Travelling waves on trees and square lattices

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CHAPTER

ONE

INTRODUCTION

In this thesis we study bistable reaction-diffusion equations on (multidimensional) lattice domains. The power of reaction-diffusion equations is that they can successfully model various natural and social phenomena with their intuitive and relatively simple (mathematical) representation. In mathematical notation, a lattice reaction-diffusion differential equation is any lattice differential equation (LDE) of the following form

$$\dot{u}_\mu(t) = d \underbrace{[\Delta u(t)]_\mu}_{\text{diffusion}} + \underbrace{g(u_\mu(t))}_{\text{reaction}}, \quad \mu \in \Lambda, \tag{1.0.1}$$

where $\Delta : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda)$ represents a diffusion operator on a lattice $\Lambda \subset \mathbb{Z}^n$ and $d > 0$ is a diffusion constant. We call $g : \mathbb{R} \to \mathbb{R}$ the reaction function. One example of such a lattice differential equation on the integer lattice $\mathbb{Z}$ is given by

$$\dot{u}_i(t) = u_{i-1}(t) - 2u_i(t) + u_{i+1}(t) + g(u_i(t)),$$

with $i \in \mathbb{Z}$ and $t \in \mathbb{R}$ or $t > 0$. Lattice equations are closely related to their continuous counterparts

$$u_t(x,t) = d\Delta_x u(x,t) + g(u(x,t)), \tag{1.0.2}$$

where x belongs to some open subset of $\mathbb{R}^n$ and Δ_x is the standard Laplace operator on $\mathbb{R}^n$. One of the main features of reaction-diffusion equations, both on discrete and continuous domains, is that they admit special solutions, so-called 'travelling waves', which we can describe as fixed profiles $\Phi : \mathbb{R} \to \mathbb{R}$ that move in a particular direction with some speed c. Depending on their shape, we can roughly divide waves into three categories:

- pulses or solitons, which can be described as local perturbations 
- periodic pulses (wave trains) 
- monotone wave fronts that connect two constant states 

In this thesis we focus on the latter type of wave and we study their existence, propagation and long term behaviour on two type of discrete domains - the two-dimensional lattice $\mathbb{Z}^2$, and infinite trees.

To guide the reader through the mechanism behind the formation of travelling waves, we will first take a separate look into the phenomena of reaction and diffusion to discover how they work together to form moving solutions. We will explain basic concepts on continuous domains and gradually extend them to the lattice domains treated in this thesis. By $\dot{u}$ we always denote a time-derivative of the function u.

1.1 Reaction equation

In this section we explore various types of reaction equations and explain how do they influence the long-term behaviour of solutions $u(t)$ to (1.0.2) when $d = 0$. In the absence of diffusion, this PDE turns into a pure ODE and therefore we can drop the variable x.

Exponential growth, $g(u) = ru$

One of the simplest reaction equations is given by

$$\dot{u} = ru, \tag{1.1.1}$$

whose solution is given by $u(t) = Ce^{rt}$, for any $C \in \mathbb{R}$. For $C = 0$ we have an *equilibrium solution* $\overline{u} = 0$ that does not change in time.

For $r < 0$ we say that $\overline{u} = 0$ is a *stable* equilibrium since the solution $u(t)$ approaches $\overline{u}$ as $t \to \infty$. For $r > 0$, $\overline{u} = 0$ is an *unstable* equilibrium since the solution $u(t)$ diverges away from $\overline{u}$ as $t \to \infty$.

Logistic growth, $g(u) = ru(1 - u)$

The logistic growth equation was first proposed by Pierre-François Verhulst in [92] to model the population growth of the species u. Namely, provided that the growth rate and maximum capacity of the population are given by the positive constants r and K respectively, we have

$$\dot{u} = ru(1 - \frac{u}{K}). \tag{1.1.2}$$

In words, when the population u is very small, it grows exponentially with some rate $r > 0$, i.e., $\dot{u} \approx ru$. However, as it grows and reaches its maximum capacity K, its growth rate is approaching 0 and we have $\dot{u} \approx 0$. The explicit solution to this equation is given by

$$u(t) = \frac{K}{1 + (K/C - 1)e^{-rt}} \tag{1.1.3}$$

where $C > 0$ is the initial population number at time $t = 0$. This equation has two equilibrium points $\overline{u} = 0$ and $\overline{u} = K$, with the later being stable since the limit $\lim_{t\to\infty} u(t) = K$ holds.

The logistic growth equation has found numerous applications beyond population growth models. One example is the **SIS epidemiological model** [59] that models diseases like common cold or influenza that do not provide long-term immunity. Namely, individuals in a population are divided into two categories, I (infected) and S (susceptible). We assume that the population size is constant in time and equal to N, i.e., $I(t) + S(t) = N$ for all $t \geq 0$. Moreover, by $\beta > 0$ we denote the average number of contacts between individuals, multiplied by the probability of a transmission in a contact. The recovery rate of infected individuals is given by γ. These assumptions lead to the following ODEs for I and S

$$\dot{I} = \frac{\beta S}{N} I - \gamma I, \qquad \dot{S} = -\frac{\beta S}{N} I + \gamma I.$$

Together with our assumption $I + S = N$, and setting $R_0 := \frac{\beta}{\gamma}$ we arrive at

$$\dot{I} = \gamma(R_0 - 1)I(1 - \frac{I}{(1 - 1/R_0)N}),$$

which is of the same form as (1.1.2). If $R_0 > 1$, then the number of infected people will grow to

$$\lim_{t \to \infty} I(t) = (1 - \frac{1}{R_0})N,$$

which one often refers to as the *herd immunity threshold*. For $R_0 < 1$, the point $\overline{I} = 0$ is a stable equilibrium since we have $\lim_{t \to \infty} I(t) = 0$ which implies that the disease is eradicated.

Bistable reaction, $g(u) = u(1 - u)(u - a)$

In many physical systems two stable equilibria compete for dominance. For example, in contrast to the logistic growth equation in which the population size always grows towards its carrying capacity, for some species undercrowding or a low density limits its growth and leads to extinction. This principle is called the Allee effect.

To model this effect, we assume that the maximum capacity is rescaled to 1 and that there exists a critical parameter $a \in (0, 1)$ such that $u < a$ implies that the population is dying out and $u > a$ implies that the population grows. Then the equation for the density $u(t)$ is given by

$$\dot{u} = u(1 - u)(u - a). \tag{1.1.4}$$

We have three equilibria $\overline{u} \in \{0, a, 1\}$, with 0 and 1 being the stable points, in the sense that $u(0) < a$ implies $\lim_{t \to \infty} u(t) = 0$ and $u(0) > a$ implies $\lim_{t \to \infty} u(t) = 1$. To justify this conclusion, it is enough to observe that $g(u) < 0$ for $u \in (0, a)$ and $g(u) > 0$ for $u \in (a, 1)$. Indeed, if $u(0) < a$, then we have $\dot{u}(t) < 0$ for small t and the population is decreasing towards 0. On the contrary, for $u(0) > a$, we have $\dot{u}(t) > 0$, which suggests that u is increasing towards 1.

