq 3$, where $I(p, T_p \cap C)$ is the intersection multiplicity of C and T_p at p. A flex point is called an $ordinary\ flex\ point$ if $I(p, T_p \cap C) = 3$, otherwise it is called a $hyperflex$.

Definition 4.6.5. *The Hessian of a polynomial $F(X,Y,Z) \in K[X,Y,Z]$ is the following matrix*

$$\mathrm{Hess}(F) \coloneqq \begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}.$$

Proposition 4.6.6. *Let K be an algebraically closed field with* $\mathrm{char}\, K = 0$. *Let C be a smooth plane curve in $\mathbb{P}^2_K$ defined by $F(X,Y,Z) = 0$. We write C_H for the plane curve defined by* $\det \mathrm{Hess}(F) = 0$. *Then*

1. *$P \in C \cap C_H$ if and only if P is a flex point.*

2. *$I(P, C \cap C_H) = 1$ if and only if P is an ordinary flex.*

Proof. See Page 116 in [27]. □

Weierstrass points on a smooth plane quartic curve are of weight 1 or 2, and they correspond to ordinary flex points and hyperflex points respectively. Thus we have an equality (see [64] Page 13, Theorem 2.2):

$$\#\{\text{ordinary flex points on } C\} + 2 \times \#\{\text{hyperflex points on } C\} = 24.$$

With the discussion above, we can calculate Weierstrass points of $\mathfrak{C}_{\mathbb{C}}$ in SageMath. The result is attached to the Appendix II.

```
x,y,z=var('x,y,z')
C=-x^3*y+x^2*y^2-x*y^2*z+y^3*z+x^2*z^2+x*z^3
M=C.hessian()
det= M.determinant()
solve([C == 0,z==1,det==0],x,y,z)
```

Weierstrass points

The code above returns the intersection points of $\mathfrak{C}_{\mathbb{C}}$ and $\mathfrak{C}_{\mathbb{C},H}$ in the affine patch $Z = 1$. Since it contains 24 points, we can conclude that these are all the Weierstrass points, and all of them are of weight 1.

Proposition 4.6.7. *If we choose* $P = (-2.000000000 : -4.214319743 : 1)$*, then we have*

$$\frac{1}{3} \sum_{W \in \mathfrak{W}} \log\|\theta\|_{g-1}(gP - W) \approx -6.817611049$$

for the curve $\mathfrak{C}_{\mathbb{C}}$*.*

The reason we choose this point P is that this is the default base point for $\mathfrak{C}_{\mathbb{C}}$ chosen by Magma. In the following sections, we will use this point several times.

Remark 4.6.8. *For a general smooth plane quartic curve* C*, the weights of Weierstrass points are computable. We give a sketch of the procedure, and details can be found in [64] Pages 7-8. Let* $\{\omega_k\}_{1 \le k \le 3}$ *be a basis for* $H^0(C, \Omega_C)$*. Then locally we can write* ω_k *as* $f_k dz$ *where* z *is a local parameter near point* P*. Now we have the Wronskian determinant around* P*:*

$$W_z(\omega_1, \omega_2, \omega_3) := \det \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''/2 & f_2''/2 & f_3''/2 \end{pmatrix}, \tag{4.24}$$

where the superscript indicates the order of differentiation with respect to z. This locally gives a non-zero rational section in $\Omega_C^{\otimes 6}$ near P by $W(\omega_1, \omega_2, \omega_3)dz^{\otimes 6}$, which can be extended to a global section. For this global section of $\Omega_C^{\otimes 6}$, we have

$$\operatorname{div} W(\omega_1, \omega_2, \omega_3) = \sum_{x \in C} w(x)x,$$

where $w(x)$ is the weight of x.

Remark 4.6.9. *If we have a sequence of positive integers $\{a_i\}_{1 \le i \le t}$ such that*

$$\sum_{i=1}^{t} a_i = g(g-1)(g+1),$$

can we always find a compact Riemann surface of genus g whose Weierstrass points have weights $\{a_i\}_{1 \le i \le t}$? The answer is no. In [64] Theorem 7.1, A.M. Vermeulen showed that there exist genus 3 curves with 0, 1, 2 *hyperflex points, but there is no genus 3 curve with* 10, 11 *hyperflex points.*

4.6.2 Computation of the volume form

Let C be a smooth plane curve of genus $g \ge 1$ defined by a homogeneous polynomial $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ of degree $d \ge 3$. For simplicity, we write $f(x, y) = F(x, y, 1)$ and $U = C|_{Z=1}$. Then we can construct an explicit basis of $H^0(C, \Omega_C^1)$ by the following theorem.

Theorem 4.6.10. *Let U_0 be the open subset of U where $\frac{\partial f}{\partial y}(x, y) \neq 0$. Then the restriction of a global holomorphic differential of C on U_0 can be written in the form $\frac{\phi(x,y)dx}{\frac{\partial f}{\partial y}(x,y)}$, where $\phi(x, y)$ is a polynomial of degree at most $d - 3$.*

Proof. See Chapter 9, Theorem 1 in [6]. □

For our curve $\mathfrak{C}_{\mathbb{C}}$, we have

$$\frac{\partial \mathfrak{f}(x, y)}{\partial y} = -x^3 + 2x^2y - 2xy + 3y^2.$$

By Theorem 4.6.10, we get a basis of $H^0(X, \Omega^1_{\mathfrak{C}_{\mathbb{C}}})$ as follows:

$$\left\{ \frac{dx}{-x^3 + 2x^2y - 2xy + 3y^2}, \frac{ydx}{-x^3 + 2x^2y - 2xy + 3y^2}, \frac{xdx}{-x^3 + 2x^2y - 2xy + 3y^2} \right\}. \tag{4.25}$$

We abbreviate these to ω_1, ω_y and ω_x respectively.

Now we can apply the Gram-Schmidt process to obtain an orthonormal basis with respect to the inner product

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \overline{\eta}.$$

The following theorem of Riemann gives us the inner product of every pair of basis elements, which will simplify our computation.

Theorem 4.6.11. *Let X be a compact Riemann surface of genus $g \geq 1$. Fix a symplectic basis for the homology $H_1(X, \mathbb{Z})$ and a basis $\omega_1, \ldots, \omega_g$ of the holomorphic differentials $H^0(X, \Omega^1_X)$. We have a period matrix $\Omega = (\Omega_1|\Omega_2)$ given by these data. Then the following matrix identity*

$$\left(\frac{i}{2} \int_X \omega_k \wedge \overline{\omega}_l \right)_{1 \leq k, l \leq g} = \frac{i}{2} (\overline{\Omega}_2 \, {}^t\Omega_1 - \overline{\Omega}_1 \, {}^t\Omega_2) = \overline{\Omega}_1 (\mathrm{Im}\tau) \, {}^t\Omega_1$$

holds.

Proof. See Pages 231-232 in [29]. □

Remark 4.6.12. *The choice of the homology basis does not affect the matrix in Theorem 4.6.11.*

The calculation of the period matrix implemented in SageMath uses the ordered basis $[\omega_1, \omega_y, \omega_x]$, which is exactly what we constructed in Equations (4.25). Thus we can carry out the Gram-Schmidt process in SageMath. The code can be found in Appendix III, and we summarize our computation as the following proposition.

Proposition 4.6.13. *We have the following orthonormal basis of differential forms:*

$$\begin{aligned}
\omega_{on1} =& 0.350487116953118 * \omega_1 \\
\omega_{on2} =& 0.358981759779085 * \omega_y + 0.119553875346235 * \omega_1 \\
\omega_{on3} =& 0.429067210690657 * \omega_x - 0.216555180015011 * \omega_y \\
& + 0.203008239643111 * \omega_1.
\end{aligned}$$

We write

$$\mu_{Ar} = \frac{i}{2 \cdot 3} \sum_{j=1}^{3} \omega_{onj} \wedge \overline{\omega}_{onj} \tag{4.26}$$

for the volume form of $\mathfrak{C}_{\mathbb{C}}$.

Remark 4.6.14. *We can get a different period matrix with Magma which also leads to a (1-1) form μ'_{Ar}. It can be checked by evaluation that the two (1-1) forms μ'_{Ar} and μ_{Ar} are identical.*

4.6.3 Computation of $\log(S(\mathfrak{C}_{\mathbb{C}}))$

Recall that in Proposition 4.6.7, we already computed the discrete sum part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$ (see Equation (4.23)) for a chosen point P. In this subsection, we approximate the integral part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in Equation (4.23) by using the Riemann sums.

First, we show that the integrated function has few singular points, otherwise the Riemann sums can have big error terms. In the expression of $\log(S(X))$ in Equation (4.23), the only possible singularities of the integration come from the zero locus of $\|\theta\|_{g-1}$. Recall that $\|\theta\|_{g-1}(D)$ vanishes if and only if D is rationally equivalent to an effective divisor of degree $g-1$. The following proposition implies that there is only 1 singular point in the integration.

Proposition 4.6.15. *Let X be a non-hyperelliptic compact Riemann surface of genus 3, i.e. a plane quartic curve. We choose two points P and Q on X. If P is a non-Weierstrass point, then $\|\theta\|_{g-1}(gP-Q)=0$ if and only if $Q=P$.*

Proof. We just need to show that $3P-Q$ is equivalent to an effective divisor exactly when $P=Q$. By Riemann-Roch, we get

$$h^0(\mathcal{O}(3P)) - h^0(\mathcal{O}(K_X-3P)) = \chi(\mathcal{O}_X) + \deg(3P) = 1. \tag{4.27}$$

Since P is not a Weierstrass point, by Definition 4.6.2, we have $h^0(K_X-3P)=0$. Thus $h^0(\mathcal{O}(3P))=1$, which means that $\Gamma(\mathcal{O}(3P),X)=\mathbb{C}$. Thus the equality $3P\sim Q+U+V$ implies that $P=Q=U=V$, otherwise we should have $h^0(\mathcal{O}(3P))\geq 2$. □

The defining polynomial $\mathfrak{f}$ of $\mathfrak{C}_{\mathbb{C}}$ on the affine patch $U_{\mathbb{C}}$ is given by

$$y^3+(x^2-x)y^2-x^3y+x^2+x=0.$$

For a generic $x\in\mathbb{C}$, there are three solutions of y such that $\mathfrak{f}(x,y)=0$. The following remark explains how we label the three solutions.

Remark 4.6.16. *(Important) We will label these y_i's by the cubic roots formula in Appendix I. This labelling is well-defined except at finitely many ramification points of the map $(x,y)\to x$. This finite set does not influence our numerical approximation. We will use this label frequently in the computation.*

Second, we show that the volume form decreases quickly, thus we can reasonably carry out the Riemann sums in a finite region. This can be summarized as the following proposition.

Proposition 4.6.17. *As $|x|\to\infty$, we have the following asymptotic approximation*

$$\mu_{Ar}\big|_{(x,y)} = O\left(\frac{1}{x^4}\right).$$

Sketch of proof : We first write $\mathfrak{f} = 0$ in the form

$$y^3 + (x^2 - x)y^2 - x^3 y + x^2 + x = 0.$$

Using the cubic equation formula in Appendix I, we can get asymptotic approximations for roots. Taking $i = 1$ in Appendix I as an example, we can subsequently get

$$\begin{aligned} u &= -\frac{x^6}{27} + \text{lower degree terms,} \\ m &= -\frac{x^2}{3} + \text{lower degree terms,} \\ n &= -\frac{x^2}{3} + \text{lower degree terms,} \\ y_1 &= -x^2 + \text{lower degree terms.} \end{aligned}$$

Substituting this into the basis of differential forms (4.25), we can get $\mu_{Ar}|_{(x,y_1)} = O(\frac{1}{x^4})$.

