On cluster algebras and topological string theory

Semenyakin, M.

Chapter 2

Cluster integrable systems and spin chains

2.1 Introduction

In the seminal paper [172] Seiberg and Witten found 'exact solution' to 4d $\mathcal{N} = 2$ super-symmetric gauge theory in the strong coupling regime. More strictly, the IR effective couplings were constructed geometrically, from the period integrals on a complex curve, whose moduli are determined by the condensates and bare couplings of the UV gauge theory. Shortly after, it has been also realized [72] that natural language for the Seiberg-Witten theory is given by classical integrable systems. In such context the pure supersymmetric gauge theories (with only $\mathcal{N} = 2$ vector supermultiplets) correspond to the Toda chains, while integrable systems for the gauge theories with fundamental matter multiplets are usually identified with classical spin chains of XXX-type.

The next important step was proposed in [164], where this picture has been lifted to $5d$. Then it has been shown that transition from $4d$ to $5d$ (actually – four plus one compact dimensions) results in 'relativization' of the integrable systems [143] (in the sense of Ruijsenaars [159]). In the simplest case of $SU(2)$ pure Yang-Mills theory, or affine Toda chain with two particles, instead of the Hamiltonian

$$H_{4d} = p^2 + e^q + Ze^{-q}, \tag{2.1}$$

corresponding to $4d$ theory, one has to consider

$$H_{5d} = e^p + e^{-p} + e^q + Ze^{-q}, \tag{2.2}$$

or the Hamiltonian of relativistic Toda chain, which describes effective theory for $5d$ pure $SU(2)$ Yang-Mills [1]. It has been also shown that $5d$ theories with fundamental matter correspond to XXZ-type spin chains (see e.g. [133] and references therein).

Relativistic Toda chains lead to natural relation of this story with the integrable systems on the Poisson submanifolds in Lie groups, or more generally to the *cluster* integrable systems – recently discovered class of integrable systems of relativistic type [71, 128, 55]. Direct relation between cluster integrable systems and $5d$ gauge theories has been proposed in [14]. It was shown there that for the case of Newton polygons with single internal point, dynamics of discrete flow is governed by q-Painlevé equations and their bilinear form is solved by Nekrasov $5d$ dual partition functions (for other examples of $5d$ gauge theories the same phenomenon was considered in [102, 16, 15])[2].

Cluster integrable systems Any convex polygon Δ with vertices in $\mathbb{Z}^2 \subset \mathbb{R}^2$ can be considered as a Newton polygon of polynomial $f_\Delta(\lambda,\mu)$, and equation

$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \tag{2.3}$$

defines a plane (noncompact) spectral curve in $\mathbb{C}^\times \times \mathbb{C}^\times$. The genus g of this curve is equal to the number of integral points strictly inside the polygon Δ.

According to [71],[55] a convex Newton polygon Δ, modulo action of $SA(2,\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, defines a cluster integrable system, i.e. an integrable system on X-cluster Poisson variety $\mathcal{X}$ of dimension $\dim_{\mathcal{X}} = 2S$, where S is area of the polygon Δ. The Poisson structure can be encoded by quiver $\mathcal{Q}$ with $2S$ vertices. Let ϵ_{ij} be the number of arrows from i-th to j-th vertex ($\epsilon_{ji} = -\epsilon_{ij}$) of $\mathcal{Q}$, then logarithmically constant Poisson bracket has the form

$$\{x_i, x_j\} = \varepsilon_{ij} x_i x_j, \quad \{x_i\} \in \left(\mathbb{C}^\times\right)^{2S}. \tag{2.4}$$

[1]The slightly misleading term 'relativistic' appears here due to formal similarity of momentum dependence to the rapidities of a massive relativistic particle in $1+1$ dimensions.

[2]Other relations between $5d$ supersymmetric gauge theories and cluster integrable systems (involving exact spectrum of quantized cluster integrable systems, BPS counting and toric Calabi-Yau quantization) were discussed in [52], [62], [151] correspondingly. They seem to be related to our case and we are going to return to these issues elsewhere.

The product of all cluster variables $\prod_i x_i$ is a Casimir for the Poisson bracket ((2.4)). Setting it to be

$$q = \prod_i x_i = 1 \tag{2.5}$$

and fixing values of other Casimirs, corresponding to the boundary points of Newton polygon $I \in \bar{\Delta}$ (their total number is $B-3$, since equation (2.3) is defined modulo multiplicative renormalization of spectral parameters λ, μ and $f_\Delta(\lambda, \mu)$ itself), one obtains symplectic leaf.

The properly normalized coefficients, corresponding to the internal points, are integrals of motion in involution

$$\{f_{a,b}(x), f_{c,d}(x)\} = 0, \qquad (a,b),(c,d) \in \Delta \tag{2.6}$$

w.r.t. the Poisson bracket (2.4). By Pick theorem one has

$$2S - 1 = (B-3) + 2g \tag{2.7}$$

where g is the number of internal points (or genus of the curve (2.3)), or the number of independent integrals of motion. So the number of independent integrals of motion is half of the dimension of symplectic leaf, and the system is integrable. One of distinguished features of the cluster integrable systems is that their integrals of motion are the Laurent polynomials of (generally – fractional powers) in the cluster variables.

There are several different ways to get explicit form of the spectral curve equation (2.3):

- Compute the dimer partition function (with signs) for a bipartite graph on a torus. One possible form of it is a characteristic equation

  $$\det \mathfrak{D}(\lambda, \mu) = 0 \tag{2.8}$$

  for the Kasteleyn-Dirac operator on a bipartite graph $\Gamma \subset \mathbb{T}^2$, depending on two 'quasimomenta' $\lambda, \mu \in \mathbb{C}^\times$;

- Alternatively, one can get the same equation (2.3) as a Lax-type equation of a spectral curve, with the Lax operator coming from affine Lie group construction, identifying cluster variety with a Poisson submanifold in the co-extended affine group.

Short exposition of the first construction of cluster integrable system, relevant for this chapter, is contained Section 1.4.

Classical integrable chains Integrability of classical $\mathfrak{gl}_M$ chains of XXZ type is based on the that their $M \times M$ Lax matrices satisfy the following classical RLL relation

$$\{L(\lambda) \otimes L(\mu)\} = \kappa[r(\lambda/\mu), L(\lambda) \otimes L(\mu)] \tag{2.9}$$

with the classical (trigonometric) r-matrix [3]

$$r(\lambda) = -\frac{\lambda^{1/2} + \lambda^{-1/2}}{\lambda^{1/2} - \lambda^{-1/2}} \sum_{i \neq j} E_{ii} \otimes E_{jj} + \frac{2}{\lambda^{1/2} - \lambda^{-1/2}} \sum_{i \neq j} \lambda^{-\frac{1}{2}s_{ij}} E_{ij} \otimes E_{ji}. \tag{2.10}$$

A classical chain of trigonometric type can be defined by the monodromy operator

$$T(\mu) = L_N(\mu/\mu_N) \dots L_1(\mu/\mu_1) \in \mathrm{End}(\mathbb{C}^M) \tag{2.11}$$

where M is called 'rank' of the chain. Integrability is guaranteed by classical RTT-relation

$$\{T(\lambda) \otimes T(\mu)\} = \kappa[r(\lambda/\mu), T(\lambda) \otimes T(\mu)] \tag{2.12}$$

for the monodromy operator that follows from (2.9), and gives rise to the integrals of motion, which can be extracted from the spectral curve equation (2.3) given explicitly by the formula

$$f_\Delta(\lambda, \mu) = \det\left(\lambda Q - T(\mu)\right) = 0. \tag{2.13}$$

where Q - diagonal twist matrix with the constant entities. Relativistic Toda system can be considered as certain degenerate case of generic XXZ chain of rank $M = 2$ (of length N for N particles).

Examples of Newton polygons In what follows we mostly consider cluster integrable systems, corresponding to the Newton polygons of the following types:

- Quadrangles with four boundary points, where all internal points are located along the same straight line, as on Fig. 2.1, left. This is the case of relativistic Toda chains, studied in [14]. The corresponding gauge theory is 5d $\mathcal{N} = 1$ Yang-Mills theory with $SU(N)$

[3] See details of derivation of Lax matrix from quantum algebra and notations in Section 1.3.

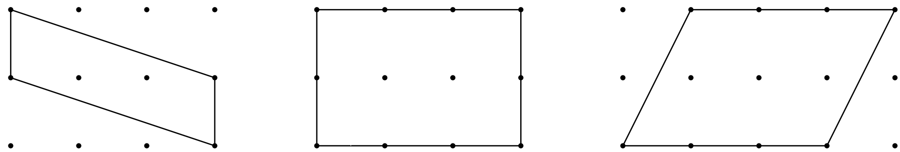

Figure 2.1. From left to right Newton polygons for: Toda chain on three sites, $\mathfrak{gl}_2$ XXZ spin chain on three sites, $\mathfrak{gl}_2$ spin chain on three sites with cyclic twist matrix.

gauge group (for $N-1$ internal points) without matter multiplets, possibly with the Chern-Simons term of level $|k| \leq N$ – in such case quadrangle is not a parallelogram.

- "Big" rectangles (modulo $SA(2,\mathbb{Z})$ transform). For the $N \times M$ rectangle (see Fig. 2.1, center) this can be alternatively described as a $\mathfrak{gl}_N$ spin chain on M sites (cf. with [21]), or vice versa. The corresponding 5d gauge theories are given by linear quivers theories with the $SU(N)$ gauge group at each of $M-1$ nodes: see Fig. 2.2.

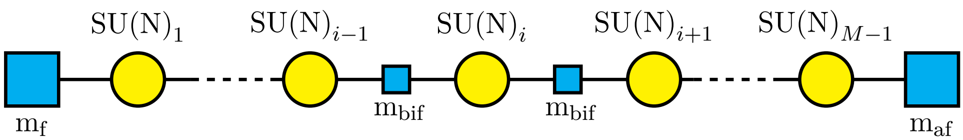

Figure 2.2. Linear quiver which defines multiplets for $\mathcal{N}=1$ gauge theory. Circles are for gauge vector multiplets, boxes are for hypermultiplets.

- "Twisted rectangles", or just the parallelograms, which are not $SA(2,\mathbb{Z})$-equivalent to the previous class (see Fig. 2.1, right), they can be alternatively formulated as spin chains with nontrivial twists. Gauge theory counterpart for this class of polygons is not yet known, except for the twisted $\mathfrak{gl}_N$ chain on one site, leading back to the basic class of Toda chains.

For all these families the spectral curve of an integrable system, determined by equation (2.3) is endowed with a pair of meromorphic differentials $\left(\frac{d\lambda}{\lambda}, \frac{d\mu}{\mu}\right)$ with the fixed $2\pi i\mathbb{Z}$-valued periods. One can also use this pair to introduce (the $SL(2,\mathbb{Z})$-invariant for our family) 2-form $\frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$ on $\mathbb{C}^\times \times \mathbb{C}^\times$, whose 'pre-symplectic' form is the SW differential.

Structure of the Chapter The main aim is to extend the correspondence between 5d theories and cluster integrable systems to wider class of models. We find isomorphism between the classes of $\mathfrak{gl}_N$ XXZ-like spin chains on M sites, corresponding to $5d$ $SU(N)$ linear quiver gauge theories (see Fig. 2.2) [21], and cluster integrable systems with $N \times M$ rectangular Newton polygons.

We start from the brief overview of classical XXZ spin chains. We illustrate with the simple example of relativistic Toda chain, how Lax operators naturally arise from the Dirac-Kasteleyn operator of cluster integrable system. Then we do this for the general case of XXZ spin chain of arbitrary length and rank. Spectral (or fiber-base) duality arises as an obvious consequence of the structure of considered bipartite graph. Spin chains with additional cyclic permutation twist matrix arise in the cluster context naturally as well.

Then we explain structure of large subgroup of cluster mapping class group $\mathcal{G}_\mathcal{Q}$. We show that in case of general rank and length of chain it contains subgroup (2.87) which act in autonomous $q = 1$ limit by permutations of inhomogeneities and diagonal twist parameters of spin chain. We also discuss issue of deautonomization and propose a way to define action of $\mathcal{G}_\mathcal{Q}$ on zig-zags in $q \neq 1$ case. Then we derive bilinear equations for the action of generators of $\mathcal{G}_\mathcal{Q}$ on A-cluster variables.

2.2 Spin chains

2.2.1 Relativistic Toda chain

Let us start with the case of relativistic Toda chain, which is known to be related to Seiberg-Witten theory in $5d$ without matter [143]. Relativistic Toda chains arise naturally on Lie groups [56], and therefore have cluster description. A typical bipartite graph of affine relativistic Toda is shown in Fig. 2.3. For the Toda system with N particles it has $2N$ vertices, $4N$ edges and $2N$ faces. Corresponding Newton polygon is shown in Fig. 2.1, left.

The cluster Poisson bracket (2.4) for the Toda face variables is

$$\{x_i^\times, x_j^\times\} = \{x_i^+, x_j^+\} = 0, \tag{2.14}$$

$$\{x_i^\times, x_j^+\} = (\delta_{i,j+1} + \delta_{i+1,j} - 2\delta_{i,j})x_i^\times x_j^+, \quad i,j \in \mathbb{Z}/N\mathbb{Z}$$

where in the non-vanishing r.h.s. one can immediately recognize the Cartan matrix of $\widehat{sl_N}$. This Poisson bracket has obviously two Casimir functions, which can be chosen, say, as[4]

$$q = \prod_j (x_j^\times x_j^+), \qquad \varkappa_1/\varkappa_2 = \prod_j x_j^+. \tag{2.15}$$

However, in what follows we are going to use the edge variables (see Section 1.4 for details), which do not have any canonical Poisson bracket, e.g. since they are not gauge invariant, when treated as elements of $\mathbb{C}^\times$-valued gauge connection on the graph. Hence, following [128], we fix the gauge and parameterize all edges by $2N$ exponentiated Darboux variables ξ_k, η_k

$$\{\xi_i, \eta_j\} = \delta_{ij}\xi_i\eta_j, \quad \{\xi_i, \varkappa_a\} = \{\eta_i, \varkappa_a\} = 0, \tag{2.16}$$

so that the face variables are expressed, as a products of oriented edge variables (see Fig. 2.3, left) by

$$x_i^\times = \frac{\xi_{i+1}}{\xi_i}(\varkappa_2/\varkappa_1)^{\delta_{iN}}, \;\; x_i^+ = \frac{\eta_i}{\eta_{i+1}}(\varkappa_1/\varkappa_2)^{\delta_{iN}}. \tag{2.17}$$

In terms of the edge variables (2.16) the monodromies over zig-zag paths (see Fig. 2.3, middle, right) can be expressed as follows

$$\alpha = \zeta/\varkappa_1, \quad \beta = \varkappa_2/\zeta, \quad \gamma = \varkappa_1\zeta, \quad \delta = 1/\varkappa_2\zeta, \quad \zeta = \prod_{k=1}^N \sqrt{\frac{\xi_k}{\eta_k}} \tag{2.18}$$

In the autonomous limit $q = 1$, there is a single independent Casimir – diagonal twist of monodromy operator $\varkappa_1/\varkappa_2$ or coupling of the affine Toda chain. Reduction from four zig-zags $\alpha, \beta, \gamma, \delta$ to single Casimir $\varkappa_1/\varkappa_2$ is a reminiscence of the freedom $\lambda \to a\lambda$, $\mu \to b\mu$ and the fact that $\alpha\beta\gamma\delta = 1$.

The Dirac-Kasteleyn operator here can be read of the left picture at Fig. 2.3, and is given by $N \times N$ matrix[5]:

$$\mathfrak{D}(\lambda,\mu) = \sum_{i=1}^N \Big((\xi_i + \mu^{-1}\eta_i)E_{ii} - \varkappa_1^{\delta_{iN}}\sqrt{\xi_i\eta_i}E_{i,i+1} + \varkappa_2^{-\delta_{iN}}\mu^{-1}\sqrt{\xi_i\eta_i}E_{i+1,i}\Big) \tag{2.19}$$

[4]Only the ratio of $\varkappa$'s is actually independent Casimir, but we introduce both of them for convenience in what follows.

[5]The spectral parameters or quasimomenta λ and μ appear due to intersection of the edge with the blue and purple cycles in $\mathrm{H}_1(\mathbb{T}^2, \mathbb{Z})$, and minuses arise due to discrete spin structure.

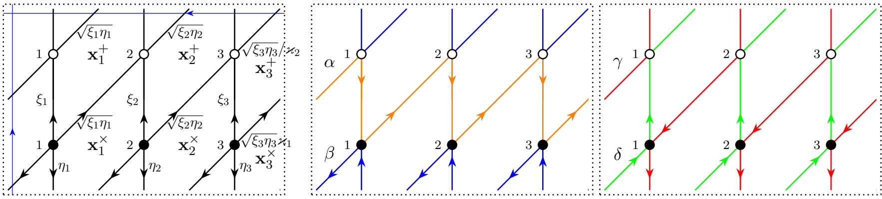

Figure 2.3. Left: Bipartite graph for the Toda chain. Center, right: zig-zag paths $\alpha, \beta, \gamma, \delta$.

where we have additionally defined

$$E_{N,N+1} = \lambda E_{N,1}, \quad E_{N+1,N} = \lambda^{-1} E_{1,N} \tag{2.20}$$

and it almost coincides here [39] with the standard $N \times N$ formalism for the spectral curve of relativistic Toda chain

$$\det\, \mathfrak{D}(\lambda, \mu) = 0 \;\Leftrightarrow\; \exists\; \mathfrak{D}(\lambda, \mu)\psi = 0 \tag{2.21}$$

with Baker-Akhiezer function $\psi \in \mathbb{C}^N$.

Now, to illustrate what is going to be done for the spin chains, let us rewrite this equation in terms of the well-known 2×2 formalism for Toda chains, but not quite in a standard way. In order to do that, we first add an additional black (white) vertex to each top (bottom) edge in left Fig. 2.3, and draw it in deformed way as in Fig. 2.4. Such operation obviously does not change the set of dimer configurations, and new dimer partition function differs from the old one only by total nonvanishing factor.

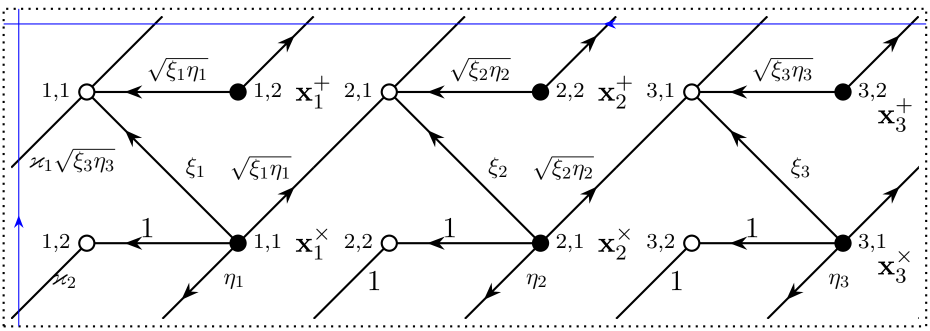

Figure 2.4. Extended and deformed bipartite graph for the Toda chain.