We also point out that the cubic nonlinearity (1.1.4) is only one of the numerous possibilities for modelling bistable reaction effects. As its name suggests, a bistable reaction term can be any function that has two stable equilibria with one unstable equilibrium point a in between. This is equivalent to the following basic assumptions we use in all chapters of this thesis, namely

$$g(u) < 0, \text{ for } u \in (0, a), \qquad g(u) > 0, \text{ for } u \in (a, 1),$$

with

$$g'(0) < 0, \qquad g'(1) < 0, \qquad g'(a) > 0.$$

To emphasize this dependence of the nonlinearity g on the bistable parameter a in the rest of this text we denote $g = g(\cdot; a)$.

1.2 Diffusion equation

In this section we study the equation (1.0.2) in the absence of reaction. For simplicity we take $d = 1$. The diffusion equation, more commonly known as the *heat equation* is one of the most studied partial differential equations. It takes the form

$$u_t(x,t) = \Delta_x u(x,t) \tag{1.2.1}$$

where $x \in \mathbb{R}^n$, $t > 0$ and Δ_x is the standard Laplace operator on $\mathbb{R}^n$, namely

$$\Delta_x u(x,t) = \sum_{i=1}^{n} u_{x_i x_i}(x,t).$$

Heat equation on $\mathbb{R}$ In the case of only one spatial variable, the heat equation is simply

$$u_t(x,t) = u_{xx}(x,t), \qquad x \in \mathbb{R},\ t > 0. \tag{1.2.2}$$

Provided that we have $u(\cdot, 0) = u_0 \in L^\infty(\mathbb{R})$, an explicit formula for the solution $u(x,t)$ is readily available, namely

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u_0(y)dy \\ &= [H(\cdot,t) * u_0](x). \end{aligned} \tag{1.2.3}$$

Here the function $H : \mathbb{R} \times (0,\infty) \to (0,\infty)$, defined by

$$H(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

is called the *fundamental solution* or the *heat kernel*. Taking derivatives in (1.2.3), and evaluating them either on H or u_0 results in the estimate

$$\sup_{x\in\mathbb{R}} |u_x(x,t)| \leq C \min\{||u_0||_{L^\infty} t^{-\frac{1}{2}}, ||u_{0,x}||_{L^\infty}\}. \tag{1.2.4}$$

This formula shows that the heat equation on $\mathbb{R}$ averages solutions, in the sense that their derivatives converge to 0 as $t \to \infty$.

1.3 Travelling waves on $\mathbb{R}$

Combining the bistable reaction function $g(u;a)$ and the diffusion operator results in an interplay between the harsh reaction jumps and the smoothening effect of diffusion. As a result, we have a special solution of (1.0.2) that we call a travelling wave, see Figure 1.1.

To find such a wave profile Φ, we assume that $u(x,t) = \Phi(x - ct)$ and we plug this Ansatz into (1.0.2). Upon substituting $\xi = x - ct$ we arrive at the second-order ODE

$$-c\Phi'(\xi) = d\Phi''(\xi) + g(\Phi(\xi);a), \tag{1.3.1}$$

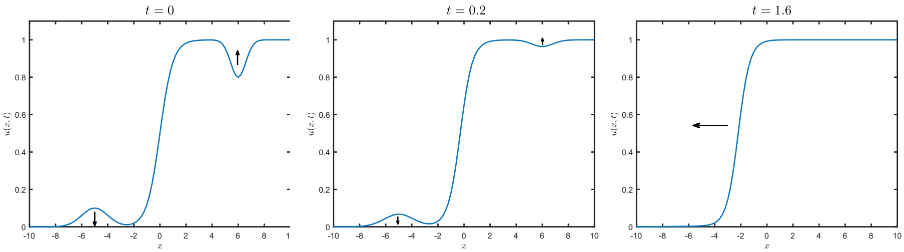


Figure 1.1: Travelling wave on $\mathbb{R}$. Under the influence of the bistable reaction term, the initial small perturbations are first pushed to either 0 or 1.

which we couple to the boundary conditions

$$\Phi(-\infty) = 0, \qquad \Phi(\infty) = 1, \tag{1.3.2}$$

since we want to connect two stable points of the nonlinearity g. The existence of a (unique up to translation) solution (Φ, c) to (1.3.1)-(1.3.2) is shown in [33] via phase-plane analysis. In particular, the authors introduce an additional function $P = \Phi'$ that transforms the second-order ODE (1.3.1) into a system of two first-order ODEs

$$\begin{cases} \Phi'(\xi) & = P(\xi), \\ P'(\xi) & = -\frac{c}{d}\Phi'(\xi) - \frac{1}{d}g(\Phi(\xi); a), \end{cases}$$

to which we add the boundary conditions $P(-\infty) = 0$, $P(\infty) = 0$. A short computation shows that $(0, 0)$ and $(1, 0)$ are two saddle equilibrium points of this system. Therefore, the solution (Φ, P) corresponds to an orbit lying in the intersection of the unstable manifold of $(0, 0)$ and the stable manifold of $(1, 0)$. As Fife shows using a geometric argument in [33], there exists a unique speed c such that these manifolds intersect in the first quadrant of the (Φ, P) plane. This result automatically shows that the wave is monotonically increasing, i.e., $\Phi' > 0$.

In the case of the standard cubic nonlinearity

$$g(u; a) = u(1 - u)(u - a) \tag{1.3.3}$$

one can check that the explicit solution to (1.3.1) is given by

$$\Phi(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\frac{\sqrt{2}\xi}{4}, \qquad c = \sqrt{2d}(a - \frac{1}{2}). \tag{1.3.4}$$

We can read off two important properties from the second formula.

1. The speed satisfies $c = 0$ if and only if $a = 1/2$;
2. Up to the sign change, the speed c is symmetric around $a = 1/2$.

This symmetry result is expected since diffusion operator does not prefer neither of the equilibrium points 0 and 1, which means that the propagation direction is in general determined by the reaction function, or more precisely, by the sign of its integral

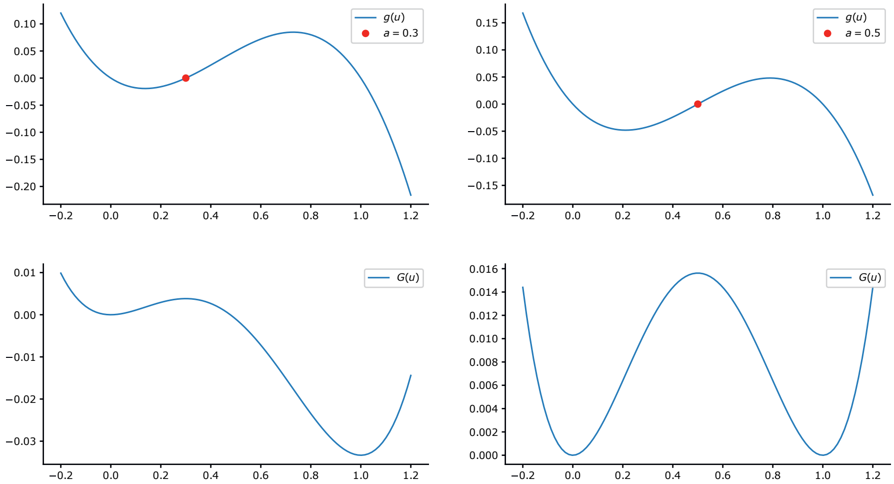


Figure 1.2: The cubic nonlinearity $g(u; a) = u(1-u)(u-a)$ with its double-well potential $G(u; a)$. On the left we have $a = 0.35$, and we observe that $G(0; a) > G(1; a)$. On the right we have $a = 0.5$ and the system is in balance as the equality $G(0; a) = G(1; a)$ holds.