For $i = 2$ or 3, the quadratic term of y_i gets cancelled, and we can show

$$y_i = c_i x + \text{lower degree terms,}$$

where c_i are constants that can be explicitly computed. Substituting this to Equations (4.25), we get $\mu_{Ar}|_{(x,y_i)} = O(\frac{1}{x^4})$. QED

Finally, we can numerically compute $\log(S(\mathfrak{C}_{\mathbb{C}}))$ with the above two propositions. Recall the expression of $\log(S(X))$ in Equation (4.23). Since $\mathfrak{C}_{\mathbb{C}} \backslash U_{\mathbb{C}}$ is a 0-measure set, we only need to compute

$$\int_{U_{\mathbb{C}}} \log\|\theta\|_{g-1}(3P - Q) \cdot \mu_{Ar}(Q).$$

With Remark 4.6.16, we denote the i-th y-coordinate over x by $y_i(x)$. We denote by $U^0_{\mathbb{C}}$ the set where the index i is well-defined ($U_{\mathbb{C}} \backslash U^0_{\mathbb{C}}$ is a 0-measure set), then our computation is reduced to

$$\sum_{i=1}^{3} \int_{(x,y_i(x)) \in U^0_{\mathbb{C}}} \log\|\theta\|_{g-1}(3P - Q_i) \cdot \mu_{Ar}(Q_i), \tag{4.28}$$

where $Q_i = (x, y_i(x))$.

If we consider the complex number x as a point $(\mathrm{Re}(x), \mathrm{Im}(x))$ in $\mathbb{R}^2$, then Equation (4.28) is actually an integration of a real-valued function (with possible singularities), denoted by F, over $\mathbb{R}^2$. We use Riemann sums to approximate the integral of F on $\mathbb{R}^2$.

By Proposition 4.6.15, the integrated function has only 1 singular point P. According to Proposition 4.6.17, the volume form μ_{Ar} decreases quickly as $|x|$ becomes large. Thus to obtain a reasonable approximation, we only need to take the Riemann sums of F on a finite region in $\mathbb{R}^2$, which contains P. We display our algorithm as follows, and the Magma code can be found in Appendix VII.

Algorithm 3 Integration part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$ in Equation (4.23)

Input: $\mathfrak{f}$: the defining polynomial of $\mathfrak{C}_{\mathbb{C}}$ on $U_{\mathbb{C}}$
X: the Riemann surface given by $f = 0$
μ_{Ar}: the volume form (considered as a function on $\mathfrak{C}_{\mathbb{C}}$)
$\|\theta\|_{g-1}$: the theta function on $\mathrm{Pic}^{g-1}(\mathfrak{C}_{\mathbb{C}})$
P_{bs}: the fixed point
Output: $-3^2 * LogS_{int}$: the integration part of $\log(S(\mathfrak{C}_{\mathbb{C}}))$

$y_i =$ the i-th root function in x for $i = 1, 2, 3$ (Appendix I)
define a function Pt which sends the tuple (a, b, i) to the point $(a + b * I, y_i(a + b * I))$ on $\mathfrak{C}_{\mathbb{C}}$
scale$= 0.1$
radius$= 50$
$LogS_{int} = 0$
for j in $[1..ceiling(2 * radius/scale)]$ **do**
  for k in $[1..ceiling(2 * radius/scale)]$ **do**
    $\mathrm{Re}x_0 = -radius + j * scale$
    $\mathrm{Im}x_0 = -radius + k * scale$
    $Q_i = Pt(\mathrm{Re}x_0, \mathrm{Im}x_0, i)$ for $i = 1, 2, 3$
    $LogS_{int} = LogS_{int} + \sum_{i=1}^{3} Log(\|\theta\|_{g-1}(3P - Q_i)) \cdot \mu_{Ar}(Q_i) * scale^2$
  end for
end for
return $-3^2 * LogS_{int}$

Remark 4.6.18. *In the algorithm, we take the Riemann sums on the region* $|\mathrm{Re}\, x| \leq 50$, $|\mathrm{Im}\, x| \leq 50$, *and choose the size of the grids (corresponds to scale in the code above) to be* $\frac{1}{10}$. *In practice, we use finer grids for the region* $|\mathrm{Re}\, x| \leq 10$, $|\mathrm{Im}\, x| \leq 10$ *(we choose the size to be* $\frac{1}{100}$*). This can improve the accuracy of our numerical approximation.*

Our computation can be summarized as follows.

Computation 4.6.19. $\log(S(\mathfrak{C}_{\mathbb{C}})) \approx 1.10$

Remark 4.6.20. *Actually, we carried out the computation for two different choices of the fixed point* P. *One is the default base point* P_{bs} *chosen by Magma (Proposition 4.6.7), and the other one can be represented as* $(\mathrm{Re}x = 1, \mathrm{Im}x = 2, i = 3)$. *They gave*

1.07 *and* 1.13 *respectively, and we choose their arithmetic mean* 1.10 *as an approximation of* $\log(S(\mathfrak{C}_{\mathbb{C}}))$. *In Section 4.9, we will show that this is at least enough for showing the positivity of* $\langle \Delta, \Delta \rangle$*, although this approximation for* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *is not that precise.*

Remark 4.6.21. *This is only a numerical approximation. It is difficult to give a theoretic bound for the error term of our numerical integration, since we do not know how the term* $\log(\|\theta\|_{g-1}(3P-Q))$ *varies.*

With the calculation we carried out so far, the computation of the Green's function $G(x, y)$ on $U_{\mathbb{C}}$ is an easy evaluation by Theorem 4.6.1.

4.7 Computation of $T(\mathfrak{C}_{\mathbb{C}})$ and $H(\mathfrak{C}_{\mathbb{C}})$

In this section, we compute two invariants $T(\mathfrak{C}_{\mathbb{C}})$ (in Subsection 4.7.1) and $H(\mathfrak{C}_{\mathbb{C}})$ (in Subsection 4.7.2), whose relation with $\delta(\mathfrak{C}_{\mathbb{C}})$ and $\varphi(\mathfrak{C}_{\mathbb{C}})$ can be found in Theorem 4.7.3 and Theorem 4.7.7. The main references for this section are [12] and [65].

4.7.1 Computation of $T(\mathfrak{C}_{\mathbb{C}})$

Let X be a compact Riemann surface of genus $g > 1$ and let z be a local coordinate near $P \in X$. we define

$$\|F_z\|(P) \coloneqq \lim_{Q \to P} \frac{\|\theta\|_{g-1}(gP - Q)}{|z(P) - z(Q)|^g}.$$

Remark 4.7.1. *See Page 31 in [25] for a discussion of the convergence of this limit.*

Let $W_z(\omega)(P)$ be the Wronskian determinant (Equation (4.24)) at P with respect to an orthonormal basis of holomorphic forms ω on X. We define

$$T(X)_{z,P} \coloneqq \|F_z\|(P)^{-(g+1)} \cdot \prod_{W \in \mathfrak{W}} \|\theta\|_{g-1}(gP - W)^{(g-1)/g^3} |W_z(\omega)(P)|^2, \tag{4.29}$$

where the product goes through the Weierstrass points on X, counted with weights.

Lemma 4.7.2. *The number* $T(X)_{z,P}$ *is an invariant of* X*, that is, it does not depend on the choice of* z *and* P.

Proof. See Theorem 2.1.3 and Proposition 2.2.7 in [12]. □

For simplicity, we write $T(X)$ for this invariant. The reason for computing $T(X)$ is the following theorem.

Theorem 4.7.3. *The Faltings* δ *invariant, and the constants* $T(X)$ *and* $S(X)$ *satisfy*

$$\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X).$$

Proof. Theorem 2.1.3 in [12]. □

Now we show how to compute $T(X)$. We are already able to compute the theta function appearing in $T(X)$, and thus only $W_z(\omega)(P)$ and $\|F_z(P)\|$ remain to be done.

For all but finitely many points on $U_{\mathbb{C}}$, the x-coordinate is a local coordinate. Fix a point P, by Appendix I we can write $P = (x_P, y_i(x_P))$ for a certain index i. If we choose a real vector (a, b), then the point

$$Q_{abn} = (x_P + (a + bI) \cdot 10^{-n}, y_i(x_P + (a + bI) \cdot 10^{-n}))$$

approaches P from the direction (a, b) as n goes to infinity. Taking x as the local coordinate, we can approximate $\|F_x\|(P)$ by

$$\|F_x\|(P) \approx \frac{\|\theta\|_{g-1}(gP - Q_{abn})}{|x(P) - x(Q_{abn})|^g} = \frac{\|\theta\|_{g-1}(gP - Q_{abn})}{|10^{-n} \cdot (a + bI)|^g}$$

for a properly chosen n.

Remark 4.7.4. *(1) In our computation, we can choose the vector (a, b) to be a point on the unit circle.*

(2) In our computation of $T(\mathfrak{C}_{\mathbb{C}})$, we choose 10^{-50} as the precision. For this precision, we can choose n in $\{4, 5, 6, 7\}$. The reason is that the Abel-Jacobi map implemented in Magma is not as precise as the chosen precision.

(3) The Wronskian determinant part decreases quickly as the coordinates of P goes away from the origin of the chosen affine patch (the denominator has a higher degree than the numerator). If we choose a point where the coordinates of P are big, the Wronskian determinant part can be smaller than the precision we set. This numerical issue in Magma can lead to unstable output.

(4) When the three requirements above are satisfied, we can find that the output does not depend significantly on the choice of the point P. Thus we have a reliable approximation of $\|F_x\|(P)$.

The computation of $W_z(\omega)$, defined in Equation (4.24), is just some lengthy but easy calculation. The main tool here is taking implicit differentiation. Recall that in Subsection 4.6.2, we have an ordered basis of holomorphic forms (they are not orthonormal) on $\mathfrak{C}_{\mathbb{C}}$, which can be written in the following form

$$\left\{ \frac{dx}{\mathfrak{f}_y}, \frac{ydx}{\mathfrak{f}_y}, \frac{xdx}{\mathfrak{f}_y} \right\}.$$

For a general point $P = (x_0, y_0)$ on $\mathfrak{C}_{\mathbb{C}}$, x is a local coordinate. We take

$$g_1 = \frac{1}{\mathfrak{f}_y}, g_2 = \frac{y}{\mathfrak{f}_y}, g_3 = \frac{x}{\mathfrak{f}_y}.$$

For points close to P, y is a function in x and it makes sense to take derivatives of g_i's with respect to x. These derivatives with respect x can be expressed as rational functions of x, y, y' and y''.

