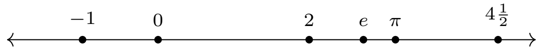
Tautological differential forms on moduli spaces of curves

Lugt, S. van der

Summary

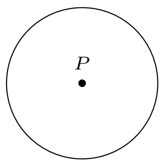
The study of moduli spaces could be viewed as the mathematical analogue of taxonomy in biology. Where a biologist would, for example, try to find all the distinct species of Galapagos finches, a mathematician would be interested in finding all the mathematical objects of a certain type. When making such classifications, one can make use of moduli spaces: geometrical objects whose points correspond one-to-one with the objects we wish to classify.

For instance, take all the numbers and arrange them in ascending order to obtain a line, the *number line*:

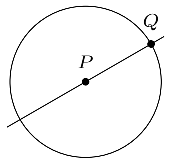


The points on the number line correspond one-to-one with the numbers. Therefore, the number line is a *moduli space* for all numbers.

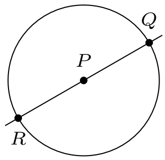
Now, let us classify all lines in the plane that go through a certain point. Suppose that P is a point in the plane, and suppose that we wish to classify all lines in the plane through P. We will therefore look for a moduli space for lines through P. Now draw a circle around P that has P as its center:



If Q is a point on this circle, then we obtain a line through P by drawing a line through P and Q:

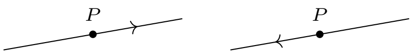


Moreover, every line through P can be obtained in this way. It appears that we have found a moduli space for all lines through P. This circle, however, is *not* a moduli space. Indeed, if it were a moduli space, its points would correspond one-to-one with lines through P. But consider two points Q and R that lie on opposite sides of the circle:

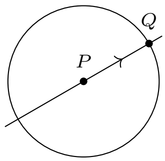


The line through P and Q and the line through P and R are the same! The correspondence between points on the circle and lines through P is 'two-to-one', instead of one-to-one: we have pairs of points that induce the same line.

So how do we get a moduli space of lines through P? One thing we can do is to add extra structure to the lines we are trying to classify. For example, we could rather try looking at lines through P with a *direction*. Every line in the plane has two directions:



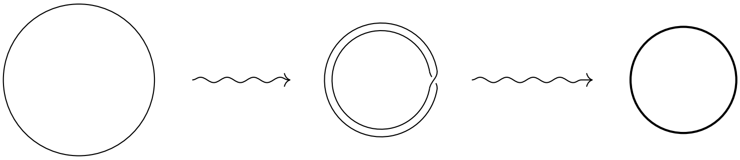
If we now have a point Q on the circle around P, then we draw the line through P and Q in the direction from P to Q:



The reader can verify that even though two points at opposite sides of the circle give rise to the same lines, the directions of these lines differ. Points on the circle therefore correspond one-to-one with lines through P with a direction. We thus find that the circle around P is a moduli space for lines through P with a direction.

This method of adding extra structure to objects so that their moduli space will be easier to describe is called *rigidification*. It is a technique that is often applied by mathematicians when they wish to study moduli spaces of more complicated objects.

But what if we do not want to add any additional structure to the objects we are classifying? If we are only interested in lines through P, and not at all in lines through P with a direction, then we can use a different technique. Manipulate the circle as follows: lift the circle from the plane, wrap it 'around itself' once, and glue the two strands together:



We thus obtain a new circle. Any two points on opposite sides of the old circle are glued together into a single point on the new circle. In other words: every point on the new circle corresponds to a pair of points on opposite sides of the old circle, and these points induce the same line through P. So there is a one-to-one correspondence between points on the new circle and lines through P, and the new circle is therefore a moduli space for the lines through P.

We invite the reader to carry out the construction described above. Take a sheet of paper, a couple of markers with different colors, and a rubber band. Draw the point P on the sheet of paper, and lay down the rubber band around it such that it forms a circle around this point. For each marker choose a pair of points on opposite sides of the rubber band, and mark these points with the same color. Now take the rubber band and wrap it around itself once, as described in the above figure. You will see that all pairs of points with the same color coincide.

'Wrapping' a geometric object around itself to obtain a new geometric object is also a technique mathematicians use often. They call this technique *taking a quotient*.

In this thesis we study the moduli space of certain geometric objects, namely compact Riemann surfaces of genus g. Moreover we discuss the differential forms that live on this moduli space. In Chapter 1 we discuss some theory about submersions of manifolds, families of compact Riemann surfaces, and hermitian line bundles on these families. We construct various canonical hermitian line bundles and give canonical isometries between these line bundles. In Chapter 2 we look at the moduli space of compact Riemann surfaces of genus g. We discuss that there is no 'nice' moduli space, but that this problem can be fixed by rigidifying or by taking quotients. In Chapter 3 we look at *marked graphs*. These are graphs of which some vertices are marked with positive integers. These graphs can be contracted, and we show that formulas can be given for the number of contracted marked graphs of any given characteristic in terms of the number of marked vertices. Finally, in Chapter 4, we study *tautological differential forms* on the moduli space. By using the marked graphs from Chapter 3 we can show that there are not 'too many' such tautological differential forms, and we compute some relations between these tautological differential forms.