The Dirac-Kasteleyn matrix, read from the Fig. 2.4, can be written in the block form

$$\mathfrak{D}(\lambda,\mu) = \sum_{i=1}^{N} \Big(E_{ii} \otimes A_i + E_{i,i+1} \otimes C_i Q^{\delta_{i,N}} \Big) = \tag{2.22}$$

$$= \sum_{i=1}^{N} \Big((\xi_i + \mu^{-1}\eta_i) E_{ii} \otimes E_{11} + E_{ii} \otimes E_{12} + \sqrt{\xi_i\eta_i} E_{ii} \otimes E_{21} -$$

$$- \varkappa_1^{\delta_{i,N}} \sqrt{\xi_i\eta_i} E_{i,i+1} \otimes E_{11} - \mu\varkappa_2^{\delta_{i,N}} E_{i,i+1} \otimes E_{22} \Big)$$

with

$$A_i = \begin{pmatrix} \xi_i + \mu^{-1}\eta_i & 1 \\ \sqrt{\xi_i\eta_i} & 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} -\sqrt{\xi_i\eta_i} & 0 \\ 0 & -\mu \end{pmatrix}, \quad Q = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}. \tag{2.23}$$

The first factor in the tensor product corresponds to the number of the particle (or of the 'site'), arising naturally in the framework of 2×2 formalism for Toda systems and spin chains below, while the second – to position of a vertex inside the 'site'. For the 'extended' (compare to (2.19)) operator (2.22) one gets the same equation (2.21), but now with $\psi \in \mathbb{C}^{2N}$, which can be written as:

$$\psi = \sum_{i=1}^{N} e_i \otimes \begin{pmatrix} \psi_{i,1} \\ \psi_{i,2} \end{pmatrix} = \sum_{i=1}^{N} e_i \otimes \psi_i. \tag{2.24}$$

For the coefficients of this expansion (2.21) gives

$$\begin{cases} \psi_{k+1} = L_k(\mu)\psi_k \\ \psi_{N+1} = \lambda Q \psi_1 \end{cases} \tag{2.25}$$

or the system of finite-difference equations on Baker-Akhiezer functions with the quasi-periodic boundary conditions, where the 2×2 Lax matrix

$$L_i(\mu) = -C_i^{-1}(\mu) A_i(\mu) = \mu^{-\frac{1}{2}} \begin{pmatrix} \mu^{\frac{1}{2}}\sqrt{\frac{\xi_i}{\eta_i}} + \mu^{-\frac{1}{2}}\sqrt{\frac{\eta_i}{\xi_i}} & \frac{\mu^{\frac{1}{2}}}{\sqrt{\eta_i\xi_i}} \\ \mu^{-\frac{1}{2}}\sqrt{\xi_i\eta_i} & 0 \end{pmatrix} \tag{2.26}$$

is equivalent to the standard Lax matrix for relativistic Toda chain (see e.g. [128]) up to conjugation by permutation matrix, and redefinition of the variables

$$\xi \mapsto \eta, \quad \eta \mapsto \xi^{-1}, \quad \mu \mapsto \mu^{-1}. \tag{2.27}$$

This Lax operator satisfies classical RLL relation

$$\{L_i(\lambda) \otimes L_j(\mu)\} = \delta_{ij}[r(\lambda/\mu), L_i(\lambda) \otimes L_j(\mu)] \tag{2.28}$$

with the classical (trigonometric) r-matrix (2.10) [6]. Compatibility condition of (2.25) gives spectral curve equation in the form

$$\det\left(\lambda Q - L_N(\mu)...L_1(\mu)\right) = 0 \tag{2.29}$$

where $Q = \mathrm{diag}\,(\varkappa_1, \varkappa_2)$ is extra twist matrix[7], and inhomogeneities $\{\mu_i\}$, which appear in the case of generic XXZ chain, are absorbed here into redefinition of dynamical variables.

2.2.2 Spin chains of XXZ type

Let us now apply the same arguments, which we used for the Toda chain, to the following class of chains: the rank M chains on N cites of XXZ-type, which means that the Poisson structure (2.28) is defined by trigonometric r-matrix. Such systems naturally arise in $q \to 1$ limit of $U_q(\mathfrak{gl}_M)$, see Appendix 1.3. We claim that such classical spin chain can be alternatively described as cluster integrable systems, constructed from 'big rectangles' of the size $N \times M$.

For a cluster integrable system with such Newton polygon (see Fig. 2.5, left) one gets a bipartite graph, drawn at Fig. 2.6. According to [71] this graph is drawn on torus $\mathbb{T}^2$, i.e. left side is glued with the right side, and top - with the bottom, we will call such graphs as $N \times M$ 'fence nets'.

The cluster coordinates $x^\times_{ia}, x^+_{ia}$, now associated with the faces of graph at Fig. 2.6, satisfy the following Poisson bracket relations

$$\{x^\times_{ia}, x^+_{jb}\} = (-\delta_{ij}\delta_{ab} + \delta_{i,j+1}\delta_{ab} + \delta_{ij}\delta_{a+1,b} - \delta_{i,j+1}\delta_{a+1,b})x^\times_{ia}x^+_{jb}, \tag{2.30}$$

$$\{x^\times_{ia}, x^\times_{jb}\} = \{x^+_{ia}, x^+_{jb}\} = 0, \quad i,j \in \mathbb{Z}/N\mathbb{Z}, \quad a,b \in \mathbb{Z}/M\mathbb{Z}$$

[6] Up to numeric rescaling, see Section 1.3 for discussion.

[7] Note that constant diagonal matrices Q satisfy $[r, Q \otimes Q] = 0$, and therefore can be also used in construction of monodromy operators.

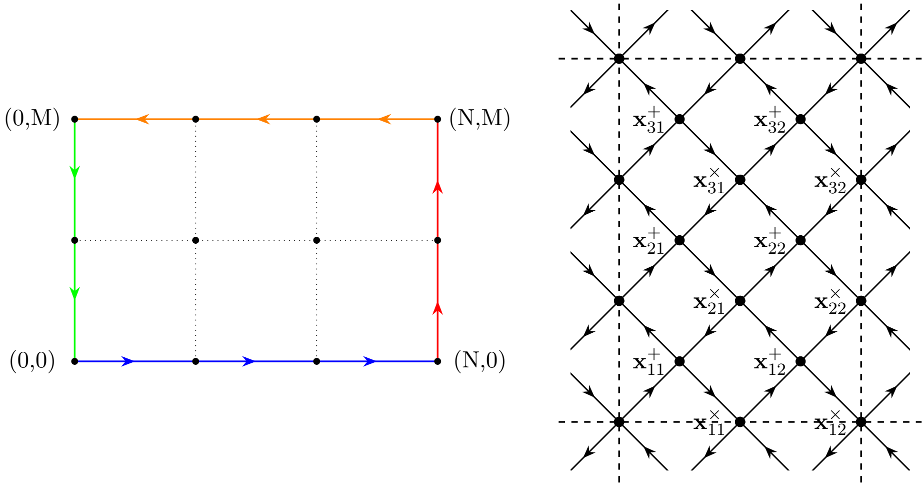

Figure 2.5. Left: Newton polygon for $(N, M) = (3, 2)$. Zig-zags from Fig. 2.6 as elements of torus first homology group are drawn by colored arrows. Right: Poisson quiver. It is drawn on the torus, so vertices lying on left-right and up-down sides have to be identified.

with two kinds of indices living 'on circles': i, j enumerating rows of bipartite graph and a, b enumerating columns. Corresponding quiver is drawn at Fig. 2.5, right. As in Toda case, 'fixing' a gauge, we pass now to the edge variables

$$x_{ia}^{\times} = \frac{\eta_{ia}^2}{\xi_{ia}^2}, \quad x_{ia}^{+} = \frac{\xi_{ia}\xi_{i+1,a-1}}{\eta_{i+1,a}\eta_{i,a-1}}(\sigma_{i+1}/\sigma_i)^{\delta_{a,1}}(\varkappa_{a-1}/\varkappa_a)^{\delta_{i,N}}. \tag{2.31}$$

with the Poisson bracket

$$\{\xi_{ia}, \eta_{jb}\} = \frac{1}{2}\delta_{ij}\delta_{ab}\xi_{ia}\eta_{jb}, \qquad i, j \in \mathbb{Z}/N\mathbb{Z}, \qquad a, b \in \mathbb{Z}/M\mathbb{Z} \tag{2.32}$$

Extra parameters in (2.31) are the Casimir functions of the bracket (2.30), together with

$$\zeta_i^h = \prod_{b=1}^{M} \frac{\xi_{ib}}{\eta_{ib}}, \quad \zeta_a^v = \prod_{j=1}^{N} \frac{\xi_{ja}}{\eta_{ja}}, \quad \{x^\times, \zeta^{h,v}\} = \{x^+, \zeta^{h,v}\} = 0. \tag{2.33}$$

It is useful to re-express them via the zig-zag variables (see the zig-zag paths on Fig. 2.6, middle and right)

$$\alpha_i = \sigma_i/\zeta_i^h, \quad \beta_i = 1/\zeta_i^h\sigma_i, \quad i = 1, \ldots, N \tag{2.34}$$

$$\gamma_a = \zeta_a^v/\varkappa_a, \quad \delta_a = \zeta_a^v \varkappa_a, \quad a = 1, \ldots, M \tag{2.35}$$

These formulas relate convenient generators of the center of cluster Poisson algebra with inhomogeneities $\{\mu_k = 1/\sigma_k\zeta_k^h = \beta_k\}$, twists $\{\kappa_a\}$, 'on-site' Casimirs $\zeta_i^h = (\alpha_i\beta_i)^{\frac{1}{2}}$ and 'projections of spins'[8] $\zeta_a^v = (\gamma_a\delta_a)^{\frac{1}{2}}$ of the chain.

Our main statement here is that the classical spin variables (for definition see Section 1.3) associated with single site of the chain could also be expressed via the edge variables ξ, η by

$$e^{S_a^0} = z_a^2, \quad S_{ab} = \frac{1}{2} z_b^{-2}(z_a^2 + z_a^{-2})\frac{\tau_a}{\tau_b}, \ a < b, \quad S_{ab} = -\frac{1}{2} z_a^2(z_a^2 + z_a^{-2})\frac{\tau_a}{\tau_b}, \ a > b, \tag{2.36}$$

where[9]

$$z_a = \sqrt{\xi_a/\eta_a}, \quad \tau_a = \sqrt{\xi_a\eta_a} \prod_{b=1}^{M} z_b^{\mathrm{sgn}(b-a)} \tag{2.37}$$

and the 'site index' $i = 1, \ldots, N$ is omitted here. Spin-variables cannot be directly expressed through the cluster variables in a natural way, but rather as a product of edge variables over some non-closed paths. However it is possible to express cluster variables via the spin variables on two adjacent sites by

$$x_{i,a}^\times = e^{-2(S_a^0)_i}, \tag{2.38}$$

$$x_{i,a}^+ = -\frac{e^{(S_a^0)_{i+1} + (S_{a-1}^0)_i}(S_{a-1}^+)_{i+1}(S_{a-1}^-)_i}{\cosh{(S_{a-1}^0)_{i+1}}\cosh{(S_a^0)_i}} \left(\frac{\sigma_{i+1}}{\sigma_i}\right)^{\delta_{a,1}} \left(\frac{\varkappa_{a-1}}{\varkappa_a}\right)^{\delta_{i,N}} \tag{2.39}$$

where index outside brackets of spin variables enumerates number of site.

The spectral curve again can be given by determinant of the Dirac-Kasteleyn operator, which is the weighted adjacency matrix of the bipartite graph. For generic (N, M) system it has the form:

$$\begin{aligned} \mathfrak{D}(\lambda, \mu) = \sum_{i=1}^{N}\sum_{a=1}^{M} \xi_{ia}(E_{i,i} \otimes E_{a,a} - \varkappa_a^{\delta_{i,1}}\sigma_i^{\delta_{M,a}} E_{i,i-1} \otimes E_{a+1,a}) + \\ + \eta_{ia}(\varkappa_a^{\delta_{1,i}} E_{i,i-1} \otimes E_{a,a} + \sigma_i^{\delta_{M,a}} E_{i,i} \otimes E_{a+1,a}) \end{aligned} \tag{2.40}$$

[8]Notice that spin's projections are not originally the Casimir functions for spin's brackets, but rather 'trivial' integrals of motion – like the total momentum of particles in Toda chains.

[9]This is basically standard bosonization formulas for the spin variables, cf. for example with [23],[134].

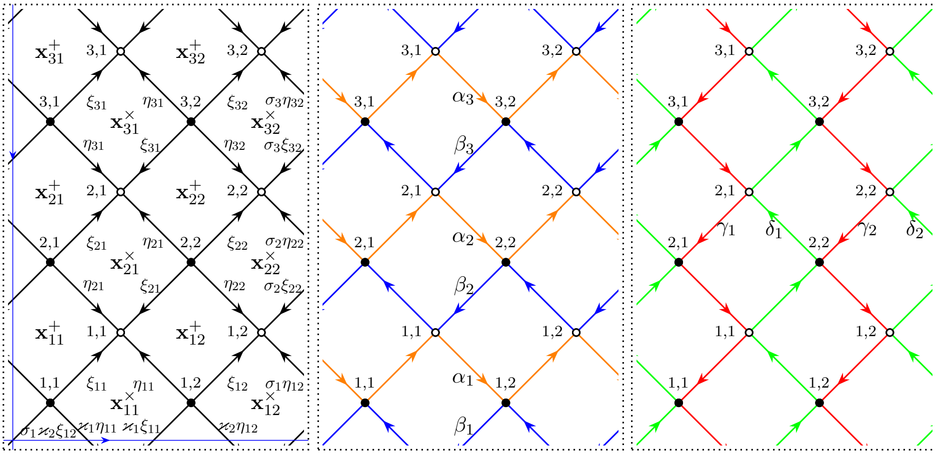

Figure 2.6. Left: bipartite graphs with labeled edges and faces: each edge, crossing purple cycle has to be multiplied by μ, each edge, crossing blue cycle – by λ. Center: horizontal zig-zag paths. Right: vertical zig-zag paths.

where the summand $E_{ij} \otimes E_{ab}$ is corresponding to the edge between black and white vertices[10] $(i, a) \to (j, b)$, and those matrices E_{ij} which get out of fundamental domain are promoted to the elements of the 'loop algebra', with the 'loop' parameters (λ, μ):

$$E_{1,0} \equiv \lambda E_{1,N}, \qquad \mathrm{E}_{M+1,M} \equiv \mu E_{1,M}. \tag{2.41}$$

Remark 2.2.1. The operator (2.40) as an element of $\mathrm{End}(\mathbb{C}^N)[[\lambda^{-1}]] \otimes \mathrm{End}(\mathbb{C}^M)[[\mu^{-1}]]$ can be naturally embedded into tensor product of evaluation representations of the loop algebras $\widetilde{\mathfrak{gl}}_N \otimes \widetilde{\mathfrak{gl}}_M$, i.e.

$$\begin{gathered}\mathfrak{D}(\lambda, \mu) = \sum_{i=1}^{N} \sum_{a=1}^{M} \xi_{ia}(h_i \otimes h_a - \varkappa_a^{\delta_{i,1}} \sigma_i^{\delta_{M,a}} f_{i-1} \otimes f_a) + \\ + \eta_{ia}(\varkappa_a^{\delta_{1,i}} f_{i-1} \otimes h_a + \sigma_i^{\delta_{M,a}} h_i \otimes f_a)\end{gathered} \tag{2.42}$$

[10]Signs '$-$' in $\mathfrak{D}$ arise in a standard way [71] due to choice of Kasteleyn marking or discrete spin structure on $\mathbb{T}^2$.

for two evaluation representations $\widetilde{\mathfrak{gl}}_K \to \mathrm{End}(\mathbb{C}^K)[[\zeta]]$:

$$\begin{aligned} &e_i = E_{i,i+1}, \quad 1 \le i \le K-1, \quad e_0 = e_K = \zeta E_{K,1} \\ &f_i = E_{i+1,i}, \quad 1 \le i \le K-1, \quad f_0 = f_K = \zeta^{-1} E_{1,K} \\ &h_i = E_{ii}, \qquad 1 \le i \le K. \end{aligned} \tag{2.43}$$

Let us now, breaking $M \leftrightarrow N$ symmetry, collect the terms, corresponding to E_{ii} and $E_{i,i-1}$ in the first tensor factor, i.e. rewrite (2.40) as:

$$\mathfrak{D}(\lambda,\mu) = \sum_{i=1}^{N} E_{i,i} \otimes A_i + E_{i,i-1} \otimes C_i(Q)^{\delta_{1,i}} \tag{2.44}$$

with

$$A_i = \sum_{b=1}^{M} \left(\xi_{ib} E_{b,b} + \eta_{ib} \sigma_i^{\delta_{M,b}} E_{b+1,b}\right), \quad C_i = \sum_{b=1}^{M} \left(\eta_{ib} E_{b,b} - \xi_{ib} \sigma_i^{\delta_{M,b}} E_{b+1,b}\right), \tag{2.45}$$

$$Q = \sum_{b=1}^{M} \varkappa_b E_{bb}$$

From the spectral curve equation $\det \mathfrak{D}(\lambda,\mu) = 0$ one finds for

$$\psi = \sum_{i=1}^{N} \psi_i e_i = \sum_{i=1}^{N} \sum_{a=1}^{M} \psi_{ia} e_i \otimes e_a \in \mathbb{C}^{MN} : \ \mathfrak{D}(\lambda,\mu)\psi = 0. \tag{2.46}$$

that

$$A_i \psi_i + C_i(Q)^{\delta_{i,1}} \psi_{i-1} = 0, \quad i = 1,\dots,N, \quad \psi_0 \equiv \lambda \psi_N. \tag{2.47}$$

Solving these equations recursively for the vectors $\psi_i = \sum_{a=1}^{M} \psi_{ia} e_a$, one finally gets

$$\left(\lambda Q - (-1)^N C_1^{-1} A_1 \dots C_N^{-1} A_N\right) \psi_N = 0 \tag{2.48}$$

with consistency condition

$$\det \left(\lambda\, Q - L_1\left(\sigma_1 \zeta_1^h \mu\right) \dots L_N\left(\sigma_N \zeta_N^h \mu\right)\right) = 0 \tag{2.49}$$

of the form (2.13), with the Lax matrices

$$L_i\left(\sigma_i \zeta_i^h \mu\right) = -C_i^{-1} A_i, \qquad i = 1,\dots,N. \tag{2.50}$$

Hence, the spectral curve $\det \mathfrak{D}(\lambda,\mu) = 0$ is represented in the form (2.11), common for the classical integrable chains with inhomogeneities $\mu_i = 1/\sigma_i\zeta_i^h = \beta_i$ and twist $Q = \sum_a \varkappa_a E_{aa} = \sum_a \sqrt{\delta_a/\gamma_a}E_{aa}$. There are also two sets of Casimirs related to spin variables: total projections of spin $\zeta_a^v = \prod_i e^{S_{ia}^0}$ and single non-trivial on-site Casimirs ζ_i^h. The Lax operators (2.50) on different sites satisfy classical RLL-relations

$$\{L_i(\mu) \otimes L_j(\mu')\} = \frac{1}{2}\delta_{ij}[r(\mu/\mu'), L_i(\mu) \otimes L_j(\mu')] \tag{2.51}$$

which coincide with (1.43) arising from the classical limit of $U_q(\mathfrak{gl}_M)$ with $q = e^{-\hbar}$ and $\kappa = \frac{1}{2}$ in (1.31), see Section 2.6 for details. In such way one gets explicit formulas (with the sign-factors (1.5)

$$(L_i)_{ab}(\mu) = \frac{1}{\mu^{\frac{1}{2}} - \mu^{-\frac{1}{2}}} \begin{cases} a = b, & \mu^{\frac{1}{2}} z_{ia}^{-2} + \mu^{-\frac{1}{2}} z_{ia}^2 \\ a \neq b, & \mu^{-\frac{s_{ab}}{2}} (z_{ib}^2 + z_{ib}^{-2}) \dfrac{\tau_{ib}}{\tau_{ia}} \end{cases}, \tag{2.52}$$

for the Lax operators (2.50) on the sites $i \in 1,...,N$ in terms of variables introduced in (2.37).

Comparing L-operator (2.52) with (1.44) one comes to the formulas (2.36), expressing the 'spin operators' on each site in terms of the edge variables. Expressions (2.36) satisfy all the relations of the classical limit of $U_q(\mathfrak{gl}_M)$ with $\kappa = \frac{1}{2}$. Note that this Lax operator is belonging to the lowest rank Kirillov orbit.