Taking y' as an example, we take the implicit derivative at both sides of $\mathfrak{f} = 0$ with respect to x:

$$-3x^2y - x^3y' + 2xy^2 + 2x^2yy' - y^2 - 2xyy' + 3y^2y' + 2x + 1 = 0.$$

This gives

$$y'(P) = \frac{3x_0^2y_0 - 2x_0y_0^2 + y_0^2 - 2x_0 - 1}{-x_0^3 + 2x_0^2y_0 - 2x_0y_0 + 3y_0^2}.$$

We can get the values of g_i's and their derivatives in similar way.

Finally, we use the coefficients in Proposition 4.6.13 to compute the Wronskian determinant with respect to the orthonormal forms $\{\omega_{onj}\}_{1\leq j\leq 3}$.

Magma code for this subsection can be found in Appendix VIII. Our computation yields the following end result.

Computation 4.7.5. $T(\mathfrak{C}_{\mathbb{C}}) \approx 0.002544$.

Remark 4.7.6. *In Section 4.9, we will see that our computation for $T(\mathfrak{C}_{\mathbb{C}})$ is stable among different choices of P.*

4.7.2 Computation of $H(\mathfrak{C}_{\mathbb{C}})$

For a principally polarized abelian variety (A, Θ) of dimension g with period matrix $(1|\tau)$, we define a 1-1 form

$$v_{(A,\Theta)} \coloneqq \frac{i}{2}\sum_{j,k=1}^{g} (\mathrm{Im}\tau)^{-1}_{jk} dz_j \wedge d\bar{z}_k. \tag{4.30}$$

We define $H(A, \Theta)$ as

$$H(A,\Theta) \coloneqq \frac{1}{g!}\int_A \log\|\theta\| v^g, \tag{4.31}$$

where $\|\theta\|$ has an explicit expression in Proposition 4.5.4. For a compact Riemann surface X, we denote $H(\mathrm{Jac}(X), \Theta_{can})$ by $H(X)$. The following theorem explains the reason we compute $H(X)$.

Theorem 4.7.7. *For any compact Riemann surface X of genus $g \geq 1$, we have*

$$\delta(X) = -24H(X) + 2\varphi(X) - 8g\log 2\pi.$$

Proof. See Theorem 5.4 in [65]. □

The following two points explain why we think it is reasonable to believe that we can approximate $H(\mathfrak{C}_{\mathbb{C}})$ to good precision by taking Riemann sums.

(1) Although $\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})$ is 6-dimensional as a real manifold, it is a relatively small torus (with respect to the default choice of the base point and the (co)homology basis implemented in Magma).

(2) The singular points of the integrated function in $H(\mathfrak{C}_{\mathbb{C}})$ are equal to the theta divisor, a compact submanifold of real codimension 2 in $\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})$. Thus it is reasonable to believe that the integration of this singular function behaves well (in an analytic sense) on $\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})$.

Now we give a description of the computation.

First, we simplify the form $\frac{1}{g!}v^g$ in Formula (4.31). This is done by

$$\begin{aligned}\frac{1}{g!}v^g &= \left(\frac{i}{2}\right)^g (\det(\mathrm{Im}\tau)^{-1})\cdot\bigwedge_{j=1}^{g}(dz_j\wedge d\bar{z}_j)\\ &= (\det(\mathrm{Im}\tau)^{-1})\cdot\bigwedge_{j=1}^{g}(dx_j\wedge dy_j),\end{aligned}$$

where $z_j = x_j + iy_j$.

Second, we calculate the volume of the complex torus $T_{\mathfrak{C}_{\mathbb{C}}} := \mathrm{Jac}(\mathfrak{C}_{\mathbb{C}}) = \mathbb{C}^3/\mathbb{Z}^3 + \tau\mathbb{Z}^3$. The volume of $T_{\mathfrak{C}_{\mathbb{C}}}$ is

$$\mathrm{Vol}(T_{\mathfrak{C}}) = \left|\det\begin{pmatrix} I & 0\\ \mathrm{Re}\tau & \mathrm{Im}\tau\end{pmatrix}\right| = \det\mathrm{Im}\tau.$$

Finally, we can take the Riemann sums. By splitting each edge of $T_{\mathfrak{C}_{\mathbb{C}}}$ into c parts, we get c^6 small polyhedrons. We approximate $H(\mathfrak{C}_{\mathbb{C}})$ by

$$\begin{aligned}&\sum_{i=1}^{c^6}\log\|\theta\|(v_i)(\det\mathrm{Im}\tau)^{-1}\mathrm{Vol}(T_{\mathfrak{C}})/c^6\\ =&\sum_{i=1}^{c^6}\log\|\theta\|(v_i)/c^6.\end{aligned}\tag{4.32}$$

where v_i is a chosen point in each small polyhedron.

Code for this subsection can be found in Appendix IX, and the result of our computation can be summarized as follows.

Computation 4.7.8. $H(\mathfrak{C}_{\mathbb{C}}) \approx -0.70356$.

Similar to the approximation of $\log(S(\mathfrak{C}_{\mathbb{C}}))$, we lack the bound of error terms. In Section 4.9, we will verify that our numerical approximation of $\int_{\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})} \|\theta\|^2 \nu^g$ is very good.

4.8 What can we get from the computation?

Our primary goal is to compute the height of a canonical Gross-Schoen cycle of a certain non-hyperelliptic genus 3 curve $\mathfrak{C}_{\mathbb{Q}}$. Summing up all the computations in this chapter, we have the following result.

Computation 4.8.1. *For the plane curve $\mathfrak{C}$ defined by*

$$-X^3Y + X^2Y^2 - XY^2Z + Y^3Z + X^2Z^2 + XZ^3 = 0,$$

we have the following results:

(1) $\delta(\mathfrak{C}_{\mathbb{C}}) \approx -24.87$*,*

(2) $\varphi(\mathfrak{C}_{\mathbb{C}}) \approx 1.17$*,*

(3) $\deg \det f_* \overline{\omega}_{\mathfrak{C}} \approx -2.9190567336$*,*

(4) $(\overline{\omega}, \overline{\omega})_{Ar} \approx 3.43$*,*

(5) $(\hat{\omega}, \hat{\omega})_{ad} \approx 1.55$*,*

(6) $\langle \Delta, \Delta \rangle \approx 0.60$*.*

Proof. (1) By Theorem 4.6.19, Proposition 4.7.3 and Proposition 4.7.5 we obtain

$$\begin{aligned}
\delta(\mathfrak{C}_{\mathbb{C}}) &= 4\left(\log(T(\mathfrak{C}_{\mathbb{C}})) - \frac{2}{9}\log(S(\mathfrak{C}_{\mathbb{C}}))\right) \\
&\approx 4 \cdot (\log(0.002544) - \frac{2}{9} \cdot 1.1) \\
&\approx -24.87.
\end{aligned}$$

(2) Then by Theorem 4.7.7 and Proposition 4.7.8, we obtain

$$\begin{aligned}
\varphi(\mathfrak{C}_{\mathbb{C}}) &= \frac{\delta(\mathfrak{C}_{\mathbb{C}}) + 24H(\mathfrak{C}_{\mathbb{C}}) + 24\log 2\pi}{2} \\
&\approx \frac{-24.87 - 24 \cdot 0.70356 + 24 \cdot \log 2\pi}{2} \\
&\approx 1.17.
\end{aligned}$$

(3) By Equation (2.8), Proposition 4.4.5 and Proposition 4.4.8, we obtain

$$\begin{aligned}
\deg \det f_*\overline{\omega}_{\mathcal{X}/B} &= \sum_{p \text{ prime}} \frac{\mathrm{ord}_p(\chi'_{18})\log p}{18} - \frac{\log\|\chi'_{18}\|_{\mathrm{Hdg}}(\mathfrak{C}_{\mathbb{C}})}{18} \\
&\approx \frac{2}{18}\log(29 \cdot 163) - 3.8591729592 \\
&\approx -2.9190567336.
\end{aligned}$$

(4) By Corollary 1.3.11 and Proposition 4.4.1, we obtain

$$\begin{aligned}
(\overline{\omega}, \overline{\omega})_{Ar} &= 12 \deg \det f_*\overline{\omega}_{\mathfrak{C}} - \sum_{p \text{ prime}} \delta(\mathfrak{C}_p)\log p - \delta(\mathfrak{C}_{\mathbb{C}}) + 4g\log 2\pi \\
&\approx -2.9190567336 \cdot 12 - \log(29 \cdot 163) + 24.87 + 12 \cdot \log(2\pi) \\
&\approx 3.43.
\end{aligned}$$

(5) By Theorem 1.5.3 and Proposition 4.4.1, we obtain

$$\begin{aligned}
(\hat{\omega}, \hat{\omega})_{ad} &= (\overline{\omega}, \overline{\omega})_{Ar} - \sum_{p \text{ prime}} \epsilon(\mathfrak{C}_{\mathbb{Z}_p})\log p \\
&\approx 3.43 - \frac{2 \cdot \log(29 * 163)}{9} \\
&\approx 1.55.
\end{aligned}$$

(6) By Theorem 1.5.6 and Proposition 4.4.1, we obtain

$$\begin{aligned}
\langle \Delta, \Delta \rangle &= \frac{7}{4}(\hat{\omega}, \hat{\omega})_{ad} - \sum_{v \in M(\mathbb{Q})} \varphi(\mathfrak{C}_v)\log(N(v)) \\
&\approx \frac{7}{4} \cdot 1.55 - \log(29 \cdot 163)/9 - 1.17 \\
&\approx 0.60.
\end{aligned}$$

□

Remark 4.8.2. *(Important) The reason we use different precisions is that some invariants (like* $\mathrm{ord}_v(\chi'_{18})$*) can be computed to fairly high precision, while others (like* $\log(S(\mathfrak{C}_{\mathbb{C}}))$*) cannot. For the former ones, we use* 10 *as the precision. For the latter ones, we choose the precisions that are stable among our computations. For example, the first six digits after the decimal point of* $T(\mathfrak{C}_{\mathbb{C}})$ *are stable among different choices of* P *(Equation (4.29)).*

Remark 4.8.3. $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *is used in the computation of* $\varphi(\mathfrak{C}_{\mathbb{C}})$ *and* $\delta(\mathfrak{C}_{\mathbb{C}})$*, thus we use the precision of* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *for* $\varphi(\mathfrak{C}_{\mathbb{C}})$ *and* $\delta(\mathfrak{C}_{\mathbb{C}})$*. In* $\deg \det f_*\overline{\omega}_{\mathcal{X}/B}$*, all components can be computed to arbitrary precision, thus we use high precision.*

4.9 Why do we think these approximations are reliable?

In this subsection, we show that these approximations are stable among choices and compatible with known facts. Recall that we defined $\log(S(X))$, $T(X)$ and $H(X)$ in Equation (4.23), Equation (4.29) and Equation (4.31).

In Theorem 3.3.2, we can find that all invariants except $\lambda(\mathfrak{C}_{\mathbb{C}})$ can easily be computed to arbitrary precision. For $\lambda(\mathfrak{C}_{\mathbb{C}})$, we decomposed it into a linear sum of other invariants: $H(\mathfrak{C}_{\mathbb{C}})$, $\log(S(\mathfrak{C}_{\mathbb{C}}))$ and $T(\mathfrak{C}_{\mathbb{C}})$. However, we can only compute these invariants by approximation (numerical integration and taking a limit in the computation of $T(\mathfrak{C}_{\mathbb{C}})$).

