On cluster algebras and topological string theory

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Summary

Partition functions in string theory and supersymmetric field theories can be often computed exactly and be shown to have rich symmetries. Often the symmetry can be presented in the form of the differential or difference equation, which the partition function solves. Among the first examples of this kind was the discovery of the relevance of classical integrable systems of particles in the context of Seiberg-Witten theory, describing low-energy dynamics of 4d $\mathcal{N} = 2$ supersymmetric gauge theories. Later it was independently observed that the Painlevé equations are solved by the instanton partition functions of those SUSY gauge theories in the self-dual Omega-background. From the point of view of integrable systems the pass from Seiberg-Witten theory to the full partition function is the“deautonomization” of them. The uplift of the story to 5d $\mathcal{N} = 1$ SUSY gauge theories compactified on a circle corresponds to the "relativisation" of integrable systems, i.e. making the momentum-dependence of Hamiltonians to be exponential. After the "deautonomization" these systems become the q-difference equations of q-Painlevé type.

The notions of cluster varieties and cluster algebras were invented in the early 2000th for the solution of the classical problem of the parametrization of the space of "totally positive" matrices, i.e. those matrices, all minors of which are strictly positive. The new notion was immediately and successively applied to the description of the moduli spaces of local systems on Riemann curves, to the theory of integrable systems, and to the description of stability conditions in algebraic geometry. The initial point for my work in this thesis was the observation that there is a natural structure of X-cluster variety on the phase space of the relativistic Toda chain, which corresponds to the 5d $\mathcal{N} = 1$ $SU(N)$ gauge theory without the matter multiplets. The discrete dynamics which appears as a result of the action of cluster mapping class group on the cluster variables, can be solved by the partition functions of these theories.

In Chapter 2 we construct the structure of the X-cluster variety on the phase space of the XXZ spin chain. This extends the class of the previously known examples of gauge theories/cluster integrable systems correspondences, since the XXZ spin chains are known to be corresponding to 5d $\mathcal{N} = 1$ quiver gauge theories, with the linear quivers of constant rank. The so-called spectral duality, interchanging the rank and the length of the spin chain, found its natural interpretation in the cluster description. We also described the structure of the large piece of the cluster mapping class group for those systems and derived the bilinear equation for the dynamic of A-cluster variables under the action of generators of cluster mapping class group.

In Chapter 3 we show that the "master" solution of Bazhanov-Sergeev to the tetrahedron equation has a clear cluster-algebraic origin. The action of tetrahedral R-matrix by conjugation appears to be equivalent to the application of four mutations; we also show how this interpretation fits into the context of application of cluster algebras to the parametrization of the double Bruhat cells in $GL(N)$. Using this interpretation of the Bashanov-Sergeev solution, we give a clear recipe how the cluster integrable system with the arbitrary symmetric Newton polygon can be constructed using it. We also prove there the combinatorial lemma, which allows us to generalize this consideration to the arbitrary Newton polygon.

In Chapter 4 we make a few steps toward understanding why it happens that the partition functions of topological string theory, generalizing the partition functions of the gauge theories, are solving the equations appearing from the discrete dynamics of the cluster variables. We claim that the box-counting of topological vertices, which is the major tool to compute the partition function of topological strings, can be obtained directly from the cluster algebras. In order to do this, one has to uplift the partition function of dimers on the bipartite graph on torus, which encodes the Hamiltonians of integrable system, to the periodic graph on the plane, and to "deautonomize" the discrete $U(1)$ connection on the graph, parametrizing the cluster variables, by applying non-zero transverse flux of purely imaginary magnetic field to it. This claim can be also viewed as a particular example of so-called Topological Strings/Spectral Theory correspondence, since the partition function of dimers on the plane can be computed as a determinant of the q-difference operator. We check the correspondence, by showing that the density of the free-energy of the model of dimers, being computed in the limit of vanishing flux, satisfies the Seiberg-Witten equations, as it is expected from the string-theoretic perspective.

Chapter 5 is devoted to a project which is not directly related to the main lines of my research in the field of integrable systems. In this project we conduct the phenomenological study of the hydrodynamic regime of the flows of electrons in graphene. There we propose the principal scheme of the experiment, which would allow us to measure the viscosity of the electronic liquid, by applying the AC current of THz frequency to the sample of the Hall bar geometry.