$G(u; a) = -\int_0^u g(u; a)du$. The function G is often called the *double-well potential*, see Figure 1.2.

To give a visual interpretation, when $a < 1/2$, the equilibrium point 1 with the lower potential energy invades the point 0 with the higher potential energy and the wave moves to the left. If $a > 1/2$, the role of the stable states 0 and 1 is reversed, and the wave propagates to the right with speed $c > 0$. For $a = 1/2$, the wave speed c is equal to 0 as a consequence of both states being in balance due to the equality $G(0; a) = G(1; a)$.

1.4 Travelling waves on lattice domains

In what follows we transfer the familiar concepts from §1.2 and §1.3 to lattice domains. In the absence of the diffusion operator, both equations (1.0.1) and (1.0.2) are systems of decoupled first order ODEs that we have already covered in §1.1.

Therefore, the first obvious difference between these two type of equations comes from the diffusion operator Δ. On the continuous domain $\mathbb{R}^n$, the Laplace operator Δ_x is a local operator, i.e., to evaluate $\Delta_x u(x_0)$ one needs to know the values of the function u in an arbitrarily small neighborhood around some point x_0. On the contrary, the discrete diffusion operator is a nonlocal operator on the lattice Λ since it couples multiple points on a lattice. Moreover, its definition differs per type of lattice that we study. We explain these concepts further in the following subsection by studying three kinds of lattice domains - the integers $\mathbb{Z}$, the two-dimensional domain $\mathbb{Z}^2$ and infinite k-ary trees $\mathcal{T}_k$.

1.5 Integer lattice $\mathbb{Z}$

On the integer domain $\mathbb{Z}$, one example of a discrete diffusion operator is given by

$$[\Delta u]_i = u_{i+1} - 2u_i + u_{i-1}, \tag{1.5.1}$$

which is considered as the standard discretization of the continuous Laplace operator. The full bistable lattice reaction-diffusion equation on $\mathbb{Z}$ now reads

$$\dot{u}_i(t) = d(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)) + g(u_i(t); a), \qquad i \in \mathbb{Z},\ t > 0. \tag{1.5.2}$$

1.5.1 Fundamental solution of the discrete heat equation

The main goal of this subsection is to draw parallels between the discrete heat equation associated to (1.5.1), namely

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t), \qquad i \in \mathbb{Z}, \quad t > 0, \tag{1.5.3}$$

and the continuous heat equation (1.2.2). We tackle (1.5.3) by applying the Fourier transform which results in the simple ODE in the Fourier space

$$\frac{d}{dt}\hat{u}(\xi) = e^{2t(\cos\omega - 1)}\hat{u}(\xi),$$

whose solution is given by

$$\hat{u}(\xi) = e^{2t(\cos\omega - 1)}\hat{u}^0(\xi),$$

where $u^0 = u(0) \in \ell^2(\mathbb{Z})$. By applying the inverse Fourier transform we derive the explicit formula for the solution

$$u_i(t) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} u_k^0 \int_{-\pi}^{\pi} \cos\left((i-k)\omega\right) e^{2t(\cos\omega - 1)} d\omega. \tag{1.5.4}$$

In this formula we recognize the integral representation

$$I_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\left(k\omega\right) e^{t\cos\omega} d\omega,$$

of the modified Bessel functions of the first kind $I_k(t)$ for $t > 0$ and $k \in \mathbb{Z}$, see Figure 1.5.1.

Setting $G_k(t) := e^{-2t}I_k(2t)$, the solution (1.5.4) can hence be written as the convolution between the sequence G and the initial condition u^0, i.e.,

$$u(t) = G * u^0.$$

This is in line with the continuous heat equation where the solution is obtained by convolving between the Gaussian kernel with the initial state, see equation (1.2.3). Our analysis in §2.6 shows that the first differences of solutions decay as

$$\sup_{i\in\mathbb{Z}} |u_{i+1}(t) - u_i(t)| = O(t^{-1/2}).$$

Therefore, the discrete heat equation on $\mathbb{Z}$ averages out its solutions, in the sense that their first differences converge to 0 as $t \to \infty$, with the same decay rate as the solutions of the continuous heat equation. We summarize the similarities between the discrete and continuous heat kernels in Table 1.1.

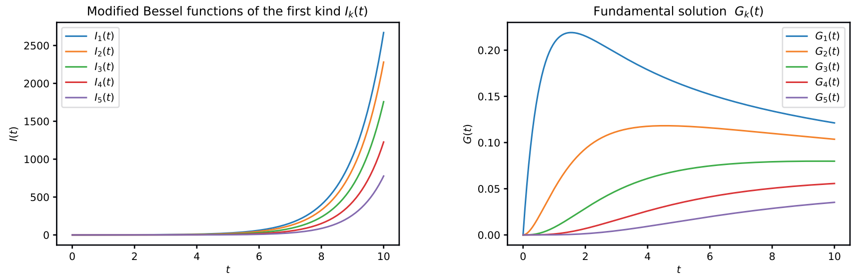


Figure 1.3: On the left we plot modified Bessel functions of the first kind for $k \in \{1, 2, 3, 4, 5\}$ and on the right we plot the corresponding fundamental solution $G_k(t) = e^{-t} I_k(t)$.

1.5.2 Wave-fronts on $\mathbb{Z}$

Similarities between the fundamental solutions of the continuous and discrete heat equation suggest that the discrete reaction-diffusion equation (1.5.2) also admits travelling wave solutions. To find such solutions, we mimic the procedure from §1.3 and we plug the Ansatz $u_i(t) = \Phi(i - ct)$ into (1.5.2). Upon substituting $\xi = i - ct$, this approach results in the following differential equation for Φ and c

$$-c\Phi'(\xi) = d\left(\Phi(\xi+1) - 2\Phi(\xi) + \Phi(\xi-1)\right) + g(\Phi(\xi); a). \qquad (1.5.5)$$

As before, we couple it to the boundary conditions that connect two stable points of the nonlinearity g, namely

$$\Phi(-\infty) = 0, \qquad \Phi(\infty) = 1. \qquad (1.5.6)$$

Already at this point we can observe the first difference between the lattice and continuous equations. Namely, the differential equation (1.5.5) involves both past and future values. We call this type of differential equation a mixed functional differential equation (MFDE). Using Brouwer's fixed point theorem, Zinner [97] was the first to show that there exists a travelling wave solution provided that the diffusion parameter d is big enough. In the seminal paper [67] Mallet-Paret gives detailed existence and uniqueness results for a much more general class of MFDEs. In particular, for every $a \in (0, 1)$ and $d > 0$ there exists a speed $c \in \mathbb{R}$ and non-decreasing profile $\Phi : \mathbb{R} \to \mathbb{R}$ that satisfy (1.5.5)-(1.5.6). In case $c \neq 0$ this wave-pair is unique upon fixing $\Phi(0) = 1/2$. Moreover, in this case both the speed c and profile $\Phi \in C^1(\mathbb{R})$ depend smoothly on the parameters a and d, and the strict inequality $\Phi' > 0$ holds. When we want to emphasize this dependence of the speed c on parameters a and d we write $c(a, d)$.