For $T(\mathfrak{C}_{\mathbb{C}})$ and $\log(S(\mathfrak{C}_{\mathbb{C}}))$, we need to fix a point P. For $H(\mathfrak{C}_{\mathbb{C}})$, we need to split each edge of the torus $T_{\mathfrak{C}_{\mathbb{C}}}$ into c segments (see Equation (4.32)). The first thing we check is to show that our computations are stable along these choices.

Using the code for $T(\mathfrak{C}_{\mathbb{C}})$ (Appendix VIII), we can find that the output does not change significantly among different choices of the fixed point P, even though $T(\mathfrak{C}_{\mathbb{C}})$ is a product of factors which depend wildly on the choice of P. For example, the Wronskian part can be smaller than 10^{-24} and bigger than 10^{-2} for different choices of the fixed point P.

For $H(\mathfrak{C}_{\mathbb{C}})$, we compute it for $c = 19$ and 23. The outputs are quite stable, giving around -0.70356438 and -0.70355787 respectively.

As we explained in Remark 4.6.20, we choose two distinct points as the point P in Equation (4.23). Since $S(\mathfrak{C}_{\mathbb{C}})$ is an invariant of $\mathfrak{C}_{\mathbb{C}}$, it should not depend on the choice of P. It turns out that our approximation for $\log(S(\mathfrak{C}_{\mathbb{C}}))$ is less precise. The two points are the default base point P_{bs} chosen by Magma and the point represented by $(\mathrm{Re}x = 1, \mathrm{Im}x = 2, \mathrm{index} = 3)$ (see Remark 4.6.16 for an explanation of the notation), where the index is explained in Appendix I. We get 1.07 and 1.13 for $\log(S(\mathfrak{C}_{\mathbb{C}}))$ respectively, and we take their arithmetic mean 1.10 as the approximation of $\log(S(\mathfrak{C}_{\mathbb{C}}))$.

By Equation (1.18), Theorem 3.3.2, Theorem 4.7.3 and Theorem 4.7.7, we can decompose $\langle \Delta, \Delta \rangle$ as follows

$$\langle \Delta, \Delta \rangle = -12H(\mathfrak{C}_{\mathbb{C}}) + 2\log(S(\mathfrak{C}_{\mathbb{C}})) - 9\log(T(\mathfrak{C}_{\mathbb{C}})) - 63.7966513771.$$

Although we cannot approximate $\log(S(\mathfrak{C}_{\mathbb{C}}))$ precisely, we can find that it contributes less to $\langle \Delta, \Delta \rangle$ than other terms. The invariants $H(\mathfrak{C}_{\mathbb{C}})$ and $\log(T(\mathfrak{C}_{\mathbb{C}}))$ contribute much more, but our approximations for them are also much more stable.

Remark 4.9.1. *(Risk) Note that the functions in the integration of* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *and* $H(\mathfrak{C}_{\mathbb{C}})$ *are singular. Thus it is still possible that our numerical approximation is far away from the correct answer.*

Summary 4.9.2. *The first part of our checking can be summarized as follows:*

(1) Our code gives relatively stable approximations for $H(\mathfrak{C}_{\mathbb{C}})$, $\log(T(\mathfrak{C}_{\mathbb{C}}))$ *and* $\log(S(\mathfrak{C}_{\mathbb{C}}))$.

(2) Among the three invariants, our approximation for $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *is less satisfying, but* $\log(S(\mathfrak{C}_{\mathbb{C}}))$ *contributes less to* $\langle \Delta, \Delta \rangle$.

The second part of our check is comparing our results with known facts.

In Theorem 1.5.6 (1), we can find $\varphi(\mathfrak{C}_{\mathbb{C}}) > 0$. This is compatible with $\varphi(\mathfrak{C}_{\mathbb{C}}) \approx 1.17$.

In Theorem 2.2.6, we have an equality about the discriminant of a plane quartic curve and the modular form $\tilde{\chi}_{18}$. Using Magma, we can get $\mathrm{Disc}(\mathfrak{F}) = 29 \cdot 163$, which shows that 29 and 163 are the only bad primes in particular. This is compatible with our computation since

$$\frac{\mathrm{Disc}(\mathfrak{F})^2}{(2\pi)^{54}\frac{\tilde{\chi}_{18}(\tau)}{\det(\Omega_1)^{18}}} \approx 0.9999991$$

The code for this can be found in Appendix X.

Proposition 4.9.3. *Let* $f : C \to \mathrm{Spec}(O_k)$ *be a semistable arithmetic surface of genus* $g \geq 1$, *where* k *is a number field. Then we have the following inequality*

$$\deg \det f_* \overline{\omega} \geq -\log(\pi\sqrt{2}) g \cdot [k : \mathbb{Q}]$$

Proof. See Equation (1.8) in [16]. □

This is compatible with our computation since

$$-2.9190567336 \geq -4.4739104284 \approx -3 \cdot \log(\pi\sqrt{2}).$$

Proposition 4.9.4. *Let* X *be a compact Riemann surface of genus* $g \geq 1$. *Then we have* $H(X) \leq -\frac{g}{4}\log 2$.

Proof. See Proposition 2.1 in [65]. □

This is compatible with our computation since

$$-0.70356 \leq -0.51986 \approx -\frac{3}{4}\log 2.$$

Remark 4.9.5. *Recall the definition of* $\|\theta\|$ *and* ν *(paragraphs around Equation (4.15)). We have the following identity*

$$\frac{1}{3!}\int_{\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})} \|\theta\|^2 \nu^3 = 2^{-3/2}.$$

This can be used to check the correctness of our code for $H(\mathfrak{C}_{\mathbb{C}})$, *since we only need to replace the integrated function* $\log\|\theta\|$ *by* $\|\theta\|^2$. *The code is almost the same as that of* $H(\mathfrak{C}_{\mathbb{C}})$, *thus we omit it. Taking* $c = 19$, *we can find that the difference between our approximations of* $\frac{1}{3!}\int_{\mathrm{Jac}(\mathfrak{C}_{\mathbb{C}})} \|\theta\|^2 v^g$ *and* $2^{-3/2}$ *is even smaller than* 10^{-10}.

According to Corollary 5.7 in [70], the pairing $(\hat{\omega}, \hat{\omega})_{ad}$ is non-negative. This is compatible with the fact $(\hat{\omega}, \hat{\omega})_{ad} \approx 1.55$ computed in Theorem 4.8.1.

Thus we believe that our approximations are reliable!

Chapter 5

Appendix

I Solving cubic equations

Given a cubic equation

$$ay^3 + by^2 + cy + d = 0,$$

we solve it in following way.

Algorithm 4 Solutions of a cubic equation

Input: a,b,c,d: coefficients of the polynomial

i: the index of expected solution, taking value in $\{1, 2, 3\}$

Output: the *i-th* solution of the polynomial

$r_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$u = \frac{9abc - 27a^2 d - 2b^3}{54a^3}$

$v = \frac{(12ac^3 - 3b^2c^2 - 54abcd + 81a^2d^2 + 12b^3d)^{1/2}}{18a^2}$

if $|u + v| \geq |u - v|$ **then**

    $m = (u + v)^{(1/3)}$

else

    $m = (u - v)^{(1/3)}$

end if

if $m = 0$ **then**

    $n = 0$

else

    $n = \frac{b^2 - 3ac}{9am}$

end if

return $r_3^{i-1} m + r_3^{2i-2} n - \frac{b}{3a}$

When we say the $1st$ root of a cubic function, we mean the output of the algorithm above when $i = 1$, similarly for other indices i. The Magma code for our curve $\mathfrak{C}_{\mathbb{C}}$ is given as follows:

```
a:=1;
b:=x^2-x;
c:=-x^3;
d:=x^2+x;
cubic_unit:=-1/2+3^(1/2)/2*I;
root:=function(u,v,i);
numa:=a;
numb:=Evaluate(b,[u+v*I,0]);
numc:=Evaluate(c,[u+v*I,0]);
preuu:=(9*a*b*c-27*a^2*d-2*b^3)/(54*a^3);
uu:=Evaluate(preuu,[u+v*I,0]);
prevv:=Evaluate((3*(4*a*c^3-b^2*c^2-18*a*b*c*d+27*a^2*d^2+4*b^3*d)),[u+
    v*I,0]);
vv:=(prevv^(1/2))/(18*a^3);
m:=(uu-vv)^(1/3);
if Abs(uu+vv) ge Abs(uu-vv) then
m:=(uu+vv)^(1/3);
end if;
n:=0;
if m ne 0 then
n:=(numb^2-3*numa*numc)/(9*numa*m);
end if;
result:=(cubic_unit^(i-1))*m+n*cubic_unit^(2*i-2)-(numb)/(3*numa);
return result;
end function;
```

In Chapter 4, we need to integrate some functions defined on a plane quartic curve. It turns out that, for these integrations, we need to label the different y's for any fixed x in the way above.

II List of Weierstrass points

The set of Weierstrass points of $\mathfrak{C}_{\mathbb{C}}$ is:

$$
\begin{aligned}
&\mathfrak{W} = \{\\
&(0.0000000000, 0.0000000000),\\
&(-2.104587156, -0.3159811808),\\
&(-1.241974125, -1.824965133),\\
&(0.3539677313, -0.7304015296),\\
&(-1.959274470, -1.410535295),\\
&(-1.000000000, 0.0000000000),\\
&(-1.455757161 - 1.277524807 * I, -0.5258271579 - 3.390009274 * I),\\
&(-1.455757161 + 1.277524807 * I, -0.5258271579 + 3.390009274 * I),\\
&(1.266455202 - 1.314861880 * I, 0.1352071928 + 2.966892635 * I),\\
&(1.266455202 + 1.314861880 * I, 0.1352071928 - 2.966892635 * I),\\
&(0.2089717154 - 1.597843531 * I, 0.1450907026 - 2.034168656 * I),\\
&(0.2089717154 + 1.597843531 * I, 0.1450907026 + 2.034168656 * I),\\
&(-1.319681529 - 0.4985373090 * I, -0.4698939284 - 0.6348619710 * I),\\
&(-1.319681529 + 0.4985373090 * I, -0.4698939284 + 0.6348619710 * I),\\
&(1.691125343 - 0.6842628628 * I, 0.6312556990 + 0.6046227212 * I),\\
&(1.691125343 + 0.6842628628 * I, 0.6312556990 - 0.6046227212 * I),\\
&(-1.125631934 - 0.2207052168 * I, -0.6596966755 + 0.07228789093 * I),\\
&(-1.125631934 + 0.2207052168 * I, -0.6596966755 - 0.07228789093 * I),\\
&(-0.5282412488 - 1.005235134 * I, -0.4177679369 + 0.3210649478 * I),\\
&(-0.5282412488 + 1.005235134 * I, -0.4177679369 - 0.3210649478 * I),\\
&(-0.3349732907 - 0.4139797966 * I, -0.4206274002 - 0.9204898342 * I),\\
&(-0.3349732907 + 0.4139797966 * I, -0.4206274002 + 0.9204898342 * I),\\
&(0.07366689191 - 0.3721976724 * I, 0.7232010036 + 0.2545999915 * I),\\
&(0.07366689191 + 0.3721976724 * I, 0.7232010036 - 0.2545999915 * I)\}
\end{aligned}
$$

