Tautological differential forms on moduli spaces of curves

Lugt, S. van der

Chapter 4

Tautological differential forms on moduli of curves

In this section we will establish a theory of tautological differential forms on families of curves, that is meant to give an analytic analogue to the theory of tautological rings and tautological cohomology. We first discuss a suitable definition for the rings of tautological forms. This definition, however, introduces exact tautological forms that cannot be detected by cohomology; it follows that the rings of tautological forms are 'bigger' than the rings of tautological classes. Next, we will describe a combinatorial framework, using marked graphs, that allows us to generate tautological forms, and prove that in fact all tautological forms can be constructed in this way, thereby showing that the rings of tautological forms are not 'too big'. Finally, we describe a method for generating relations in the rings of tautological forms and fully compute the degree 2 parts of these rings.

4.1 Tautological morphisms and submersions

Fix an integer $g \geq 2$. In Chapter 2 we have defined the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ of genus g curves. Although $\mathcal{M}_g$ and $\mathcal{C}_g$ are not complex manifolds but merely differentiable stacks, we will often treat these spaces as if they were honest manifolds. The reader should understand that statements about this universal family of genus g curves can in that case be interpreted as statements that hold universally among all families of genus g curves. In Chapter 2 we have clarified this correspondence between statements for the universal family and universal statements for families.

Let us briefly recall the tautological morphisms we constructed in Chapter 2. Let $f : \mathcal{C} \to S$ be a family of genus g curves. Recall that to each pair of integers $r, s \geq 0$ and each map of sets $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ we have associated a morphism

$$f^\phi : \mathcal{C}^r \to \mathcal{C}^s : (x_1, \ldots, x_r) \mapsto (x_{\phi(1)}, \ldots, x_{\phi(s)}),$$

where $\mathcal{C}^r$ and $\mathcal{C}^s$ denote the r-fold and s-fold fiber products of $\mathcal{C}$ over S. This morphism is a submersion if and only if ϕ is injective. Universally, we obtain a morphism

$$f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$$

and morphisms of this form are called tautological morphisms. The tautological morphism f^ϕ is a submersion if and only if ϕ is injective.

The following examples list some tautological morphisms that we will be using often.

Example 4.1.1. The tautological morphism associated to the unique map $\{1,2\} \to \{1\}$ is the diagonal morphism $\Delta : \mathcal{C}_g \to \mathcal{C}_g^2$.

Example 4.1.2. The tautological submersion associated to the unique map $\emptyset \to \{1,\dots,r\}$ is the projection morphism $\mathcal{C}_g^r \to \mathcal{M}_g$.

Example 4.1.3. If $1 \leq i \leq r$ is an integer, the map $\{1\} \to \{1,\dots,r\}$ given by $1 \mapsto i$ induces the map $\mathcal{C}_g^r \to \mathcal{C}_g$ that projects onto the ith coordinate. We denote this map by p_i.
More generally, if $1 \leq i_1,\dots,i_s \leq r$ are integers, we denote by

$$p_{i_1,\dots,i_s} : \mathcal{C}_g^r \to \mathcal{C}_g^s$$

the tautological morphism associated to $\phi : \{1,\dots,s\} \to \{1,\dots,r\} : k \mapsto i_k$.

Example 4.1.4. Let $1 \leq i_1 < \dots < i_s \leq r$ be integers. Consider the unique increasing map $\phi : \{1,\dots,r-s\} \to \{1,\dots,r\}$ whose image is $\{1,\dots,r\} \setminus \{i_1,\dots,i_s\}$. Denote by

$$p_{(i_1,\dots,i_s)} : \mathcal{C}_g^r \to \mathcal{C}_g^{r-s}$$

the tautological morphism associated to ϕ (notice the parentheses!). Then $p_{(i_1,\dots,i_s)}$ is the tautological submersion that 'forgets the coordinates $i_1,\dots,i_s$'. For instance, the map $p_{(2)} : \mathcal{C}_g^2 \to \mathcal{C}_g$ equals the map $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$.

Consider a commutative diagram of sets, together with the associated diagram of moduli stacks.

$$\begin{array}{ccc} \{1,\dots,u\} & \xleftarrow{\eta} & \{1,\dots,s\} \\ \uparrow \chi & & \uparrow \phi \\ \{1,\dots,t\} & \xleftarrow{\psi} & \{1,\dots,r\} \end{array} \qquad \begin{array}{ccc} \mathcal{C}_g^u & \xrightarrow{f^\eta} & \mathcal{C}_g^s \\ \downarrow f^\chi & & \downarrow f^\phi \\ \mathcal{C}_g^t & \xrightarrow{f^\psi} & \mathcal{C}_g^r \end{array}$$

As we have seen in Section 2.4, the diagram of moduli stacks is cartesian if and only if the diagram of sets is a pushout diagram. We will be using such cartesian diagrams often.

4.2 Tautological classes

This section serves as a short introduction to tautological rings of moduli spaces of curves. We will recall the definition of the tautological ring $R^*(\mathcal{C}_g^r)$ of $\mathcal{C}_g^r$, which is a subring of the Chow ring $\mathrm{CH}^*(\mathcal{C}_g^r)$ of $\mathcal{C}_g^r$ with rational coefficients.

Let $g \geq 2$ be an integer, and consider the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ of genus g curves. We let $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ denote the relative cotangent bundle, and $K \in \mathrm{CH}^1(\mathcal{C}_g)$ its first Chern class in the Chow ring with rational coefficients. For $d \geq 0$ we define the dth *Mumford–Morita–Miller class* κ_d by

$$\kappa_d = p_* K^{d+1} \in \mathrm{CH}^d(\mathcal{M}_g).$$

The *tautological ring* on $\mathcal{M}_g$, defined by Mumford [Mum83], is the sub-$\mathbb{Q}$-algebra $R^*(\mathcal{M}_g) \subseteq \mathrm{CH}^*(\mathcal{M}_g)$ generated by these κ-classes. Mumford proved that the tautological ring is generated by the tautological classes $\kappa_1, \ldots, \kappa_{g-2}$. He also proved that all Chern classes of the Hodge bundle $p_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ lie in the tautological ring.

The Chow ring and the tautological ring vanish in degrees higher than $\dim(\mathcal{M}_g) = 3g - 3$. Looijenga [Loo95] proved the stronger statement that $R^*(\mathcal{M}_g)$ vanishes in degrees higher than $g - 2$, and that $R^{g-2}(\mathcal{M}_g)$ is at most one-dimensional, spanned by the class κ_{g-2}. Faber [Fab97] then proved that κ_{g-2} is nonzero, so $R^{g-2}(\mathcal{M}_g)$ is one-dimensional. Faber also conjectured that the tautological ring is a *Gorenstein algebra*.

4

Conjecture 4.2.1 ([Fab99])**.** For any $g \geq 2$ the following holds.

1. $R^d(\mathcal{M}_g) = 0$ for $d > g - 2$;
2. $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$;
3. Multiplication in the Chow ring gives a perfect pairing

$$R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

for all $0 \leq d \leq g - 2$.

This conjecture has been verified by Faber [Fab13] for all $g \leq 23$, but not enough relations have been found in genus 24 to verify the conjecture there.

More generally, the *tautological ring* $R^*(\mathcal{C}_g^r)$ of $\mathcal{C}_g^r$ (introduced in [Loo95]) is defined to be the $\mathbb{Q}$-subalgebra of $\mathrm{CH}^*(\mathcal{C}_g^r)$ generated by the following classes:

- the classes κ_d (obtained from $\mathcal{M}_g$ by pullback);
- the classes $K_i = p_i^* K$ for $1 \leq i \leq r$;
- the diagonal classes $\Delta_{ij} = p_{ij}^* \Delta$, with $\Delta \subseteq \mathcal{C}_g^2$ the diagonal, for $1 \leq i < j \leq r$.

Note that the classes K_i can also be defined as follows: if $p_{(i)} : \mathcal{C}_g^r \to \mathcal{C}_g^{r-1}$ is the projection map that forgets the ith coordinate, then K_i is the first Chern class of the relative cotangent bundle of this projection.

In the ring $R^*(\mathcal{C}_g^r)$ we have the relations:

$$\begin{aligned}\Delta_{ij}\Delta_{jk} &= \Delta_{ij}\Delta_{ik}\\ \Delta_{ij}^2 &= -\Delta_{ij}K_i.\end{aligned}$$

where i, j, k are pairwise distinct. If $\Delta : \mathcal{C}_g \to \mathcal{C}_g^2$ denotes the diagonal map, we have:

$$\Delta^*\Delta = -K.$$

Looijenga proved in [Loo95] that $R^*(\mathcal{C}_g^r)$ vanishes in degree $d > g + r - 2$.

Using the above relations, one can deduce that the pullbacks along tautological morphisms of tautological classes are again tautological classes. Moreover, it is straightforward to verify using the above relations and the projection formula that the pushforward of every tautological class along every tautological morphism is a tautological class. In other words: the system of $\mathbb{Q}$-algebras $\{R^*(\mathcal{C}_g^r) : r \geq 0\}$ is closed under pushforward and pullback along tautological morphisms. If $\{S^*(\mathcal{C}_g^r) : r \geq 0\}$ is another system of $\mathbb{Q}$-subalgebras of the Chow rings that is closed under pushforwards and pullbacks along tautological morphisms, then

$$\begin{aligned}\Delta &= \Delta_*(1) \in S^*(\mathcal{C}_g^2)\\ K &= -\Delta^*\Delta \in S^*(\mathcal{C}_g)\\ \kappa_d &= p_*K^{d+1} \in S^*(\mathcal{M}_g).\end{aligned}$$

It follows that the classes κ_d, K_i and Δ_{ij} lie in $S^*(\mathcal{C}_g^r)$, and therefore $R^*(\mathcal{C}_g^r) \subseteq S^*(\mathcal{C}_g^r)$. We obtain the following.

Proposition 4.2.2. The system of $\mathbb{Q}$-subalgebras $R^*(\mathcal{C}_g^r) \subseteq \mathrm{CH}^*(\mathcal{C}_g^r)$ (with $r \geq 0$) is the smallest system of $\mathbb{Q}$-subalgebras that is closed under pullbacks and pushforwards along tautological morphisms. □

In fact, we can slightly rephrase this proposition to the following. It will be this formulation that allows us to translate the language of tautological classes to a language of tautological differential forms.

Proposition 4.2.3. The system of $\mathbb{Q}$-subalgebras $R^*(\mathcal{C}_g^r) \subseteq \mathrm{CH}^*(\mathcal{C}_g^r)$ (with $r \geq 0$) satisfies:

1. $\Delta \in R^*(\mathcal{C}_g^2)$;
2. the system is closed under pullbacks along tautological morphisms;
3. the system is closed under pushforwards along tautological *submersions*;
4. the system is the smallest system that satisfies 1–3. □

Analogously, the *tautological cohomology ring* $RH^*(\mathcal{C}_g^r)$ is a subring of the cohomology ring of $\mathcal{C}_g^r$ with rational coefficients. It is defined as the image of the canonical map

$$R^*(\mathcal{C}_g^r) \to H^{2*}(\mathcal{C}_g^r, \mathbb{Q}).$$

Notice that the grading on cohomology is twice the grading in the Chow ring, and that the tautological cohomology ring does not contain any odd-degree cohomology classes. So far, it seems to be unknown whether the canonical map from tautological Chow classes to tautological cohomology classes is an isomorphism.