Provided that the nonlinearity g is the standard cubic we have just as in (1.3.4) the symmetry relation

$$c(1/2 + a, d) = -c(1/2 - a, d),$$

for every $a \in (0, 1/2)$ and $d > 0$. This relation shows that the speed c is, up to the sign change, symmetrical around the axis $a = 1/2$. Moreover, we have $c(1/2, d) = 0$ for every $d > 0$.

Heat equation on $\mathbb{R}$ and $\mathbb{Z}$

Domain	Continuous domain $\mathbb{R}$	Discrete domain $\mathbb{Z}$
Fundamental solution	$H(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$	$G_k(t) = e^{-2t} I_k(2t)$
Singularity at $t = 0$	$\lim_{t\to 0} H(0,t) = \infty$	$G_0(0) = 0$
Integral of the kernel	$\dfrac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{x^2}{4t}}\, dx = 1$	$\sum_{k\in\mathbb{Z}} e^{-2t} I_k(2t) = 1$
Decay rate of solutions	$\sup_{x\in\mathbb{R}} \lvert u_x(x,t)\rvert \le Ct^{-1/2}$	$\sup_{i\in\mathbb{Z}} \lvert u_{i+1}(t) - u_i(t)\rvert \le Ct^{-1/2}$

Table 1.1: In this table we draw some parallels between the (fundamental) solutions of the continuous and discrete heat equations.

Before we delve into the further analysis of waves on lattices, we want to point out some crucial differences between equation (1.5.5) and its counterpart on a continuous domain.

1.5.3 Pinning on lattices

Turning back to lattice domains, we point out one of the key differences between the MFDE (1.5.5) and ODE (1.3.1). In particular, setting $c = 0$ in (1.5.5) results in a difference equation

$$0 = d(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}) + g(\Phi_i; a), \tag{1.5.7}$$

to which we add the boundary conditions

$$\lim_{i\to-\infty} \Phi_i = 0, \qquad \lim_{i\to\infty} \Phi_i = 1. \tag{1.5.8}$$

This change from a differential to a difference equation is an underlying mechanism that causes the pinning of waves. Namely, we say that *pinning* or *propagation failure* occurs when the equality $c = 0$ holds for a range of bistable parameters a in some nontrivial interval $[a_-, a_+]$. This is in stark contrast with the continuous reaction-diffusion equation, in which we have $c = 0$ for only one value of the bistable parameter a, see Figure 1.4. The first systematic study of this phenomenon was performed by Keener in [55]. In this paper, Keener embeds the difference equation (1.5.7) into the framework set up by Moser in [71] to show the existence of infinitely many chaotic solutions that block the propagation of waves. Since we also employ this theory in Chapter 4, we give this construction some attention here.

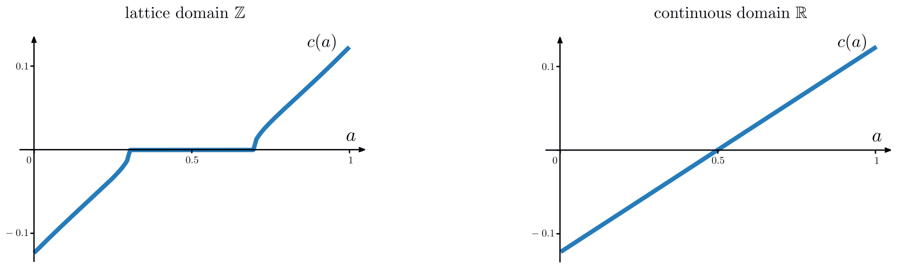


Figure 1.4: Speed c as the function of a. In this example we take $d = 0.025$ and the standard cubic nonlinearity (1.3.3). On the left we plot numerical solutions of the MFDE (1.5.5), and on the right we plot the analytical solution $c(a)$ for the ODE (1.3.1).

Spatial chaos The substitution $P_i := \Phi_{i-1}$ transforms the difference equation (1.5.7) into the two-dimensional recurrence relation

$$\begin{cases} \Phi_{i+1} &= 2\Phi_i - P_i - \frac{1}{d}g(\Phi_i; a), \\ P_{i+1} &= \Phi_i, \end{cases} \tag{1.5.9}$$

for $i \in \mathbb{Z}$. We define a mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\phi(u, v) = \left(2u - v - \frac{1}{d}g(u; a), u\right),$$

together with its inverse

$$\phi^{-1}(u, v) = \left(v, 2v - u - \frac{1}{d}g(v; a)\right).$$

We can now take any $(\Phi_0, P_0) \in \mathbb{R}^2$ to define a bi-infinite sequence $(\Phi_i, P_i)_{i\in\mathbb{Z}}$ by setting

$$(\Phi_i, P_i) := \phi^i(\Phi_0, P_0), \qquad i \in \mathbb{Z}. \tag{1.5.10}$$

By construction this sequence satisfies the system (1.5.9). However, taking for example $(\Phi_0, P_0) = (0.5, 0.7)$ and the cubic nonlinearity (4.1.2) with $a = 0.2$ we soon arrive at $\Phi_{12} = 4.89 \times 10^{11}$. Due to its very large values, this sequence does not correspond with the physical notion of a travelling wave. Therefore, at this point it is not immediately clear that one can actually find a bounded sequence that satisfies (1.5.9).