Remark 2.2.2. An equivalent construction of the cluster integrable systems is based on the Poisson submanifolds or double Bruhat cells in $\widehat{\mathrm{PGL}}$, endowed with the usual r-matrix Poisson structure [49, 55]. For the family of systems we consider here, given by the $SA(2,\mathbb{Z})$-orbit of rectangular $N \times M$ Newton polygons, one gets in such way a double Bruhat cell of $\widehat{\mathrm{PGL}}(N+M)$, given by the word

$$u = (s_M \bar{s}_M \dots s_1 \bar{s}_1 \Lambda)^N \tag{2.53}$$

in the co-extended double Weyl group $\widetilde{W}(A_K^{(1)} \times A_K^{(1)})$ (here with $K = N + M$) with the generators $s_i, s_{\bar{i}}, \Lambda$ satisfying relations

$$\begin{aligned} &s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad s_i s_j = s_j s_i, \quad \text{for} \quad |i-j| > 1 \\ &\bar{s}_i^2 = 1, \quad (\bar{s}_i \bar{s}_{i+1})^3 = 1, \quad \bar{s}_i \bar{s}_j = \bar{s}_j \bar{s}_i, \quad \text{for} \quad |i-j| > 1 \qquad i,j = 1,\dots,K \\ &\Lambda^K = 1, \quad \Lambda s_{i+1} = s_i \Lambda, \quad \Lambda \bar{s}_{i+1} = \bar{s}_i \Lambda \end{aligned} \tag{2.54}$$

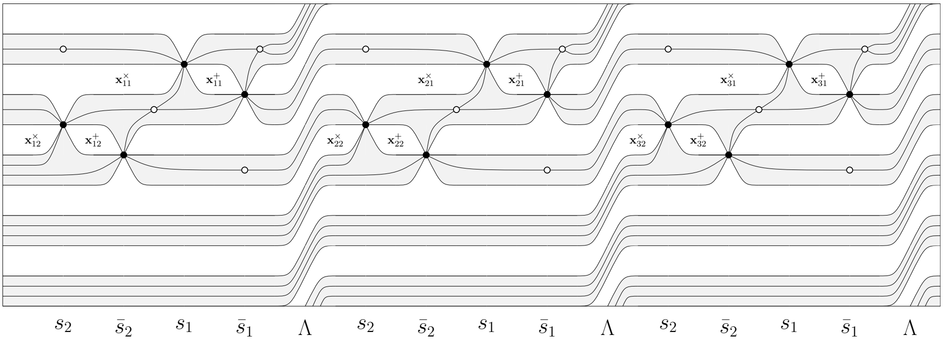

Figure 2.7. Thurston diagram in the $(3,2)$ case, which appears from $u = (s_2\bar{s}_2s_1\bar{s}_1\Lambda)^3$.

We are not going to repeat here all steps of the construction in detail, and just present the main ingredient – the Thurston diagram for (2.53), drawn for $(N,M) = (3,2)$ at Fig. 2.7. The corresponding bipartite graph (see Fig. 2.7) differs from the discussed above 'fence-net' by additional horizontal twist of the cylinder by 2π, which does not affect an integrable system, since it corresponds to the $SL(2,\mathbb{Z})$ transformation of the spectral parameters $(\lambda,\mu) \to (\lambda, \mu\lambda^{-1})$.

Example. SU(2) theory with $N_f = 4$ The most well-known case of the system we consider here corresponds to the five-dimensional supersymmetric gauge theory with the $SU(2)$ gauge group and $N_f = 4$ fundamental multiplets. The corresponding Newton polygon is a square with sides of length $N = M = 2$ (see Fig. 2.8), and as a spin chain this is just common XXZ-model on two sites with the Lax operator[11] (see e.g. [133])

$$L(\mu) = \begin{pmatrix} \mu e^{S^0} - \mu^{-1}e^{-S^0} & 2S^- \\ 2S^+ & \mu e^{-S^0} - \mu^{-1}e^{S^0} \end{pmatrix}, \qquad Q = \begin{pmatrix} \varkappa & 0 \\ 0 & \varkappa^{-1} \end{pmatrix}. \tag{2.55}$$

Spectral curve for the system is given by

$$\det\left(L\left(\mu/\mu_1\right)L\left(\mu/\mu_2\right)Q - \lambda\right) = 0. \tag{2.56}$$

[11]This form is slightly different from (1.52) arising from the classical limit of $U_q(\mathfrak{gl}_2)$. However, in 2×2 case these two forms are equivalent.

The Poisson brackets of spin operators are given by classical trigonometric r-matrix and written as:

$$\{S^0,\ S^{\pm}\} = \pm S^{\pm},\ \ \{S^+,\ S^-\} = \sinh 2S^0 \tag{2.57}$$

for the S-variables on the same site, and zero for the variables on the different sites. Such bracket has one natural Casimir function

$$K = -\zeta^h - (\zeta^h)^{-1} = \frac{1}{2}\cosh 2S^0 + S^+S^-. \tag{2.58}$$

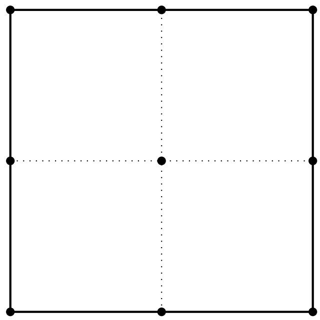

Figure 2.8. Newton polygon for $(N, M) = (2, 2)$.

As a cluster integrable system it lives on X-variety with the quiver corresponding to $A_3^{(1)}$-type system from figure 2 in [14], and its deautonomization leads to the Painlevé VI equation, solvable by conformal blocks, or equivalently topological strings amplitudes [102]. We derive Lax operator for this system from Kasteleyn operator in details in the next example, which is simply generalization of this example to three sites.

Example. SU(3) theory with $N_f = 6$. This case is corresponding to the word $u = (2\bar{2}1\bar{1}\Lambda)^3$ in double Weyl group of $\widehat{\mathrm{PGL}}(5)$. Bipartite graph is drawn on Fig. 2.6. Kasteleyn operator is 6×6 matrix

$$\mathfrak{D} = \begin{array}{c|cccccc} bw & 11 & 12 & 21 & 22 & 31 & 32 \\ \hline 11 & \xi_{11} & \mu\sigma_1\eta_{12} & 0 & 0 & \lambda\varkappa_1\eta_{11} & -\lambda\mu\varkappa_2\sigma_1\xi_{12} \\ 12 & \eta_{11} & \xi_{12} & 0 & 0 & -\lambda\varkappa_1\xi_{11} & \lambda\varkappa_2\eta_{12} \\ 21 & \eta_{21} & -\mu\sigma_2\xi_{22} & \xi_{21} & \mu\sigma_2\eta_{22} & 0 & 0 \\ 22 & -\xi_{21} & \eta_{22} & \eta_{21} & \xi_{22} & 0 & 0 \\ 31 & 0 & 0 & \eta_{31} & -\mu\sigma_2\xi_{32} & \xi_{31} & \mu\sigma_2\eta_{32} \\ 32 & 0 & 0 & -\xi_{31} & \eta_{32} & \eta_{31} & \xi_{32} \end{array} = \tag{2.59}$$

$$= \begin{pmatrix} A_1 & 0 & \lambda C_1 Q \\ C_2 & A_2 & 0 \\ 0 & C_3 & A_3 \end{pmatrix}.$$

Spectral curve is given by condition

$$\det \mathfrak{D}(\lambda,\mu) = 0 \quad \Leftrightarrow \quad \exists\ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} : \tag{2.60}$$

$$\mathfrak{D}(\lambda,\mu)\psi = 0 \qquad \Leftrightarrow \quad \begin{cases} \lambda Q\psi_3 = L_1(\sigma_1\zeta_1^h\mu)\psi_1 \\ \psi_1 = L_2(\sigma_2\zeta_2^h\mu)\psi_2 \\ \psi_2 = L_3(\sigma_3\zeta_3^h\mu)\psi_3 \end{cases}$$

$$L_i(\mu) = \frac{1}{\mu^{\frac{1}{2}} - \mu^{-\frac{1}{2}}} \begin{pmatrix} \mu^{-\frac{1}{2}}\frac{\xi_{i1}}{\eta_{i1}} + \mu^{\frac{1}{2}}\frac{\eta_{i1}}{\xi_{i1}} & \mu^{\frac{1}{2}}\frac{\eta_{i2}}{\xi_{i1}}\left(\frac{\xi_{i2}}{\eta_{i2}} + \frac{\eta_{i2}}{\xi_{i2}}\right) \\ \mu^{-\frac{1}{2}}\frac{\xi_{i1}}{\eta_{i2}}\left(\frac{\xi_{i1}}{\eta_{i1}} + \frac{\eta_{i1}}{\xi_{i1}}\right) & \mu^{-\frac{1}{2}}\frac{\xi_{i2}}{\eta_{i2}} + \mu^{\frac{1}{2}}\frac{\eta_{i2}}{\xi_{i2}} \end{pmatrix} \tag{2.61}$$

$$\zeta_i^h = \frac{\xi_{i1}\xi_{i2}}{\eta_{i1}\eta_{i2}}, \quad Q = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$$

which could be rewritten using monodromy operator

$$\begin{gathered}\left(\lambda Q - T_3^{2\times 2}(\mu)\right)\psi_3 = 0 \quad \Leftrightarrow \quad \det\left(\lambda Q - T_3^{2\times 2}(\mu)\right) = 0,\\ T_3^{2\times 2}(\mu) = L_1(\sigma_1\zeta_1^h\mu)L_2(\sigma_2\zeta_2^h\mu)L_3(\sigma_3\zeta_3^h\mu).\end{gathered} \tag{2.62}$$

Lax operator (2.61) is of $\mathfrak{gl}_2$ type, so can be mapped to (1.52). To transform it in $\mathfrak{sl}_2$ form (2.55) we have to apply transformations like (1.54)

$$\mu \mapsto -\mu\,\frac{\xi_1\xi_2}{\eta_1\eta_2}, \tag{2.63}$$

$$L(\mu) \mapsto \left(\sqrt{\frac{\xi_1\xi_2}{\eta_1\eta_2}}\mu^{\frac{1}{2}} - \sqrt{\frac{\eta_1\eta_2}{\xi_1\xi_2}}\mu^{-\frac{1}{2}}\right)\begin{pmatrix}\mu^{-1/2} & 0\\ 0 & 1\end{pmatrix}\cdot L(\mu)\cdot\begin{pmatrix}\mu^{1/2} & 0\\ 0 & 1\end{pmatrix}$$

so it becomes

$$L(\mu) = \begin{pmatrix}\mu^{\frac{1}{2}}\sqrt{\dfrac{\eta_1\xi_2}{\xi_1\eta_2}} - \mu^{-\frac{1}{2}}\sqrt{\dfrac{\xi_1\eta_2}{\eta_1\xi_2}} & \sqrt{\dfrac{\xi_2\eta_2}{\xi_1\eta_1}}\left(\dfrac{\xi_2}{\eta_2} + \dfrac{\eta_2}{\xi_2}\right)\\ -\sqrt{\dfrac{\xi_1\eta_1}{\xi_2\eta_2}}\left(\dfrac{\xi_1}{\eta_1} + \dfrac{\eta_1}{\xi_1}\right) & \mu^{\frac{1}{2}}\sqrt{\dfrac{\xi_1\eta_2}{\eta_1\xi_2}} - \mu^{-\frac{1}{2}}\sqrt{\dfrac{\eta_1\xi_2}{\xi_1\eta_2}}\end{pmatrix}. \tag{2.64}$$

Defining classical $\mathfrak{sl}_2$ spin variables by

$$S^- = \frac{1}{2}\sqrt{\frac{\xi_2\eta_2}{\xi_1\eta_1}}\left(\frac{\xi_2}{\eta_2} + \frac{\eta_2}{\xi_2}\right), \quad S^+ = -\frac{1}{2}\sqrt{\frac{\xi_1\eta_1}{\eta_2\xi_2}}\left(\frac{\xi_1}{\eta_1} + \frac{\eta_1}{\xi_1}\right), \quad e^{S^0} = \sqrt{\frac{\xi_1\eta_2}{\eta_1\xi_2}} \tag{2.65}$$

we see that Lax operator (2.64) coincides with the (2.55) up to replacement $\mu^{1/2} \to \mu$ and $S^0 \mapsto -S^0$. The latter is a consequence of the fact that (2.64) is coming from $q = e^{-\hbar}$ prescription, but (2.55) - from the usual $q = e^{\hbar}$. Poisson brackets of spin variables coming from edge variables bracket $\{\xi_i, \eta_j\} = \frac{1}{2}\delta_{ij}\xi_i\eta_j$ are

$$\{S^0,\ S^\pm\} = \pm\frac{1}{2}\,S^\pm,\ \ \{S^+,\ S^-\} = \frac{1}{2}\sinh 2S^0 \tag{2.66}$$

which differs from (2.57) by factor 1/2, appearing from $\kappa = \frac{1}{2}$ in the prescription for the classical limit of commutators (1.31). For details see Section 1.3. Spectral curve (2.56) could be obtained from (2.49) by transformation $\lambda \mapsto \lambda(\varkappa_1\varkappa_2)^{-\frac{1}{2}}$ with identification of parameters $\varkappa = (\varkappa_1/\varkappa_2)^{\frac{1}{2}}$, $\mu_i = (\varkappa_1\varkappa_2)^{\frac{1}{2}}(\sigma_i\zeta_i^h)^{-1}$.

2.3 Dualities and twists

2.3.1 Spectral duality

For some integrable chains special kind of duality could be observed both on the classical and on the quantum level: namely system with N-dimensional auxiliary space on M sites share Hamiltonians with some other system with M-dimensional auxiliary space on N sites. Under duality spectral parameter which monodromy operator depends on, and spectral parameter of characteristic equation exchange, so this duality is often called spectral duality (however, sometimes referred as 'level-rank' or 'fiber-base' duality, see [134] and references therein).

In the case of our interest, system doesn't change its type: XXZ classical spin chain of $\mathfrak{gl}_M$ type on N sites is dual to the XXZ chain of the $\mathfrak{gl}_N$ type on M sites [134], [23]. Looking at $M \times N$ fence-net bipartite graph, it becomes obvious: graph keeps its structure under 90-degree rotation. On the level of Kasteleyn operator, this corresponds to exchange of factors in tensor product, and using different expressions for spin variables.

SU(2) theory with $N_f = 4$ and one bi-fundamental multiplet. We start discussion of spectral duality in our context from simplest non-trivial example. Let us consider $\mathfrak{gl}_3$ spin chain on two sites, which is dual to $\mathfrak{gl}_2$ chain on three sites, considered in Section 2.2.2. To derive dual Lax operators, we should permute some rows and columns of Kasteleyn operator (2.59), which is exchanging of factors in tensor product $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^3) = \mathrm{End}(\mathbb{C}^3 \otimes \mathbb{C}^2)$:

$$
\mathfrak{D} = \begin{array}{c|cccccc}
 & 11 & 21 & 31 & 12 & 22 & 32 \\
\hline
11 & \xi_{11} & 0 & \lambda\varkappa_1\eta_{11} & \mu\sigma_1\eta_{12} & 0 & -\lambda\mu\varkappa_2\sigma_1\xi_{12} \\
21 & \eta_{21} & \xi_{21} & 0 & -\mu\sigma_2\xi_{22} & \mu\sigma_2\eta_{22} & 0 \\
31 & 0 & \eta_{31} & \xi_{31} & 0 & -\mu\sigma_3\xi_{32} & \mu\sigma_3\eta_{32} \\
12 & \eta_{11} & 0 & -\lambda\varkappa_1\xi_{11} & \xi_{12} & 0 & \lambda\varkappa_2\eta_{12} \\
22 & -\xi_{21} & \eta_{21} & 0 & \eta_{22} & \xi_{22} & 0 \\
32 & 0 & -\xi_{31} & \eta_{31} & 0 & \eta_{32} & \xi_{32}
\end{array} = \tag{2.67}
$$

$$= \begin{pmatrix} \tilde{A}_1 & \mu\tilde{Q}\tilde{C}_2 \\ \tilde{C}_1 & \tilde{A}_2 \end{pmatrix}$$

Spectral curve is given by condition

$$\det \mathfrak{D}(\lambda,\mu) = 0 \quad \Leftrightarrow \quad \exists\ \tilde{\psi} = \begin{pmatrix} \tilde{\psi}_1 & \tilde{\psi}_2 \end{pmatrix} : \tag{2.68}$$

$$\tilde{\psi}\mathfrak{D}(\lambda,\mu) = 0 \qquad \Leftrightarrow \qquad \begin{cases} \tilde{\psi}_2 = \tilde{\psi}_1 \tilde{L}_1(\varkappa_1 \zeta_1^v \lambda) \\ \mu\tilde{\psi}_1\tilde{Q} = \tilde{\psi}_2 \tilde{L}_2(\varkappa_2\zeta_2^v\lambda) \end{cases}$$

$$\tilde{L}_k(\lambda) = \frac{1}{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}\cdot \tag{2.69}$$

$$\cdot \begin{pmatrix} \lambda^{-\frac{1}{2}}\frac{\xi_{1k}}{\eta_{1k}} + \lambda^{\frac{1}{2}}\frac{\eta_{1k}}{\xi_{1k}} & \lambda^{\frac{1}{2}}\frac{\eta_{1k}}{\xi_{2k}}\left(\frac{\xi_{1k}}{\eta_{1k}} + \frac{\eta_{1k}}{\xi_{1k}}\right) & \lambda^{\frac{1}{2}}\frac{\eta_{1k}\eta_{2k}}{\xi_{2k}\xi_{3k}}\left(\frac{\xi_{1k}}{\eta_{1k}} + \frac{\eta_{1k}}{\xi_{1k}}\right) \\ \lambda^{-\frac{1}{2}}\frac{\xi_{2k}}{\eta_{1k}}\left(\frac{\xi_{2k}}{\eta_{2k}} + \frac{\eta_{2k}}{\xi_{2k}}\right) & \lambda^{-\frac{1}{2}}\frac{\xi_{2k}}{\eta_{2k}} + \lambda^{\frac{1}{2}}\frac{\eta_{2k}}{\xi_{2k}} & \lambda^{\frac{1}{2}}\frac{\eta_{2k}}{\xi_{3k}}\left(\frac{\xi_{2k}}{\eta_{2k}} + \frac{\eta_{2k}}{\xi_{2k}}\right) \\ \lambda^{-\frac{1}{2}}\frac{\xi_{2k}\xi_{3k}}{\eta_{1k}\eta_{2k}}\left(\frac{\xi_{3k}}{\eta_{3k}} + \frac{\eta_{3k}}{\xi_{3k}}\right) & \lambda^{-\frac{1}{2}}\frac{\xi_{3k}}{\eta_{2k}}\left(\frac{\xi_{3k}}{\eta_{3k}} + \frac{\eta_{3k}}{\xi_{3k}}\right) & \lambda^{-\frac{1}{2}}\frac{\xi_{3k}}{\eta_{3k}} + \lambda^{\frac{1}{2}}\frac{\eta_{3k}}{\xi_{3k}} \end{pmatrix}$$

$$\zeta_k^v = \frac{\xi_{1k}\xi_{2k}\eta_{3k}}{\eta_{1k}\eta_{2k}\eta_{3k}}, \quad \tilde{Q} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{2.70}$$

which could be rewritten using monodromy operator

$$\tilde{\psi}_1\left(\mu\tilde{Q} - \tilde{T}_2^{3\times3}(\lambda)\right) = 0 \quad \Leftrightarrow \quad \det\left(\mu\tilde{Q} - \tilde{T}_2^{3\times3}(\lambda)\right) = 0, \tag{2.71}$$

$$\tilde{T}_2^{3\times3}(\lambda) = \tilde{L}_1(\varkappa_1\zeta_1^v\lambda)\tilde{L}_2(\varkappa_2\zeta_2^v\lambda).$$

It is indeed spectral dual to the curve (2.62). One can check by direct calculation that

$$\begin{aligned} &(1-\varkappa_1\zeta_1^v\lambda)(1-\varkappa_2\zeta_2^v\lambda)\det\left(\mu\tilde{Q} - \tilde{T}_2^{3\times3}(\lambda)\right) = \\ &= (1-\sigma_1\zeta_1^h\mu)(1-\sigma_2\zeta_2^h\mu)(1-\sigma_3\zeta_3^h\mu)\det\left(\lambda Q - T_3^{2\times2}(\mu)\right). \end{aligned} \tag{2.72}$$