We will define a third tautological ring of a more analytical nature, the ring of tautological differential forms, which is a subring of the ring of real differential forms on $\mathcal{C}_g^r$. These forms will be closed differential forms and we can take their cohomology classes in $H^*(\mathcal{C}_g^r, \mathbb{R})$. When we compare rings of tautological differential forms with tautological cohomology rings, we should consider cohomology with real coefficients.

4.3 Rings of tautological differential forms

Fix an integer $g \geq 2$. In Section 4.2 we have seen that there are multiple equivalent ways to define the rings of tautological Chow or cohomology classes on the moduli stacks $\mathcal{C}_g^r$. A priori, these rings are defined to be the sub-$\mathbb{Q}$-algebras of the Chow or cohomology rings that are generated by the classes Δ_{ij}, K_i and κ_d, and Propositions 4.2.2 and 4.2.3 yield two more equivalent definitions. In this section, we will attempt to translate these definitions to an analytical setting. Rather than Chern or cohomology classes, we will consider differential forms. Of the three equivalent definitions for the ring of tautological forms given in Section 4.2, the one given by Proposition 4.2.3 can be translated directly to the analytical setting, and we will be using this translation to define rings of tautological differential forms on the moduli stacks $\mathcal{C}_g^r$.

4

Let us endow the line bundle $O(\Delta)$ on $\mathcal{C}_g^2$ with its canonical metric (see Section 1.4), and take the first Chern form of the resulting hermitian bundle to obtain a closed real 2-form

$$h = c_1(O(\Delta))$$

on $\mathcal{C}_g^2$ that represents the diagonal. Let $\omega = \omega_{\mathcal{C}_g/\mathcal{M}_g}$ be the relative cotangent bundle of the universal family of genus g curves $\mathcal{C}_g \to \mathcal{M}_g$, endowed with its canonical metric; recall from Section 1.4 that we have a canonical isometry

$$\omega^{\otimes -1} \simeq \Delta^* O(\Delta)$$

of hermitian vector bundles on $\mathcal{C}_g$. We therefore have

$$c_1(\omega) = -\Delta^* c_1(O(\Delta)) = -e^A \in A^2(\mathcal{C}_g)$$

where e^A is defined to be the first Chern form of the relative tangent bundle

$$T_{\mathcal{C}_g/\mathcal{M}_g} \simeq \omega^{\otimes -1} \simeq \Delta^* O(\Delta)$$

with the metric induced by the canonical metric on ω.

Recall that in the Chow ring $\mathrm{CH}^*(\mathcal{M}_g)$ we have constructed the kappa-classes κ_d by pushing forward powers of the canonical class K on $\mathcal{C}_g$ along the universal family $\mathcal{C}_g \to \mathcal{M}_g$. Analogously we define forms $e_d^A \in A^{2d}(\mathcal{M}_g)$ for all $d \geq 0$ by

$$e_d^A := \int_{\mathcal{C}_g/\mathcal{M}_g} (e^A)^{d+1}.$$

Let us consider the sub-$\mathbb{R}$-algebras of $A^*(\mathcal{C}_g^r)$ generated by forms $p_{ij}^* h$, $p_i^* e^A$ and e_d^A. Certainly, we want to consider forms in these rings to be tautological. However, a problem arises: this system of rings is not closed under fiber integrals along projection maps. For instance, consider the differential form

$$\nu := \int_{\mathcal{C}_g^2/\mathcal{M}_g} h^3 \in A^2(\mathcal{M}_g).$$

We have [dJon16]:

$$\nu - e_1^A = \frac{\partial\bar{\partial}\varphi}{\pi\sqrt{-1}},$$

where $\varphi \in A^0(\mathcal{M}_g)$ is the Kawazumi-Zhang invariant, introduced by Kawazumi [Kaw08; Kaw09] and Zhang [Zha10] in different contexts. Later we will see that for $g \geq 3$ the forms ν and e_1^A are linearly independent (whereas for $g = 2$ there is a linear relation), and thus we find that ν is not in the subring of $A^*(\mathcal{M}_g)$ generated by the classes e_d^A for all $g \geq 3$.

A second problem is the fact that, in the context of differential forms, proper pushforwards or fiber integrals can only be taken along submersions. While the tautological class Δ on $\mathcal{C}_g^2$ can be obtained by taking the pushforward of 1 along the diagonal map $\mathcal{C}_g \to \mathcal{C}_g^2$, we can not obtain the corresponding form h in an analogous way.

4

The following definition, based on Proposition 4.2.3, solves both our problems.

Definition 4.3.1. The *rings of tautological forms* $\mathcal{R}^*(\mathcal{C}_g^r)$ $(r \geq 0)$ are the unique sub-$\mathbb{R}$-algebras $\mathcal{R}^*(\mathcal{C}_g^r) \subseteq A^*(\mathcal{C}_g^r)$ such that the following holds:

1. $h \in \mathcal{R}^*(\mathcal{C}_g^2)$;
2. If $f : \mathcal{C}_g^r \to \mathcal{C}_g^s$ is a tautological morphism, then $f^*(\mathcal{R}^*(\mathcal{C}_g^s)) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$;
3. If $f : \mathcal{C}_g^r \to \mathcal{C}_g^s$ is a tautological submersion, then $\int_f(\mathcal{R}^*(\mathcal{C}_g^r)) \subseteq \mathcal{R}^*(\mathcal{C}_g^s)$;
4. $\mathcal{R}^*(\mathcal{C}_g^r)$ are minimal: if $S^*(\mathcal{C}_g^r) \subseteq A^*(\mathcal{C}_g^r)$ $(r \geq 0)$ is another collection of sub-$\mathbb{R}$-algebras that satisfies 1–3, then $\mathcal{R}^*(\mathcal{C}_g^r) \subseteq S^*(\mathcal{C}_g^r)$ for all $r \geq 0$.

Elements of these rings are called *tautological (differential) forms.*

Notice that this definition implies that there are no tautological forms of odd degree. Indeed, taking pullbacks and fiber integrals of differential forms along morphisms of complex manifolds changes the degrees of these differential forms by an even number; see Proposition 1.3.19. It follows that removing the summands of odd degree from the rings

$$\mathcal{R}^*(\mathcal{C}_g^r) = \bigoplus_{d \geq 0} \mathcal{R}^d(\mathcal{C}_g^r)$$

still yields a system that satisfies properties 1–3, which is smaller than, and hence equal to, the system of tautological differential forms.

Definition 4.3.1 implies that $e^A = \Delta^* h$ is a tautological form on $\mathcal{C}_g$, and $e_d^A = \int_p (e^A)^{d+1}$ is a tautological form on $\mathcal{M}_g$ for all $d \geq 0$. It follows that passing to cohomology yields a surjective map

$$\mathcal{R}^*(\mathcal{C}_g^r) \to RH^*(\mathcal{C}_g^r) \otimes_{\mathbb{Q}} \mathbb{R}.$$

However, as opposed to the settings of Chow rings and cohomology rings, the tautological rings are not generated by pullbacks of classes h, e^A and e_d^A. For instance, the real 2-form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}$$

is a tautological form on $\mathcal{M}_g$ that is not in the subring generated by the e_d^A if $g \geq 3$. Such 'extra' tautological forms are introduced by the homogeneous ideal $I^*(\mathcal{C}_g^r)$ of exact tautological forms:

$$0 \to I^*(\mathcal{C}_g^r) \to \mathcal{R}^*(\mathcal{C}_g^r) \to RH^*(\mathcal{C}_g^r) \otimes_{\mathbb{Q}} \mathbb{R} \to 0.$$

By Looijenga's result [Loo95] we know that all tautological forms of degree $d > 2(g+r-2)$ are exact.

Of particular interest is the degree 2 part

$$I^2(\mathcal{M}_g) \subseteq I^*(\mathcal{M}_g).$$

If $g \geq 3$ then exact forms in this space can be written in the form

$$\frac{\partial\overline{\partial}\alpha}{\pi\sqrt{-1}}$$

with α a real-valued smooth function on $\mathcal{M}_g$ defined uniquely up to an additive constant; see [Kaw09, Lemma 8.1]. For example, the Kawazumi–Zhang invariant φ arises from the exact tautological form $\nu - e_1^A$ in this way. One might wonder if it is possible to obtain more such invariants for genus g curves from exact tautological 2-forms on $\mathcal{M}_g$. As it will turn out, this is not the case. In Corollary 4.8.4 we will find that $I^2(\mathcal{M}_g)$ is spanned by

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}},$$

and that the Kawazumi–Zhang invariant is the only invariant, up to additive and multiplicative constants, that arises in this way.

Next, we will prove some elementary equalities of tautological differential forms, which we will use in the proof of Proposition 4.6.2.

Lemma 4.3.2. Let $p : \mathcal{C}_g \to \mathcal{M}_g$ be the universal family of genus g curves, and let e^A be the first Chern form of the relative tangent bundle $T_{\mathcal{C}_g/\mathcal{M}_g} \simeq \omega^{\otimes -1}$ with its canonical metric. Then

$$\int_p e^A = 2 - 2g \in A^0(\mathcal{M}_g).$$

Proof. Recall that the cotangent bundle of any genus g curve has degree $2g-2$. Applying Lemma 1.4.10 therefore gives the desired result. □

Lemma 4.3.3. Consider the tautological submersion $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$. Then

$$\int_{p_1} h = 1 \in A^0(\mathcal{C}_g).$$

Proof. This identity, too, follows immediately from Lemma 1.4.10. □

Lemma 4.3.4. Consider the tautological submersion $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$. If L is a hermitian line bundle on $\mathcal{C}_g^2$ which is fiberwise admissible with respect to p_1, then

$$\int_{p_1} h \wedge c_1(L) = \Delta^* c_1(L) \in A^2(\mathcal{C}_g).$$

In particular, we have:

$$\int_{p_1} h^2 = e^A$$

and for $i = 1, 2$ we have

$$\int_{p_1} h \wedge p_i^* e^A = e^A.$$

Proof. From Proposition 1.4.13 we obtain

$$\int_{p_1} h \wedge c_1(L) = \int_{p_1} c_1(O(\Delta)) \wedge c_1(L) = c_1(\langle O(\Delta), L\rangle_{p_1}) = c_1(\Delta^* L) = \Delta^* c_1(L),$$

where the third equality follows from the fact that the canonical metric on $O(\Delta)$ has the useful property that the canonical isomorphism

$$\langle O(\Delta), L\rangle_{p_1} \xrightarrow{\sim} \Delta^* L$$

is an isometry; see Section 1.4. The other identities now follow from:

$$h = c_1(O(\Delta)), \quad \text{and} \quad p_i^* e^A = p_i^* c_1(\omega^{\otimes -1}) = c_1(p_i^* \omega^{\otimes -1}).$$ □

Lemma 4.3.5. Let $p_{12}, p_{13}, p_{23} : \mathcal{C}_g^3 \to \mathcal{C}_g^2$ be the three tautological submersions. Then

$$\int_{p_{12}} p_{13}^* h \wedge p_{23}^* h = h \in A^2(\mathcal{C}_g^2).$$

Proof. Let $\sigma_1, \sigma_2 : \mathcal{C}_g^2 \to \mathcal{C}_g^3$ be the two canonical sections of p_{12}, such that $p_3 \circ \sigma_i = p_i : \mathcal{C}_g^2 \to \mathcal{C}_g$ for $i = 1, 2$. Notice that $p_{13} \circ \sigma_2 : \mathcal{C}_g^2 \to \mathcal{C}_g^2$ is the identity. Endow the induced line bundles $O(\sigma_1), O(\sigma_2)$ on $\mathcal{C}_g^3$ with their canonical metrics. We use

Proposition 1.4.13 to obtain

$$\begin{aligned}\int_{p_{12}} p_{13}^* h \wedge p_{23}^* h &= \int_{p_{12}} p_{13}^* c_1(O(\Delta)) \wedge p_{23}^* c_1(O(\Delta)) \\ &= \int_{p_{12}} c_1(O(\sigma_1)) \wedge c_1(O(\sigma_2)) \\ &= c_1(\langle O(\sigma_1), O(\sigma_2)\rangle_{p_1}) \\ &= \sigma_2^* c_1(O(\sigma_1)) \\ &= \sigma_2^* p_{13}^* c_1(O(\Delta)) \\ &= c_1(O(\Delta)) \\ &= h.\end{aligned}$$

□

Recall from Section 1.4 that for each family $f : \mathcal{C} \to S$ of genus g curves with Jacobian family $\mathcal{J} \to S$ we have canonical morphisms $\kappa : \mathcal{C} \to \mathcal{J}$ and $\delta : \mathcal{C}^2 \to \mathcal{J}$. The morphism κ takes a point x in a fiber $\mathcal{C}_s$ and maps it to the class of the degree 0 line bundle $O((2g-2)x) \otimes \omega_{\mathcal{C}_s}^{\otimes -1}$ in $\mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)$. The morphism δ is the Abel–Jacobi morphism: it maps a pair $(x, y) \in \mathcal{C}_s^2$ to the class of the line bundle $O(y - x)$ in $\mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)$.