To prove that such bounded sequences indeed exist, Keener applied results from the field of Symbolic Dynamics, in particular the Moser theorem [71]. This result implies that for every small diffusion d there exist a correspondence between sequences in the set

$$S := \{(\dots, s_{-1}, s_0, s_1, \dots) : s_i \in \{0, 1\}\}$$

and bounded solutions to (1.5.9). Specifically, there exist $x_0 \in [0, a)$ and $x_1 \in (a, 1]$ such that for each $(s_i)_{i\in\mathbb{Z}} \subset S$ we can find a sequence $(\Phi_i, P_i)_{i\in\mathbb{Z}} \subset [0, 1]^2$ that satisfies (1.5.9), together with

$$\Phi_i \in [0, x_0), \quad \text{if } s_i = 0, \qquad \Phi_i \in (x_1, 1], \quad \text{if } s_i = 1,$$

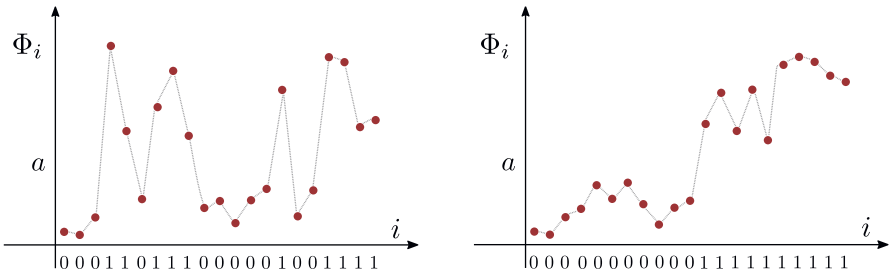


Figure 1.5: Two possible stationary solutions of (1.5.5) and their corresponding sequences $(s_i)_{i\in\mathbb{Z}}$.

see Figure 1.5. To conclude, for small d one can construct a rich variety of solutions in $[0,1]^2$. Moreover, using the comparison principle, Keener shows that intervals $[0, x_0)$ and $(x_1, 1]$ are invariant for (1.5.2) in the sense that for all $i \in \mathbb{Z}$ we have

$$u_i(0) \in [0, x_0) \Longrightarrow u_i(t) \in [0, x_0) \text{ for all } t > 0,$$
$$u_i(0) \in (x_1, 1] \Longrightarrow u_i(t) \in (x_1, 1] \text{ for all } t > 0.$$

Therefore, solutions to the LDE (1.5.3) are blocked from propagating in any direction.

Pinning in systems like (1.5.9) can also be studied from the Dynamical Systems point of view. For example, in [47] the authors characterize pinned fronts as intersection points of stable and unstable manifold of saddle equilibrium points. We describe this construction in the next paragraph.

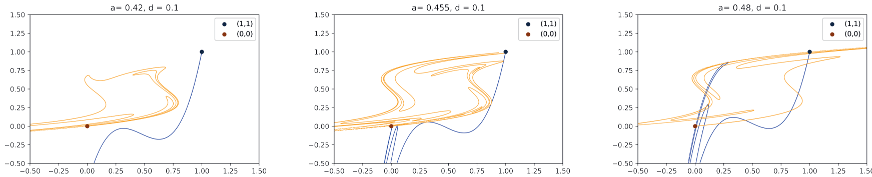
Stable and unstable manifolds Equilibrium points $(\overline{\Phi}, \overline{P})$ of the system (1.5.9) are defined as scalar solutions to

$$\overline{\Phi} = \overline{P}, \qquad \overline{\Phi} = 2\overline{\Phi} - \overline{P} - d^{-1} g(\overline{\Phi}; a). \tag{1.5.11}$$

One can verify that $(0,0)$ and $(1,1)$ are saddle equilibrium points to which we associate the sets

$$W^u(0,0) = \{(u,v) : \phi^{-n}(u,v) \to (0,0), \text{ as } n \to \infty\},$$
$$W^s(1,1) = \{(u,v) : \phi^{n}(u,v) \to (1,1), \text{ as } n \to \infty\}.$$

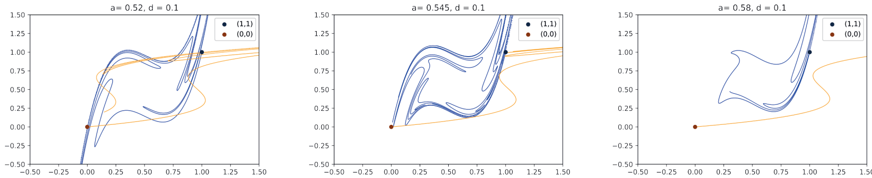
These sets $W^u(0,0)$ and $W^s(1,1)$ are called the *stable* and *unstable manifold* of points $(0,0)$ and $(1,1)$, respectively. If a point (Φ_0, P_0) lies in the intersection of these two sets then its associated sequence (1.5.10) also belongs to $W^u(0,0) \cap W^s(1,1)$ and it satisfies the boundary conditions (1.5.8). We call such solutions *pinned fronts* since they connect two stable points of the nonlinearity g. Provided that g is the cubic function, a result by Qin and Xian from [76] implies that at least two sequences lie the intersection $W^u(0,0) \cap W^s(1,1)$. If these manifolds intersect transversely for some parameter a_0, then these pinned waves persist as we vary a around a_0, so we obtain pinned fronts in some nonempty interval $[a_-, a^+]$ around a_0. At $a = a_-$ and $a = a_+$ the manifolds intersect tangentially, and finally, for $a \notin [a_-, a_+]$ the manifolds are disjoint, which means that no pinned fronts exist. To visualize this process, we implemented a numerical algorithm from [41] and we show our results in Figure 1.6.



(a) Manifolds are disjoint, a pinned front does not exist.

(b) Manifolds touch in a tangential fashion, $a = a_-$.

(c) Manifolds intersect transversely, we have a pinned front.



(d) Manifolds are intersect transversely, we have a pinned front.

(e) Manifolds touch in a tangential fashion, $a = a_+$.

(f) Manifolds are completely separated, a pinned front does not exist.

Figure 1.6: This sequence of panels shows the formation and disappearance of a pinned front as we increase the bistable parameter a. We plot $W^u(0,0)$ and $W^s(1,1)$ in orange and blue, respectively. A pinned front exists if these two manifolds intersect. Based on these numerical simulations, for $d = 0.1$ we can find pinned fronts for a approximately in $[0.455, 0.545]$. At the end points of this interval the manifold intersection is tangential.

To conclude this section, we summarize the similarities and some basic differences between the waves on continuous and lattice domains in Table 1.2.

1.6 Infinite k-ary trees

Chapter 4 of this thesis is concerned with the propagation and pinning of waves on infinite k-ary trees $\mathcal{T}_k$. Infinite k-ary trees are undirected graphs in which the neighbourhood of each node consists of one parent with coordinates $(i-1, j)$, and k children

$$(i+1, kj), (i+1, kj+1), \ldots, (i+1, kj+k-1),$$

see Figure 1.7. Assuming that the diffusion parameter d is equal between all nodes, the bistable reaction-diffusion equation takes the form

$$\dot{u}_{i,j} = d\Big(\sum_{l=0}^{k-1} u_{i+1,kj+l} - u_{i,j}\Big) + d(u_{i-1,j} - u_{i,j}) + g(u_{i,j}; a), \tag{1.6.1}$$

for all $(i, j) \in \mathbb{Z} \times \mathbb{N}_0$. In this thesis we focus on so-called 'layer' solutions, i.e., solutions for which we have $u_{i,j}(t) = u_i(t)$ for all (i, j) and $t > 0$. Such solutions

Travelling waves on $\mathbb{R}$ and $\mathbb{Z}$

Domain	Continuous domain $\mathbb{R}$	Discrete domain $\mathbb{Z}$
Type of the equation	a second-order ODE for all $c \in \mathbb{R}$	MFDE for $c \neq 0$ and a difference equation for $c = 0$.
Singular perturbation problem as $c \to 0$	No, setting $c = 0$ does not change the nature of the equation.	Yes, since the MFDE transforms into a difference equation.
Uniqueness of solutions for $(a, d) \in (0, 1) \times \mathbb{R}_{>0}$	A solution pair (Φ, c) is unique.	A solution pair (Φ, c) is unique only when $c \neq 0$.
Smoothness of solutions	Both Φ and Φ' are in $C(\mathbb{R})$	The wave and its derivative are in $C(\mathbb{R})$ provided that $c \neq 0$.
Pinning	There is no pinning; $c = 0 \iff a = \frac{1}{2}$	Yes; analytical results for $d \ll 1$ and open problem for large d.