III Code for canonical form

```
from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
R.<x,y> = QQ[]
f = -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x

S = RiemannSurface(f)
M = S.period_matrix()
Omega1 = M[[0,1,2],[0,1,2]]
Omega2 = M[[0,1,2],[3,4,5]]
A = Omega1.apply_map(real)
B = Omega1.apply_map(imag)
tau = S.riemann_matrix()
Omega1bar= A-B*I
Imtau = tau.apply_map(imag)
RieSecBil = Omega1bar*Imtau*Omega1.transpose()
RieSecBil
[ 8.14058999836226 - 4.44089209850063e-16*I -2.71111012021317 - 9.21762666195036e-14*I -5.21995559097122 - 5.85642645489770e-14*I ]
[-2.71111012021315 + 6.52811138479592e-14*I 8.66278158093292 + 8.32667268468867e-14*I 5.65493670273040 + 1.56541446472147e-13*I ]
[-5.21995559097118 + 1.45994327738208e-14*I 5.65493670273037 - 1.75831571525009e-14*I 10.7557438794561 + 1.18932641512970e-14*I ]

w11 = RieSecBil[0][0]
w1y = RieSecBil[0][1]
w1x = RieSecBil[0][2]
wyy = RieSecBil[1][1]
wyx = RieSecBil[1][2]
wxx = RieSecBil[2][2]
k = (wyx-w1x*w1y/w11)/(wyy-w1y*w1y/w11)
norm1 = RieSecBil[0][0]**(1/2)
norm2 = (wyy-(2*w1y*w1y/w11)+w1y*w1y/w11)**(1/2)
norm3 = (wxx+w1x*w1x/w11+k*k*wyy+k*k*w1y*w1y/w11-2*w1x*w1x/w11-2*k*wyx+2*k*w1y*w1x/w11+2*k*w1x*w1y/w11-2*k*w1x*w1y/w11-2*k*k*w1y*w1y/w11)**(1/2)
```

Gram-Schmidt process

In the code, the symbol wmn denotes the integration of 1-1 forms, for example, $w1y$ represents $\frac{i}{2}\int_X \omega_1 \wedge \overline{\omega}_y$. The Gram-Schmidt process is actually done by hand, and we just use SageMath for evaluation.

IV Code for semistability

```
from collections import defaultdict
PP.<x,y,z> = ProjectiveSpace(QQ,2)
Poly.<U,V> = PolynomialRing(ZZ, 2)
RR.<X, Y, Z> = PolynomialRing(ZZ, 3)
C = Curve(-x^3*y+x^2*y^2-x*y^2*z+y^3*z+x^2*z^2+x*z^3, PP)
inj1 = Poly.hom([X, Y], RR)


def IsDegree1(I):    #checking the ideal I has degree 1 generators
out = True
for f in I.gens():
if not f.degree() == 1:
out = False
return out


def MyBadPrimes(C):    #finding out primes with bad reduction
f = C.defining_polynomial()
RZ.<xZ,yZ,zZ> = PolynomialRing(ZZ, 3)
coeffs = f.coefficients()
dens = [c.denominator() for c in coeffs]
den = lcm(dens)
F = RZ(f*den)
Fx = F.derivative(xZ)
Fy = F.derivative(yZ)
Fz = F.derivative(zZ)
NaiveDisc = 1
for P in [[xZ,yZ,1],[xZ,1,zZ],[1,yZ,zZ]]:
I = ideal([g(P) for g in [F,Fx,Fy,Fz]])
G = I.groebner_basis()
n = G[len(G)-1]
NaiveDisc = lcm(n, NaiveDisc)
return [a[0] for a in factor(NaiveDisc)]


def MyBadPoints(C, default_prime):    #CORE FUNCTION: finding out all singular points over certain prime number and the tangent lines
f = C.defining_polynomial()
RZ.<xZ,yZ,zZ> = PolynomialRing(ZZ, 3)
coeffs = f.coefficients()
dens = [c.denominator() for c in coeffs]
den = lcm(dens)
F = RZ(f*den)
```

```
Fx = F.derivative(xZ)
Fy = F.derivative(yZ)
Fz = F.derivative(zZ)
Out = ['Dimention of singular locus is positive.']
GG = []                #list of associate primes (after field extension)
sing_set = []          #list of degree 2 parts of singular points after transition
factor_sing = []       #list of factorizations of degree 2 parts
degree = 1
SingDim = 0
found = False
while found == False:
Rp = RZ.change_ring(GF(default_prime^degree))
win = True
S.<u, v> = PolynomialRing(GF(default_prime^degree), 2)
for P in [ [u, v, 1], [u, 1, v], [1, u, v] ]:
affine_patch_map = RZ.hom(P, S)
I_2 = Ideal(S,[ affine_patch_map(g) for g in [F, Fx, Fy, Fz]])
G_2 = I_2.associated_primes()
for I in G_2:
VV = I.gens()
if Set(VV).cardinality()==1:
SingDim = 1
if SingDim == 1:
break
for I in G_2:
if IsDegree1(I) == False:
win = False          #This means that the singular points don't have coordinates in this base field and we still need to extend it
if SingDim == 1:
break
Out = []
if win == True:
for P in [ [u, v, 1], [u, 1, v], [1, u, v] ]:
affine_patch_map = RZ.hom(P, S)
I_2 = Ideal(S,[affine_patch_map(g) for g in [F, Fx, Fy, Fz]])
G_2 = I_2.associated_primes()
GG += G_2
for AP in G_2:           #in this for loop, we find out the coordinate of singular points
T.<x, y> = S.quotient(ideal(S, [w for w in AP.gens()]))
Quot = T.cover()
if Quot(1) == 0:
continue
for m in GF(default_prime^degree):
if Quot(u-m)==0:
```

```
SingX = m
break
for n in GF(default_prime^degree):
if Quot(v-n)==0:
SingY = n
break
Trans = S.hom([u+SingX, v+SingY])
change_to_S = RZ.hom(P, S)
TranP = Trans(change_to_S(F))    #our curve after transition
homog_part = defaultdict(TranP.parent())     #the following 4 lines
    help us to get the degree 2 part
for coeff,monom in TranP:
homog_part[monom.degree()] += coeff * monom
sing_set.append(homog_part[2])
found = True
Out += GG
for w in sing_set:    #find out the singularity type
SS = S.change_ring(GF(default_prime^(2*degree)))
if w!=0:
factor_sing.append(factor(SS(w)))
else:
factor_sing.append('HIGHER_SINGULARITY')
else: degree += 1
return Out, factor_sing
```

V Code for semi-canonical divisor

In Magma, a base point P_{bs} is already chosen and $AbelJacobi(D)$ means $AJ(D-\deg(D)[P_{bs}])$.

```
PeriodMatrix:=Matrix([
[-0.0756487726827658975 + 0.82850793518670070913*I,
0.48656585589140586329 - 0.07531944436044814614*I,
0.53238504944901350223 + 0.23143384347205016268*I],
[0.48656585589140586329 - 0.07531944436044814614*I,
0.500584921111822623585 + 0.48874080037323974253*I,
0.137880100924435244693 - 0.05703730011758514006*I],
[0.53238504944901350223 + 0.23143384347205016268*I,
0.137880100924435244693 - 0.05703730011758514006*I,
0.689717878372433609278 + 0.69264774350470313589*I]])
```

```
C<I> := ComplexField(30);
Qxy<x, y> := PolynomialRing(Rationals(), 2);
f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;    //defining polynomial
X := RiemannSurface(f);    //create the riemann surface
tau:=SmallPeriodMatrix(X);

Pt := X ! [0,0];    // a point on X
Div:=Divisor([Pt],[2]);    //effective divisor of degree 2
Div2:=Divisor([BasePoint(X)],[2]);
Div3:=Divisor([BasePoint(X),Pt],[1,1]);

asd := InfinitePoints(X);
U:=[asd[1],asd[2],asd[3]];
V:=[2,1,1];
CanDiv:=Divisor(U,V);    //canonical divisor

MatrixIm := function(M)    //function to take the imaginary part of a
    matrix.
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

Columns := function(M)    //function to extract columns as a list of
    matrices
```

```
out := [];
for i in [1..Ncols(M)] do
out := out cat [ColumnSubmatrix(M, i, 1)];
end for;
return out;
end function;

char:=Matrix([[0],[0],[0],[0],[0],[0]]);
Imtau:=MatrixIm(tau);
detimtau:=Determinant(Imtau);
LeftPeriod:=RemoveColumn(RemoveColumn(RemoveColumn(BigPeriodMatrix(X)
    ,4),4),4);
ID3 := DiagonalMatrix(C, [1,1,1]);
columnlist := Columns(ID3) cat Columns(tau);
AJCD := Matrix(AbelJacobi(CanDiv));    //the image of canonical
    divisor under AJ map

AJ_2_torsion := [];
for v in CartesianPower([0,1], 6) do
new := ZeroMatrix(C, 3, 1);
for i in [1..6] do
new := new + v[i] * columnlist[i];
end for;
AJ_2_torsion := AJ_2_torsion cat [(1/2) * new];
end for;

translates_to_try := [];    //this will be a list of all 64 square
    roots of CanDiv
for V in AJ_2_torsion do
translates_to_try := translates_to_try cat [(1/2) * AJCD + V];
end for;

torus_theta_function := function(D)// D is a divisor of degree g-1
myinput := Matrix(AbelJacobi(D));
myabstheta:=Abs(Theta(char, myinput, tau));
myiminput:= MatrixIm(myinput);
myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^(-1))*
    myiminput;
return Exp(myexponent[1,1])*myabstheta*detimtau^(1/4);
end function;

find_correct_theta_translate := function(D)    //find out the correct
    translation
P := AbelJacobi(D);
out := [];
for V in translates_to_try do
myinput := Matrix(P) - V;
```

```
myabstheta:=Abs(Theta(char, myinput, tau));
myiminput:= MatrixIm(myinput);
myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^(-1))*
    myiminput;
if Abs(Exp(myexponent[1,1])*myabstheta*detimtau^(1/4)) le 0.00000001
    then
return V;
end if;
end for;
end function;

//we check the result by choosing 3 different effective divisors
ThetaTranslate1 := find_correct_theta_translate(Div);
ThetaTranslate1;
ThetaTranslate2 := find_correct_theta_translate(Div2);
ThetaTranslate2;
ThetaTranslate3 := find_correct_theta_translate(Div3);
ThetaTranslate3;
```