Universally we obtain morphisms $\kappa : \mathcal{C}_g \to \mathcal{J}_g$ and $\delta : \mathcal{C}_g^2 \to \mathcal{J}_g$. Recall from Section 2.7 that on the universal Jacobian $\mathcal{J}_g$ we have constructed a canonical hermitian line bundle $\mathcal{B}$. We denote by $2\omega_0$ the first Chern form of $\mathcal{B}$. As the form $2\omega_0$ and the morphisms κ, δ are completely canonical, it makes sense to expect that the forms $2\kappa^*\omega_0$ and $2\delta^*\omega_0$ are tautological. Indeed, this is the case, as the following proposition shows.

Proposition 4.3.6. The forms $\kappa^*\omega_0 \in A^2(\mathcal{C}_g)$ and $\delta^*\omega_0 \in A^2(\mathcal{C}_g^2)$ are tautological. More precisely, we have the following identities of 2-forms:

$$\begin{aligned}-2\kappa^*\omega_0 &= 2g(2g-2)e^A + p^* e_1^A \\ -2\delta^*\omega_0 &= p_1^* e^A + p_2^* e^A - 2h.\end{aligned}$$

Note that these identities match identities (K1) and (K3) in [dJon16, Theorem 1.4].

Proof. Denote by $p : \mathcal{C}_g \to \mathcal{M}_g$ the universal family of genus g curves. Recall from Proposition 1.4.15 that we have canonical isometries

$$\begin{aligned}\kappa^* \mathcal{B}^{\otimes -1} &\simeq \omega^{-2g(2g-2)} \otimes p^* \langle \omega, \omega \rangle_p \\ \delta^* \mathcal{B}^{\otimes -1} &\simeq p_1^* \omega^{\otimes -1} \otimes p_2^* \omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2}\end{aligned}$$

Taking first Chern classes and applying Proposition 1.4.13 then yields the desired result. □

One could argue, in fact, that the 2-form $2\delta^*\omega_0$ is the 'prototypical' tautological form on $\mathcal{C}_g^2$, more so than h, and replace h by $2\delta^*\omega_0$ in Definition 4.3.1. We claim that this does not affect the resulting system of tautological rings. Indeed, Proposition 4.3.6 states that $2\delta^*\omega_0$ is tautological. Conversely, it is possible to obtain h from $2\delta^*\omega_0$ by using pullbacks and fiber integrals as follows. Squaring $-2\delta^*\omega_0 = p_1^*e^A + p_2^*e^A - 2h$ and integrating the result along the fibers of $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$ yields:

$$\begin{aligned}\int_{p_1} (-2\delta^*\omega_0)^2 &= \int_{p_1} c_1(\delta^*\mathcal{B}^{\otimes -1})^2 = c_1(\langle \mathcal{B}^{\otimes -1}, \mathcal{B}^{\otimes -1}\rangle_{p_1}) \\ &= c_1(\omega^{\otimes 4g} \otimes p^*\langle \omega, \omega\rangle_p) = -4ge^A + p^*e_1^A,\end{aligned}$$

where the second and third equalities follow from Propositions 1.4.13 and 1.4.15, respectively. Squaring the resulting form and integrating it along the fibers of $p : \mathcal{C}_g \to \mathcal{M}_g$ then gives:

$$\int_p (-4ge^A + p^*e_1^A)^2 = \int_p (16g^2(e^A)^2 - 8ge^A \wedge p^*e_1^A + p^*(e_1^A)^2)$$

We have:

$$\int_p 16g^2(e^A)^2 = 16g^2e_1^A,$$

and applying the projection formula and Lemma 4.3.2 yields

$$\int_p -8ge^A \wedge p^*e_1^A = -8ge_1^A \wedge \int_p e^A = -8g(2-2g)e_1^A.$$

Another application of the projection formula gives

$$\int_p p^*(e_1^A)^2 = (e_1^A)^2 \int_p 1 = 0.$$

We conclude:

$$\int_p (-4ge^A + p^*e_1^A)^2 = 16g(2g-1)e_1^A.$$

We thus find that we can obtain e_1^A, e^A, and finally h from $2\delta^*\omega_0$ by taking fiber integrals and pullbacks.

4

4.4 Tautological forms associated to marked graphs

Now that we have defined the rings of tautological forms, we need a method to generate lots of tautological forms in order to be able to study relations of these forms. We can start with some 'basic' tautological forms like h and e^A and take pullbacks, fiber integrals, and wedge products in order to generate more tautological forms. The theory of marked graphs gives us a combinatorial framework for generating such forms, and it will turn out that this framework is able to give us all tautological forms.

In this section, we fix an integer $g \geq 2$, and we will describe an operation that takes an r-marked graph and outputs a tautological form on $\mathcal{C}_g^r$.

Let $\Gamma = (V, E, m)$ be an r-marked graph, and let u be the number of unmarked vertices of Γ. Choose a bijective extension

$$\bar{m} : \{1, \ldots, r + u\} \xrightarrow{\sim} V$$

of the marking $m : \{1, \ldots, r\} \to V$. We will define a differential form μ_Γ on $\mathcal{C}_g^{r+u}$ that will depend on the choice of this extension $\bar{m}$.

First, we associate to every edge $e \in E$ a 2-form h_e on $\mathcal{C}_g^{r+u}$. This form is defined as follows. Suppose that the endpoints of e are $\bar{m}(i)$ and $\bar{m}(j)$. We define

$$h_e = p_{i,j}^* h \in \mathcal{R}^2(\mathcal{C}_g^{r+u}),$$

where $p_{i,j} : \mathcal{C}_g^{r+u} \to \mathcal{C}_g^2$ is the projection on the ith and jth coordinate. If e is a loop based at vertex $\bar{m}(i)$, then

$$h_e = p_{i,i}^* h = p_i^* \Delta^* h = p_i^* e^A,$$

where $p_i : \mathcal{C}_g^{r+u} \to \mathcal{C}_g$ is the projection on the ith coordinate, and $\Delta : \mathcal{C}_g \to \mathcal{C}_g^2$ is the diagonal morphism. Notice that h_e does not depend on the order of i and j as the form h is symmetric in the two coordinates of $\mathcal{C}_g^2$.

Now, we let μ_Γ denote the product of all these 2-forms:

$$\mu_\Gamma = \bigwedge_{e \in E} h_e \in \mathcal{R}^{2|E|}(\mathcal{C}_g^{r+u}).$$

This form depends on the choice of $\bar{m}$. However, the form obtained from a different choice of $\bar{m}$ only differs from μ_Γ by permutation of the last u coordinates of $\mathcal{C}_g^{r+u}$. Therefore, by Fubini's theorem, the fiber integral

$$\alpha_\Gamma := \int_{p_{1,\ldots,r} : \mathcal{C}_g^{r+u} \to \mathcal{C}_g^r} \mu_\Gamma \in \mathcal{R}^{2(|E|-u)}(\mathcal{C}_g^r) \qquad (4.4.1)$$

does not depend on the choice of $\bar{m}$.

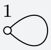
Definition 4.4.2. Let Γ be an r-marked graph. The form α_Γ on $\mathcal{C}_g^r$ defined in Equation 4.4.1 is *the (tautological) form associated to* Γ.

As the following examples show, many of the tautological differential forms we found before can be expressed as tautological forms associated to marked graphs.

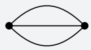
Example 4.4.3. Consider the unique 2-marked graph Γ with no unmarked vertices and a single edge between the two marked vertices. The associated 2-form α_Γ on $\mathcal{C}_g^2$ is h.

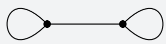
$\Gamma =$ 

Example 4.4.4. Consider the unique 1-marked graph Γ with no unmarked vertices and a single loop based at the unique vertex of Γ. The associated 2-form α_Γ on $\mathcal{C}_g$ is $\Delta^* h = e^A$.

$\Gamma =$ 

Example 4.4.5. Consider the two 0-marked graphs in the following picture.

$\Gamma_1 =$ 

$\Gamma_2 =$ 

The associated forms on $\mathcal{M}_g$ are

$$\alpha_{\Gamma_1} = \int_{\mathcal{C}_g^2/\mathcal{M}_g} h^3 =: \nu$$

and

$$\begin{aligned}
\alpha_{\Gamma_2} &= \int_{\mathcal{C}_g^2/\mathcal{M}_g} h \wedge p_1^* e^A \wedge p_2^* e^A \\
&= \int_{\mathcal{C}_g/\mathcal{M}_g} \int_{p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g} h \wedge p_1^* e^A \wedge p_2^* e^A \\
&= \int_{\mathcal{C}_g/\mathcal{M}_g} \left(e^A \wedge \int_{p_1} h \wedge p_2^* e^A \right) \\
&= \int_{\mathcal{C}_g/\mathcal{M}_g} (e^A)^2 \\
&= e_1^A,
\end{aligned}$$

where we have used the projection formula and Lemma 4.3.4. We therefore see that the tautological form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}} = \nu - e_1^A,$$

while not being associated to a graph itself, is in the linear span of forms on $\mathcal{M}_g$ associated to 0-marked graphs.

In the next section we will prove that, in fact, every tautological form on $\mathcal{C}_g^r$ can be written as the linear combination of forms associated to r-marked graphs.

4.5 Graph operations and tautological forms

In the previous section we introduced a combinatorial method of defining tautological forms on $\mathcal{C}_g^r$ for all $r \geq 0$ by associating them to r-marked graphs.

In this section we will study the various operations on marked graphs introduced in Chapter 3 and observe how the corresponding differential forms are affected. It turns out that these forms behave rather nicely with respect to pullbacks, pushforwards, and coproducts of marked graphs. By using this fact, we will be able to prove the following theorem.

Theorem 4.5.1. For every integer $r \geq 0$, the ring of tautological differential forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is spanned as an $\mathbb{R}$-vector space by forms α_Γ associated to r-marked graphs Γ.

By Definition 4.3.1 it suffices to prove that the system of linear subspaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ generated by forms associated to r-marked graphs is a system of sub-$\mathbb{R}$-algebras (that is: closed under wedge products and containing 1), that the system is closed under pullbacks and fiber integrals, and that h is contained in $S^*(\mathcal{C}_g^2)$.