Table 1.2: In this table we summarize some similarities and basic differences between the travelling wave solutions of the continuous equation (1.3.1) and the lattice equation (4.1.6).

satisfy the simplified version of (1.6.1), namely

$$\dot{u}_i = d\,(ku_{i+1} - (k+1)u_i + u_{i-1}) + g(u_i; a). \tag{1.6.2}$$

Searching for a travelling wave solution of the form $u_i(t) = \Phi(i - ct)$ we arrive at the MFDE

$$-c\Phi'(\xi) = d\,(k\Phi(\xi+1) - (k+1)\Phi(\xi) + \Phi(\xi-1)) + g(\Phi(\xi); a), \tag{1.6.3}$$

coupled with the boundary conditions (1.5.8). Since this MFDE falls under the general framework of Mallet-Paret, there exists a travelling wave solution (Φ, c) for every $k > 0$. For $k = 1$ we recover the standard bistable reaction diffusion equation on $\mathbb{Z}$. We notice that the future and the past terms in MFDE (1.6.3) are asymmetrical in the parameter k. Indeed, we can also rewrite this equation as a reaction-diffusion equation with a convection term $d(k-1)(\Phi(\xi+1) - \Phi(\xi))$ on the lattice $\mathbb{Z}$, namely

$$\begin{aligned} -c\Phi'(\xi) = d\,(\Phi(\xi+1) - 2\Phi(\xi) + \Phi(\xi-1)) + g(\Phi(\xi); a) \\ + d(k-1)(\Phi(\xi+1) - \Phi(\xi)). \end{aligned} \tag{1.6.4}$$

For $k \neq 1$, the convection term also contributes to the speed of the wave c and it is expected that the speed c will not be symmetric anymore around the axis $a = 1/2$, also in the case of the standard cubic nonlinearity.

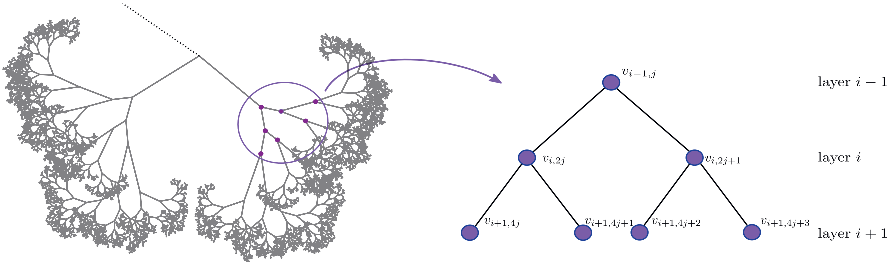


Figure 1.7: Infinite k-ary tree with $k = 2$ and indicated layers.

Motivation and main questions Our work is inspired by the study of Kouvaris, Kori and Mikhailov [62], in which the authors perform a non-rigorous analysis of the bistable reaction-diffusion equation on semi-finite k-ary trees. They discover that the direction of wave propagation largely depends on the branch factor k.

Therefore, our main questions in this work are the following.

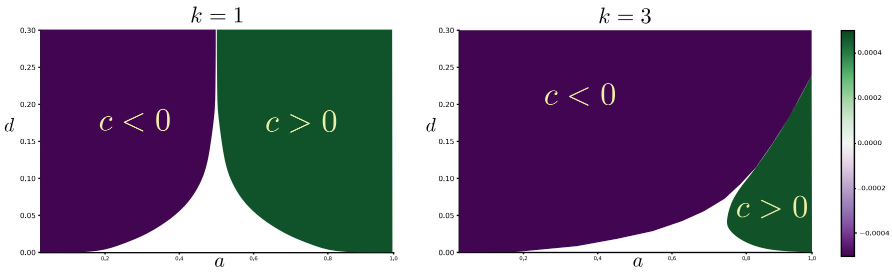
- Q1 Do travelling waves on k-ary trees admit both positive and negative speeds?
- Q2 Does the asymmetry in the equation cause a preferred direction for a travelling wave?
- Q3 What is the shape of the pinning region?
- Q4 Can the increase in diffusion parameter d cause the wave to change its direction of propagation for a fixed bistable parameter a and tree parameter k?

The initial numerical observations indeed suggest that the region in which $c = 0$ is now finite and asymmetric, in contrary to the case $k = 1$, see Figure 1.8.

These numerical findings motivated us to perform a rigorous mathematical analysis to answer the above-mentioned questions. Our methods rely on the construction of two different types of sub-solutions that help us to detect the regions in which we have $c < 0$. Moreover, exploiting certain parameter transformations we can also detect regions in which $c > 0$. We sum up our results into the following answers to our initial questions.

- A1 Yes, for every bistable nonlinearity and every $k \neq 1$, there exists a parameter regime (a, d) close to $a = 0$ for which we have $c < 0$. Similarly, $c > 0$ holds for some parameter regime (a, d) close to $a = 1$.
- A2 For $k > 1$, the wave prefers to retreat on the k-ary tree. In particular, for each bistable parameter $a \in (0, 1)$ there exists a parameter $d^*(a)$ such that the travelling wave solution of (1.6.3) for $d > d^*(a)$ travels with the strictly negative speed $c < 0$ [Theorem 4.2.5 in Chapter 4].
- A3 We show that the pinning region exists for $d \ll 1$. The previous answer implies that the pinning region is finite.
- A4 Yes. For each $k > 1$ and parameter a close to one the wave experiences at least once the following changes as we increase d from 0 to $+\infty$:

$$\text{pinning } (c = 0) \to \text{spreading } (c > 0) \to \text{pinning } (c = 0) \to \text{retreating } (c < 0).$$



on

Figure 1.8: Initial numerical observations for the wave propagation on k-ary trees for the standard cubic nonlinearity (1.3.3). For $k = 1$(left) we have the standard LDE on $\mathbb{Z}$ whose wave solution has no preferred direction. In other words, the speed c is, up to the sign change, symmetrical around the axis $a = \frac{1}{2}$. On the right we show the direction of the wave propagation on the k-ary tree with $k = 3$.

In addition, our methods provide more than abstract existence results. For instance, our results include an analytical description of regions where we surely have $c < 0$, $c = 0$ and $c > 0$ for the standard cubic nonlinearity.