General case. If the order of factors in tensor product in (2.44) had been chosen in the other way, we would get M matrices A_k and C_k of size $N \times N$:

$$\mathfrak{D}(\lambda,\mu) = \sum_{m=1}^{M} \tilde{A}_m \otimes E_{m,m} + (\tilde{Q})^{\delta_{M,m}} \tilde{C}_m \otimes E_{m+1,m} \tag{2.73}$$

$$\tilde{A}_m = \sum_{n=1}^{N} \xi_{nm} E_{n,n} + \eta_{nm} \varkappa_m^{\delta_{1,n}} E_{n,n-1}, \; \tilde{C}_m = \sum_{n=1}^{N} \eta_{nm} E_{n,n} - \xi_{nm} \varkappa_m^{\delta_{n,1}} E_{n,n-1}, \tag{2.74}$$

$$\tilde{Q} = \sum_{n=1}^{N} \sigma_n E_{nn}.$$

Again, we present spectral curve as condition

$$\exists\, \tilde{\psi} = \sum_{n=1}^{N} \sum_{m=1}^{M} \tilde{\psi}_{nm} e_n \otimes e_m \in \mathbb{C}^{MN} : \; \tilde{\psi} \mathfrak{D}(\lambda,\mu) = 0 \tag{2.75}$$

which gives for the spectral curve

$$\det (\tilde{L}_1(\varkappa_1 \zeta_1^v \lambda) ... \tilde{L}_M(\varkappa_M \zeta_M^v \lambda) - \mu \tilde{Q}) = 0, \quad \tilde{L}_k(\varkappa_k \zeta_k^v \lambda) = -\tilde{A}_k \tilde{C}_k^{-1}. \tag{2.76}$$

Using variables (2.37) we can write dual Lax operator

$$(\tilde{L}_m)_{ij}(\lambda) = \frac{1}{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}} \begin{cases} i \neq j, & \lambda^{-\frac{s_{ij}}{2}} (z_{im}^2 + z_{im}^{-2}) \dfrac{\tilde{\tau}_{im}}{\tilde{\tau}_{jm}} \\ i = j, & \lambda^{\frac{1}{2}} z_{im}^{-2} + \lambda^{-\frac{1}{2}} z_{im}^2 \end{cases}, \tag{2.77}$$

$$\tilde{\tau}_{nm} = w_{nm} \prod_{i=1}^{N} z_{im}^{-s_{in}} \; .$$

We can relate them to L-operators (2.52) of the same size

$$L(z,w,\mu) = \tilde{L}(z \to z^{-1}, w, \lambda \to \mu^{-1})^{\top}. \tag{2.78}$$

Noting that for the classical r-matrix

$$r(a^{-1})^{\top} = -r(a) \tag{2.79}$$

where transposition is taken in each tensor multiplier, we can deduce from (2.51) that

$$\{\tilde{L}(\lambda) \otimes \tilde{L}(\mu)\} = \frac{1}{2} [\tilde{L}(\lambda) \otimes \tilde{L}(\mu), r(\lambda/\mu)]. \tag{2.80}$$

To obtain explicit relation for the dual spectral curves, we have to come back to the Kasteleyn operator of the system, and consider its determinant. In terms of $M \times M$ blocks A_k, C_k defined by (2.74) spectral curve is given by

$$\det \mathfrak{D}(\lambda,\mu) = \begin{vmatrix} A_1 & 0 & ... & 0 & \lambda C_1 Q \\ C_2 & A_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & A_{N-1} & 0 \\ 0 & 0 & ... & C_N & A_N \end{vmatrix} = \tag{2.81}$$

$$= \prod_i (\det C_i) \cdot \begin{vmatrix} C_1^{-1}A_1 & 0 & ... & 0 & \lambda Q \\ \mathbf{1} & C_2^{-1}A_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & C_{N-1}^{-1}A_{N-1} & 0 \\ 0 & 0 & ... & \mathbf{1} & C_N^{-1}A_N \end{vmatrix} =$$

$$= ... = \prod_i (\det C_i) \cdot \begin{vmatrix} \mathbf{1} & 0 & ... & 0 & \lambda Q \\ \mathbf{1} & \mathbf{1} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \mathbf{1} & 0 \\ 0 & 0 & ... & \mathbf{1} & (-1)^N T_N^{M\times M} \end{vmatrix},$$

$$T_N^{M\times M} = L_1 ... L_N, \quad L_k = -C_k^{-1} A_k,$$

and subtracting consequentially lines from first to last

$$\det \mathfrak{D}(\lambda,\mu) = (-1)^{NM} \det\,(C_1 ... C_N) \det\,(T_N^{M\times M}(\mu) - \lambda Q). \tag{2.82}$$

Acting in the same way, we get for the dual spectral curve

$$\det \mathfrak{D}(\lambda,\mu) = (-1)^{NM} \det\,\left(\tilde{C}_1 ... \tilde{C}_M\right) \det\,(\tilde{T}_M^{N\times N}(\lambda) - \mu\tilde{Q}), \tag{2.83}$$

$$\tilde{T}_M^{N\times N} = \tilde{L}_1 ... \tilde{L}_M, \quad \tilde{L}_k = -\tilde{A}_k \tilde{C}_k^{-1}$$

so, precise relation between curves is

$$\det\,(C_1...C_N)\det\,(T_N^{M\times M}(\mu)-\lambda Q)=\det\,\left(\tilde{C}_1...\tilde{C}_M\right)\det\,(\tilde{T}_M^{N\times N}(\lambda)-\mu\tilde{Q}) \tag{2.84}$$

Note that the relation of pre-factors is Casimir of the bracket

$$\frac{\det\,(C_1...C_N)}{\det\,\left(\tilde{C}_1...\tilde{C}_N\right)}=\frac{\mu^{\frac{N}{2}}}{\lambda^{\frac{M}{2}}}\left(\frac{\sigma_1...\sigma_N}{\varkappa_1...\varkappa_M}\right)^{1/2}\frac{\prod\limits_{n=1}^{N}(\sigma_n\zeta_n^h\mu)^{-1/2}-(\sigma_n\zeta_n^h\mu)^{1/2}}{\prod\limits_{m=1}^{M}(\varkappa_m\zeta_m^v\lambda)^{-1/2}-(\varkappa_m\zeta_m^v\mu)^{1/2}}. \tag{2.85}$$

2.3.2 Twisted chains

A diagonal twist matrix is not the only one, commuting with r-matrices. A cyclic twist

$$Q_\Lambda(\lambda)=\sum_{i=1}^{N}E_{i+1,i}=\sum_{i=1}^{N-1}\mathrm{E}_{i+1,i}+\lambda\mathrm{E}_{1,N} \tag{2.86}$$

also satisfies $[r(\lambda/\mu),Q_\Lambda(\lambda)\otimes Q_\Lambda(\mu)]=0$. In terms of bipartite graphs it corresponds to the twist on a cycle of the torus, where the bipartite graph is drawn on, or the gluing condition for the sides of fundamental domain, see Fig. 2.6. Such twist also changes a Poisson quiver, even though the edge variables are not affected themselves.

The twist of a bipartite graph results further in change of the zig-zag's structure. Several parallel zig-zags now join into 'longer sequences' with non-trivial winding so that rectangle Newton polygon undergoes a 'shear shift' – see examples on Fig. 2.9.

In the context of such transformations one can expect nontrivial consequences for spectral duality. Consider the trivial case of $\mathfrak{gl}_N$ chain on a single site, which is dual to rank 1 chain on N sites, and apply the cyclic twist along the longer side of a bipartite graph. In original picture this is just a multiplication of a single $N\times N$ Lax operator by cyclic permutation matrix. However in the dual setup, this results in passing from trivial $\mathfrak{gl}_1$ chain to the Toda chain on the same number of sites, which can be verified by comparing Fig. 2.9 and Fig. 2.3. After such procedure the number of Casimirs drops by $2N-2$, while number of Hamiltonians jumps from 0 to $N-1$.

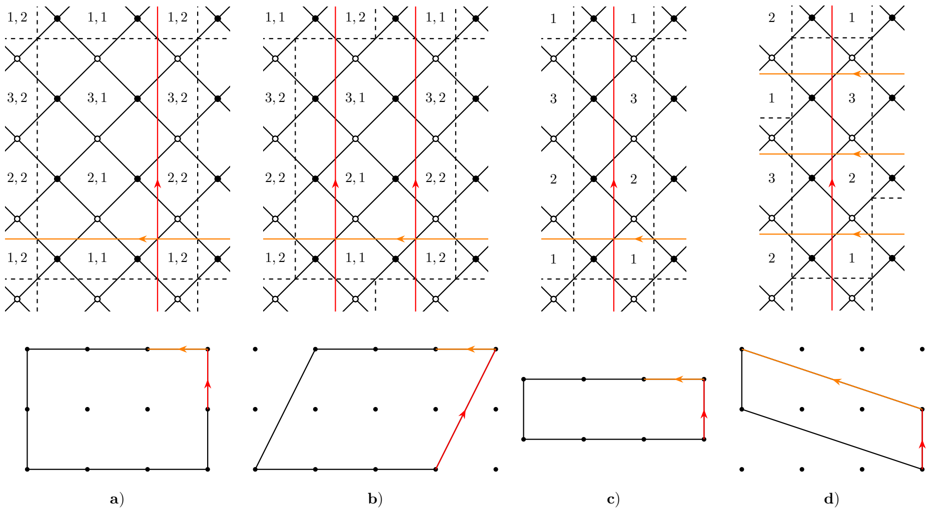

Figure 2.9. Examples of twisted $\mathfrak{gl}_2$ chains. Dashed lines bound fundamental domains. We use different notations for zig-zags here, comparing to the pictures above. Edges crossed by red arrows belong to γ_2 zig-zag, orange arrows are for α_1. a,b) XXZ chain of rank two and its twisted cousin. Note that the twisted twice chain is equivalent up to $\mathrm{SL}(2,\mathbb{Z})$ transformation $\lambda \to \lambda\mu$ to the untwisted chain, as $Q_\Lambda^2 = \mu\,\mathbf{1}$, like in Remark 2.2.2. c,d) Making Toda chain by twisting $\mathfrak{gl}_N$ chain dual to $\mathfrak{gl}_1$ chain.

For supersymmetric theories such transformation turns the theory of a single $\mathrm{SU}(N)$ hypermultiplet with only $\mathrm{SU}(N) \times \mathrm{SU}(N)$ flavor symmetry into pure $\mathrm{SU}(N)$ gauge theory.

2.4 Discrete dynamics

The cluster mapping class group $\mathcal{G}_\mathcal{Q}$ consists of sequences of mutations and permutations of quiver vertices, which maps quiver to itself, but acts in general non-trivially to the cluster variables (see Section 1.4 for details). As a simplification one can restrict the action of $\mathcal{G}_\mathcal{Q}$ to the set of Casimirs of the Poisson bracket. Each monomial Casimir maps to the monomial in Casimir functions. When the necessary for integrability condition $\prod_i x_i = 1$ is relaxed to $\prod_i x_i = q$ (which is called as deautonomization), these flows act on the set of Casimirs, inducing non-trivial $q-$dynamics.

In [14] the cluster mapping class groups for the quivers, corresponding to Newton polygons with a single internal point, were identified with the symmetry groups of q-Painlevé equations[12]. Passing from X-cluster to A-cluster variety, the q-Painlevé equations acquire bilinear form for the tau-functions, and can be solved via the dual Nekrasov partition functions for 5d supersymmetric SU(2) gauge theories [24, 14, 102, 16], which is a natural '5d uplift' of '4d' isomonodromic/CFT correspondence [68]. In [15] the cluster description was further applied to discrete dynamics of relativistic Toda chains of arbitrary lengths, where the solutions of non-autonomous versions are given by $SU(N)$ partition functions with the $|k| \leq N$ Chern-Simons terms. Recently, cluster realization of generalized q-Painlevé VI system was also observed in [153]. Note that for $q = 1$ case with trivial Casimirs solution of discrete dynamics for arbitrary bipartite graph can be written in terms of θ-functions [44].

Below in this section we discuss the cluster mapping class groups and non-autonomous bilinear equations, arising for generic rectangle Newton polygons. We present their explicit construction in the example, which will illustrate the following results:

Structure of the group $\mathcal{G}_\mathcal{Q}$.
For the $SA(2,\mathbb{Z})$-class of $N \times M$ rectangular Newton polygon, the MCG $\mathcal{G}_\mathcal{Q}$ always contains a subgroup of the form

$$\widetilde{W}\left(A^{(1)}_{N-1} \times A^{(1)}_{N-1}\right) \times \widetilde{W}\left(A^{(1)}_{M-1} \times A^{(1)}_{M-1}\right) \rtimes \mathbb{Z} \subset \mathcal{G}_\mathcal{Q}. \tag{2.87}$$

where $\widetilde{W}\left(A^{(1)}_{N-1} \times A^{(1)}_{N-1}\right)$ is a co-extended double Weyl group (2.54).

The generators of each subgroup are naturally labeled by intervals on sides of a Newton polygon, or subset of 'parallel' zig-zag paths (in the same homology class) on a bipartite graph:

$$\widetilde{W}\left(A^{(1)}_{N-1} \times A^{(1)}_{N-1}\right): \ \{s_{\alpha_i,\alpha_{i+1}}\}, \quad \{s_{\beta_i,\beta_{i+1}}\}, \quad i \in \mathbb{Z}/N\mathbb{Z} \tag{2.88}$$

$$\widetilde{W}\left(A^{(1)}_{M-1} \times A^{(1)}_{M-1}\right): \ \{s_{\gamma_a,\gamma_{a+1}}\}, \quad \{s_{\delta_a,\delta_{a+1}}\}, \quad a \in \mathbb{Z}/M\mathbb{Z} \tag{2.89}$$

where subscripts $\alpha, \beta, \gamma, \delta$ label the corresponding group of paths, see Fig. 2.6 middle and right. The group being extended by the additional generator ρ contains lattice of the rank $2N + 2M - 3$ of $q-$difference flows of integrable system.

[12] Such relation for particular cases was earlier mentioned in [90, 146, 24, 147].

Moreover, in special cases there is an obvious symmetry enhancement: for example, for $N = M$ an additional 'external' generator appears, which rotates the whole picture by $\pi/2$. However, sometimes this enhancement is more essential: if any of the sides is of length 2, two rest Weyl groups can be 'glued' together by additional permutation, so the known subgroup of $\mathcal{G}_{\mathcal{Q}}$ becomes

$$\widetilde{W}\left(A_{2N-1}^{(1)}\right) \times \widetilde{W}\left(A_1^{(1)} \times A_1^{(1)}\right) \subset \mathcal{G}_{\mathcal{Q}} \tag{2.90}$$

This enhancement is closely related to the fact that spectral curves with the $N \times 2$ rectangular Newton polygon can be mapped to the curves with the triangular Newton polygon with the integer sides $2N \times 2 \times 2$ (see e.g. (3.70) in [62]). If both $N = M = 2$ one finds the extra enhancement from $\widetilde{W}(A_1^{(1)} \times A_1^{(1)}) \times \widetilde{W}(A_1^{(1)} \times A_1^{(1)})$ to $\widetilde{W}(D_5^{(1)})$, see below.

Action on spin chain Casimirs.
Inhomogeneities, total spins, on-site Casimirs and twists of spin chain are permuted under the action of different components of $\mathcal{G}_{\mathcal{Q}}$.

Inhomogeneities are given by single zig-zags $\mu_i = \beta_i$, while on-site Casimirs are given by products of zig-zags $\zeta_i^h = (\alpha_i\beta_i)^{\frac{1}{2}}$. So the well defined transformation of them, which 'permutes sites' of spin chain are products of primitive permutations

$$\mathrm{s}_{\alpha_i,\alpha_{i+1}}\mathrm{s}_{\beta_i,\beta_{i+1}}: \quad \mu_i \mapsto \mu_{i+1},\ \mu_{i+1} \mapsto \mu_i, \quad \zeta_i^h \mapsto \zeta_{i+1}^h,\ \zeta_{i+1}^h \mapsto \zeta_i^h. \tag{2.91}$$

Permutations of twists $\varkappa_a = (\delta_a/\gamma_a)^{\frac{1}{2}}$ and projections of spins $\zeta_a^v = (\gamma_a\delta_a)^{\frac{1}{2}}$ by products

$$\mathrm{s}_{\gamma_a,\gamma_{a+1}}\mathrm{s}_{\delta_a,\delta_{a+1}}: \quad \varkappa_a \mapsto \varkappa_{a+1},\ \varkappa_{a+1} \mapsto \varkappa_a, \quad \zeta_a^v \mapsto \zeta_{a+1}^v,\ \zeta_{a+1}^v \mapsto \zeta_a^v. \tag{2.92}$$

can be viewed as an action of the Weyl group by permutations on the maximal torus of Lie group.

Bilinear equations.
Equations defining the action of each single generator of $\mathcal{G}_{\mathcal{Q}}$ on A-cluster variables $(\tau_{ij}^{\times}, \tau_{ij}^{+})$ could be rewritten in the form of bilinear equations. Evolution of coefficients can be encapsulated into the transformations of frozen variables $\{\mathbf{u}_{\alpha_i}, \mathbf{u}_{\beta_i}, \mathbf{u}_{\gamma_a}, \mathbf{u}_{\delta_a}\}$, which are evolving in the same way

as Casimirs in $\mathcal{X}$-variables.

For example τ-variables $\bar{\tau}^{\times}_{k,a}, \bar{\tau}^{+}_{k,a}$ transformed under the action of generator $\mathrm{s}_{\beta_i,\beta_{i+1}}$ satisfy bilinear equations

$$\begin{aligned}
&(\mathbf{u}_{\beta_{i+1}} - q^{\frac{1}{N}}\mathbf{u}_{\beta_i})(\mathbf{u}_\delta \mathbf{u}_{\gamma_a})^{\frac{1}{N}} \tau^{+}_{i-1,a}\tau^{\times}_{i+1,a} = \\
&= \mathbf{u}^{\frac{1}{M}}_{\beta_{i+1}} \bar{\tau}^{+}_{i,a}\tau^{\times}_{i,a} - q^{\frac{1}{NM}} \mathbf{u}^{\frac{1}{M}}_{\beta_i} \bar{\tau}^{\times}_{i,a}\tau^{+}_{i,a} \\
&(\mathbf{u}_{\beta_{i+1}} - q^{\frac{1}{N}}\mathbf{u}_{\beta_i})(\mathbf{u}_\delta / \mathbf{u}_{\delta_a})^{\frac{1}{N}} \tau^{+}_{i-1,a+1}\tau^{\times}_{i+1,a} = \\
&\mathbf{u}^{-\frac{1}{M}}_{\alpha_i} \bar{\tau}^{\times}_{i,a}\tau^{+}_{i,a+1} - q^{\frac{1}{NM}} \mathbf{u}^{-\frac{1}{M}}_{\alpha_i} \bar{\tau}^{+}_{i,a+1}\tau^{\times}_{i,a}
\end{aligned} \tag{2.93}$$

for all $a \in \mathbb{Z}/M\mathbb{Z}$, where $\mathbf{u}_\delta = \prod_a \mathbf{u}_{\delta_a}$. Frozen variables are transforming as

$$\mathrm{s}_{\beta_i,\beta_{i+1}}: \quad \mathbf{u}_{\beta_i} \mapsto q^{-\frac{1}{N}}\mathbf{u}_{\beta_{i+1}}, \ \mathbf{u}_{\beta_{i+1}} \mapsto q^{\frac{1}{N}}\mathbf{u}_{\beta_i}. \tag{2.94}$$

Bilinear equations for the action of generators $\mathrm{s}_{\alpha_i,\alpha_{i+1}}, \mathrm{s}_{\gamma_a,\gamma_{a+1}}, \mathrm{s}_{\delta_a,\delta_{a+1}}$ are similar.