We start by proving that $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is a subring for every $r \geq 0$. First of all, the form associated to the unique r-marked graph consisting of r vertices and no edges is 1. The following proposition implies that $S^*(\mathcal{C}_g^r)$ is closed under wedge products and therefore a subring of $\mathcal{R}^*(\mathcal{C}_g^r)$.

Proposition 4.5.2. Let $\Gamma = (V, E, m)$ and $\Gamma' = (V', E', m')$ be two r-marked graphs, and let α_Γ and $\alpha_{\Gamma'}$ be the associated tautological forms on $\mathcal{C}_g^r$. Then

$$\alpha_\Gamma \wedge \alpha_{\Gamma'} = \alpha_{\Gamma \sqcup_r \Gamma'}.$$

Proof. Assume that Γ and Γ' have respectively u and u' unmarked vertices. Choose bijective extensions

$$\bar{m} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$$
$$\bar{m}' : \{1, \ldots, r+u'\} \xrightarrow{\sim} V'$$

of m and m'. Let $\phi : \{1, \ldots, r+u\} \to \{1, \ldots, r+u+u'\}$ be the inclusion, and define the map

$$\psi : \{1, \ldots, r+u'\} \to \{1, \ldots, r+u+u'\} : k \mapsto \begin{cases} k & \text{if } k \leq r \\ k+u & \text{if } k > r. \end{cases}$$

It follows that the diagram

$$\begin{array}{ccc} \{1, \ldots, r+u+u'\} & \xleftarrow{\ \phi\ } & \{1, \ldots, r+u\} \\ \uparrow{\scriptstyle \psi} & & \uparrow \\ \{1, \ldots, r+u'\} & \hookleftarrow & \{1, \ldots, r\} \end{array}$$

is a pushout diagram of sets, so we have the associated cartesian diagram of moduli stacks

$$\begin{array}{ccc} \mathcal{C}_g^{r+u+u'} & \xrightarrow{p_{1,\dots,r+u}} & \mathcal{C}_g^{r+u} \\ {\scriptstyle p_{1,\dots,r,r+u+1,\dots,r+u+u'}}\downarrow & & \downarrow{\scriptstyle p_{1,\dots,r}} \\ \mathcal{C}_g^{r+u'} & \xrightarrow{p_{1,\dots,r}} & \mathcal{C}_g^{r}. \end{array}$$

Now let $\Gamma'' = (V'', E'', m'') = \Gamma \sqcup_r \Gamma'$. By the universal property of the pushout, we have an induced $r + u + u'$-marking

$$\bar{m}'' : \{1, \dots, r + u + u'\} \xrightarrow{\sim} V''$$

of the set of vertices V'' of Γ'' that extends m''. If $e \in E$ is an edge in Γ between vertices $\bar{m}(i)$ and $\bar{m}(j)$, then the corresponding edge in Γ'' has endpoints $\bar{m}''(\phi(i))$ and $\bar{m}''(\phi(j))$. Similarly, if $e \in E'$ is an edge in Γ' between vertices $\bar{m}'(i)$ and $\bar{m}'(j)$, then the corresponding edge in Γ'' has endpoints $\bar{m}''(\psi(i))$ and $\bar{m}''(\psi(j))$. It follows that

$$\begin{aligned} \mu_{\Gamma''} &= \bigwedge_{e \in E''} h_e \\ &= \bigwedge_{e \in E} p^*_{1,\dots,r+u} h_e \wedge \bigwedge_{e \in E'} p^*_{1,\dots,r,r+u+1,\dots,r+u+u'} h_e \\ &= p^*_{1,\dots,r+u} \mu_\Gamma \wedge p^*_{1,\dots,r,r+u+1,\dots,r+u+u'} \mu_{\Gamma'}. \end{aligned}$$

Using the base change formula 1.3.14 and the projection formula 1.3.1, we find that the fiber integral $\alpha_{\Gamma \sqcup_r \Gamma'}$ equals $\alpha_\Gamma \wedge \alpha_{\Gamma'}$. ☐

Next, we will show that the system of vector spaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is closed under pullbacks along tautological morphisms. Let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be a tautological morphism, induced by a map $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$. Recall from Chapter 3 that ϕ induces a pushforward operator $\phi_* : \mathcal{G}_r \to \mathcal{G}_s$ from r-marked graphs to s-marked graphs. The following proposition implies that the pullback map $f^{\phi,*}$ on differential forms is compatible with the pushforward map on graphs. From this one easily deduces that the system of forms $S^*(\mathcal{C}_g^r)$ is closed under pullbacks along tautological maps.

Proposition 4.5.3. Let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be the tautological morphism associated to a map $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$. Suppose that $\alpha_\Gamma \in S^*(\mathcal{C}_g^s)$ is the form associated to an s-marked graph Γ. Then

$$f^{\phi,*} \alpha_\Gamma = \alpha_{\phi_* \Gamma}$$

with $\phi_* \Gamma$ the pushforward of Γ along ϕ.

Proof. The proof is similar to the proof of Proposition 4.5.2, so only a short sketch is given here. We extend the labeling on Γ to an $(s+u)$-labeling, with u the number of unmarked vertices of Γ. This induces an $(r + u)$-labeling of $\phi_* \Gamma$, and it follows that the pullback of μ_Γ along the induced map $\mathcal{C}_g^{r+u} \to \mathcal{C}_g^{s+u}$ equals $\mu_{\phi_*\Gamma}$. By the base change formula the desired result follows. $\square$

Now, let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be a tautological submersion, associated to an injective map $\phi : \{1, \ldots, s\} \hookrightarrow \{1, \ldots, r\}$. In Chapter 3 we introduced a pullback map $\phi^* : \mathcal{G}_r \to \mathcal{G}_s$. The following proposition shows that, analogously to the pushforward map, the pullback map on graphs is compatible with the fiber integral map on differential forms. This implies that the system $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is closed under fiber integrals.

Proposition 4.5.4. Let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be an injective map, and let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be the associated tautological submersion. Let $\Gamma \in \mathcal{G}_r$ be an r-marked graph, and let $\phi^*\Gamma$ be the s-marked graph induced by ϕ. Then

$$\int_{f^\phi} \alpha_\Gamma = \alpha_{\phi^*\Gamma}$$

Proof. Let u be the number of unmarked vertices in Γ. Extend the inclusion $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ to a permutation $\{1, \ldots, r\} \to \{1, \ldots, r\}$, and then join this map with the identity on $\{r+1, \ldots, r+u\}$ to obtain a bijective map

$$\bar{\phi} : \{1, \ldots, r+u\} \xrightarrow{\sim} \{1, \ldots, r+u\}$$

that extends ϕ.

Moreover, choose a bijective extension $\bar{m} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$ of the marking m of Γ. We immediately obtain an extension

$$\overline{m\phi} = \bar{m} \circ \bar{\phi} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$$

of the marking $m\phi$ of the s-marked graph $\phi^*\Gamma = (V, E, m\phi)$. We have a commutative diagram of sets, inducing a commutative diagram of moduli stacks:

$$\begin{array}{ccc} \{1, \ldots, r+u\} & \xleftarrow{\supseteq} & \{1, \ldots, r\} \\ \uparrow \bar{\phi} & & \uparrow \phi \\ \{1, \ldots, r+u\} & \xleftarrow{\supseteq} & \{1, \ldots, s\} \end{array} \qquad \begin{array}{ccc} \mathcal{C}_g^{r+u} & \xrightarrow{p_{1,\ldots,r}} & \mathcal{C}_g^r \\ \downarrow f^{\bar{\phi}} & & \downarrow f^\phi \\ \mathcal{C}_g^{r+u} & \xrightarrow{p_{1,\ldots,s}} & \mathcal{C}_g^s \end{array}$$

If e is an edge in Γ with endpoints $\bar{m}(i), \bar{m}(j)$, then the corresponding edge ϕ^*e in $\phi^*\Gamma$ has endpoints $\overline{m\phi}(\bar{\phi}^{-1}(i))$ and $\overline{m\phi}(\bar{\phi}^{-1}(j))$. It follows that the corresponding 2-forms on $\mathcal{C}_g^{r+u}$ are related as follows:

$$h_e = f^{\bar{\phi},*} h_{\phi^* e}.$$

From this, we find that

$$\mu_\Gamma = f^{\bar{\phi},*} \mu_{\phi^*\Gamma},$$

so

$$\mu_{\phi^*\Gamma} = \int_{f^{\bar{\phi}}} \mu_\Gamma.$$

We therefore have:

$$\int_{f^\phi} \alpha_\Gamma = \int_{f^\phi} \int_{p_{1,\ldots,r}} \mu_\Gamma = \int_{p_{1,\ldots,s}} \int_{f^{\bar{\phi}}} \mu_\Gamma = \int_{p_{1,\ldots,s}} \mu_{\phi^*\Gamma} = \alpha_{\phi^*\Gamma},$$

proving the proposition. □

We have seen that differential forms associated to graphs are quite well-behaved with respect to the graph operations defined in Chapter 3. Using this, we can quite easily prove the main theorem of this section.

Proof of Theorem 4.5.1. By Proposition 4.5.2, and the fact that the form associated to the r-marked graph with no edges and no unmarked vertices equals 1, we find that the subspaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ are, in fact, sub-$\mathbb{R}$-algebras. Propositions 4.5.3 and 4.5.4 show that the system of subspaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is closed under taking pullbacks along tautological morphisms and fiber integrals along tautological submersions. Example 4.4.3 shows that h is an element of $S^2(\mathcal{C}_g^2)$.

But as the system of rings $\mathcal{R}^*(\mathcal{C}_g^r)$ is defined in Definition 4.3.1 to be the smallest system that satisfies these properties, we find that the two systems must be equal. □

4

4.6 Graph contractions and tautological forms

In the last section, we proved that every tautological form is a linear combination of tautological forms associated to graphs. We did so by observing the behavior of the resulting tautological forms when manipulating the marked graphs using the pushforward, pullback, and gluing operations in Chapter 3. In Section 3.6 we defined contraction operations on r-marked graphs. In this section we will show that these contractions are well-behaved with respect to taking associated tautological forms. This will allow us to prove the following theorem.

Theorem 4.6.1. For all $r \geq 0$ and $g \geq 2$, the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is finite-dimensional.

In the following proposition, we consider the various graph contraction operations defined in Chapter 3, and see how contracting vertices on an r-marked graph Γ influences the associated tautological form α_Γ. The proposition will be proved at the end of this section.

Proposition 4.6.2. Let $\Gamma = (V, E, m)$ be an r-marked graph, and suppose that Γ has an unmarked vertex v, such that either $\deg(v) \leq 2$, or $\deg(v) = 3$ and v is incident to precisely two edges. Let Γ' be the graph obtained from Γ by contracting v.

0. If $\deg v = 0$, then $\alpha_\Gamma = 0$.
1. If $\deg v = 1$, then $\alpha_\Gamma = \alpha_{\Gamma'}$.

2a. Suppose that $\deg v = 2$ and that v has two distinct neighbors $w \neq w'$. Then

$\alpha_\Gamma = \alpha_{\Gamma'}$.

2b. Suppose that $\deg v = 2$ and that v has a single neighbor $w \neq v$. Then $\alpha_\Gamma = \alpha_{\Gamma'}$.

2c. Suppose that $\deg v = 2$ and that v is its own neighbor; that is: there is a loop based at v. Then $\alpha_\Gamma = (2-2g)\alpha_{\Gamma'}$.