1.7 Two-dimensional lattice $\mathbb{Z}^2$

In Chapters 2 and 3 of this thesis we study the bistable reaction-diffusion equation on the two-dimensional lattice $\mathbb{Z}^2$. For the discrete diffusion operator we take the plus-shaped Laplacian operator that takes into account the four closest neighbours of each point $(i, j) \in \mathbb{Z}^2$, i.e.,

$$[\Delta u_{i,j}] = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}, \tag{1.7.1}$$

and set the diffusion coefficient to 1, i.e., $d = 1$. Therefore, the central object of our study in these chapters is the LDE

$$\dot{u}_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} + g(u_{i,j}; a), \tag{1.7.2}$$

with $(i, j) \in \mathbb{Z}^2$. The function $g : \mathbb{R} \to \mathbb{R}$ is a bistable reaction function. We couple this LDE with an initial condition

$$u(0) = u^0 \in \ell^\infty(\mathbb{Z}^2). \tag{1.7.3}$$

Due to the anisotropy of the lattice, wavefront solutions to this LDE experience yet another feature that is uncharacteristic for waves on continuous domains, namely, the wave-pair (Φ, c) depends on the direction of propagation. To elaborate, as we have a two-dimensional lattice, it is natural to search for travelling wave solutions that move in the specific direction $(\sigma_h, \sigma_v) \in \mathbb{R}^2$, with the normalization condition $\sigma_h^2 + \sigma_v^2 = 1$. In particular, we plug the Ansatz

$$u_{i,j}(t) = \Phi(i\sigma_h + j\sigma_v - ct)$$

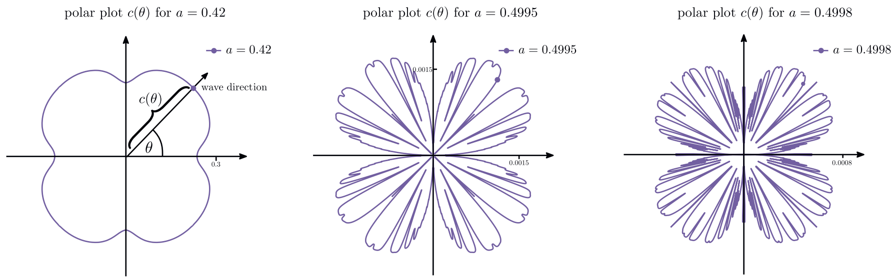


Figure 1.9: These plots depict the directional dependence of speed c on the angle $\theta = \arctan \sigma_v/\sigma_h$ of the wave-propagation for three different values of parameter a. The first image shows the smoothness of the graph $\theta \mapsto c(\theta)$ when far from the pinning region. On the second and third image we see the situation in which the waves propagate in some directions but are pinned in other.

into (1.7.2), which results in the MFDE

$$-c\Phi'(\xi) = \Phi(\xi + \sigma_h) + \Phi(\xi - \sigma_h) + \Phi(\xi + \sigma_v) + \Phi(\xi - \sigma_v) - 4\Phi(\xi) \\ + g(\Phi(\xi); a), \tag{1.7.4}$$

coupled with the boundary conditions

$$\lim_{\xi \to -\infty} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Phi(\xi) = 1.$$

The novelty compared to the previous sections is that the wave-pair solution (c, Φ) now depends also on the chosen direction (σ_h, σ_v), see Figure 1.9. This phenomenon does not occur for the standard waves on continuous domain $\mathbb{R}^n$, as the analogous wave Ansatz $u(x, y, t) = \Phi(x\sigma_h + y\sigma_v - ct)$ would again result in the second-order ODE (1.3.1).

1.7.1 Stability of travelling waves on $\mathbb{Z}^2$

In this work we are interested in the stability and long-term behaviour of the moving waves, far from the pinning regime. For that reason we are not interested in small $d \approx 0$, but we fix $d = 1$. In general, the main stability question (SQ) can be paraphrased as following.

SQ Given an initial condition $u^0 \in X$, where X is a normed space, under which assumptions on u^0 does the solution $u(t)$ of the initial value problem (1.7.2)-(1.7.3) converge in X to the travelling wave solution Φ as $t \to \infty$?

This question has already been the main topic of two prequel papers [44] and [43]. To demonstrate their results, we assume that $(\sigma_h, \sigma_v) = (1, 0)$ and consider the initial perturbations of the form

$$u^0_{i,j} = \Phi(i) + v^0_{i,j}. \tag{1.7.5}$$

In [44] the authors assume that v^0 is an arbitrarily big, but localized sequence, i.e.,

$$\lim_{|i|+|j| \to \infty} v^0_{i,j} = 0.$$

They show that this assumption is sufficient to guarantee that

$$\sup_{(i,j)\in\mathbb{Z}^2} |u_{i,j}(t) - \Phi(i-ct)| \to 0, \qquad \text{as } t\to\infty.$$

In their companion paper [43] the authors consider initial perturbations v^0 as elements of the space $\ell^p(\ell^1(\mathbb{Z});\mathbb{R})$, for any $p\in[1,\infty]$. They prove that there exists $\delta>0$ such that

$$||v^0||_{\ell^p(\mathbb{Z};\ell^1(\mathbb{Z}))} \le \delta$$

implies the algebraic convergence of the solution $u(t)$ to the travelling wave solution in $\ell^p(\ell^1(\mathbb{Z});\mathbb{R})$. Taking $p=\infty$, we see that the absolute value of the initial perturbation necessarily needs to be bounded by a constant δ, and localized only in the j-direction. However, due to technical obstacles, this result is shown exclusively for rational directions, that is, for $(\sigma_h,\sigma_v)\in\mathbb{Z}^2$, whereas [44] handles both rational and irrational directions.

Chapters 2 and 3 of this thesis concern with the stability question of the travelling waves, but now with arbitrarily large perturbations in $\ell^\infty(\mathbb{Z}^2)$. Adapting the framework and methods developed by Matano and Nara in [69] for the bistable reaction-diffusion equation on $\mathbb{R}^n$, we extend stability results to bounded perturbations that need not to be small or localized. However, to compromise for this generality, we assume that v^0 is a localized perturbation from a sequence that is periodic in the variable j, i.e., the initial perturbation u^0 satisfies the following two conditions (C):

(C_1) We have the spatially uniform bounds when $i\to\pm\infty$

$$\limsup_{i\to-\infty}\sup_{(i,j)\in\mathbb{Z}^2} u^0_{i,j} < a \qquad \liminf_{i\to+\infty}\inf_{(i,j)\in\mathbb{Z}^2} u^0_{i,j} > a. \tag{1.7.6}$$

(C_2) There exists $P\in\mathbb{Z}$ such that

$$\lim_{|i|+|j|\to\infty} |u^0_{i,j+P} - u^0_{i,j}| = 0, \tag{1.7.7}$$

see Figure 2.3 in Chapter 2. We show that these two conditions are enough to guarantee the orbital stability of the travelling wave.