2.4.1 Structure of $\mathcal{G}_\mathcal{Q}$

Now we present generators of $\mathcal{G}_\mathcal{Q}$ in terms of the quiver mutations[13] $\{\mu^{\times}_{ij}, \mu^{+}_{ij}\}$ (in the vertices, initially assigned with $\{x^{\times}_{ij}, x^{+}_{ij}\}$) and permutations of the vertices $\{s^{\lambda_a,\lambda_b}_{ij,kl}\}$. Consider for simplicity the $(3,2)$-example, which already illustrates how the explicit formulas look like in generic case. Here $2(N+M)=10$ generators (2.88) can be realized as

$$\begin{aligned}
\mathrm{s}_{\beta_1,\beta_2} &= s^{\lambda_a,\lambda_b}_{12,12}\mu^{+}_{11}\mu^{\times}_{11}\mu^{\times}_{12}\mu^{+}_{12}\mu^{\times}_{11}\mu^{+}_{11} \qquad & \mathrm{s}_{\alpha_3,\alpha_1} &= s^{\lambda_a,\lambda_b}_{12,31}\mu^{+}_{32}\mu^{\times}_{11}\mu^{\times}_{12}\mu^{+}_{31}\mu^{\times}_{11}\mu^{+}_{32} \\
\mathrm{s}_{\beta_2,\beta_3} &= s^{\lambda_a,\lambda_b}_{22,22}\mu^{+}_{21}\mu^{\times}_{21}\mu^{\times}_{22}\mu^{+}_{22}\mu^{\times}_{21}\mu^{+}_{21} \qquad & \mathrm{s}_{\alpha_1,\alpha_2} &= s^{\lambda_a,\lambda_b}_{22,11}\mu^{+}_{12}\mu^{\times}_{21}\mu^{\times}_{22}\mu^{+}_{11}\mu^{\times}_{21}\mu^{+}_{12} \\
\mathrm{s}_{\beta_3,\beta_1} &= s^{\lambda_a,\lambda_b}_{32,32}\mu^{+}_{31}\mu^{\times}_{31}\mu^{\times}_{32}\mu^{+}_{32}\mu^{\times}_{31}\mu^{+}_{31} \qquad & \mathrm{s}_{\alpha_2,\alpha_3} &= s^{\lambda_a,\lambda_b}_{32,21}\mu^{+}_{22}\mu^{\times}_{31}\mu^{\times}_{32}\mu^{+}_{21}\mu^{\times}_{31}\mu^{+}_{22}
\end{aligned} \tag{2.95}$$

and

$$\begin{aligned}
\mathrm{s}_{\delta_2,\delta_1} &= s^{\lambda_a,\lambda_b}_{31,31}\mu^{+}_{21}\mu^{\times}_{21}\mu^{+}_{11}\mu^{\times}_{11}\mu^{\times}_{31}\mu^{+}_{31}\mu^{\times}_{11}\mu^{+}_{11}\mu^{\times}_{21}\mu^{+}_{21} \\
\mathrm{s}_{\gamma_1,\gamma_2} &= s^{\lambda_a,\lambda_b}_{21,12}\mu^{+}_{22}\mu^{\times}_{31}\mu^{+}_{32}\mu^{\times}_{11}\mu^{\times}_{21}\mu^{+}_{12}\mu^{\times}_{11}\mu^{+}_{32}\mu^{\times}_{31}\mu^{+}_{22}
\end{aligned} \tag{2.96}$$

[13]For the definitions on cluster algebras see Section 1.4.

$$\mathrm{s}_{\delta_1,\delta_2} = s^{\lambda_a,\lambda_b}_{32,32}\mu^{+}_{22}\mu^{\times}_{22}\mu^{+}_{12}\mu^{\times}_{12}\mu^{\times}_{32}\mu^{+}_{32}\mu^{\times}_{12}\mu^{+}_{12}\mu^{\times}_{22}\mu^{+}_{22}$$

$$\mathrm{s}_{\gamma_2,\gamma_1} = s^{\lambda_a,\lambda_b}_{22,11}\mu^{+}_{21}\mu^{\times}_{32}\mu^{+}_{31}\mu^{\times}_{12}\mu^{\times}_{22}\mu^{+}_{11}\mu^{\times}_{12}\mu^{+}_{31}\mu^{\times}_{32}\mu^{+}_{21}$$

which are sequences of mutations in the vertices along zig-zags in the forward and then backward directions. One can check that each generator here is involution i.e. $\mathrm{s}^2 = 1$, and acts by rational transformation on X-cluster variables: e.g. for $\mathrm{s}_{\beta_2,\beta_3} = s^{\lambda_a,\lambda_b}_{22,22}\mu^{+}_{21}\mu^{\times}_{21}\mu^{\times}_{22}\mu^{+}_{22}\mu^{\times}_{21}\mu^{+}_{21}$ one can explicitly write:

$$x^{\times}_{31} \mapsto x^{\times}_{31}\cdot x^{+}_{22}x^{\times}_{21}\frac{[x^{\times}_{22}, x^{+}_{21}, x^{\times}_{21}]}{[x^{\times}_{21}, x^{+}_{22}, x^{\times}_{22}]}, \quad x^{\times}_{32} \mapsto x^{\times}_{32}\cdot x^{+}_{21}x^{\times}_{22}\frac{[x^{\times}_{21}, x^{+}_{22}, x^{\times}_{22}]}{[x^{\times}_{22}, x^{+}_{21}, x^{\times}_{21}]}, \quad (2.97)$$

$$x^{+}_{21} \mapsto \frac{1}{x^{\times}_{21}}\cdot\frac{[x^{+}_{21}, x^{\times}_{21}, x^{+}_{22}]}{[x^{+}_{22}, x^{\times}_{22}, x^{+}_{21}]}, \quad x^{+}_{22} \mapsto \frac{1}{x^{\times}_{22}}\cdot\frac{[x^{+}_{22}, x^{\times}_{22}, x^{+}_{21}]}{[x^{+}_{21}, x^{\times}_{21}, x^{+}_{22}]},$$

$$x^{\times}_{21} \mapsto \frac{1}{x^{+}_{22}}\cdot\frac{[x^{\times}_{21}, x^{+}_{22}, x^{\times}_{22}]}{[x^{\times}_{22}, x^{+}_{21}, x^{\times}_{21}]}, \quad x^{\times}_{22} \mapsto \frac{1}{x^{+}_{21}}\cdot\frac{[x^{\times}_{22}, x^{+}_{21}, x^{\times}_{21}]}{[x^{\times}_{21}, x^{+}_{22}, x^{\times}_{22}]},$$

$$x^{+}_{11} \mapsto x^{+}_{11}\cdot x^{\times}_{21}x^{+}_{21}\frac{[x^{+}_{22}, x^{\times}_{22}, x^{+}_{21}]}{[x^{+}_{21}, x^{\times}_{21}, x^{+}_{22}]}, \quad x^{+}_{12} \mapsto x^{+}_{12}\cdot x^{\times}_{22}x^{+}_{22}\frac{[x^{+}_{21}, x^{\times}_{21}, x^{+}_{22}]}{[x^{+}_{22}, x^{\times}_{22}, x^{+}_{21}]},$$

while all the other variables remain unchanged. Here we have used the notation

$$[x_1, x_2, .., x_n] = 1 + x_1 + x_1\cdot x_2 + ... + x_1\cdot ...\cdot x_n = \quad (2.98)$$

$$= 1 + x_1(1 + x_2(.... + x_{n-1}(1 + x_n)...)).$$

Notice also that the result of zig-zag mutation sequences actually do not depends on the point of the 'zig-zag strip' one starts with the first mutation and direction of the jumps along/across given zig-zag. Note that the [ ]-function possesses nice 'inversion' property

$$[x_1, ..., x_n] = x_1...x_n\cdot[x_n^{-1}, ..., x_1^{-1}] \quad (2.99)$$

which allows to write equivalently, for example

$$x^{\times}_{21} \mapsto \frac{1}{x^{+}_{22}}\cdot\frac{[x^{\times}_{21}, x^{+}_{22}, x^{\times}_{22}]}{[x^{\times}_{22}, x^{+}_{21}, x^{\times}_{21}]} = \frac{1}{x^{+}_{21}}\cdot\frac{[(x^{\times}_{22})^{-1}, (x^{+}_{22})^{-1}, (x^{\times}_{21})^{-1}]}{[(x^{\times}_{21})^{-1}, (x^{+}_{21})^{-1}, (x^{\times}_{22})^{-1}]}. \quad (2.100)$$

Each set of permutations $\mathrm{s}_{\zeta_i,\zeta_{i+1}}$ with similar ζ constitute affine Weyl group of $A^{(1)}$-type. The groups for different z are commuting, so they satisfy usual relations

$$\begin{cases} \mathrm{s}^2_{\zeta_i,\zeta_{i+1}} = 1, \\ (\mathrm{s}_{\zeta_i,\zeta_{i+1}}\mathrm{s}_{\zeta_{i+1},\zeta_{i+2}})^3 = 1 \\ \mathrm{s}_{\zeta_i,\zeta_{i+1}}\mathrm{s}_{\zeta_j,\zeta_{j+1}} = \mathrm{s}_{\zeta_j,\zeta_{j+1}}\mathrm{s}_{\zeta_i,\zeta_{i+1}}, \quad |i-j|>1 \end{cases} \tag{2.101}$$

$\zeta = \alpha, \beta$ with $i, j \in \mathbb{Z}/3\mathbb{Z}$

$$\mathrm{s}^2_{\zeta_i,\zeta_{a+1}} = 1$$

$\zeta = \gamma, \delta$ with $i, j \in \mathbb{Z}/2\mathbb{Z}$.

$$\mathrm{s}_{\zeta_i,\zeta_{i+1}}\mathrm{s}_{\zeta'_j,\zeta'_{j+1}} = \mathrm{s}_{\zeta'_j,\zeta'_{j+1}}\mathrm{s}_{\zeta_i,\zeta_{i+1}},$$

$\zeta, \zeta' = \alpha, \beta, \gamma, \delta$ such that $\zeta \neq \zeta'$. There are two more 'external' automorphisms preserving bipartite graph

$$\begin{aligned} \Lambda_h : \quad & x^{\times}_{ia} \mapsto x^{\times}_{i,a-1}, \quad x^{+}_{ia} \mapsto x^{+}_{i,a-1} \\ \Lambda_v : \quad & x^{\times}_{ia} \mapsto x^{\times}_{i-1,a}, \quad x^{+}_{ia} \mapsto x^{+}_{i-1,a} \end{aligned} \tag{2.102}$$

which satisfy obvious relations

$$\Lambda_h\Lambda_v = \Lambda_v\Lambda_h, \quad \Lambda_h^2 = 1, \quad \Lambda_v^3 = 1, \tag{2.103}$$

$$\Lambda_h \mathrm{s}_{\zeta_a,\zeta_{a+1}} = \mathrm{s}_{\zeta_{a-1},\zeta_a}\Lambda_h, \quad \text{for } \zeta = \gamma, \delta, \tag{2.104}$$

$$\Lambda_h \mathrm{s}_{\zeta_i,\zeta_{i+1}} = \mathrm{s}_{\zeta_i,\zeta_{i+1}}\Lambda_h, \quad \text{for } \zeta = \alpha, \beta, \tag{2.105}$$

$$\Lambda_v \mathrm{s}_{\zeta_i,\zeta_{i+1}} = \mathrm{s}_{\zeta_{i-1},\zeta_i}\Lambda_v, \quad \text{for } \zeta = \alpha, \beta \ , \tag{2.106}$$

$$\Lambda_v \mathrm{s}_{\zeta_a,\zeta_{a+1}} = \mathrm{s}_{\zeta_a,\zeta_{a+1}}\Lambda_v, \quad \text{for } \zeta = \gamma, \delta \ , \tag{2.107}$$

and promote affine Weyl groups to extended affine Weyl groups. There is also one more generator of infinite order

$$\rho = s^{\lambda_b\lambda_a}\mu^{\lambda_b} : \quad \mu^{\lambda_b} = \prod_{i,a}\mu^{\lambda_b}_{ia}, \quad s^{\lambda_b\lambda_a} : \quad x^{+}_{ia} \mapsto x^{\times}_{ia}, \quad x^{\times}_{ia} \mapsto x^{+}_{i-1,a+1}, \tag{2.108}$$

satisfying relations

$$\rho\, \mathrm{s}_{\alpha_{i-1},\alpha_i} = \mathrm{s}_{\alpha_i,\alpha_{i+1}}\, \rho, \quad \rho\, \mathrm{s}_{\beta_i,\beta_{i+1}} = \mathrm{s}_{\beta_i,\beta_{i+1}}\, \rho, \tag{2.109}$$

$$\rho\, \mathrm{s}_{\gamma_i,\gamma_{i+1}} = \mathrm{s}_{\gamma_{i-1},\gamma_i}\, \rho, \quad \rho\, \mathrm{s}_{\delta_i,\delta_{i+1}} = \mathrm{s}_{\delta_i,\delta_{i+1}}\, \rho,$$

so the cluster mapping class group contains

$$\widetilde{W}\left(A_2^{(1)} \times A_2^{(1)}\right) \times \widetilde{W}\left(A_1^{(1)} \times A_1^{(1)}\right) \rtimes \mathbb{Z} \subset \mathcal{G}_{\mathcal{Q}}. \tag{2.110}$$

We conjecture that for general rectangular $N \times M$ Newton polygon, cluster mapping class group contains subgroup (2.87). Construction of generators for general N and M is straightforward, by 'jumps over zig-zags' as in example.

In the case $N = M$ there is also an additional 'external' generator $R_{\pi/2}$ of order 4, which rotates bipartite graph by $\pi/2$

$$R_{\pi/2}: \; x^{\times}_{i,a} \mapsto x^{+}_{-a,i}, \quad x^{+}_{i,a} \mapsto x^{\times}_{1-a,i}. \tag{2.111}$$

In the case $N = 2K$ or $M = 2K$ there is another additional 'external' generator, which flips the rectangle.

Discrete flows. The group $\mathcal{G}_{\mathcal{Q}}$ contains lattice L of discrete flows of rank $B-3$, where $B = 2N + 2M$ is the number of boundary integral points of Newton polygon. It consists of four pairwise commuting lattices contained in two copies of $W(A^{(1)}_{N-1}) = \mathbb{Z}^{N-1} \rtimes W(A_{N-1})$ and two copies of $W(A^{(1)}_{M-1}) = \mathbb{Z}^{M-1} \rtimes W(A_{M-1})$, and generator $(\rho)^{\mathrm{lcm}(N,M)}$ where $\mathrm{lcm}(N,M)$ is the least common multiple of N and M. The lattice is generated by elements $\mathrm{T}_{\zeta_i,\zeta_{i+1}}$ which take pair of adjacent strands, wind them up in opposite directions over cylinder and put on the initial places, if one imagine $W(A^{(1)}_{N-1}), W(A^{(1)}_{M-1})$ as a groups acting by permutations of strands on cylinder. For $(3,2)$ example β-piece of $\mathcal{G}_{\mathcal{Q}}$ can be presented as $W(A_2^{(1)}) = \mathbb{Z}^2 \rtimes W(A_2)$ with $\mathbb{Z}^2$ and $W(A_2)$ generated by

$$\mathrm{T}_{\beta_1,\beta_2} = \mathrm{s}_{\beta_1,\beta_2}\mathrm{s}_{\beta_2,\beta_3}\mathrm{s}_{\beta_3,\beta_1}\mathrm{s}_{\beta_2,\beta_3}, \quad \mathrm{T}_{\beta_2,\beta_3} = \mathrm{s}_{\beta_2,\beta_3}\mathrm{s}_{\beta_3,\beta_1}\mathrm{s}_{\beta_1,\beta_2}\mathrm{s}_{\beta_3,\beta_1} \tag{2.112}$$

and by

$$\mathrm{s}_{\beta_1,\beta_2}, \mathrm{s}_{\beta_2,\beta_3} \tag{2.113}$$

correspondingly.

One can find a homomorphism of the lattice L of the shifts (2.112) into the group of discrete flows $\mathcal{G}'_\Delta$ (defined as in [55] to be an additive group of integral valued functions on boundary vertices of Newton polygon modulo sub-group A generated by the restrictions from $\mathbb{Z}^2$ to the boundary of Newton polygon of affine functions $f(i,j) = ai + bj + c$). For the case of rectangular Newton polygons one can easily finds that $\mathcal{G}'_\Delta = \mathbb{Z}^{B-3}$. Embedding of L to $\mathcal{G}'_\Delta$ actually comes from consideration of the action of $\mathcal{G}_\mathcal{Q}$ on zig-zags presented in the next section, and results in the image $\mathbb{Z}^{B-3}$. However, the factor is $\mathcal{G}'_\Delta/L = \mathbb{Z}/\mathrm{lcm}(N,M)\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/M\mathbb{Z}$. The non-trivial index appears due to the functions on the corners of Newton polygon. It can be also seen that the image of generator $(\rho)^{\mathrm{lcm}(N,M)}$ coincides with the image of generator τ from [55].

2.4.2 Monomial dynamics of Casimirs

According to [71] the lattice of Casimir functions x_γ is generated by zig-zag paths[14]

$$\mathsf{Z} = \{\gamma \in \mathrm{H}_1(\Gamma, \mathbb{Z}) \mid \varepsilon(\gamma, \cdot) = 0\}. \tag{2.114}$$

As the skew-symmetric form ε is intersection form on dual surface, this condition is equivalent to being trivial in dual surface $\hat{S}$ homologies. In order to be expressed in terms of cluster variables $\{x^\times_{ij}, x^+_{ij}\}$ Casimir should be also trivial in torus homologies, i.e. we are interested in subset

$$\mathsf{C} = \{\gamma \in \mathrm{H}_1(\Gamma, \mathbb{Z}) \mid [\gamma] = 0 \in \mathrm{H}_1(\hat{S}, \mathbb{Z}), \quad [\gamma] = 0 \in \mathrm{H}_1(\mathbb{T}^2, \mathbb{Z})\}. \tag{2.115}$$

As zig-zags and faces are drawn on torus $\mathsf{Z}, \mathsf{F} \subset \mathrm{H}_1(\Gamma, \mathbb{Z})$, they are constrained by $\prod_i x_{\zeta_i} = 1$, where the product goes over all zig-zag paths and $\prod_i x_{f_i} = 1$, where the product goes over all faces of bipartite graph on torus. To obtain non-trivial q-dynamic these constraints have to be relaxed to $\prod_i x_{f_i} = q \neq 1$ so that x_γ now is an element of extension $\mathrm{H}_1(\tilde{\Gamma}, \mathbb{Z}) = \mathrm{H}_1(\Gamma, \mathbb{Z}) \oplus \mathbb{Q}^2_{\langle\omega,\hat{\omega}\rangle}$ with the relations $\sum_i f_i = \omega,\ \sum_i \zeta_i = \hat{\omega}$. In multiplicative notations this reads

$$\prod_i x_{f_i} = q,\ \prod_i x_{\zeta_i} = \hat{q} \tag{2.116}$$

where we have additionally defined $q = x_\omega$, $\hat{q} = x_{\hat{\omega}}$. Introduction of $q \neq 1$ can be considered by lifting of bipartite graph to universal cover of $\mathbb{T}^2$ which is $\mathbb{R}^2$.

[14]For details on definitions see Section 1.4.