VI Code for $\|\chi'_{18}\|_{\mathbf{Hdg}}$

```
C<I> := ComplexField(50);
Qxy<x, y> := PolynomialRing(Rationals(), 2);
f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;     //defining polynomial
X := RiemannSurface(f);     //create the riemann surface

tau1:=SmallPeriodMatrix(X);     //period matrix implemented in Magma
tau2:=Matrix(C,3,[  0.855638485763810 + 1.03033263075936*I,
    0.380495294693576 + 0.00436338826915927*I,   0.506949568788359 -
    0.178115589372895*I,
0.380495294693584 + 0.00436338826915617*I ,  0.401072147476828 +
    0.694848752465586*I , -0.580339267402228 - 0.147446853346675*I,
0.506949568788374 - 0.178115589372884*I,  -0.580339267402254 -
    0.147446853346687*I,   0.219761917471621 + 0.625482641303850*I
]);     //period matrix implemented in SageMath

//the following set S contains all the even characteristics
S:=[Matrix(RationalField(),1,[0,0,0,0,0,0]),Matrix(RationalField()
    ,1,[0,0,0,0,0,1]),Matrix(RationalField(),1,[0,0,0,0,1,0]),Matrix(
    RationalField(),1,[0,0,0,0,1,1]),Matrix(RationalField()
    ,1,[0,0,0,1,0,0]),Matrix(RationalField(),1,[0,0,0,1,0,1]),Matrix(
    RationalField(),1,[0,0,0,1,1,0]),Matrix(RationalField()
    ,1,[0,0,0,1,1,1]),Matrix(RationalField(),1,[0,0,1,0,0,0]),Matrix(
    RationalField(),1,[0,0,1,0,1,0]),Matrix(RationalField()
    ,1,[0,0,1,1,0,0]),Matrix(RationalField(),1,[0,0,1,1,1,0]),Matrix(
    RationalField(),1,[0,1,0,0,0,0]),Matrix(RationalField()
    ,1,[0,1,0,0,0,1]),Matrix(RationalField(),1,[0,1,0,1,0,0]),Matrix(
    RationalField(),1,[0,1,0,1,0,1]),Matrix(RationalField()
    ,1,[0,1,1,0,0,0]),Matrix(RationalField(),1,[0,1,1,1,0,0]),Matrix(
    RationalField(),1,[0,1,1,1,1,1]),Matrix(RationalField()
    ,1,[0,1,1,0,1,1]),Matrix(RationalField(),1,[1,0,0,0,0,0]),Matrix(
    RationalField(),1,[1,0,0,0,0,1]),Matrix(RationalField()
    ,1,[1,0,0,0,1,0]),Matrix(RationalField(),1,[1,0,0,0,1,1]),Matrix(
    RationalField(),1,[1,0,1,0,0,0]),Matrix(RationalField()
    ,1,[1,0,1,0,1,0]),Matrix(RationalField(),1,[1,0,1,1,0,1]),Matrix(
    RationalField(),1,[1,0,1,1,1,1]),Matrix(RationalField()
    ,1,[1,1,0,0,0,0]),Matrix(RationalField(),1,[1,1,0,0,0,1]),Matrix(
    RationalField(),1,[1,1,0,1,1,0]),Matrix(RationalField()
    ,1,[1,1,0,1,1,1]),Matrix(RationalField(),1,[1,1,1,0,0,0]),Matrix(
    RationalField(),1,[1,1,1,0,1,1]),Matrix(RationalField()
    ,1,[1,1,1,1,1,0]),Matrix(RationalField(),1,[1,1,1,1,0,1])];

MatrixIm := function(M)     //imaginary part of a matrix
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
```

```
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

compute:=function(period)    //compute ||theta||
chitilde:=1;
zerovector:=Matrix(C,1,[0,0,0]);
for ele in S do
chitilde:=chitilde*Theta((1/2)*ele,zerovector,period);
end for;
abschitilde:=Abs(chitilde);
Imtau:=MatrixIm(period);
det:=Determinant(Imtau);
prepreresult:=Log((det^9)*abschitilde);
preresult:=26*Log(2)+54*Log(Pi(C))+prepreresult;
result:=-(21/18)*preresult;    //contribution of \chi_18 at the
    infinite place
return result;
end function;

//we compute the result using two period matrices implemented in
    Magma and SageMath
compute(tau1);
compute(tau2);
```

VII Code for the integration part in log($S(\mathfrak{C}_{\mathbb{C}})$)

```
C<I> := ComplexField(30);
Cxy<x, y> := PolynomialRing(RationalField(), 2);
f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;    //defining polynomial
X := RiemannSurface(f : Precision:=30);    //create riemann surface
tau:=SmallPeriodMatrix(X);
BP:=BasePoint(X);

MatrixIm := function(M)    //take the imaginary part of a matrix
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

Imtau:=MatrixIm(tau);
detimtau:=Determinant(Imtau);
ThetaTranslate:=Matrix([[0.479250542651680186758281300774 -
    0.00334176833187451614116524746349*I],
[0.698684877508432322290935180841 + 0.199495723882563563098727841525*
    I],
[0.00722266620787249384535143960054 -
    0.0430102069343208149623250436805*I]]);

normalizedtheta := function(D);    //compute ||theta||_{g-1} at D
P := AbelJacobi(D);
myinput := Matrix(P) - ThetaTranslate;
char:=Matrix([[0],[0],[0],[0],[0],[0]]);
myabstheta:=Abs(Theta(char, myinput, tau));
myiminput:= MatrixIm(myinput);
myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^-1)*myiminput
    ;
return Abs(Exp(myexponent[1,1])*myabstheta*detimtau^(1/4));
end function;

a:=1;
b:=x^2-x;
c:=-x^3;
d:=x^2+x;
cubic_unit:=-1/2+3^(1/2)/2*I;
```

```
        root:=function(u,v,i);     //solve the cubic equation with
            coefficients a, b, c and d.
        numa:=a;
        numb:=Evaluate(b,[u+v*I,0]);
        numc:=Evaluate(c,[u+v*I,0]);
        preuu:=(9*a*b*c-27*a^2*d-2*b^3)/(54*a^3);
        uu:=Evaluate(preuu,[u+v*I,0]);
        prevv:=Evaluate((3*(4*a*c^3-b^2*c^2-18*a*b*c*d+27*a^2*d^2+4*b^3*d)),[
            u+v*I,0]);
        vv:=(prevv^(1/2))/(18*a^2);
        m:=(uu-vv)^(1/3);
        if Abs(uu+vv) ge Abs(uu-vv) then
        m:=(uu+vv)^(1/3);
        end if;
        n:=0;
        if (Abs(m) ge 0.00001) then
        n:=(numb^2-3*numa*numc)/(9*numa*m);
        else n:=0;
        end if;
        result:=(cubic_unit^(i-1))*m+n*cubic_unit^(2*i-2)-(numb)/(3*numa);
        return result;
        end function;

        embedding:=function(u,v,i);     //find out the embedding index in
            Magma
        zz,pt1:=IsPoint(X,<[u,v],1>);
        zz,pt2:=IsPoint(X,<[u,v],2>);
        zz,pt3:=IsPoint(X,<[u,v],3>);
        if (Abs(Coordinates(pt1)[2]-root(u,v,i)) le 0.0001) then result:=1;
        elif (Abs(Coordinates(pt2)[2]-root(u,v,i)) le 0.0001) then result:=2;
        elif (Abs(Coordinates(pt3)[2]-root(u,v,i)) le 0.0001) then result:=3;
        end if;
        return result;
        end function;

        theta_wrt_yi:=function(u,v,i);     //for a fixed point BP, evaluate
            theta_{g-1}(gP-Q) at the point Q=(Rex=u, Imx=v, index=i)
        zz, pt_in_cover_i:=IsPoint(X,<[u,v],embedding(u,v,i)>);
        targetdivisor:=Divisor([ BP,pt_in_cover_i],[3,-1]);
        return normalizedtheta(targetdivisor);
        end function;

        omega1 := function(u,v,i)     // first differential form, i denotes
            index of y
        u:=u+0.000001;
        v:=v+0.000001;
        x:=u+v*I;
```

```
y:=root(u,v,i);
return 1/(-x^3+2*x^2*y-2*x*y+3*y^2);
end function;

omegax := function(u,v,i)    //second differential form, i denotes index of y
u:=u+0.000001;
v:=v+0.000001;
x:=u+v*I;
y:=root(u,v,i);
return x/(-x^3+2*x^2*y-2*x*y+3*y^2);
end function;

omegay := function(u,v,i)     //third differential form, i denotes index of y
u:=u+0.000001;
v:=v+0.000001;
x:=u+v*I;
y:=root(u,v,i);
return y/(-x^3+2*x^2*y-2*x*y+3*y^2);
end function;

canonicalform:=function(u,v,i)    //the volume form,i denotes index of y
ort_nor_w1:=0.350487116953118*omega1(u,v,i);
ort_nor_wy:=(0.358981759779085*omegay(u,v,i)+0.119553875346235*omega1(u,v,i));
ort_nor_wx:=(0.203008239643111*omega1(u,v,i)-0.216555180015011*omegay(u,v,i)+0.429067210690657*omegax(u,v,i));
result:=(ort_nor_w1*Conjugate(ort_nor_w1)+ort_nor_wy*Conjugate(ort_nor_wy)+ort_nor_wx*Conjugate(ort_nor_wx))/3;
return result;
end function;

ff:=function(u,v)    //compute three i in one time
result:= Log(theta_wrt_yi(u,v,1))*(canonicalform(u,v,1))+Log(theta_wrt_yi(u,v,2))*(canonicalform(u,v,2))+Log(theta_wrt_yi(u,v,3))*(canonicalform(u,v,3));
return result;
end function;

SX:=0;    //take Riemann sum
scale:=0.01;
rex_start:=-10;
imx_start:=-10;
for p in [1..200] do
for q in [1..2000] do
```

```
    SX:=SX+ff(p*scale+rex_start+0.005,q*scale+imx_start+0.005)*scale*
        scale*(-9);
    end for;
    end for;
    SXn10:=SX;
    SXn10;
```

We can change the *rex_start* and *imx_start* in the code to get Riemann sums for a selected area. The edge length of small squares is *scale*. We choose $scale = 0.01$ when $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$, and choose $scale = 0.1$ for other region in $-50 \leq x \leq 50$ and $-50 \leq y \leq 50$.

In this code, we only compute the Riemann sums for points with coordinates in $-10 \leq x \leq -8$ and $-10 \leq y \leq 10$ (we choose $scale = 0.01$). We need to change *rex_start* and *imx_start* manually to get the Riemann sums of other region.

The reason of doing this is that we can split the computation into small pieces, so that we can parallelly compute them in Magma. This can reduce the computing time significantly. The codes above takes around 40 hours. If we carry out all the computation ($-50 \leq x \leq 50$ and $-50 \leq y \leq 50$) in one time, it will take more than 600 hours. We also use this trick in the computation of $H(\mathfrak{C}_{\mathbb{C}})$.