1.8 Graphs and lattices

To conclude this introduction, we provide some insight to shed light on our choices of discrete diffusion operators. For lattices $\mathbb{Z}$ and $\mathbb{Z}^2$, these choices can be considered as merely standard discretizations of the continuous Laplace operator. On the other hand, discrete structures such as k-ary trees $\mathcal{T}_k$ have no continuous analogue, and it is not immediately clear how to derive a diffusion operator on such a structure.

1.8.1 Graph Laplacian

To explain the process of a discrete diffusion, let us first consider an undirected graph $G=(V,E)$ with vertices V, edges E and adjacency matrix A, i.e., $A_{I,J}=1$ if the nodes I and J are connected and $A_{I,J} = 0$ otherwise. We denote the neighbourhood of the node I by $\mathcal{N}(I)$, i.e., $\mathcal{N}(I) = \{J \in G : A_{I,J} = 1\}$. Moreover, to each node $I \in G$ we assign a function $u_I(t)$. Provided that the rate of diffusion along the edge of two connected nodes is equal to $d > 0$, the rate of change in time of the substance u at the node I is given by

$$\dot{u}_I(t) = d \sum_{J \in \mathcal{N}(I)} (u_J(t) - u_I(t)).$$

This equation is called *the graph heat equation* and the operator

$$[\Delta u]_I = \sum_{J \in \mathcal{N}(I)} (u_J - u_I)$$

is called the discrete Laplace operator or the *graph Laplacian*.

Lattice domains can be considered as special cases of undirected graphs. For example, we can see the integer lattice $\mathbb{Z}$ as a graph with vertices $V = \mathbb{Z}$. Provided that $E = \{\{i, i \pm 1\} : i \in \mathbb{Z}\}$, the discrete Laplace operator reads

$$[\Delta u]_i = (u_{i+1} - u_i) + (u_{i-1} - u_i) = u_{i+1} - 2u_i + u_{i-1}. \tag{1.8.1}$$

It is also possible to consider infinite range-interactions on $\mathbb{Z}$. For example, in [8], the authors take the discrete Laplacian operator on $\mathbb{Z}$ to be

$$[\Delta u]_i = \sum_{k \in \mathbb{Z}} \alpha_k (u_{i+k} - u_i) \tag{1.8.2}$$

where the coefficients $(\alpha_k)_{k \in \mathbb{Z}}$ satisfy some symmetry and localization conditions, such as $\sum_{k \in \mathbb{Z}} |\alpha_k| k^2 < \infty$ and $\alpha_k = \alpha_{-k}$ for $k \geq 0$. We can see these coefficients as diffusion weights between each edge $\{u_i, u_{i+k}\}$.

For $\Lambda = \mathbb{Z}^2$, the set V is naturally $V = \mathbb{Z}^2$. However, there are many possibilities for the set E, which result in different representations of discrete Laplace operators. For instance, if we take

$$E := \big\{\{(i,j),(i \pm 1, j)\}, \{(i,j),(i, j \pm 1)\} : (i,j) \in \mathbb{Z}^2\big\},$$

then we obtain the plus-shaped discrete Laplacian (1.7.1). On the other hand, one could also consider the set

$$E = \big\{\{(i,j),(i \pm 1, j \pm 1)\}, \{(i,j),(i \mp 1, j \mp 1)\} : (i,j) \in \mathbb{Z}^2\big\}$$

that gives us the cross-shaped discrete Laplacian $\Delta^{\times}$, namely

$$[\Delta^{\times} u]_{i,j} = u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} - 4u_{i,j}.$$

For the infinite k-ary tree $\mathcal{T}_k$, the graph structure is already ingrained in its definition. By assuming that the weights between the nodes are equal, we arrive at the discrete diffusion operator from (1.6.1). However, it would still be interesting to see what happens should the weights between the node and its parent be different than the weight between the node and its child and how that would change the dynamics between the retreating, pinned and spreading waves.

This freedom of choice of the diffusion operator results in infinite possibilities to model various phenomena with underlying discrete structures using graphs and lattices. As we have seen during the examples of the spatial chaos, pinning, directional-dependency, diffusion-induced propagation-reversal, and many more that we did not even tackle in this introduction, this variety makes lattices an interesting and rich field of research.

1.9 Overview of the thesis

Here we present the content of the chapters in this thesis. Chapters 2 and 3 can be regarded as companion chapters as they both study the bistable LDE on $\mathbb{Z}^2$, whereas Chapter 4 concerns the bistable LDE on the infinite k-ary tree $\mathcal{T}_k$.

Chapter 2 In this chapter we consider the bistable reaction-diffusion equation on the lattice $\mathbb{Z}^2$. This equation is also often called *the Allen-Cahn equation.* Our basic assumption is that the initial condition u^0 is a perturbation from the wave that moves in the horizontal direction. In the first part of this work, we do not assume that u^0 is a localized or 'small' perturbation from the wave, but that it only satisfies the condition $(C1)$; see (1.7.6). Already this assumption is enough to guarantee that there exists a one-dimensional differential equation which governs the flow of the zero-level surface $\gamma(t)$ of our solution $u(t)$. We call this governing equation *a discrete mean curvature flow* with a drift term. Using the Cole-Hopf transformation, we are able to transform this equation to the discrete heat equation on $\mathbb{Z}$ to show that the zero-level surface $\gamma(t)$ smoothens out over time and that the long-term behaviour of our solution is determined by the travelling wave $\Phi(\cdot - \gamma(t))$. In the second part of this work, by adding the assumption $(C2)$ from (1.7.7) we show that $\gamma(t) \to ct + \mu$, for some $\mu \in \mathbb{R}$, which ensures the orbital stability of the horizontal travelling wave.

Chapter 3 This chapter is a generalization of our work from Chapter 2 to rational directions on $\mathbb{Z}^2$, i.e., now we assume that u^0 is a perturbation of the wave that moves in some direction $(\sigma_h, \sigma_v) \in \mathbb{Z}^2$. The framework is similar to the one in Chapter 2; however, due to the fact that our wave is not aligned with the lattice anymore, we encounter more technical difficulties. One of these difficulties is that the governing equation for the zero-level surface $\gamma(t)$ does not transform via the Cole-Hopf transformation to the discrete heat equation but to a linear lattice equation that has both negative and asymmetrical coefficients. We treat this equation and its decay estimates in detail in Section §3.5 , which can also be seen as a section independent of this chapter.

Chapter 4 In this chapter we step away from the travelling waves on the two - dimensional lattice to study a wave-propagation and pinning on infinite k-ary trees. To show the existence of the pinning region that comprises chaotic solutions, we use the Moser Theorem from the field of Symbolic Dynamics. On the other hand, we are also interested in which parameter regimes the moving waves retreat $(c < 0)$ or spread

$(c > 0)$. Therefore, we employ the comparison principle to two types of sub-solutions: steep, step-like profiles that approximate the waves closer to the pinning region, and wide-profiles which show that for $d \gg 0$ the wave always retreats, irrespective of the value of the bistable parameter a.