Any variable x_γ, $\gamma \in \mathsf{C}$ can be expressed via face variables x_{f_i}, which are cluster variables, and can be mutated by usual rules (1.82). However, there is no generic rule for mutation of variable associated with a single zig-zag, except for mutation in four-valent vertex identified with a 'spider move' [71]. We propose here the generic rule for transformation of zig-zags [15] under the action of generators (2.94), namely, for the $N \times M$ rectangle:

$$\begin{aligned}
\mathrm{s}_{\alpha_i,\alpha_{i+1}}: &\quad \alpha_i \mapsto q^{\frac{1}{N}}\alpha_{i+1}, \quad \alpha_{i+1} \mapsto q^{-\frac{1}{N}}\alpha_i,\\
\mathrm{s}_{\beta_i,\beta_{i+1}}: &\quad \beta_i \mapsto q^{-\frac{1}{N}}\beta_{i+1}, \quad \beta_{i+1} \mapsto q^{\frac{1}{N}}\beta_i,\\
\mathrm{s}_{\gamma_a,\gamma_{a+1}}: &\quad \gamma_a \mapsto q^{\frac{1}{M}}\gamma_{a+1}, \quad \gamma_{a+1} \mapsto q^{-\frac{1}{M}}\gamma_a,\\
\mathrm{s}_{\delta_a,\delta_{a+1}}: &\quad \delta_a \mapsto q^{-\frac{1}{M}}\delta_{a+1}, \quad \delta_{a+1} \mapsto q^{\frac{1}{M}}\delta_a,
\end{aligned} \tag{2.117}$$

where $i = 1, \ldots, N$, $a = 1, \ldots, M$. The group $\mathcal{G}_\mathcal{Q}$ acts on the elements of C, embedded in multiplicative lattice generated by zig-zags, precisely as Coxeter groups of A_{K-1}-type act on the root lattices embedded into $\mathbb{Z}^K$ (c.f. [153, 95]).

These rules basically come just from consistency with mutation transformations for the elements of C. There is a two-parametric family of transformations for zig-zag variables

$$\zeta \mapsto \zeta a^{[\zeta]_A} b^{[\zeta]_B}, \text{ if } [\zeta] = ([\zeta]_\mathrm{A}, [\zeta]_\mathrm{B}) - \text{class of } \zeta \text{ in } \mathrm{H}_1(\mathbb{T}^2, \mathbb{Z}) \tag{2.118}$$

which do not affect C, since C consists of the combinations of zig-zags with zero class in torus homology. This ambiguity is fixed using the 'locality assumption' that zig-zags not adjacent to the transformed faces are not changed.

Let us now demonstrate, how formulas (2.117) come for $(N, M) = (3, 2)$ from consistency with transformations of C, where one can introduce the following over-determined set of generators

$$\begin{aligned}
&Z_{\beta_1,\alpha_1} = x_{11}^\times x_{12}^\times, &&Z_{\beta_2,\alpha_2} = x_{21}^\times x_{22}^\times, &&Z_{\beta_3,\alpha_3} = x_{31}^\times x_{32}^\times,\\
&Z_{\alpha_1,\beta_2} = (x_{11}^+ x_{12}^+)^{-1}, &&Z_{\alpha_2,\beta_3} = (x_{21}^+ x_{22}^+)^{-1}, &&Z_{\alpha_3,\beta_1} = (x_{31}^+ x_{32}^+)^{-1}
\end{aligned} \tag{2.119}$$

[15] We abuse notations, denoting $x_\zeta = \zeta$ for zig-zags.

$$\begin{aligned} &Z_{\gamma_1,\delta_1} = (x^\times_{11}x^\times_{21}x^\times_{31})^{-1}, \quad Z_{\delta_1,\gamma_2} = x^+_{12}x^+_{22}x^+_{32}, \\ &Z_{\gamma_2,\delta_2} = (x^\times_{12}x^\times_{22}x^\times_{32})^{-1}, \quad Z_{\delta_2,\gamma_1} = x^+_{11}x^+_{21}x^+_{31} \end{aligned} \tag{2.120}$$

satisfying

$$\begin{aligned} &Z_{\beta_1,\alpha_1}Z_{\beta_2,\alpha_2}Z_{\beta_3,\alpha_3}Z_{\gamma_1,\delta_1}Z_{\gamma_2,\delta_2} = 1 \\ &Z_{\alpha_1,\beta_2}Z_{\alpha_2,\beta_3}Z_{\alpha_3,\beta_1}Z_{\delta_1,\gamma_2}Z_{\delta_2,\gamma_1} = 1 \\ &Z_{\beta_1,\alpha_1}Z_{\beta_2,\alpha_2}Z_{\beta_3,\alpha_3}(Z_{\alpha_1,\beta_2}Z_{\alpha_2,\beta_3}Z_{\alpha_3,\beta_1})^{-1} = q = 1. \end{aligned} \tag{2.121}$$

so that the number of independent Casimirs is seven. In the autonomous limit, these Casimirs reduce to $Z_{\zeta,\zeta'} = \zeta\cdot\zeta'$, where ζ,ζ' correspond to zig-zags $\{\alpha,\beta,\gamma,\delta\}$, expressed via the edge variables. The transformation, for example, $\mathrm{s}_{\beta_1,\beta_2}$ acts by

$$\mathrm{s}_{\beta_1,\beta_2}: \quad \begin{aligned} &Z_{\beta_1,\alpha_1} \mapsto Z_{\alpha_1,\beta_2}, \quad Z_{\beta_2,\alpha_2} \mapsto \frac{Z_{\beta_2,\alpha_2}Z_{\beta_1,\alpha_1}}{Z_{\alpha_1,\beta_2}}, \quad Z_{\beta_3,\alpha_3} \mapsto Z_{\beta_3,\alpha_3}, \\ &Z_{\alpha_1,\beta_2} \mapsto Z_{\beta_1,\alpha_1}, \quad Z_{\alpha_2,\beta_3} \mapsto Z_{\alpha_2,\beta_3}, \quad Z_{\alpha_3,\beta_1} \mapsto \frac{Z_{\alpha_3,\beta_1}Z_{\alpha_1,\beta_2}}{Z_{\beta_1,\alpha_1}}. \end{aligned} \tag{2.122}$$

and substituting here $Z_{\zeta,\zeta'} = \zeta\cdot\zeta'$ one finds that the action of $\mathrm{s}_{\beta_1,\beta_2}$ reduces just to permutation of β_1 and β_2, the same is true for the other generators $\mathrm{s}_{\zeta_1,\zeta_2}$.

For $q\neq 1$ consider the generators $\mathrm{T}_{\beta_i,\beta_{i+1}}$ (2.112) which act trivially on C at all in the autonomous limit. One gets now

$$\mathrm{T}_{\beta_1,\beta_2}: \quad \begin{aligned} &Z_{\beta_1,\alpha_1} \mapsto q^{-1}Z_{\beta_1,\alpha_1}, \quad Z_{\beta_2,\alpha_2} \mapsto qZ_{\beta_2,\alpha_2} \\ &Z_{\alpha_1,\beta_2} \mapsto qZ_{\alpha_1,\beta_2}, \quad Z_{\alpha_3,\beta_1} \mapsto q^{-1}Z_{\alpha_3,\beta_1} \end{aligned} \tag{2.123}$$

where $q = \prod_{i,j} x^\times_{ij}x^+_{ij}$. Again, after expressing the Casimirs via zig-zags, the action of $\mathrm{T}_{\beta_1,\beta_2}$ is equivalent to $\beta_1 \mapsto q^{-1}\beta_1,\ \beta_2 \mapsto q\beta_2$. These formulas suggest that at $q\neq 1$ one can express generators of C via zig-zags and q by [16]

$$\begin{aligned} &Z_{\beta_1,\alpha_1} = q^{\frac16}\beta_1\alpha_1, \quad Z_{\beta_2,\alpha_2} = q^{\frac16}\beta_2\alpha_2, \quad Z_{\beta_3,\alpha_3} = q^{\frac16}\beta_3\alpha_3, \\ &Z_{\alpha_1,\beta_2} = q^{-\frac16}\alpha_1\beta_2, \quad Z_{\alpha_2,\beta_3} = q^{-\frac16}\alpha_2\beta_3, \quad Z_{\alpha_3,\beta_1} = q^{-\frac16}\alpha_3\beta_1 \end{aligned} \tag{2.124}$$

[16]The fractional powers of q in these formulas can be restored using the 'magnetic flux' interpretation for $q\neq 1$ in non-autonomous case. This interpretation is also consistent with the fact that zig-zags with the different orientations collect fluxes of different signs.

$$Z_{\gamma_1,\delta_1} = q^{-\frac{1}{4}}\gamma_1\delta_1, \quad Z_{\delta_1,\gamma_2} = q^{\frac{1}{4}}\delta_1\gamma_2, \quad Z_{\gamma_2,\delta_2} = q^{-\frac{1}{4}}\gamma_2\delta_2, \quad Z_{\delta_2,\gamma_1} = q^{\frac{1}{4}}\delta_2\gamma_1 \tag{2.125}$$

which are consistent with constraints (2.121) with $q \neq 1$ if one assumed[17] $\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\gamma_1\gamma_2\delta_1\delta_2 = \hat{q} = 1$. Comparison of transformation (2.122) with (2.124) and (2.125) leads to the formulas (2.117) for $(N, M) = (3, 2)$. The action of remaining generators is defined by

$$\begin{array}{lllll}
\Lambda_h: & \alpha_i \mapsto \alpha_i, & \beta_i \mapsto \beta_i, & \gamma_a \to \gamma_{a+1}, & \delta_a \mapsto \delta_{a+1}, \\
\Lambda_v: & \alpha_i \mapsto \alpha_{i+1}, & \beta_i \to \beta_{i+1}, & \gamma_a \mapsto \gamma_a, & \delta_a \mapsto \delta_a, \\
\rho: & \alpha_i \mapsto q^{-\frac{1}{N}}\alpha_{i-1}, & \beta_i \mapsto \beta_i, & \gamma_a \to q^{\frac{1}{M}}\gamma_{a+1}, & \delta_a \mapsto \delta_a.
\end{array} \tag{2.128}$$

Remark 2.4.1. Specialities of $N = 2$ or $M = 2$ case.
It is well known (see e.g. [62], eq.(3.70)) that spectral curves with a Newton polygon being $2 \times N$ rectangle can be mapped to the 'triangle ones' with the catheti of lengths 2 and $2N$ (see Fig. 2.10) just by change of variables. Namely, equation

$$S(\lambda, \mu) = P_N^+(\mu)\lambda^2 + P_N(\mu)\lambda + P_N^-(\mu) = 0 \tag{2.129}$$

under $\lambda \mapsto P_N^-(\mu) \cdot \lambda^{-1}$ than $S(\lambda, \mu) \mapsto \lambda^2 P_N^-(\mu)^{-1} S(\lambda, \mu)$ turns into

$$S(\lambda, \mu) = \lambda^2 + P_N(\mu)\lambda + P_N^+(\mu)P_N^-(\mu) = 0. \tag{2.130}$$

For a corresponding cluster integrable system the Poisson quiver from Fig. 2.5 can be transformed into the form drawn at Fig. 2.11 – more common for 'triangular' polygons [18], studied in detail in [153]. This corre-

[17] One can incorporate $\hat{q} \neq 1$ consistently modifying formulas (2.119) and (2.120) by

$$\begin{gathered}
Z_{\beta_1,\alpha_1} = \hat{q}^{\frac{1}{5}} x_{11}^\times x_{12}^\times, \quad Z_{\beta_2,\alpha_2} = \hat{q}^{\frac{1}{5}} x_{21}^\times x_{22}^\times, \quad Z_{\beta_3,\alpha_3} = \hat{q}^{\frac{1}{5}} x_{31}^\times x_{32}^\times, \\
Z_{\alpha_1,\beta_2} = \hat{q}^{\frac{1}{5}} (x_{11}^+ x_{12}^+)^{-1}, \quad Z_{\alpha_2,\beta_3} = \hat{q}^{\frac{1}{5}} (x_{21}^+ x_{22}^+)^{-1}, \quad Z_{\alpha_3,\beta_1} = \hat{q}^{\frac{1}{5}} (x_{31}^+ x_{32}^+)^{-1}
\end{gathered} \tag{2.126}$$

$$\begin{gathered}
Z_{\gamma_1,\delta_1} = \hat{q}^{\frac{1}{5}} (x_{11}^\times x_{21}^\times x_{31}^\times)^{-1}, \quad Z_{\delta_1,\gamma_2} = \hat{q}^{\frac{1}{5}} x_{12}^+ x_{22}^+ x_{32}^+, \\
Z_{\gamma_2,\delta_2} = \hat{q}^{\frac{1}{5}} (x_{12}^\times x_{22}^\times x_{32}^\times)^{-1}, \quad Z_{\delta_2,\gamma_1} = \hat{q}^{\frac{1}{5}} x_{11}^+ x_{21}^+ x_{31}^+
\end{gathered} \tag{2.127}$$

However, as a meaning of this extension is not clear, we will assume $\hat{q} = 1$ in the following.

[18] For generic triangular Newton polygon each node of quiver is connected to six arrows (and corresponding dimer lattice is hexagonal). However, in $2 \times 2N$ case a partial cancelation happens: the arrows directed from $\mathbf{x}_{i1}^\times$ to $\mathbf{x}_{i2}^\times$ annihilate the arrows from $\mathbf{x}_{i2}^\times$ to $\mathbf{x}_{i1}^\times$, and the same happens with $\mathbf{x}_{i1}^+$ and $\mathbf{x}_{i2}^+$, so only four arrows at each node remain.

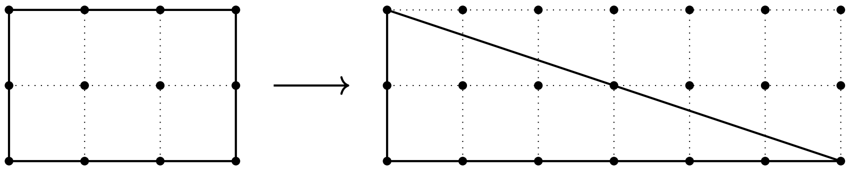

Figure 2.10. Transformation from rectangle to triangle for $(3,2)$ case.

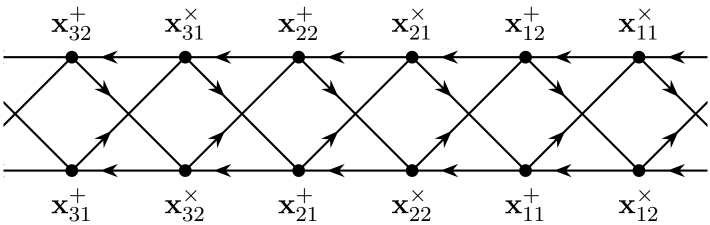

Figure 2.11. Quiver for $(3,2)$ case represented in 'triangular' form.

spondence results in the 'enhancement' of the symmetry group [19]: a pair of commuting Weyl groups $A_{N-1}^{(1)} \times A_{N-1}^{(1)}$ is now embedded into larger group $A_{2N-1}^{(1)}$ with the generators

$$\mathrm{s}_{\alpha_i\beta_{i+1}} = s_{i1,i2}^{\lambda_b,\lambda_b}\ \mu_{i1}^{\lambda_b}\mu_{i2}^{\lambda_b}, \quad \mathrm{s}_{\beta_i\alpha_i} = s_{i1,i2}^{\lambda_a,\lambda_a}\ \mu_{i1}^{\lambda_a}\mu_{i2}^{\lambda_a}, \qquad i = 1, \dots, N \tag{2.131}$$

Embedding $A_{N-1}^{(1)} \times A_{N-1}^{(1)} \to A_{2N-1}^{(1)}$ is provided by

$$\mathrm{s}_{\beta_i,\beta_{i+1}} = \mathrm{s}_{\beta_i\alpha_i}\mathrm{s}_{\alpha_i\beta_{i+1}}\mathrm{s}_{\beta_i\alpha_i}, \quad \mathrm{s}_{\alpha_i,\alpha_{i+1}} = \mathrm{s}_{\alpha_i\beta_{i+1}}\mathrm{s}_{\beta_{i+1}\alpha_{i+1}}\mathrm{s}_{\alpha_i\beta_{i+1}} \tag{2.132}$$

and commutativity of $\mathrm{s}_{\alpha_i,\alpha_{i+1}}$ and $\mathrm{s}_{\beta_i,\beta_{i+1}}$ just follows form the relations on 'elementary' generators $\mathrm{s}_{\beta_i\alpha_i}$, $\mathrm{s}_{\alpha_i\beta_{i+1}}$. The generators of $A_{2N-1}^{(1)}$ also commute with $\mathrm{s}_{\delta_i,\delta_{i+1}}$, $\mathrm{s}_{\gamma_i,\gamma_{i+1}}$. The generator ρ is also absorbed. Now it is not a primitive one, but can be presented as a composition

$$\rho = \Lambda_h\tilde{\Lambda}_v \prod_{i=1}^{N} \mathrm{s}_{\alpha_i,\beta_{i+1}} \tag{2.133}$$

[19] We are grateful to Y.Yamada for clarification of this point.

where we used 'root' from Λ_v

$$\tilde{\Lambda}_v : \; x^{\times}_{ia} \mapsto x^{+}_{i-1,a}, \quad x^{+}_{ia} \mapsto x^{\times}_{i,a}, \quad \text{so} \quad \Lambda_v = (\tilde{\Lambda}_v)^2 \tag{2.134}$$

so there are no extra 'dimensions' in the lattice of the flows.

The action of the enhanced group on Casimirs can be constructed in a way similar to generic case. For example, for the generator s_{α_1,β_2} in $(N,M) = (3,2)$ case from

$$s_{\alpha_1,\beta_2} : \begin{array}{ll} Z_{\beta_1,\alpha_1} \mapsto \dfrac{Z_{\beta_1,\alpha_1}}{Z_{\alpha_1,\beta_2}} & Z_{\alpha_1,\beta_2} \mapsto \dfrac{1}{Z_{\alpha_1,\beta_2}}, \\ Z_{\beta_2,\alpha_2} \mapsto \dfrac{Z_{\beta_2,\alpha_2}}{Z_{\alpha_1,\beta_2}} & Z_{\gamma_1,\delta_1} \mapsto Z_{\alpha_1,\beta_2} Z_{\gamma_1,\delta_1}, \\ Z_{\delta_1,\gamma_2} \mapsto Z_{\alpha_1,\beta_2} Z_{\delta_1,\gamma_2}, & Z_{\gamma_2,\delta_2} \mapsto Z_{\alpha_1,\beta_2} Z_{\gamma_2,\delta_2}, \\ Z_{\delta_2,\gamma_1} \mapsto Z_{\alpha_1,\beta_2} Z_{\delta_2,\gamma_1} & \end{array} \tag{2.135}$$

one gets for the zig-zags

$$s_{\alpha_1,\beta_2} : \quad \alpha_1 \mapsto q^{\frac{1}{6}} \beta_2^{-1}, \quad \beta_2 \mapsto q^{\frac{1}{6}} \alpha_1^{-1}, \quad \gamma_a \delta_a \mapsto q^{-\frac{1}{6}} \alpha_1 \beta_2 \gamma_a \delta_a. \tag{2.136}$$

which contains now 'inversion' of zig-zag, since α_i and β_i correspond to the opposite classes in $H_1(\mathbb{T}^2, \mathbb{Z})$. Generally, for the action of $A_5^{(1)}$ on zig-zags one gets

$$\begin{aligned} s_{\alpha_i\beta_{i+1}} : \quad & \alpha_i \mapsto q^{\frac{1}{6}} \beta_{i+1}^{-1}, \quad \beta_{i+1} \mapsto q^{\frac{1}{6}} \alpha_i^{-1}, \quad \gamma_a \delta_a \mapsto q^{-\frac{1}{6}} \alpha_i \beta_{i+1} \gamma_a \delta_a \\ s_{\beta_i\alpha_i} : \quad & \alpha_i \mapsto q^{-\frac{1}{6}} \beta_i^{-1}, \quad \beta_i \mapsto q^{-\frac{1}{6}} \alpha_i^{-1}, \quad \gamma_a \delta_a \mapsto q^{\frac{1}{6}} \alpha_i \beta_i \gamma_a \delta_a. \end{aligned} \tag{2.137}$$

Remark 2.4.2. Further enhancement for $N = M = 2$ 'small square'.
The group $\mathcal{G}_\mathcal{Q}$ for this case can be identified with the q-Painlevé VI symmetry group $W(D_5^{(1)})$ (see e.g. [14]). It corresponds naively to the 'double' symmetry enhancement

$$A^{(1)}_{1,\alpha} \times A^{(1)}_{1,\beta} \to A^{(1)}_{3,\alpha,\beta}, \quad A^{(1)}_{1,\gamma} \times A^{(1)}_{1,\delta} \to A^{(1)}_{3,\gamma,\delta}. \tag{2.138}$$

but it turns out moreover that generators of the 'new' extended groups do not commute. For example the generators $s_{\alpha_1\beta_2}$ and $s_{\delta_1\gamma_2}$ satisfy

$$(s_{\alpha_1,\beta_2} s_{\gamma_1,\delta_1})^3 = 1 \tag{2.139}$$

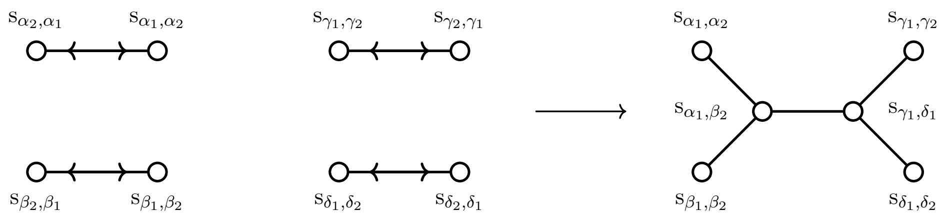

Figure 2.12. Symmetry enhancement from $W\left(A_1^{(1)} \times A_1^{(1)} \times A_1^{(1)} \times A_1^{(1)}\right)$ to $W(D_5^{(1)})$.

and this non-commutativity results in gluing of Dynkin quivers as shown on Fig. 2.12.