VIII Code for $T(X)$

```
C<I> := ComplexField(50);    //we need high precision here, since components of T(X) can be very small
Cxy<x, y> := PolynomialRing(RationalField(), 2);
f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;    //defining polynomial
X := RiemannSurface(f : Precision:=15);    //create riemann surface
tau:=SmallPeriodMatrix(X);

MatrixIm := function(M)    //function to take the imaginary part of a matrix with entries in the field C defined just above.
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

Imtau:=MatrixIm(tau);
detimtau:=Determinant(Imtau);
ThetaTranslate:=Matrix([[0.4792505426516801867582813007740 - 0.00334176833187451614116524746349*I],
[0.6986848775084323222909351808410 + 0.1994957238825635630987278415 25*I],
[0.00722266620787249384535143960054 - 0.04301020693432081496232504368 05*I]]);

normalizedtheta := function(D);    // ||theta||_{g-1}
P := AbelJacobi(D);
myinput := Matrix(P) - ThetaTranslate;
char:=Matrix([[0],[0],[0],[0],[0],[0]]);
myabstheta:=Abs(Theta(char, myinput, tau));
myiminput:= MatrixIm(myinput);
myexponent:=-3.141592653589*Transpose(myiminput)*(Imtau^-1)*myiminput;
return Abs(Exp(myexponent[1,1])*myabstheta*detimtau^(1/4));
end function;

a:=1;    //the following lines are the coefficient of the defining polynomial in y
b:=x^2-x;
c:=-x^3;
d:=x^2+x;
cubic_unit:=-1/2+3^(1/2)/2*I;
```

```
    root:=function(u,v,i);    //solve equation with coefficients a,b,c,d
    numa:=a;
    numb:=Evaluate(b,[u+v*I,0]);
    numc:=Evaluate(c,[u+v*I,0]);
    preuu:=(9*a*b*c-27*a^2*d-2*b^3)/(54*a^3);
    uu:=Evaluate(preuu,[u+v*I,0]);
    prevv:=Evaluate((3*(4*a*c^3-b^2*c^2-18*a*b*c*d+27*a^2*d^2+4*b^3*d)),[u+v*I,0]);
    vv:=(prevv^(1/2))/(18*a^2);
    m:=(uu-vv)^(1/3);
    if Abs(uu+vv) ge Abs(uu-vv) then
    m:=(uu+vv)^(1/3);
    end if;
    n:=0;
    if m ne 0 then
    n:=(numb^2-3*numa*numc)/(9*numa*m);
    end if;
    result:=(cubic_unit^(i-1))*m+n*cubic_unit^(2*i-2)-(numb)/(3*numa);
    return result;
    end function;

    wronskisquare:=function(x,y)
    fy:=-x^3+2*x^2*y-2*x*y+3*y^2;    //partial derivative of defining polynomial wrt y
    dy:=(3*x^2*y-2*x*y^2+y^2-2*x-1)/(-x^3+2*x^2*y-2*x*y+3*y^2);    //implicit derivative of y wrt x
    ddy:=(6*x*y+3*x^2*dy+3*x^2*dy-2*y^2-4*x*y*dy-4*x*y*dy-2*x^2*dy^2+2*y*dy+2*y*dy+2*x*dy^2-6*y*dy^2-2)/(-x^3+2*x^2*y-2*x*y+3*y^2);    //second derivative
    dfy:=-3*x^2+4*x*y+2*x^2*dy-2*y-2*x*dy+6*y*dy;    //implicit derivative of fy wrt x
    ddfy:=-6*x+4*y+4*x*dy+4*x*dy+2*x^2*ddy-2*dy-2*dy-2*x*ddy+6*dy^2+6*y*ddy;    //second derivative
    w1:=1/fy;    //differential forms
    wy:=y/fy;
    wx:=x/fy;
    dw1:=-dfy/fy^2;    //implicit derivative of w1 wrt x
    dwy:=(dy*fy-dfy*y)/(fy^2);
    dwx:=(fy-x*dfy)/(fy^2);
    ddw1:=-(ddfy*fy^2-2*fy*dfy^2)/(fy^4);    //second derivative
    ddwy:=((ddy*fy+dy*dfy-ddfy*y-dfy*dy)*fy^2-2*fy*dfy*(dy*fy-dfy*y))/(fy^4);
    ddwx:=((dfy-dfy-x*ddfy)*fy^2-2*fy*dfy*(fy-x*dfy))/(fy^4);
    c11:=0.350487116953118;    //orthogonalizing matrix
    c21:=0.119553875346235;
    c22:=0.358981759779085;
```

```
c31:=0.203008239643111;
c32:=−0.216555180015011;
c33:=0.429067210690657;
W11:=c11*w1;      //following lines gives the wronskian matrix wrt a
    orthonormal basis of differential form.
W12:=c21*w1+c22*wy;
W13:=c31*w1+c32*wy+c33*wx;
W21:=c11*dw1;
W22:=c21*dw1+c22*dwy;
W23:=c31*dw1+c32*dwy+c33*dwx;
W31:=c11*ddw1;
W32:=c21*ddw1+c22*ddwy;
W33:=c31*ddw1+c32*ddwy+c33*ddwx;
Wronski:=Matrix([[W11,W12,W13],[W21,W22,W23],[W31/2,W32/2,W33/2]]);
detwro:=Abs(Determinant(Wronski));
return detwro^2;
end function;

embedding:=function(u,v,i);      //this gives the embedding index of
    point (u+v*I,y_i) on \cC  implemented in Magma
zz,pt1:=IsPoint(X,<[u,v],1>);
zz,pt2:=IsPoint(X,<[u,v],2>);
zz,pt3:=IsPoint(X,<[u,v],3>);
if (Abs(Coordinates(pt1)[2]−root(u,v,i)) le 0.01) then result:=1;
elif (Abs(Coordinates(pt2)[2]−root(u,v,i)) le 0.01) then result:=2;
elif (Abs(Coordinates(pt3)[2]−root(u,v,i)) le 0.01) then result:=3;
end if;
return result;
end function;

WeierPts_coor:=[[ 0, 0], [ −2.104587155963303, −0.3159811807558051],
    [ (−1.455757161115023 − 1.277524806650578*I),
    (−0.5258271579143416 − 3.390009274518869*I)], [
    (−1.455757161115023 + 1.277524806650578*I), (−0.5258271579143403
    + 3.390009274518867*I)], [ −1.95927446954141,
    −1.410535295372148], [ (1.266455202041324 − 1.31486188027894*I),
    (0.1352071928093522 + 2.966892634578606*I)], [ (1.266455202041324
     + 1.31486188027894*I), (0.1352071928093521 − 2.966892634578606*I
    )], [ (0.2089717154013823 − 1.59784353074737*I),
    (0.1450907026008794 − 2.034168656250672*I)], [
    (0.2089717154013823 + 1.59784353074737*I), (0.1450907026008798 +
    2.034168656250671*I)], [ (−1.31968152920268 − 0.4985373089678906*
    I), (−0.4698939283630282 − 0.634861971042462*I)], [
    (−1.31968152920268 + 0.4985373089678906*I), (−0.4698939283630282
    + 0.6348619710424618*I)], [ (1.691125343336006 −
    0.6842628628388769*I), (0.6312556990319607 + 0.6046227212351708*I
    )], [ (1.691125343336006 + 0.6842628628388769*I),
```

```
    (0.6312556990319608 - 0.6046227212351708*I)], [
    (-1.125631933711411 - 0.2207052167744392*I), (-0.6596966755302974
    + 0.07228789092962729*I)], [ (-1.125631933711411 +
    0.2207052167744392*I), (-0.6596966755302974 -
    0.07228789092962723*I)], [ (-0.5282412487764151 -
    1.005235133687909*I), (-0.4177679368816311 + 0.3210649477940537*I
    )], [ (-0.5282412487764151 + 1.005235133687909*I),
    (-0.4177679368816311 - 0.3210649477940537*I)], [
    -1.241974125539051, -1.824965132496513], [ 0.3539677313137965,
    -0.7304015296367112], [ (-0.3349732907033383 -
    0.4139797966480703*I), (-0.4206274001987688 - 0.9204898341402288*
    I)], [ (-0.3349732907033383 + 0.4139797966480703*I),
    (-0.4206274001987688 + 0.9204898341402289*I)], [
    (0.0736668919074069 - 0.3721976723848182*I), (0.7232010036495048
    + 0.2545999914748455*I)], [ (0.0736668919074069 +
    0.3721976723848182*I), (0.7232010036495049 - 0.2545999914748455*I
    )], [ -1, 0]];
WeierPts:=[];

for i in WeierPts_coor do    //make a lst of Wpoints from coordinates
zz,pt1:=IsPoint(X,<[Re(i[1]),Im(i[1])],1>);
zz,pt2:=IsPoint(X,<[Re(i[1]),Im(i[1])],2>);
zz,pt3:=IsPoint(X,<[Re(i[1]),Im(i[1])],3>);
if (Abs(Coordinates(pt1)[2]-i[2]) le 0.0001) then embed:=1;
elif (Abs(Coordinates(pt2)[2]-i[2]) le 0.0001) then embed:=2;
elif (Abs(Coordinates(pt3)[2]-i[2]) le 0.0001) then embed:=3;
end if;
zz, wpt:=IsPoint(X,<[Re(i[1]),Im(i[1])],embed>);
WeierPts:=Append(WeierPts,wpt);    //the list of W-points
end for;

Fz:=function(P,n,dire)    //Fz with n the 'distance' scale and dire
    the relative direction

for i in [1,2,3] do
rootQ:=root(Re(Coordinates(P)[1]),Im(Coordinates(P)[1]),i);
if (Abs(rootQ-Coordinates(P)[2]) le 0.01) then    //since Q is close
    to P, their y-coord are close.
rinde:=i;    //the index of the y-coordinate for Q
end if;
end for;

P_x_r:=Real(Coordinates(P)[1]);
P_x_i:=Imaginary(Coordinates(P)[1]);
Q_x_r:=P_x_r+dire[1]*10^(-n);    //real part of the x coordinate of Q
Q_x_i:=P_x_i+dire[2]*10^(-n);
```

```
einde:=embedding(Q_x_r,Q_x_i,rinde);
zz,Q:=IsPoint(X,<[Q_x_r,Q_x_i],einde>);
gPQ:=Divisor([P,Q],[3,-1]);
distPQ:=Abs(10^(-n)*(dire[1]+dire[2]*I));

result:=normalizedtheta(gPQ)/(distPQ^3);
return result;
end function;

TX:=function(P,n,dire) //T(X)
brick1:=1;
for ele in WeierPts do    //theta part
WDIV:=Divisor([P,ele],[3,-1]);
brick1:=brick1*Abs(normalizedtheta(WDIV)^(2/27));
end for;
brick2:=Fz(P,n,dire)^(-4);
brick3:=wronskisquare(Coordinates(P)[1],Coordinates(P)[2]);
result:=brick1*brick2*brick3;
return result;
end function;

avWe:=[0.1,0.1];     //avoid Weierstrass point
dire:=[-1,0];     //initializing direction
deltapq:=6;     //initializing the 'distance' scale between P and Q
zz,PointP:=IsPoint(X,<[1+avWe[1],0+avWe[2]],1>);     //initializing
    the point P
TX(PointP,deltapq,dire)
```