3. Suppose that $\deg v = 3$ and that v is incident to precisely two edges. Then $\alpha_\Gamma = \alpha_{\Gamma'}$.

Proposition 4.6.2 shows that we can contract every r-marked graph to a contracted r-marked graph while leaving the resulting tautological form the same up to multiplication by zero or a power of $(2-2g)$. Therefore, we find that the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is the linear span of the tautological forms associated to contracted r-marked graphs.

Suppose that Γ is an r-marked graph with u unmarked vertices and e edges. The Euler characteristic of Γ is $\chi(\Gamma) = r+u-e$. After extending the marking of Γ to an $(r+u)$-marking, we obtain the form μ_Γ that lives on $\mathcal{C}_g^{r+u}$ and has degree $2e$. Now α_Γ is the fiber integral of μ_Γ along the projection $\mathcal{C}_g^{r+u} \to \mathcal{C}_g^r$, whose fibers are of real dimension $2u$, and hence the degree of α_Γ is $2e - 2u = 2r - 2\chi(\Gamma)$. We obtain the following.

Lemma 4.6.3. Let $d \geq 0$ and $r \geq 0$ be integers. The space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ of tautological forms of degree $2d$ on $\mathcal{C}_g^r$ is the linear span of the forms α_Γ associated to contracted r-marked graphs Γ with Euler characteristic $\chi(\Gamma) = r - d$. □

In Theorem 3.7.1, we proved that there are (up to isomorphism) only finitely many contracted r-marked graphs of any given characteristic $\chi \in \mathbb{Z}$. By combining Lemma 4.6.3 with Theorem 3.7.1, we obtain the following.

Theorem 4.6.4. Let $g \geq 2$. For all integers $r \geq 0$ and $d \geq 0$, the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ of tautological forms of degree $2d$ on $\mathcal{C}_g^r$ is finite-dimensional. More precisely: the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by forms α_Γ, where Γ ranges over all contracted r-marked graphs of characteristic $r-d$. These graphs have at most $2d$ unmarked vertices, and there are only finitely many such graphs up to isomorphism. □

The main theorem of this section is now a simple consequence of the previous theorem.

Proof of Theorem 4.6.1. Recall from Section 2.5 that there exists an inclusion $A^*(\mathcal{C}_g^r) \to A^*(\mathcal{X}_g^r)$ where $\mathcal{X}_g \to \mathcal{T}_g$ is the universal family of genus g curves with Teichmüller structure and $\mathcal{X}_g^r$ denotes the r-fold fiber product of $\mathcal{X}_g$ over $\mathcal{T}_g$. As $\mathcal{X}_g^r$ is a manifold of (real) dimension $6g-6+2r$, it follows that $A^d(\mathcal{X}_g^r)$ is zero for all $d > 2r+6g-6$, and the same is true for $A^d(\mathcal{C}_g^r)$ and hence for $\mathcal{R}^d(\mathcal{C}_g^r)$. Moreover, the odd-degree subspaces $\mathcal{R}^{2d+1}(\mathcal{C}_g^r)$ are zero. Therefore

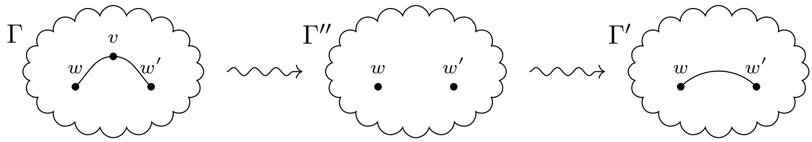
$$\mathcal{R}^*(\mathcal{C}_g^r) = \bigoplus_{d\geq 0} \mathcal{R}^*(\mathcal{C}_g^r) = \bigoplus_{d=0}^{3g-3+r} \mathcal{R}^{2d}(\mathcal{C}_g^r)$$

is a direct sum of finitely many finite-dimensional subspaces, and therefore it is itself finite-dimensional. □

We devote the remainder of this section to proving Proposition 4.6.2. The proof is merely technical, and does not introduce any new concepts.

Proof of Proposition 4.6.2. Let $\Gamma = (V, E, m)$ be an r-marked graph, and let v be an unmarked vertex of degree ≤ 2, or an unmarked vertex of degree ≤ 3 with a loop. Define a graph Γ'' by removing v, and all edges emanating from v, from Γ. Moreover, we have the graph Γ' that is obtained from Γ by contracting v.

The graph Γ'' represents an 'intermediate step' in obtaining Γ' from Γ. The following picture describes the situation in the case where v has two distinct neighbors.



Let $u \geq 0$ be such that Γ has $u+1$ unmarked points. Fix an extension of m to an $(r+u+1)$-marking

$$\bar{m} : \{1, \ldots, r+u+1\} \xrightarrow{\sim} V,$$

such that $\bar{m}(r+u+1) = v$.

Restricting $\bar{m}$ to $\{1, \ldots, r+u\}$ induces an $(r+u)$-marking on Γ' and Γ'' that extends the r-marking on these graphs. We obtain differential forms μ_Γ, $\mu_{\Gamma'}$, and $\mu_{\Gamma''}$ that live on $\mathcal{C}_g^{r+u+1}$, $\mathcal{C}_g^{r+u}$, and $\mathcal{C}_g^{r+u}$, respectively.

The inclusions $\{1, \ldots, r\} \subseteq \{1, \ldots, r+u\} \subseteq \{1, \ldots, r+u+1\}$ induce tautological submersions

$$\begin{array}{ccc} \mathcal{C}_g^{r+u+1} & \xrightarrow{q} & \mathcal{C}_g^{r+u} \\ & \searrow^{pq} & \downarrow p \\ & & \mathcal{C}_g^{r} \end{array}$$

We have

$$\alpha_\Gamma = \int_{pq} \mu_\Gamma \quad \text{and} \quad \alpha_{\Gamma'} = \int_p \mu_{\Gamma'}$$

If we can prove that $\int_q \mu_\Gamma = 0$ in case 0, $\int_q \mu_\Gamma = \mu_{\Gamma'}$ in cases 1, 2a, 2b, and 3, and $\int_q \mu_\Gamma = (2-2g)\mu_{\Gamma'}$ in case 2c, we are done.

0. Suppose v has degree 0. The set of edges of Γ is equal to the set of edges of Γ', so we obtain

$$\mu_\Gamma = q^* \mu_{\Gamma'}.$$

Taking fiber integrals and applying the projection formula yields:

$$\int_q \mu_\Gamma = \mu_{\Gamma'} \int_q 1 = 0,$$

and we find that $\alpha_\Gamma = 0$.

1. Suppose v has degree 1; let $i \in \{1, \ldots, r+u\}$ be such that $\bar{m}(i)$ is the neighbor of v. The graph Γ is obtained from Γ' by adding the vertex v and the edge between v and $\bar{m}(i)$. We therefore have:

$$\mu_\Gamma = q^* \mu_{\Gamma'} \wedge p_{i,r+u+1}^* h,$$

so

$$\int_q \mu_\Gamma = \mu_{\Gamma'} \wedge \int_q p_{i,r+u+1}^* h.$$

By using the base change formula with the cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_g^{r+u+1} & \xrightarrow{p_{i,r+u+1}} & \mathcal{C}_g^2 \\ \downarrow{\scriptstyle q} & \square & \downarrow{\scriptstyle p_1} \\ \mathcal{C}_g^{r+u} & \xrightarrow[p_i]{} & \mathcal{C}_g, \end{array}$$

we find:

$$\int_q p_{i,r+u+1}^* h = p_i^* \int_{p_1} h = 1,$$

where the latter equality follows from Lemma 4.3.3. This shows that $\int_q \mu_\Gamma = \mu_{\Gamma'}$, so $\alpha_\Gamma = \alpha_{\Gamma'}$.

2a. Suppose v has degree 2, and that v has two distinct neighbors w and w'. Let $i, j \in \{1, \ldots, r+u\}$ be such that $\bar{m}(i) = w$ and $\bar{m}(j) = w'$. In this case, we find

$$\mu_\Gamma = \mu_{\Gamma''} \wedge p_{i,r+u+1}^* h \wedge p_{j,r+u+1}^* h,$$

and

$$\mu_{\Gamma'} = \mu_{\Gamma''} \wedge p_{i,j}^* h.$$

In this case, another application of the base change formula, together with the identity of forms

$$\int_{p_{12}} p_{13}^* h \wedge p_{23}^* h = h$$

from Lemma 4.3.5 shows that $\int_q \mu_\Gamma = \mu_{\Gamma'}$, and hence $\alpha_\Gamma = \alpha_{\Gamma'}$.

2b. The proof in this case is very similar to the proofs for cases 1 and 2. In this case, we use the identity

$$\int_{p_1} h^2 = e^A$$

from Lemma 4.3.4.

2c. Again, the proof of this case is similar to that of the previous cases. The identity used here is

$$\int_{\mathcal{C}_g/\mathcal{M}_g} e^A = (2 - 2g),$$

see Lemma 4.3.2.

3. Finally, the proof in case 3 is analogous to that of earlier cases, where we use the identity
$$\int_{p_1} h \wedge p_2^* e^A = e^A$$
from Lemma 4.3.4.

□

4.7 The ring of tautological forms as a quotient algebra

In the previous sections we proved that the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is the linear span of forms associated to r-marked graphs. In this section we will exploit this and show that we can write the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ as a quotient algebra of a graded $\mathbb{R}$-algebra whose summands are effectively computable.

Recall from Chapter 3 that $\mathrm{G}(r)$ denotes the set of isomorphism classes of r-marked graphs. The coproduct $\sqcup_r$ on the category $\mathcal{G}_r$ of r-marked graphs induces a binary operator on $\mathrm{G}(r)$, which gives $\mathrm{G}(r)$ the structure of a commutative monoid. There is a homomorphism of monoids

4

$$\bar{\chi}_r : \mathrm{G}(r) \to \mathbb{Z} : \Gamma \mapsto r - \chi(\Gamma).$$

If $d \in \mathbb{Z}$ is an integer, the inverse image $\bar{\chi}_r^{-1}(d)$ is the set

$$\mathrm{G}(r, r-d) = \{r\text{-marked graphs of characteristic } r-d\}/\cong.$$

The *monoid ring* of $\mathrm{G}(r)$ over $\mathbb{R}$ is the $\mathbb{R}$-algebra

$$\mathbb{R}[\mathrm{G}(r)]$$

that has as the underlying $\mathbb{R}$-module the vector space with the elements of $\mathrm{G}(r)$ as its basis, and whose multiplication is defined uniquely by demanding it extends the binary operator on $\mathrm{G}(r)$, where we view $\mathrm{G}(r)$ as a subset of $\mathbb{R}[\mathrm{G}(r)]$ via the map $\Gamma \mapsto 1 \cdot \Gamma$. The homomorphism $\bar{\chi}_r : \mathrm{G}(r) \to \mathbb{Z}$ and the corresponding partition of $\mathrm{G}(r)$ induce a grading on $\mathbb{R}[\mathrm{G}(r)]$ whose degree d summand is spanned by the graphs of characteristic $r - d$:

$$\mathbb{R}[\mathrm{G}(r)] = \bigoplus_{d \in \mathbb{Z}} \mathbb{R}[\mathrm{G}(r, r-d)].$$

The method described in Section 4.4 of taking a graph $\Gamma \in \mathrm{G}(r)$ and assigning to it a tautological form $\alpha_\Gamma \in \mathcal{R}^*(\mathcal{C}_g^r)$ induces a map of sets $\mathrm{G}(r) \to \mathcal{R}^*(\mathcal{C}_g^r)$. This map is in fact a homomorphism of monoids by Proposition 4.5.2, where the monoid structure on $\mathcal{R}^*(\mathcal{C}_g^r)$ is given by the wedge product. This monoid homomorphism induces a homomorphism of $\mathbb{R}$-algebras

$$\alpha : \mathbb{R}[G(r)] \to \mathcal{R}^*(\mathcal{C}_g^r).$$

By Theorem 4.5.1 it holds that this homomorphism is surjective.