Another cluster realization of $W\left(D_5^{(1)}\right)$ has been proposed in [14], given by generators

$$\begin{array}{lll} s_0 = s_{11,22}^{\lambda_b,\lambda_b}, & s_1 = s_{12,21}^{\lambda_b,\lambda_b}, & s_2 = s_{11,12}^{\lambda_b,\lambda_b}\mu_{11}^{\lambda_b}\mu_{12}^{\lambda_b} \\ s_5 = s_{21,12}^{\lambda_a,\lambda_a}, & s_4 = s_{11,22}^{\lambda_a,\lambda_a}, & s_3 = s_{11,21}^{\lambda_a,\lambda_a}\mu_{11}^{\lambda_a}\mu_{21}^{\lambda_a} \end{array} . \tag{2.140}$$

in terms of mutations of the same bipartite graph. In our notation this generators are

$$\begin{array}{lll} s_0 = s_{\alpha_1\beta_2}s_{\delta_1\gamma_1}s_{\gamma_1\gamma_2}s_{\delta_1\gamma_1}s_{\alpha_1\beta_2}, & s_1 = s_{\alpha_1\beta_2}s_{\delta_1\gamma_1}s_{\delta_1\delta_2}s_{\delta_1\gamma_1}s_{\alpha_1\beta_2}, & s_2 = s_{\alpha_1\beta_2} \\ s_5 = s_{\gamma_1\delta_1}s_{\alpha_1\beta_2}s_{\beta_1\beta_2}s_{\alpha_1\beta_2}s_{\gamma_1\delta_1}, & s_4 = s_{\gamma_1\delta_1}s_{\alpha_1\beta_2}s_{\alpha_1\alpha_2}s_{\alpha_1\beta_2}s_{\gamma_1\delta_1}, & s_3 = s_{\gamma_1\delta_1}. \end{array} \tag{2.141}$$

Two presentations can be mapped one to another by conjugation by $s_{\alpha_1\beta_2}s_{\gamma_1\delta_1}s_{\alpha_1\beta_2}$.

2.4.3 Towards bilinear equations

Let us finally turn to the issue of bilinear equations for the cluster tau-functions or A-cluster variables. We postpone rigorous discussion of this issue for a separate publication, but demonstrate here, how Hirota bilinear equations can arise in the systems, corresponding to rectangle Newton polygons.

The simplest example of bilinear equations is provided by spider moves, or mutations in a four-valent vertex of the Poisson quiver, see also Fig. 1.5

in Appendix for the transformation of corresponding piece of a bipartite graph. Such transformation induce the only change in τ-variables, which (for all unit coefficients)

$$\tau_0 \mapsto \bar{\tau}_0 = \frac{\tau_1\tau_3 + \tau_2\tau_4}{\tau_0} \quad \text{or} \quad \tau_0\bar{\tau}_0 = \tau_1\tau_3 + \tau_2\tau_4. \tag{2.142}$$

obviously leads to bilinear equation. However, there is no *a priori* reason to get bilinear equations from generic action by an element of $\mathcal{G}_{\mathcal{Q}}$. For example, a single mutation in a six-valent vertex rather leads to relation, which symbolically has form

$$\tau\bar{\tau} = \tau^3 + \tau^3 \tag{2.143}$$

instead of bilinear. Sometimes one can get nevertheless a bilinear relation for a sequence of mutations without no *a priori* reason for them to hold, see e.g. Section. 2.8 of [15]. We are going to show in this section that the same happens for the transformations, induced by the zig-zag permutations (e.g. $\{\mathrm{s}_{\beta_i,\beta_{i+1}}\}$ or $\{\mathrm{s}_{\gamma_a,\gamma_{a+1}}\}$), constructing their explicit action on tau-variables.

For A-cluster algebras[20] the role of Casimir functions is played by 'coefficients' [59], taking values in some tropical semi-field $\mathbb{P}$, see also discussion in [15]. For the case of rectangle Newton polygons we label the generators of $\mathbb{P}$ by zig-zags (together with q), i.e.

$$\mathbb{P} = \mathrm{Trop}(q, \{\mathbf{u}_{\alpha_i}, \mathbf{u}_{\beta_i}\}_{i=1,\dots,N}, \{\mathbf{u}_{\gamma_a}, \mathbf{u}_{\delta_a}\}_{i=1,\dots,M}). \tag{2.144}$$

so that the coefficients are expressed by

$$\mathbf{y}^{\times}_{ia} = q^{\frac{1}{NM}} \frac{(\mathbf{u}_{\alpha_i}\mathbf{u}_{\beta_i})^{\frac{1}{M}}}{(\mathbf{u}_{\gamma_a}\mathbf{u}_{\delta_a})^{\frac{1}{N}}}, \quad \mathbf{y}^{+}_{ia} = q^{\frac{1}{NM}} \frac{(\mathbf{u}_{\gamma_a}\mathbf{u}_{\delta_{a-1}})^{\frac{1}{N}}}{(\mathbf{u}_{\alpha_i}\mathbf{u}_{\beta_{i+1}})^{\frac{1}{M}}}. \tag{2.145}$$

The action of transformations $\mathrm{s}_{\zeta_i,\zeta_{i+1}}$ on coefficients in this basis is equivalent to the action on generators of $\mathbb{P}$ like in (2.117) on zig-zags, i.e.

$$\begin{aligned}
\mathrm{s}_{\alpha_i,\alpha_{i+1}}: &\quad \mathbf{u}_{\alpha_i} \mapsto q^{\frac{1}{N}}\mathbf{u}_{\alpha_{i+1}}, &\quad \mathbf{u}_{\alpha_{i+1}} \mapsto q^{-\frac{1}{N}}\mathbf{u}_{\alpha_i},\\
\mathrm{s}_{\beta_i,\beta_{i+1}}: &\quad \mathbf{u}_{\beta_i} \mapsto q^{-\frac{1}{N}}\mathbf{u}_{\beta_{i+1}}, &\quad \mathbf{u}_{\beta_{i+1}} \mapsto q^{\frac{1}{N}}\mathbf{u}_{\beta_i},\\
\mathrm{s}_{\gamma_a,\gamma_{a+1}}: &\quad \mathbf{u}_{\gamma_a} \mapsto q^{\frac{1}{M}}\mathbf{u}_{\gamma_{a+1}}, &\quad \mathbf{u}_{\gamma_{a+1}} \mapsto q^{-\frac{1}{M}}\mathbf{u}_{\gamma_a},\\
\mathrm{s}_{\delta_a,\delta_{a+1}}: &\quad \mathbf{u}_{\delta_a} \mapsto q^{-\frac{1}{M}}\mathbf{u}_{\delta_{a+1}}, &\quad \mathbf{u}_{\delta_{a+1}} \mapsto q^{\frac{1}{M}}\mathbf{u}_{\delta_a}.
\end{aligned} \tag{2.146}$$

[20]For the definition of A-cluster algebra with coefficients and transition from $\mathcal{X}$ to A-cluster algebra see Section 1.4.3.

Coefficients could be encoded by 'frozen' vertices of quiver. This suggests principle that we assign frozen variables to faces of dual surface, corresponding to zig-zag variables, while mutable variables - to faces of original torus.

Let us now present an example of the action of the generator $\mathrm{s}_{\beta_1,\beta_2}$ on τ-variables in $(N,M)=(3,2)$ case. An explicit computation gives

$$\begin{pmatrix} \frac{\bar{\tau}^+_{11}}{\tau^+_{11}} \\ \frac{\bar{\tau}^\times_{11}}{\tau^\times_{11}} \\ \frac{\bar{\tau}^+_{12}}{\tau^+_{12}} \\ \frac{\bar{\tau}^\times_{12}}{\tau^\times_{12}} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_{\beta_2}^{\frac{1}{2}} & q^{\frac{1}{12}}(\mathbf{u}_{\beta_1}\mathbf{u}_{\beta_2})^{\frac{1}{2}} & q^{\frac{2}{12}}\mathbf{u}_{\beta_1}^{\frac{1}{2}} & q^{\frac{3}{12}}\mathbf{u}_{\beta_1} \\ q^{\frac{3}{12}}\mathbf{u}_{\beta_1}^{\frac{1}{2}} & \mathbf{u}_{\beta_2} & q^{\frac{1}{12}}\mathbf{u}_{\beta_2}^{\frac{1}{2}} & q^{\frac{2}{12}}(\mathbf{u}_{\beta_1}\mathbf{u}_{\beta_2})^{\frac{1}{2}} \\ q^{\frac{2}{12}}\mathbf{u}_{\beta_1}^{\frac{1}{2}} & q^{\frac{3}{12}}\mathbf{u}_{\beta_1} & \mathbf{u}_{\beta_2}^{\frac{1}{2}} & q^{\frac{1}{12}}(\mathbf{u}_{\beta_1}\mathbf{u}_{\beta_2})^{\frac{1}{2}} \\ q^{\frac{1}{12}}\mathbf{u}_{\beta_2}^{\frac{1}{2}} & q^{\frac{2}{12}}(\mathbf{u}_{\beta_1}\mathbf{u}_{\beta_2})^{\frac{1}{2}} & q^{\frac{3}{12}}\mathbf{u}_{\beta_1}^{\frac{1}{2}} & \mathbf{u}_{\beta_2} \end{pmatrix} \cdot C \cdot \begin{pmatrix} \frac{\tau^+_{31}\tau^\times_{21}}{\tau^+_{11}\tau^\times_{11}} \\ \frac{\tau^+_{32}\tau^\times_{21}}{\tau^+_{12}\tau^\times_{11}} \\ \frac{\tau^+_{32}\tau^\times_{22}}{\tau^+_{12}\tau^\times_{12}} \\ \frac{\tau^+_{31}\tau^\times_{22}}{\tau^+_{11}\tau^\times_{12}} \end{pmatrix} \tag{2.147}$$

where $C = \mathrm{diag}\left((\mathbf{u}_{\gamma_1}\mathbf{u}_\delta)^{\frac{1}{3}}, \mathbf{u}_{\alpha_1}^{\frac{1}{2}}(\mathbf{u}_\delta/\mathbf{u}_{\delta_1})^{\frac{1}{3}}, (\mathbf{u}_{\gamma_2}\mathbf{u}_\delta)^{\frac{1}{3}}, \mathbf{u}_{\alpha_1}^{\frac{1}{2}}(\mathbf{u}_\delta/\mathbf{u}_{\delta_2})^{\frac{1}{3}}\right)$, $\mathbf{u}_\delta = \mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}$. The main point is that the matrix in the r.h.s. is nicely invertible so that these equations can be rewritten in *bilinear form*

$$\begin{cases} (\mathbf{u}_{\beta_2} - q^{\frac{1}{3}}\mathbf{u}_{\beta_1})(\mathbf{u}_\delta\mathbf{u}_{\gamma_1})^{\frac{1}{3}}\tau^+_{31}\tau^\times_{21} = \mathbf{u}_{\beta_2}^{\frac{1}{2}}\bar{\tau}^+_{11}\tau^\times_{11} - q^{\frac{1}{12}}\mathbf{u}_{\beta_1}^{\frac{1}{2}}\bar{\tau}^\times_{11}\tau^+_{11} \\ (\mathbf{u}_{\beta_2} - q^{\frac{1}{3}}\mathbf{u}_{\beta_1})(\mathbf{u}_\delta/\mathbf{u}_{\delta_1})^{\frac{1}{3}}\tau^+_{32}\tau^\times_{21} = \mathbf{u}_{\alpha_1}^{-\frac{1}{2}}\bar{\tau}^\times_{11}\tau^+_{12} - q^{\frac{1}{12}}\mathbf{u}_{\alpha_1}^{-\frac{1}{2}}\bar{\tau}^+_{12}\tau^\times_{11} \\ (\mathbf{u}_{\beta_2} - q^{\frac{1}{3}}\mathbf{u}_{\beta_1})(\mathbf{u}_\delta\mathbf{u}_{\gamma_2})^{\frac{1}{3}}\tau^+_{32}\tau^\times_{22} = \mathbf{u}_{\beta_2}^{\frac{1}{2}}\bar{\tau}^+_{12}\tau^\times_{12} - q^{\frac{1}{12}}\mathbf{u}_{\beta_1}^{\frac{1}{2}}\bar{\tau}^\times_{12}\tau^+_{12} \\ (\mathbf{u}_{\beta_2} - q^{\frac{1}{3}}\mathbf{u}_{\beta_1})(\mathbf{u}_\delta/\mathbf{u}_{\delta_2})^{\frac{1}{3}}\tau^+_{31}\tau^\times_{22} = \mathbf{u}_{\alpha_1}^{-\frac{1}{2}}\bar{\tau}^\times_{12}\tau^+_{11} - q^{\frac{1}{12}}\mathbf{u}_{\alpha_1}^{-\frac{1}{2}}\bar{\tau}^+_{11}\tau^\times_{12} \end{cases} . \tag{2.148}$$

This is actually a generic phenomenon for the zig-zag generators: the same happens, for example, for the generator $\mathrm{s}_{\delta_1,\delta_2}$ from another component of $\mathcal{G}_\mathcal{Q}$. One gets explicitly for the transformation of A-cluster variables

$$t_1 = C_1 \cdot C_2 \cdot t_2, \tag{2.149}$$

where

$$
\begin{aligned}
t_1 &= \begin{pmatrix} \frac{\bar\tau^{+}_{32}}{\tau^{+}_{32}} & \frac{\bar\tau^{\times}_{32}}{\tau^{\times}_{32}} & \frac{\bar\tau^{+}_{22}}{\tau^{+}_{22}} & \frac{\bar\tau^{\times}_{22}}{\tau^{\times}_{22}} & \frac{\bar\tau^{+}_{12}}{\tau^{+}_{12}} & \frac{\bar\tau^{\times}_{12}}{\tau^{\times}_{12}} \end{pmatrix}^T \\
t_2 &= \begin{pmatrix} \frac{\tau^{+}_{31}\tau^{\times}_{31}}{\tau^{+}_{32}\tau^{\times}_{32}} & \frac{\tau^{+}_{21}\tau^{\times}_{31}}{\tau^{+}_{22}\tau^{\times}_{32}} & \frac{\tau^{+}_{21}\tau^{\times}_{21}}{\tau^{+}_{22}\tau^{\times}_{22}} & \frac{\tau^{+}_{11}\tau^{\times}_{21}}{\tau^{+}_{12}\tau^{\times}_{22}} & \frac{\tau^{+}_{11}\tau^{\times}_{11}}{\tau^{+}_{12}\tau^{\times}_{12}} & \frac{\tau^{+}_{31}\tau^{\times}_{11}}{\tau^{+}_{32}\tau^{\times}_{12}} \end{pmatrix}^T
\end{aligned}
\tag{2.150}
$$

$$
C_1 = \begin{pmatrix}
\mathbf{u}_{\delta_2} & q^{\frac{1}{12}}\mathbf{u}_{\delta_2}^{\frac{2}{3}} & q^{\frac{2}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} & q^{\frac{3}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{4}{12}}(\mathbf{u}_{\delta_1}^2\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{5}{12}}\mathbf{u}_{\delta_1}^{\frac{2}{3}} \\
q^{\frac{5}{12}}\mathbf{u}_{\delta_1} & \mathbf{u}_{\delta_2}^{\frac{2}{3}} & q^{\frac{1}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} & q^{\frac{2}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{3}{12}}(\mathbf{u}_{\delta_1}^2\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{4}{12}}\mathbf{u}_{\delta_1}^{\frac{2}{3}} \\
q^{\frac{4}{12}}(\mathbf{u}_{\delta_1}^2\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{5}{12}}\mathbf{u}_{\delta_1}^{\frac{2}{3}} & \mathbf{u}_{\delta_2} & q^{\frac{1}{12}}\mathbf{u}_{\delta_2}^{\frac{2}{3}} & q^{\frac{2}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} & q^{\frac{3}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} \\
q^{\frac{3}{12}}(\mathbf{u}_{\delta_1}^2\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{4}{12}}\mathbf{u}_{\delta_1}^{\frac{2}{3}} & q^{\frac{5}{12}}\mathbf{u}_{\delta_1} & \mathbf{u}_{\delta_2}^{\frac{2}{3}} & q^{\frac{1}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} & q^{\frac{2}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2})^{\frac{1}{3}} \\
q^{\frac{2}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} & q^{\frac{3}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{4}{12}}(\mathbf{u}_{\delta_1}^2\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{5}{12}}\mathbf{u}_{\delta_1}^{\frac{2}{3}} & \mathbf{u}_{\delta_2} & q^{\frac{1}{12}}\mathbf{u}_{\delta_2}^{\frac{2}{3}} \\
q^{\frac{1}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2}^2)^{\frac{1}{3}} & q^{\frac{2}{12}}(\mathbf{u}_{\delta_1}\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{3}{12}}(\mathbf{u}_{\delta_1}^2\mathbf{u}_{\delta_2})^{\frac{1}{3}} & q^{\frac{4}{12}}\mathbf{u}_{\delta_1}^{\frac{2}{3}} & q^{\frac{5}{12}}\mathbf{u}_{\delta_1} & \mathbf{u}_{\delta_2}^{\frac{2}{3}}
\end{pmatrix}
\tag{2.151}
$$

$$
C_2 = \mathrm{diag}\left((\mathbf{u}_\alpha/\mathbf{u}_{\alpha_3})^{\frac{1}{2}}\mathbf{u}_{\gamma_2}^{\frac{1}{3}}, (\mathbf{u}_\alpha\mathbf{u}_{\beta_3})^{\frac{1}{2}}, (\mathbf{u}_\alpha/\mathbf{u}_{\alpha_2})^{\frac{1}{2}}\mathbf{u}_{\gamma_2}^{\frac{1}{3}}, (\mathbf{u}_\alpha\mathbf{u}_{\beta_2})^{\frac{1}{2}}, (\mathbf{u}_\alpha/\mathbf{u}_{\alpha_1})^{\frac{1}{2}}\mathbf{u}_{\gamma_2}^{\frac{1}{3}}, (\mathbf{u}_\alpha\mathbf{u}_{\beta_1})^{\frac{1}{2}}\right)
\tag{2.152}
$$

with $\mathbf{u}_\alpha = \mathbf{u}_{\alpha_1}\mathbf{u}_{\alpha_2}\mathbf{u}_{\alpha_3}$. Again, inverting matrix C_1 we end up with the set of bilinear equations