IX Code for $H(X)$

```
C<I> := ComplexField(30);
Cxy<x, y> := PolynomialRing(RationalField(), 2);
f := -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;    //defining polynomial
X := RiemannSurface(f : Precision:=30);    //riemann surface
tau:=SmallPeriodMatrix(X);
Imtau:=MatrixIm(tau);
detimtau:=Determinant(Imtau);

//the following rows are rows of the big period matrix
tau1:=[-0.0756487726827658975623807266149
    ,0.828507935186700709139824389302,
0.486565855891405863290312449179 ,-
    0.0753194443604481461426310259273,
0.532385049449013502234878708914 , 0.231433843472050162680353039649];
tau2:=[0.486565855891405863290312449179, -
    0.0753194443604481461426310259273,
0.500584921111822623585859133424, + 0.488740800373239742539460301357,
0.137880100924435244693062999773, -
    0.0570373001175851400662331352196];
tau3:=[0.532385049449013502234878708914, +
    0.231433843472050162680353039649,
0.137880100924435244693062999773, -
    0.0570373001175851400662331352196,
0.689717878372433609278715552080, +
    0.692647743504703135897856925333];

MatrixIm := function(M)    //take the imaginary part of a matrix
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

rectan:=Matrix([[1.0,0.0,0.0,0.0,0.0,0.0],
    [0.0,0.0,1.0,0.0,0.0,0.0],
[0.0,0.0,0.0,0.0,1.0,0.0],
tau1,tau2,tau3]);
volume:=Abs(Determinant(rectan));

TorusTheta := function(V);    //the function ||theta|| on a torus
char:=Matrix([[0],[0],[0],[0],[0],[0]]);
```

```
myabstheta:=Abs(Theta(char, V, tau));
myexponent:=-3.141592653589*Transpose(MatrixIm(V))*(Imtau^-1)*
    MatrixIm(V);
return Abs(Exp(myexponent[1,1])*myabstheta*detimtau^(1/4));
end function;

A:=Matrix(C,3,1,[0,0,1]);
B:=Matrix(C,3,1,[0,1,0]);
C:=Matrix(C,3,1,[1,0,0]);
D:=Transpose(Matrix(tau[1]));
E:=Transpose(Matrix(tau[2]));
F:=Transpose(Matrix(tau[3]));

c:=19;     \\the Riemann sum
result:=0;
for ii in [1..2] do
for jj in [1..c] do
for kk in [1..c] do
for ll in [1..c] do
for mm in [1..c] do
for nn in [1..c] do
vector:=(ii/c)*A+(jj/c)*B+(kk/c)*C+(ll/c)*D+(mm/c)*E+(nn/c)*F;
result:=result+Log(TorusTheta(vector))/(c^6);
end for;
end for;
end for;
end for;
end for;
end for;
result;
```

The code above computes the Riemann sum for $2 \times c^5$ small polyhedrons. See the end of Appendix VII for further explanation.

X Code for Klein's formula

```
C<I> := ComplexField(50);

P<U,V,W>:= PolynomialRing(Rationals(), 3);
F:= -U^3*V+U^2*V^2-U*V^2*W+V^3*W+U^2*W^2+U*W^3;
Qxy<x, y>:= PolynomialRing(Rationals(), 2);
f:= -x^3*y+x^2*y^2-x*y^2+y^3+x^2+x;     //defining polynomial
X := RiemannSurface(f);     //create the riemann surface
tau1:=SmallPeriodMatrix(X);//the period matrix

Omega1:=Matrix([[3.2745858887873855986305431993 6E-40 +
    1.40623766693192062162024765467*I,
-1.7995250123240345889112214124 6 + 1.38907786473238785380622191148*I,
-2.53736487575268201398849418408 + 0.422954571624542218773280089228*I
    ],[-3.32482770801607988312171989709E-40 +
    2.21299876803863343807211504318*I,
0.520294432989390175634113180014 + 0.311971662365700775264258133464*I
    ,
2.49655614620140646500159641736 - 1.72853703668803372011916427569*I
    ],[-6.11781900650324543346854241722E-40 -
    0.519248888908945152419998545118*I,
-0.778001857456629823002912917512 + 1.46695721833231037837976066119*I
    ,
3.20106058440048591056866925446 - 1.29453940681916549786848960725*I
    ]]);

//the following set S contains all the even characteristics
S:=[Matrix(RationalField(),1,[0,0,0,0,0,0]),Matrix(RationalField()
    ,1,[0,0,0,0,0,1]),Matrix(RationalField(),1,[0,0,0,0,1,0]),Matrix(
    RationalField(),1,[0,0,0,0,1,1]),Matrix(RationalField()
    ,1,[0,0,0,1,0,0]),Matrix(RationalField(),1,[0,0,0,1,0,1]),Matrix(
    RationalField(),1,[0,0,0,1,1,0]),Matrix(RationalField()
    ,1,[0,0,0,1,1,1]),Matrix(RationalField(),1,[0,0,1,0,0,0]),Matrix(
    RationalField(),1,[0,0,1,0,1,0]),Matrix(RationalField()
    ,1,[0,0,1,1,0,0]),Matrix(RationalField(),1,[0,0,1,1,1,0]),Matrix(
    RationalField(),1,[0,1,0,0,0,0]),Matrix(RationalField()
    ,1,[0,1,0,0,0,1]),Matrix(RationalField(),1,[0,1,0,1,0,0]),Matrix(
    RationalField(),1,[0,1,0,1,0,1]),Matrix(RationalField()
    ,1,[0,1,1,0,0,0]),Matrix(RationalField(),1,[0,1,1,1,0,0]),Matrix(
    RationalField(),1,[0,1,1,1,1,1]),Matrix(RationalField()
    ,1,[0,1,1,0,1,1]),Matrix(RationalField(),1,[1,0,0,0,0,0]),Matrix(
    RationalField(),1,[1,0,0,0,0,1]),Matrix(RationalField()
    ,1,[1,0,0,0,1,0]),Matrix(RationalField(),1,[1,0,0,0,1,1]),Matrix(
    RationalField(),1,[1,0,1,0,0,0]),Matrix(RationalField()
    ,1,[1,0,1,0,1,0]),Matrix(RationalField(),1,[1,0,1,1,0,1]),Matrix(
```

```
    RationalField() ,1 ,[1 ,0 ,1 ,1 ,1 ,1]) ,Matrix(RationalField()
    ,1 ,[1 ,1 ,0 ,0 ,0 ,0]) ,Matrix(RationalField() ,1 ,[1 ,1 ,0 ,0 ,0 ,1]) ,Matrix(
    RationalField() ,1 ,[1 ,1 ,0 ,1 ,1 ,0]) ,Matrix(RationalField()
    ,1 ,[1 ,1 ,0 ,1 ,1 ,1]) ,Matrix(RationalField() ,1 ,[1 ,1 ,1 ,0 ,0 ,0]) ,Matrix(
    RationalField() ,1 ,[1 ,1 ,1 ,0 ,1 ,1]) ,Matrix(RationalField()
    ,1 ,[1 ,1 ,1 ,1 ,1 ,0]) ,Matrix(RationalField() ,1 ,[1 ,1 ,1 ,1 ,0 ,1])];

MatrixIm := function(M) // Imaginary part of a matrix
N := ZeroMatrix(C, Nrows(M), Ncols(M));
for i in [1..Nrows(M)] do
for j in [1..Ncols(M)] do
N[i,j] := Imaginary(M[i,j]);
end for;
end for;
return N;
end function;

Klein_ratio:=function(period)//compute the ratio of Klein formula
chitilde:=1;
zerovector:=Matrix(C,1,[0,0,0]);
for ele in S do
chitilde:=chitilde*Theta((1/2)*ele,zerovector,period);
end for;
left:=DiscriminantOfTernaryQuartic(F)^2;
right:=chitilde*(2*3.1415926)^(54)/(2^(28)*Determinant(Omega1)^(18));
return right/left;
end function;

Klein_ratio(tau1);
```

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Summary

Explicit Computation of the Height of a Gross-Schoen Cycle

In this thesis, we study the Beilinson-Bloch height of the Gross-Schoen cycle on a curve over a field.

Let X be a genus g smooth curve over a field k (a number field or a function field). To an element $e \in \mathrm{Div}^1(X)_{\mathbb{Q}}$, we can associate a Gross-Schoen cycle Δ_e in $\mathrm{CH}^2(X^3)$. The cycle is an alternating sum of small diagonals on X^3. When k is a global field, the height (studied by A. Beilinson, S. Bloch, B. Gross and C. Schoen) of Δ_e can be used to measure the non-triviality of Δ_e.

In Chapter 1, we review Arakelov theory and Zhang's work on the heights of Gross-Schoen cycles. The main result of this chapter is Theorem 1.5.16, in which we show that the height for genus $g \geq 3$ curves over $\mathbb{Q}$ is unbounded. The proof relies on the Northcott property for Gross-Schoen cycles proved by S. Zhang.

In Chapter 2, we recall some moduli properties of genus 3 curves and Klein's formula for smooth plane quartic curves.

In Chapter 3, we focus on Arakelov geometry of genus 3 curves. We explain how to explicitly compute the admissible invariants of genus 3 pm-graphs. The main result of this chapter is a sufficient condition for the heights of a family of genus 3 curves to go to infinity.

In Chapter 4, we numerically compute the height of a canonical Gross-Schoen cycle of a particular plane quartic curve over $\mathbb{Q}$.

Samenvatting

Explicit Computation of the Height of a Gross-Schoen Cycle

In dit proefschrift bestuderen we de Beilinson-Bloch-hoogte van de Gross-Schoen-cykels op een kromme over een lichaam.

Laat X een geslacht g gladde kromme zijn over een lichaam k (een getallenlichaam of een functielichaam). Aan een element $e \in \mathrm{Div}^1(X)_{\mathbb{Q}}$ kunnen we een Gross-Schoen-cykel Δ_e associëren in $\mathrm{CH}^2(X^3)$. Deze cykel is een alternerende som van kleine diagonalen op X^3. Wanneer k een globaal lichaam is, kan de hoogte (bestudeerd door A. Beilinson, S. Bloch, B. Gross en C. Schoen) van Δ_e worden gebruikt om de niet-trivialiteit van Δ_e te meten.

In Hoofdstuk 1 bespreken we de theorie van Arakelov en het werk van Zhang over de hoogte van de Gross-Schoen-cykels. Het belangrijkste resultaat van dit hoofdstuk is Stelling 1.5.16, waarin we laten zien dat de hoogte voor krommen over $\mathbb{Q}$ met geslacht $g \geq 3$ onbegrensd is. Dit is gebaseerd op de Northcott-eigenschap van de Gross-Schoen-cykels, bewezen door S. Zhang.

In Hoofdstuk 2 bespreken we enkele moduli-eigenschappen van geslacht 3 krommen en de Klein-formule voor gladde vlakke vierdegraadskrommen.

In Hoofdstuk 3 leggen we onze focus op de Arakelov-meetkunde van geslacht 3-krommen. We leggen uit hoe je expliciet de toelaatbare invarianten van geslacht 3 pm-grafen kan berekenen. Het belangrijkste resultaat van dit hoofdstuk is een voldoende voorwaarde voor de hoogten van een familie van geslacht 3 om naar oneindig te gaan.

In Hoofdstuk 4 berekenen we numeriek de hoogte van een canonieke Gross-Schoen-cykel van een bepaalde vlakke vierdegraadskromme over $\mathbb{Q}$.

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Curriculum Vitae

Ruihua Wang was born on Feb 29^{th} 1992 in Zaozhuang, China. He became interested in mathematics in junior high school and then won a second award in the national mathematics contest for high school students.

In 2011, he was admitted to the mathematics department of Shandong University. In 2014-2015, he studied number theory at the Chinese Academy of Science. In April 2015, he got a postgraduate recommendation from Shandong University. He studied analytic number theory under the supervision of Jianya Liu. In 2018, he started his PhD studies in Arakelov geometry under the supervision of Robin de Jong and David Holmes.

After earning his PhD, he plans to find a position in finance or wireless communication.