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Random walks on Arakelov class groups

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Random Walks on Arakelov Class Groups

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Centrum Wiskunde & Informatica



Universiteit
Leiden

Koen de Boer

Random Walks on Arakelov Class Groups
and Their Applications in Cryptography
and Algorithmic Number Theory

To my family.

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