If Γ is an r-marked graph of characteristic $r-d$, the corresponding form α_Γ is of degree $2d$, and hence the above $\mathbb{R}$-algebra homomorphism is in fact a homomorphism of graded $\mathbb{R}$-algebras

$$\bigoplus_{d\in\mathbb{Z}} \mathbb{R}[\mathrm{G}(r, r-d)] \to \bigoplus_{d\in\mathbb{Z}} \mathcal{R}^{2d}(\mathcal{C}_g^r).$$

Next, consider the submonoid $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$ and the subsets $\mathrm{CG}(r, r-d) \subseteq \mathrm{G}(r, r-d)$ consisting of graphs that are contracted. The inclusion map $\mathrm{CG}(r) \to \mathrm{G}(r)$ induces a homomorphism of (graded) $\mathbb{R}$-algebras

$$\mathbb{R}[\mathrm{CG}(r)] \to \mathbb{R}[\mathrm{G}(r)],$$

and the composition of this map with α yields another graded homomorphism

$$\alpha' : \mathbb{R}[\mathrm{CG}(r)] \to \mathcal{R}^{2*}(\mathcal{C}_g^r).$$

Theorem 4.6.4 says that this homomorphism is surjective.

By Theorem 3.8.1 the set $\mathrm{CG}(r, r-d)$ is empty for all $d < 0$. For each $d \geq 0$ the set $\mathrm{CG}(r, r-d)$ is effectively computable by using the algorithm found in Section 3.7. Moreover, for $d > r + 3g - 3$ the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is trivial, so for these d the degree d summand of $\mathbb{R}[\mathrm{CG}(r, r-d)]$ is contained in the kernel of α'. It follows that, in order to compute $\mathcal{R}^*(\mathcal{C}_g^r)$, we need to compute the kernel of the linear map

$$\mathbb{R}[\mathrm{CG}(r, r-d)] \to \mathcal{R}^{2d}(\mathcal{C}_g^r)$$

for all $0 \leq d \leq r + 3g - 3$. In any case, we obtain the following.

Theorem 4.7.1. The graded $\mathbb{R}$-algebra $\mathcal{R}^{2*}(\mathcal{C}_g^r)$ is a quotient of the monoid ring $\mathbb{R}[\mathrm{CG}(r)]$. More precisely, it is a quotient of the quotient ring

$$\frac{\mathbb{R}[\mathrm{CG}(r)]}{\mathbb{R}[\mathrm{CG}(r)]_{>(r+3g-3)}} = \frac{\mathbb{R}[\mathrm{CG}(r)]}{\bigoplus_{d>r+3g-3} \mathbb{R}[\mathrm{CG}(r, r-d)]}.$$

and that quotient ring is effectively computable; it is isomorphic as a vector space to

$$\bigoplus_{d=0}^{r+3g-3} \mathbb{R}[\mathrm{CG}(r, r-d)] \qquad \square$$

The inclusion $\mathrm{CG}(r) \to \mathrm{G}(r)$ has a retraction $\varrho : \mathrm{G}(r) \to \mathrm{CG}(r)$ which is the contraction map. Therefore, the induced homomorphism $\mathbb{R}[\mathrm{G}(r)] \to \mathbb{R}[\mathrm{CG}(r)]$ is a retraction of the inclusion $\mathbb{R}[\mathrm{CG}(r)] \to \mathbb{R}[\mathrm{G}(r)]$. However, it turns out that this retraction is not the right retraction for our purposes: this retraction is incompatible with the homomorphisms from these monoid rings to the ring of tautological forms. For instance, if Γ denotes the 0-marked graph with one vertex and one loop, then the associated form α_Γ is the constant function $(2-2g)$. The contracted graph $\varrho(\Gamma)$ is the empty graph, and the associated form is the constant function 1; we therefore see that $\alpha_\Gamma \neq \alpha_{\varrho(\Gamma)}$. It is more natural to define an $\mathbb{R}$-algebra homomorphism

$$\tilde{\varrho}_g : \mathbb{R}[\mathrm{G}(r)] \to \mathbb{R}[\mathrm{CG}(r)],$$

that depends on g, as follows.

Let Γ be an r-marked graph, and let $\varrho(\Gamma)$ be the corresponding contracted r-marked graph. Define the integer $\lambda_{\Gamma,g}$ as:

$$\lambda_{\Gamma,g} = 0^a \cdot (2-2g)^b,$$

where a and b denote the number of contractions of type 0 and 2c, respectively, in the contraction procedure. Equivalently, a and b equal the number of connected components of Γ, without marked vertices, of characteristic 1 and 0, respectively. It follows from Proposition 4.6.2 that

$$\alpha_\Gamma = \lambda_{\Gamma,g} \cdot \alpha_{\varrho(\Gamma)}.$$

We define the $\mathbb{R}$-algebra homomorphism $\tilde{\varrho}_g : \mathbb{R}[\mathrm{G}(r)] \to \mathbb{R}[\mathrm{CG}(r)]$ by setting

$$\tilde{\varrho}_g(\Gamma) = \lambda_{\Gamma,g} \cdot \varrho(\Gamma) \quad \text{for all } \Gamma \in \mathrm{G}(r).$$

As $\lambda_{\Gamma,g} = 1$ for all contracted graphs, it follows that $\tilde{\varrho}_g$ is a retraction of the inclusion map $\mathbb{R}[\mathrm{CG}(r)] \to \mathbb{R}[\mathrm{G}(r)]$. Moreover, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{R}[\mathrm{G}(r)] & & \\ & \searrow\!\!\!\!\searrow & \\ \downarrow \tilde{\varrho}_g & & \mathcal{R}^{2*}(\mathcal{C}_g^r) \\ & \nearrow\!\!\!\!\nearrow & \\ \mathbb{R}[\mathrm{CG}(r)] & & \end{array} \tag{4.7.2}$$

4.8 Tautological 2-forms

In this section we give a description of the vector spaces $\mathcal{R}^2(\mathcal{C}_g^r)$ of tautological two-forms on the spaces $\mathcal{C}_g^r$ for all $r \geq 0$. Recall that we have seen some examples of these 2-forms already: on $\mathcal{C}_g^2$ we have the 2-form h associated to the diagonal, on $\mathcal{C}_g$ we have $e^A = \Delta^* h$ associated to the tangent bundle, and on $\mathcal{M}_g$ we found two more 2-forms

$$e_1^A := \int_{\mathcal{C}_g/\mathcal{M}_g} (e^A)^2 \quad \text{and} \quad \nu := \int_{\mathcal{C}_g^2/\mathcal{M}_g} h^3.$$

We will prove that these 2-forms are 'all there is': the tautological ring $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by pullbacks of h, e^A, e_1^A, and ν along tautological submersions.

Let $r \geq 0$ be an integer. We wish to compute generators for $\mathcal{R}^2(\mathcal{C}_g^r)$. By Theorem 4.6.4, we find that this space is spanned by forms α_Γ, where Γ ranges over all contracted r-marked graphs of characteristic $r-1$. In Example 3.7.2 we have computed the set $\mathrm{CG}(r, r-1)$. We found the following graphs:

- Graphs Γ with r marked vertices, no unmarked vertices, and a single edge. If this edge is a loop based at vertex i then the associated form is
$$\alpha_\Gamma = p_i^* e^A.$$
If the edge is not a loop, and its endpoints are vertices i and j, then the associated form is
$$\alpha_\Gamma = p_{ij}^* h.$$
- The graph Γ with r marked vertices, one unmarked vertex, and two loops based at the unmarked vertex. The associated form is
$$\alpha_\Gamma = \int_{p_{1,\dots,r}:\mathcal{C}_g^{r+1}\to\mathcal{C}_g^r} p_{r+1}^*(e^A)^2 = e_1^A$$
by the base change formula. Note the slight abuse of notation here: we write e_1^A for the pullback of e_1^A along the tautological morphism $\mathcal{C}_g^r \to \mathcal{M}_g$.
- The graph Γ with r marked vertices, two unmarked vertices, and three edges between the unmarked vertices. By using the base change formula we obtain
$$\alpha_\Gamma = \int_{p_{1,\dots,r}:\mathcal{C}_g^{r+2}\to\mathcal{C}_g^r} p_{r+1,r+2}^* h^3 = \nu$$
where we again abuse the notation by writing ν for the pullback of ν along $\mathcal{C}_g^r \to \mathcal{M}_g$.

We find that $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by the following collection of 2-forms:
$$\{p_{ij}^* h : 1 \le i < j \le r\} \cup \{p_i^* e^A : 1 \le i \le r\} \cup \{e_1^A, \nu\}. \qquad (4.8.1)$$
In the remainder of this section, we prove that, in fact, these 2-forms form a basis of $\mathcal{R}^2(\mathcal{C}_g^r)$ if $g > 2$, and there is only one relation among these forms if $g = 2$.

Theorem 4.8.2. For all $g \ge 2$ and $r \ge 0$, we have
$$\dim \mathcal{R}^2(\mathcal{C}_g^r) = \tfrac{1}{2} r(r+1) + 2 - \varepsilon(g),$$
where
$$\varepsilon(g) = \begin{cases} 1 & \text{if } g = 2 \\ 0 & \text{if } g \ge 3. \end{cases}$$
If $g \ge 3$ a basis is given by the 2-forms
$$\{p_{ij}^* h : 1 \le i < j \le r\} \cup \{p_i^* e^A : 1 \le i \le r\} \cup \{e_1^A, \nu\}.$$
If $g = 2$ a basis is given by the 2-forms
$$\{p_{ij}^* h : 1 \le i < j \le r\} \cup \{p_i^* e^A : 1 \le i \le r\} \cup \{e_1^A\},$$
and e_1^A and ν are linearly dependent: we have
$$-8\nu - 12 e_1^A = 0.$$

We will prove this theorem by induction on r. We start with the following proposition.

Proposition 4.8.3. If $g = 2$, then $\mathcal{R}^2(\mathcal{M}_g)$ is one-dimensional, and spanned by e_1^A. If $g \geq 3$, then $\mathcal{R}^2(\mathcal{M}_g)$ is two-dimensional, and spanned by e_1^A and ν.

Proof. Recall that the following identity holds:

$$\nu - e_1^A = \frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}.$$

Before stating Theorem 4.8.2, we showed for all $r \geq 0$ that $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by the forms listed in Equation 4.8.1. In particular, $\mathcal{R}^2(\mathcal{M}_g)$ is spanned by ν and e_1^A. Therefore, the dimension of $\mathcal{R}^2(\mathcal{M}_g)$ is at most two.

Suppose, first, that $g = 2$. In Example 4.10.5, which does not depend on any of the material treated in this section, we obtain the relation

$$-8\nu - 12e_1^A = 0.$$

Moreover, the real 2-form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}$$

is nonzero; see [DG14]. We conclude that $\mathcal{R}^2(\mathcal{M}_2)$ is one-dimensional.