$$
\begin{cases}
(\mathbf{u}_{\delta_2} - q^{\frac{1}{2}}\mathbf{u}_{\delta_1})(\mathbf{u}_\alpha/\mathbf{u}_{\alpha_3})^{\frac{1}{2}}\tau^{+}_{31}\tau^{\times}_{31} = \mathbf{u}_{\gamma_2}^{-\frac{1}{3}}\bar\tau^{+}_{32}\tau^{\times}_{32} - q^{\frac{1}{12}}\mathbf{u}_{\gamma_2}^{-\frac{1}{3}}\bar\tau^{\times}_{32}\tau^{+}_{32} \\
(\mathbf{u}_{\delta_2} - q^{\frac{1}{2}}\mathbf{u}_{\delta_1})(\mathbf{u}_\alpha\mathbf{u}_{\beta_3})^{\frac{1}{2}}\tau^{+}_{21}\tau^{\times}_{31} = \mathbf{u}_{\delta_2}^{\frac{1}{3}}\bar\tau^{\times}_{32}\tau^{+}_{22} - q^{\frac{1}{12}}\mathbf{u}_{\delta_1}^{\frac{1}{3}}\bar\tau^{+}_{22}\tau^{\times}_{32} \\
(\mathbf{u}_{\delta_2} - q^{\frac{1}{2}}\mathbf{u}_{\delta_1})(\mathbf{u}_\alpha/\mathbf{u}_{\alpha_2})^{\frac{1}{2}}\tau^{+}_{21}\tau^{\times}_{21} = \mathbf{u}_{\gamma_2}^{-\frac{1}{3}}\bar\tau^{+}_{22}\tau^{\times}_{22} - q^{\frac{1}{12}}\mathbf{u}_{\gamma_2}^{-\frac{1}{3}}\bar\tau^{\times}_{22}\tau^{+}_{22} \\
(\mathbf{u}_{\delta_2} - q^{\frac{1}{2}}\mathbf{u}_{\delta_1})(\mathbf{u}_\alpha\mathbf{u}_{\beta_2})^{\frac{1}{2}}\tau^{+}_{11}\tau^{\times}_{21} = \mathbf{u}_{\delta_2}^{\frac{1}{3}}\bar\tau^{\times}_{22}\tau^{+}_{12} - q^{\frac{1}{12}}\mathbf{u}_{\delta_1}^{\frac{1}{3}}\bar\tau^{+}_{12}\tau^{\times}_{22} \\
(\mathbf{u}_{\delta_2} - q^{\frac{1}{2}}\mathbf{u}_{\delta_1})(\mathbf{u}_\alpha/\mathbf{u}_{\alpha_1})^{\frac{1}{2}}\tau^{+}_{11}\tau^{\times}_{11} = \mathbf{u}_{\gamma_2}^{-\frac{1}{3}}\bar\tau^{+}_{12}\tau^{\times}_{12} - q^{\frac{1}{12}}\mathbf{u}_{\gamma_2}^{-\frac{1}{3}}\bar\tau^{\times}_{12}\tau^{+}_{12} \\
(\mathbf{u}_{\delta_2} - q^{\frac{1}{2}}\mathbf{u}_{\delta_1})(\mathbf{u}_\alpha\mathbf{u}_{\beta_1})^{\frac{1}{2}}\tau^{+}_{31}\tau^{\times}_{11} = \mathbf{u}_{\delta_2}^{\frac{1}{3}}\bar\tau^{\times}_{12}\tau^{+}_{32} - q^{\frac{1}{12}}\mathbf{u}_{\delta_1}^{\frac{1}{3}}\bar\tau^{+}_{32}\tau^{\times}_{12}
\end{cases}
\tag{2.153}
$$

It remains yet unclear, how to derive bilinear equations systematically for compositions of elements of $\mathcal{G}_\mathcal{Q}$. We are going to return to this issue together with discussion of their solutions elsewhere.

2.5 Conclusion

In this chapter we have presented extra evidence that cluster integrable systems provide convenient framework for the description of 5d supersymmetric Yang-Mills theory. It has been shown that cluster integrable systems with the Newton polygons $SA(2,\mathbb{Z})$-equivalent to the $N \times M$ rectangles are isomorphic to the XXZ-like spin chains of rank M on N sites (or vice versa) on the 'lowest orbit'. Due to special symmetry of the Kasteleyn operators, defining spectral curves of these systems, it turns to be possible to express the Lax operators of spin chain in terms of the X-cluster variables. Inhomogeneities and twists of the chain can be expressed via (part of) the zig-zag paths on the Goncharov-Kenyon bipartite graphs.

Rectangle Newton polygons generally correspond to linear quiver gauge theories [21] so that inhomogeneities, 'on-site' Casimirs and twists define the fundamental and bi-fundamental masses together with the bare couplings on the Yang-Mills side. The proposed cluster description possesses obvious symmetry between the structure in horizontal and vertical directions so that one gets a natural spectral (or fiber-base or length-rank) duality, interchanging also the rank and length of spin chains. Shear shift of one side of a Newton polygon to the shape of $N \times M$ parallelogram results in the multiplication of the monodromy operator of the spin chain by the cyclic twist matrix.

We have found that the cluster mapping class group $\mathcal{G}_\mathcal{Q}$ for the 'spin-chain class' always contains a subgroup isomorphic to

$$\widetilde{W}\left(A^{(1)}_{N-1,\alpha} \times A^{(1)}_{N-1,\beta}\right) \times \widetilde{W}\left(A^{(1)}_{M-1,\gamma} \times A^{(1)}_{M-1,\delta}\right) \rtimes \mathbb{Z} \tag{2.154}$$

whose generators act on zig-zag paths by permutations. Moreover, their action on the A-cluster variables gives rise to the $q-$difference bilinear relations. The symmetry enhancement happens in the case $N = 2$ (or $M = 2$) and results in 'gluing' of two copies of $A^{(1)}_{N-1}$ into $A^{(1)}_{2N-1}$. If both $N = M = 2$ the symmetry $\widetilde{W}\left(A^{(1)}_1 \times A^{(1)}_1\right) \times \widetilde{W}\left(A^{(1)}_1 \times A^{(1)}_1\right) \rtimes \mathbb{Z}$ enhances to the $D^{(1)}_5$ symmetry group of q-PVI equation.

Our first results in this direction actually produce more question than give answers. The following obvious questions (at least!) can be addressed for the further investigations:

- Trivial rank-N spin chain on a single site once twisted becomes spectrally dual to relativistic Toda chain, see Section 2.3.2. Can we similarly identify the spectral duals of the twisted chains of arbitrary lengths and twists, whose Newton polygons are generic parallelograms – or even extend this to generic four-gons? This question is also very interesting on the gauge-theory side, where by now only the hyperelliptic case of 'generalized Toda' (four boundary points and all internal points are lying on one line – pure $SU(N)$ theory with the CS term) was studied in [15].

- We have derived in Section 2.4.3 the bilinear relations, coming out of the action of a single 'permutation' generator of $\mathcal{G}_\mathcal{Q}$ on A-cluster variables, acting by transpositions on zig-zags. Is there any systematic principle to derive bilinear equations for compositions of such transformations?

- In [24], [14], [102], [15], [25] and [136] the solutions for q-difference bilinear equations and their degenerations, arising from certain cluster integrable systems, were found in terms of Fourier-transformed Nekrasov functions for the corresponding 5d gauge theories. As partition functions for the 5d linear quiver gauge theories are well known, a natural further step is to show that they solve the bilinear equations found here (and their hypothetical generalizations!).

2.6 Appendix. Proof of the RLL relation for cluster L-matrices

Here some details of proof of (2.51) are collected. Recall the definitions (2.52) (here and below $i, j = 1, \ldots, M$)

$$L_{ij}(\mu) = \frac{1}{\mu^{\frac{1}{2}} - \mu^{-\frac{1}{2}}} \begin{cases} i = j, & \mu^{\frac{1}{2}} z_i^{-2} + \mu^{-\frac{1}{2}} z_i^2 \\ i \neq j, & \mu^{-\frac{s_{ij}}{2}} (z_j^2 + z_j^{-2}) \dfrac{\tau_j}{\tau_i} \end{cases}, \quad \tau_i = w_i \prod_{k=1}^{M} z_k^{s_{ki}} . \tag{2.155}$$

where the variables z_i, w_i have Poisson brackets

$$\{z_i, w_j\} = \frac{1}{4} \delta_{ij} z_i w_j, \quad \{z_i, z_j\} = \{w_i, w_j\} = 0. \tag{2.156}$$

It is useful to note that

$$\{z_i, \tau_j\} = \frac{1}{4} \delta_{ij} z_i \tau_j, \quad \{\tau_i, \tau_j\} = -\frac{1}{2} s_{ij} \tau_i \tau_j. \tag{2.157}$$

In addition to the sign-factors (1.5) we also introduce [21]

$$s^k_{ij} = \begin{cases} +1, & k \in (ij) \\ -1, & k \in (ji) \\ 0, & k = i, j \end{cases} \tag{2.158}$$

which satisfies

$$s^k_{ij} = -s^k_{ji}, \quad s^k_{ij} = s^i_{jk}, \quad s^k_{ij} = s_{ij} + s_{jk} + s_{ki}. \tag{2.159}$$

From definitions (2.155)

$$z_k^2 = -\frac{L_{kk}(\lambda)\sqrt{\mu} - L_{kk}(\mu)\sqrt{\lambda}}{\sqrt{\lambda/\mu} - \sqrt{\mu/\lambda}}, \quad z_k^{-2} = \frac{L_{kk}(\lambda)/\sqrt{\mu} - L_{kk}(\mu)/\sqrt{\lambda}}{\sqrt{\lambda/\mu} - \sqrt{\mu/\lambda}} \tag{2.160}$$

$$L_{ij}(\lambda)L_{kl}(\mu) = \lambda^{-\frac{1}{2}s_{ij}+\frac{1}{2}s_{kl}}\mu^{\frac{1}{2}s_{ij}-\frac{1}{2}s_{kl}}L_{ij}(\mu)L_{kl}(\lambda), \quad i \neq j, \quad k \neq l.$$

We take an anzatz

$$\tilde{r}(a) = \sum_{k=1}^{M} f_k(a) E_{kk} \otimes E_{kk} + \sum_{m \neq n} g_{mn}(a) E_{mn} \otimes E_{nm} \tag{2.161}$$

and show that one can choose f_k and g_{mn} such that equation

$$\{L(\lambda) \otimes L(\mu)\} = [\tilde{r}(\lambda/\mu), L(\lambda) \otimes L(\mu)] \tag{2.162}$$

holds. By direct computation it can be shown that ($a \neq i \neq j \neq k \neq l$):

	a. $\{L(\lambda) \otimes L(\mu)\}$	*b.* $[\tilde{r}(\lambda/\mu), L(\lambda) \otimes L(\mu)]$
1. $E_{ii} \otimes E_{jj}$	0	0
2. $E_{aa} \otimes E_{ij}$	0	$g_{ai}L_{ia}(\lambda)L_{aj}(\mu) - g_{ja}L_{aj}(\lambda)L_{ia}(\mu)$
3. $E_{aa} \otimes E_{aj}$	$AL_{aa}(\lambda)L_{aj}(\mu) - B_{aj}L_{aj}(\lambda)L_{aa}(\mu)$	$f_aL_{aa}(\lambda)L_{aj}(\mu) - g_{ja}L_{aj}(\lambda)L_{aa}(\mu)$
4. $E_{aa} \otimes E_{ia}$	$-AL_{aa}(\lambda)L_{ia}(\mu) + B_{ia}L_{ia}(\lambda)L_{aa}(\mu)$	$-f_aL_{aa}(\lambda)L_{ia}(\mu) + g_{ai}L_{ia}(\lambda)L_{aa}(\mu)$
5. $E_{ij} \otimes E_{ji}$	$B_{ji}(L_{jj}(\lambda)L_{ii}(\mu) - L_{ii}(\lambda)L_{jj}(\mu))$	$g_{ij}L_{jj}(\lambda)L_{ii}(\mu) - g_{ij}L_{ii}(\lambda)L_{jj}(\mu)$
6. $E_{ij} \otimes E_{kl}$	$\frac{1}{2}(s^k_{ij} + s^l_{ji})L_{ij}(\lambda)L_{kl}(\mu)$	$g_{ik}L_{kj}(\lambda)L_{il}(\mu) - g_{lj}L_{il}(\lambda)L_{kj}(\mu)$
7. $E_{ij} \otimes E_{ia}$	$-\frac{1}{2}s^a_{ij}L_{ij}(\lambda)L_{ia}(\mu)$	$f_iL_{ij}(\lambda)L_{ia}(\mu) - g_{aj}L_{ia}(\lambda)L_{ij}(\mu)$
8. $E_{ij} \otimes E_{aj}$	$\frac{1}{2}s^a_{ij}L_{ij}(\lambda)L_{aj}(\mu)$	$-f_jL_{ij}(\lambda)L_{aj}(\mu) + g_{ia}L_{aj}(\lambda)L_{ij}(\mu)$
9. $E_{ij} \otimes E_{ja}$	$B_{ji}L_{jj}(\lambda)L_{ia}(\mu) - B_{ja}L_{ia}(\lambda)L_{jj}(\mu)$	$g_{ij}L_{jj}(\lambda)L_{ia}(\mu) - g_{aj}L_{ia}(\lambda)L_{jj}(\mu)$
10. $E_{ij} \otimes E_{ai}$	$-B_{ji}L_{ii}(\lambda)L_{aj}(\mu) + B_{ai}L_{aj}(\lambda)L_{ii}(\mu)$	$-g_{ij}L_{ii}(\lambda)L_{aj}(\mu) + g_{ia}L_{aj}(\lambda)L_{ii}(\mu)$

(2.163)

[21] Notation $k \in (ij)$ means that we consider i, j, k on the circle $\mathbb{Z}/M\mathbb{Z}$, with k in the oriented interval from i to j.

with

$$A = A(\sqrt{\lambda/\mu}) = \frac{1}{2}\frac{\sqrt{\lambda/\mu}+\sqrt{\mu/\lambda}}{\sqrt{\lambda/\mu}-\sqrt{\mu/\lambda}}, \quad B_{ij} = B_{ij}(\sqrt{\lambda/\mu}) = \frac{(\lambda/\mu)^{\frac{1}{2}s_{ij}}}{\sqrt{\lambda/\mu}-\sqrt{\mu/\lambda}}. \tag{2.164}$$

Computations in 1,2,7,8.a) are straightforward. In 3, 4, 5.a) relation (2.160) has to be used. 9,10.a) can be obtained by application of (2.160) and (2.159):

$$\{L_{ij}(\lambda), L_{ja}(\mu)\} = \tag{2.165}$$

$$= -\frac{1}{2}\lambda^{-\frac{1}{2}s_{ij}}\mu^{-\frac{1}{2}s_{ja}}\frac{\tau_a}{\tau_i}(s^a_{ij}(z_j^2+z_j^{-2})(z_a^2+z_a^{-2})+(z_j^2-z_j^{-2})(z_a^2+z_a^{-2})) =$$

$$= -\frac{1}{2}\lambda^{-\frac{1}{2}s_{ij}}\mu^{-\frac{1}{2}s_{ja}}(z_a^2+z_a^{-2})\frac{\tau_a}{\tau_i}((s^a_{ij}+1)z_j^2+(s^a_{ij}-1)z_j^{-2})) =$$

$$= -s^a_{ij}\lambda^{-\frac{1}{2}s_{ij}}\mu^{-\frac{1}{2}s_{ja}}(z_a^2+z_a^{-2})\frac{\tau_a}{\tau_i}z_j^{2s^a_{ij}} =$$

$$= \frac{\lambda^{-\frac{1}{2}s_{ij}+\frac{1}{2}s_{ia}}\mu^{-\frac{1}{2}s_{ja}}}{\sqrt{\lambda/\mu}-\sqrt{\mu/\lambda}}L_{ia}(\lambda)\left[L_{jj}(\lambda)\mu^{\frac{1}{2}s^a_{ij}} - L_{jj}(\mu)\lambda^{\frac{1}{2}s^a_{ij}}\right] =$$

$$= \frac{(\lambda/\mu)^{\frac{1}{2}s_{ji}}L_{jj}(\lambda)L_{ia}(\mu) - (\lambda/\mu)^{\frac{1}{2}s_{ja}}L_{ia}(\lambda)L_{jj}(\mu)}{\sqrt{\lambda/\mu}-\sqrt{\mu/\lambda}}$$

Looking at the table (2.163) we can suggest that the last two columns are equal, if we put

$$f_i = A(\sqrt{\lambda/\mu}), \quad g_{ij} = B_{ji}(\sqrt{\lambda/\mu}) \tag{2.166}$$

For 1-5 and 9-10 it is obvious. For 6, 7, 8 it is easier to move from the right to the left. For 6, using (2.159):

$$g_{ik}L_{kj}(\lambda)L_{il}(\mu) - g_{lj}L_{il}(\lambda)L_{kj}(\mu) = \tag{2.167}$$

$$= \frac{\lambda^{-\frac{1}{2}s_{ik}-\frac{1}{2}s_{kj}}\mu^{-\frac{1}{2}s_{ki}-\frac{1}{2}s_{il}} - \lambda^{-\frac{1}{2}s_{lj}-\frac{1}{2}s_{il}}\mu^{-\frac{1}{2}s_{jl}-\frac{1}{2}s_{kj}}}{\sqrt{\lambda/\mu}-\sqrt{\mu/\lambda}}\frac{\tau_j}{\tau_k}\frac{\tau_l}{\tau_i}(z_j^2+z_j^{-2})(z_l^2+z_l^{-2}) =$$

$$= \frac{\lambda^{-\frac{1}{2}s^j_{ik}}\mu^{-\frac{1}{2}s^l_{ki}} - \lambda^{-\frac{1}{2}s^i_{lj}}\mu^{-\frac{1}{2}s^k_{jl}}}{\sqrt{\lambda/\mu}-\sqrt{\mu/\lambda}}L_{ij}(\lambda)L_{kl}(\mu)$$

All possible relative positions of the indices i, j, k, l can be encoded in the table

s^j_{ik}	s^l_{ki}	s^i_{lj}	s^k_{jl}	$s^k_{ij} + s^l_{ji}$
$+1$	$+1$	$+1$	$+1$	0
$+1$	-1	$+1$	-1	0
$+1$	-1	-1	$+1$	-2
-1	$+1$	-1	$+1$	0
-1	$+1$	$+1$	-1	$+2$
-1	-1	-1	-1	0

(2.168)

which shows that 6.a) and 6.b) from (2.163) are equal. For 7.b):

$$f_i L_{ij}(\lambda) L_{ia}(\mu) - g_{aj} L_{ia}(\lambda) L_{ij}(\mu) = \tag{2.169}$$

$$= \frac{1}{2} \frac{(\sqrt{\lambda/\mu} + \sqrt{\mu/\lambda})\lambda^{-\frac{1}{2}s_{ij}}\mu^{-\frac{1}{2}s_{ia}} - 2\lambda^{-\frac{1}{2}s_{aj}-\frac{1}{2}s_{ia}}\mu^{-\frac{1}{2}s_{ja}-\frac{1}{2}s_{ij}}}{\sqrt{\lambda/\mu} - \sqrt{\mu/\lambda}} \cdot$$

$$\cdot \frac{\tau_j}{\tau_i}\frac{\tau_a}{\tau_i}(z_j^2 + z_j^{-2})(z_a^2 + z_a^{-2}) =$$

$$= \frac{1}{2} \frac{\sqrt{\lambda/\mu} + \sqrt{\mu/\lambda} - 2(\lambda/\mu)^{-\frac{1}{2}s^j_{ia}}}{\sqrt{\lambda/\mu} - \sqrt{\mu/\lambda}} L_{ij}(\lambda) L_{ia}(\mu) = -\frac{1}{2} s^a_{ij} L_{ij}(\lambda) L_{ia}(\mu)$$

which is equal to 7.a). Similarly for 8 a) and b). To show that (2.161) is equal to (1.32) multiplied by $\frac{1}{2}$, we have to note that

$$\sum_{k=1}^{M} E_{kk} \otimes E_{kk} = \mathbf{1} \otimes \mathbf{1} - \sum_{i \neq j} E_{ii} \otimes E_{jj} \tag{2.170}$$

and $\mathbf{1} \otimes \mathbf{1}$ is commuting with anything, so can be always added to the r-matrix with the arbitrary coefficient, without any change of the relations.