Now, suppose that $g \geq 3$. By observing the asymptotic behavior of φ around the boundary of $\mathcal{M}_g$ studied in [dJon14], we find in particular that φ is not constant. Using [Kaw09, Lemma 8.1] we deduce:

4

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}} \neq 0$$

Moreover, the cohomology class κ_1 associated to e_1^A does not vanish. Indeed, one can show (see, for instance, [Mum83]) that $\kappa_1 = 12\lambda_1$ with λ_1 the first Chern class of the Hodge bundle on $\mathcal{M}_g$, and in [AC87] it is proved that λ_1 freely generates the Picard group of $\mathcal{M}_g$, and is in particular not torsion. Consequently, e_1^A is not an exact form; we find therefore that e_1^A and $\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}$ are linearly independent. ☐

Proof of Theorem 4.8.2. The case $r = 0$ is proved in Proposition 4.8.3. For the case $r = 1$: by Lemma 2.5.4 the morphism $p : \mathcal{C}_g \to \mathcal{M}_g$ induces an inclusion $p^* : \mathcal{R}^2(\mathcal{M}_g) \to \mathcal{R}^2(\mathcal{C}_g)$. Moreover, forms pulled back from $\mathcal{M}_g$ are in the kernel of the fiber integral along p: for each $\alpha \in A^*(\mathcal{M}_g)$ we have by the projection formula:

$$\int_p p^*\alpha = \alpha \cdot \int_p 1 = 0$$

As

$$\int_p e^A = (2 - 2g) \neq 0,$$

we find that e^A is not an element of $p^*\mathcal{R}^2(\mathcal{M}_g)$. As $\mathcal{R}^2(\mathcal{C}_g)$ is spanned by the forms e^A, e_1^A, and ν, we obtain:

$$\dim \mathcal{R}^2(\mathcal{C}_g) = \dim \mathcal{R}^2(\mathcal{M}_g) + 1 = 3 - \varepsilon(g).$$

Now let $r \geq 2$, and assume that

$$\dim \mathcal{R}^2(\mathcal{C}_g^s) = \tfrac{1}{2}s(s+1) + 2 - \varepsilon(g)$$

for all $0 \leq s < r$. Consider the following three tautological morphisms:

$$\begin{aligned} p_{(r)} &: \mathcal{C}_g^r \to \mathcal{C}_g^{r-1} : (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_{r-1}); \\ p_{(r-1)} &: \mathcal{C}_g^r \to \mathcal{C}_g^{r-1} : (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_{r-2}, x_r); \\ q_{(r-1)} &: \mathcal{C}_g^{r-1} \to \mathcal{C}_g^{r-2} : (x_1, \ldots, x_{r-1}) \mapsto (x_1, \ldots, x_{r-2}). \end{aligned}$$

We have a cartesian square

$$\begin{array}{ccc} \mathcal{C}_g^r & \xrightarrow{p_{(r)}} & \mathcal{C}_g^{r-1} \\ {\scriptstyle p_{(r-1)}}\downarrow & \square & \downarrow{\scriptstyle q_{(r-1)}} \\ \mathcal{C}_g^{r-1} & \xrightarrow{q_{(r-1)}} & \mathcal{C}_g^{r-2}. \end{array}$$

These maps induce linear subspaces $W_1 := \operatorname{Im} p_{(r)}^*$, $W_2 := \operatorname{Im} p_{(r-1)}^*$, and $W_{12} := W_1 \cap W_2$ of $\mathcal{R}^2(\mathcal{C}_g^r)$. The forms e_1^A, ν, $p_i^* e^A$, and $p_{ij}^* h$, (possibly) except for the form $p_{r-1,r}^* h$, all lie in W_1 or W_2. It follows that

$$\mathcal{R}^2(\mathcal{C}_g^r) = (W_1 + W_2) + \mathbb{R} \cdot p_{r-1,r}^* h.$$

Obviously the pullback of each form on $\mathcal{C}_g^{r-2}$ along the composition $q_{(r-1)} \circ p_{(r-1)} = q_{(r-1)} \circ p_{(r)}$ is an element of W_{12}. Conversely, we claim that each form in W_{12} is the pullback along this composition of some form on $\mathcal{C}_g^{r-2}$. Indeed, let $\alpha \in W_{12}$ be any form; we may write $\alpha = p_{(r)}^* \beta = p_{(r-1)}^* \gamma$ for forms $\beta, \gamma \in \mathcal{R}^2(\mathcal{C}_g^{r-1})$. Let $\mu \in \mathcal{R}^2(\mathcal{C}_g)$ be the 2-form given by $\mu = e^A/(2-2g)$; it follows that $\int_{\mathcal{C}_g/\mathcal{M}_g} \mu = 1$, and by the base change formula we obtain

$$\int_{p_{(r)}} p_r^* \mu = 1.$$

We then find by repeatedly using the base change formula and the projection formula:

$$\begin{aligned} \beta &= \beta \wedge \int_{p_{(r)}} p_r^* \mu \\ &= \int_{p_{(r)}} p_{(r)}^* \beta \wedge p_r^* \mu \\ &= \int_{p_{(r)}} p_{(r-1)}^* \gamma \wedge p_r^* \mu \\ &= \int_{p_{(r)}} p_{(r-1)}^* (\gamma \wedge p_{r-1}^* \mu) \\ &= q_{(r-1)}^* \int_{q_{(r-1)}} \gamma \wedge p_{r-1}^* \mu, \end{aligned}$$

and therefore

$$\alpha = p_{(r)}^* \beta = p_{(r)}^* q_{(r-1)}^* \int_{q_{(r-1)}} \gamma \wedge p_{r-1}^* \mu,$$

which proves our claim.

As pullbacks along tautological submersions are injective, we obtain the following equalities from the induction hypothesis:

$$\begin{aligned} \dim W_1 = \dim W_2 = \dim \mathcal{R}^2(\mathcal{C}_g^{r-1}) &= \tfrac{1}{2}r^2 - \tfrac{1}{2}r + 2 - \varepsilon(g) \\ \dim W_{12} = \dim \operatorname{Im}(p_{(r)}^* \circ q_{(r-1)}^*) = \dim \mathcal{R}^2(\mathcal{C}_g^{r-2}) &= \tfrac{1}{2}r^2 - \tfrac{3}{2}r + 3 - \varepsilon(g) \\ \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_{12} &= \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 1 - \varepsilon(g). \end{aligned}$$

If we can prove that $p_{r-1,r}^* h \notin W_1 + W_2$ then $\dim \mathcal{R}^2(\mathcal{C}_g^r) = \frac{1}{2}r^2 + \frac{1}{2}r + 2 - \varepsilon(g)$ and we are done. Suppose, therefore, that $p_{r-1,r}^* h \in W_1 + W_2$; we can write $p_{r-1,r}^* h = p_{(r)}^* \alpha + p_{(r-1)}^* \beta$ for some 2-forms α, β on $\mathcal{C}_g^{r-1}$. As h is symmetric, we may even assume with no loss of generality that $\alpha = \beta$:

$$p_{r-1,r}^* h = p_{(r)}^* \alpha + p_{(r-1)}^* \alpha.$$

Consider the map

$$f : \mathcal{C}_g^{r-1} \to \mathcal{C}_g^r : (x_1, \ldots, x_{r-1}) \mapsto (x_1, \ldots, x_{r-1}, x_{r-1});$$

this map is a section of both $p_{(r)}$ and $p_{(r-1)}$ and fits in a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_g^{r-1} & \xrightarrow{f} & \mathcal{C}_g^r \\ \downarrow{\scriptstyle p_{r-1}} & & \downarrow{\scriptstyle p_{r-1,r}} \\ \mathcal{C}_g & \xrightarrow{\Delta} & \mathcal{C}_g^2. \end{array}$$

We then find:

$$p_{r-1}^* e^A = p_{r-1}^* \Delta^* h = f^* p_{r-1,r}^* h = 2\alpha;$$

so $\alpha = \frac{1}{2} p_{r-1}^* e^A$, and

$$p_{r-1,r}^* h = \tfrac{1}{2} p_{r-1}^* e^A + \tfrac{1}{2} p_r^* e^A \in \mathcal{R}^2(\mathcal{C}_g^r).$$

Integration along the fibers of the morphism $p_{(r)} : \mathcal{C}_g^r \to \mathcal{C}_g^{r-1}$ then yields:

$$1 = \int_{p_{(r)}} p_{r-1,r}^* h = \int_{p_{(r)}} \tfrac{1}{2}(p_{r-1}^* e^A + p_r^* e^A) = 0 + \tfrac{1}{2}(2 - 2g),$$

which contradicts with our assumption that $g \geq 2$. We conclude that $p_{r-1,r}^* h$ is not in the span of the subspaces W_1 and W_2, so we find:

$$\dim \mathcal{R}^2(\mathcal{C}_g^r) = \dim(W_1 + W_2) + 1 = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2 - \varepsilon(g).$$

The theorem follows by induction. □

Let us return to the discussion we started in Section 4.3.

Corollary 4.8.4. The subspace of exact 2-forms $I^2(\mathcal{M}_g) \subseteq \mathcal{R}^2(\mathcal{M}_g)$ is one-dimensional. It is spanned by the form

$$\frac{\partial\bar{\partial}\varphi}{\pi\sqrt{-1}} = \nu - e_1^A$$

where $\varphi : \mathcal{M}_g \to \mathbb{R}$ denotes the Kawazumi–Zhang invariant.

Proof. The cohomology classes of ν and e_1^A are equal: if $p : \mathcal{C}_g \to \mathcal{M}_g$, $q : \mathcal{C}_g^2 \to \mathcal{M}_g$, and $p_1, p_2 : \mathcal{C}_g^2 \to \mathcal{C}_g$ denote the tautological submersions, then

$$[\nu] = q_*(\Delta^3) = q_*(p_1^* K^2 \cdot \Delta) = p_*(K^2 \cdot p_{1,*}\Delta) = p_* K^2 = \kappa_1 = [e_1^A] \in RH^2(\mathcal{M}_g).$$

It follows that $\nu - e_1^A$ is exact. It is moreover a nonzero form, as we saw in the proof of Proposition 4.8.3, so the dimension of $I^2(\mathcal{M}_g)$ is positive. If $g = 2$ this concludes our proof, since $\dim \mathcal{R}^2(\mathcal{M}_g) = 1$. If $g \geq 3$ then we saw in the proof of 4.8.3 that the class κ_1 of e_1^A does not vanish, so $I^2(\mathcal{M}_g)$ is a proper subspace of the two-dimensional space $\mathcal{R}^2(\mathcal{M}_g)$. □

We therefore find that the Kawazumi–Zhang invariant is the only invariant (up to additive and multiplicative constants) that arises on $\mathcal{M}_g$ from tautological forms.

4

4.9 Tautological $2d$-forms

In the previous section we have given a complete description of the vector space $\mathcal{R}^2(\mathcal{C}_g^r)$ of tautological 2-forms on the space $\mathcal{C}_g^r$. We observed that for high values of r no 'new' tautological forms appear; that is: for $r > 2$ the space $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by pullbacks of 2-forms in $\mathcal{R}^2(\mathcal{C}_g^2)$ along tautological submersions.

This observation generalizes to higher degrees, too.

Theorem 4.9.1. Let $d \geq 0$ be an integer. For all $r > 2d$ the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by pullbacks of tautological $2d$-forms on $\mathcal{C}_g^{2d}$ along tautological submersions $\mathcal{C}_g^r \to \mathcal{C}_g^{2d}$.

Proof. Let $r > 2d$ be given. By Theorem 4.6.4, it holds that the vector space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by tautological forms associated to contracted r-marked graphs Γ of characteristic $r - d$. Let Γ be such a graph. Lemma 3.8.3 implies that the number of marked vertices of positive degree is at most $2d$. Let $\phi : \{1, \ldots, 2d\} \to \{1, \ldots, r\}$ be an injective map, such that every $i \in \{1, \ldots, r\}$ with $\deg(m(i)) > 0$ lies in the image of ϕ. By Lemma 3.8.5 it follows that Γ is in the image of the pushforward map

$$\phi_* : \mathrm{CG}(2d, 2d - d) \to \mathrm{CG}(r, r - d),$$

and Proposition 4.5.3 thus implies that α_Γ lies in the image of the pullback map

$$f^{\phi,*} : \mathcal{R}^{2d}(\mathcal{C}_g^{2d}) \to \mathcal{R}^{2d}(\mathcal{C}_g^r).$$ □

In Theorem 4.8.2 we gave a closed formula for the dimension of $\mathcal{R}^2(\mathcal{C}_g^r)$ in terms of r and g. In particular, we saw that the growth rate of $\dim \mathcal{R}^2(\mathcal{C}_g^r)$ as r tends to infinity is quadratic. This latter statement can be generalized to arbitrary degree.

Theorem 4.9.2. Let $d \geq 0$. There exists a polynomial f_d of degree $2d$ (that does not depend on g), whose leading coefficient equals $1/(2^d \cdot d!)$, such that

$$\dim \mathcal{R}^{2d}(\mathcal{C}_g^r) \leq f_d(r) \quad \text{for all } r \geq 0, g \geq 2.$$

Proof. By Theorem 4.6.4 $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by forms associated to contracted r-marked graphs of characteristic $r - d$. Using Theorem 3.8.1 we find that the number of such graphs in terms of r is given by a degree $2d$ polynomial f_d with leading coefficient equal to $1/(2^d \cdot d!)$. □

In the case $d = 1$, we have seen in the last section that at most one relation appears among the tautological forms associated to graphs that span $\mathcal{R}^2(\mathcal{C}_g^r)$, and that this relation can be obtained from $\mathcal{R}^2(\mathcal{M}_g)$ via pullback. In other words: the linear relations among tautological forms associated to graphs in $\mathcal{R}^2(\mathcal{C}_g^r)$ for low values of r determine the linear relations among these forms in $\mathcal{R}^2(\mathcal{C}_g^r)$ for general r. This allows us to prove that the dimension of $\mathcal{R}^2(\mathcal{C}_g^r)$ is given by a quadratic polynomial. This polynomial does depend on g, but stabilizes for $g > 2$.

It seems natural that this result would generalize as follows. For any $d \geq 1$, any linear relations among forms associated to graphs in $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ for high r (say, $r > 2d$) would be obtained by pulling back such relations from $\mathcal{R}^{2d}(\mathcal{C}_g^{2d})$. Then, by combining arguments from sections 4.8 and 3.8, one might be able to prove that the dimension of $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is given by a polynomial of degree $2d$. The polynomial would depend on g, but might stabilize for high values of g. One of the main problems the author encounters is that the inclusion-exclusion principle, that aids us in proving that the number of r-marked graphs of a certain characteristic is given by a polynomial, does not translate well into the language of vector spaces we use in this section: while taking the intersection of sets is distributive over taking unions, the same cannot be said about taking intersections of vector subspaces and spans of vector subspaces.

4.10 Relations induced by Abel–Jacobi maps

In [Ran12] Randal-Williams constructs cohomology classes $\Omega_A \in H^2(\mathcal{C}_g^r; \mathbb{Z})$ whose $(g + 1)$st power is torsion and hence trivial when passed to cohomology with rational coefficients. These cohomology classes are tautological and can therefore be expressed as linear combinations of the 'standard' tautological classes Δ_{ij}, K_i, and κ_i. Taking the $(g+1)$st power, then, yields relations between these tautological classes. Moreover, fiber integrating these relations then gives relations between tautological classes on $\mathcal{M}_g$.

In this section we will take a similar approach to obtain relations between tautological differential forms. Recall from Sections 1.4 and 2.7 that on the universal

Jacobian we have a canonical line bundle $\mathcal{B}$, equipped with a canonical admissible metric. We denote by $2\omega_0$ the first Chern form of this hermitian line bundle. In this section we will construct morphisms $\mathcal{C}_g^r \to \mathcal{J}_g$ and show that the pullbacks of $2\omega_0$ along these morphisms are tautological differential forms. Moreover, we will see that the $(g+1)$st power of ω_0 vanishes, and we will use this to generate relations among tautological forms.

Let $f : \mathcal{C} \to S$ be a family of curves of genus $g \geq 2$, and let $\mathcal{J} \to S$ denote the associated Jacobian family. Let $r \geq 0$ be any integer, and let $m = (m_1, \ldots, m_r)$ be an r-tuple of integers whose sum equals zero. Consider the submersion

$$p = p_{(r)} : \mathcal{C}^{r+1} \to \mathcal{C}^r : (x_1, \ldots, x_{r+1}) \mapsto (x_1, \ldots, x_r)$$

and its r sections

$$\sigma_i : \mathcal{C}^r \to \mathcal{C}^{r+1} : (x_1, \ldots, x_r) \to (x_1, \ldots, x_r, x_i) \quad (1 \leq i \leq r)$$

Now consider the following line bundle on $\mathcal{C}^{r+1}$:

$$L_m = O(m_1\sigma_1 + \cdots + m_r\sigma_r) = O(\sigma_1)^{\otimes m_1} \otimes \cdots \otimes O(\sigma_r)^{\otimes m_r}$$

The restriction of this line bundle to each fiber of p has degree 0, and this line bundle hence determines a section of the Jacobian family $\mathcal{J} \times_S \mathcal{C}^r \to \mathcal{C}^r$ associated to p. The composition of this section with the projection $\mathcal{J} \times_S \mathcal{C}^r \to \mathcal{J}$ is the morphism

$$f_m : \mathcal{C}^r \to \mathcal{J} : ((x_1, \ldots, x_r) \in \mathcal{C}_s^r) \mapsto ([O(m_1x_1 + \cdots + m_rx_r)] \in \mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)).$$

We obtain from Proposition 1.4.14 a canonical isometry

$$f_m^*\mathcal{B}^{\otimes -1} \xrightarrow{\sim} \langle L_m, L_m \rangle_p.$$

As the Deligne pairing is bilinear, we find another canonical isometry

$$\langle L_m, L_m \rangle_p \xrightarrow{\sim} \bigotimes_{i=1}^{r} \bigotimes_{j=1}^{r} \langle O(\sigma_i), O(\sigma_j) \rangle^{\otimes m_i m_j}.$$

Notice that for all $1 \leq j \leq r$ we have a canonical isometry

$$O(\sigma_j) \xrightarrow{\sim} p_{j,r+1}^* O(\Delta)$$

so taking the pullback along σ_i yields yet another canonical isometry

$$\sigma_i^* O(\sigma_j) \simeq \sigma_i^* p_{j,r+1}^* O(\Delta) \simeq \begin{cases} p_{ji}^* O(\Delta) = p_{ij}^* O(\Delta) & \text{if } i \neq j \\ p_j^* \Delta^* O(\Delta) = p_j^* \omega^{\otimes -1} & \text{if } i = j. \end{cases}$$

By combining the above canonical isometries we obtain

$$f_m^* \mathcal{B}^{\otimes -1} \simeq \bigotimes_{1 \leq i < j \leq r} p_{ij}^* O(\Delta)^{\otimes 2m_i m_j} \otimes \bigotimes_{i=1}^{r} p_i^* \omega^{\otimes -m_i^2}.$$

Universally we obtain the following result.

Proposition 4.10.1. Let $r \geq 0$ be an integer, and let $(m_1, \dots, m_r)$ be a tuple of integers whose sum equals zero. Consider the morphism of stacks

$$f_m : \mathcal{C}_g^r \to \mathcal{J}_g$$

that takes a family $f : \mathcal{C} \to S$ with sections $\sigma_1, \dots, \sigma_r$ and maps it to the pair (f, σ), with σ the following section of the Jacobian family $\mathcal{J}_f \to S$:

$$\sigma : S \to \mathcal{J}_f : s \mapsto [O(m_1\sigma_1(s) + \dots + m_r\sigma_r(s))] \in \mathrm{Jac}(\mathcal{C}_s).$$

Then we have a canonical isometry of line bundles on $\mathcal{C}_g^r$:

$$f_m^* \mathcal{B}^{\otimes -1} \simeq \bigotimes_{1 \leq i < j \leq r} p_{ij}^* O(\Delta)^{\otimes 2m_im_j} \otimes \bigotimes_{i=1}^{r} p_i^* \omega^{\otimes -m_i^2}. \qquad \square$$

Taking first Chern forms then yields:

Corollary 4.10.2. Let $r \geq 0$ be an integer, and let $m = (m_1, \dots, m_r)$ be a tuple of integers whose sum equals zero. Consider the induced morphism of stacks $f_m : \mathcal{C}_g^r \to \mathcal{J}_g$ as described in Proposition 4.10.1. Then we have the following equality of 2-forms on $\mathcal{C}_g^r$:

$$-2f_m^*\omega_0 = \sum_{1 \leq i < j \leq r} 2m_im_jp_{ij}^*h + \sum_{i=1}^{r} m_i^2 p_i^* e^A \in A^2(\mathcal{C}_g^r).$$

In particular the form $f_m^*\omega_0$ is tautological. $\square$

Example 4.10.3. Set $r = 2$ and $m = (-1, 1)$, then the associated morphism

$$f_{(-1,1)} : \mathcal{C}_g^2 \to \mathcal{J}_g$$

equals the Abel–Jacobi morphism δ defined in Section 1.4 and Section 2.7. From Proposition 4.10.1 we retrieve the canonical isometry

$$\delta^*\mathcal{B}^{\otimes -1} \xrightarrow{\sim} O(\Delta)^{\otimes -2} \otimes p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1}$$

that we already encountered in Proposition 1.4.15. Taking Chern forms, then, yields the identity

$$-2\delta^*\omega_0 = -2h + p_1^*e^A + p_2^*e^A$$

that was proved in Proposition 4.3.6 and [dJon16, Theorem 1.4].

The following proposition will be used to obtain relations among tautological differential forms.

Proposition 4.10.4 ([dJon20, Proposition 5.1])**.** Let $2\omega_0 \in A^2(\mathcal{J}_g)$ be the first Chern form of the canonical line bundle $\mathcal{B}$ on $\mathcal{J}_g$ with its canonical admissible metric. Then we have

$$\omega_0^{g+1} = 0 \in A^{2g+2}(\mathcal{J}_g). \qquad \square$$

For instance, the $(g+1)$st power of the form induced by the Abel–Jacobi map in Example 4.10.3 is zero, and we can express this $(g+1)$st power in terms of tautological forms. In this example we will compute the resulting relation in the case $g = 2$, as this is still feasible to do by hand.

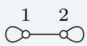
Example 4.10.5. Suppose that $g = 2$ and $r = 2$. In this case, we obtain from Example 4.10.3 and Corollary 4.10.4:

$$(-2h + p_1^* e^A + p_2^* e^A)^3 = 0 \in \mathcal{R}^6(\mathcal{C}_2^2).$$

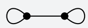
By expanding parentheses we obtain a linear combination of 10 tautological forms on $\mathcal{C}_g^2$ associated to 2-marked graphs. Of these tautological forms we can take the fiber integral along the map $\mathcal{C}_g^2 \to \mathcal{M}_g$. For instance, consider the form

$$\alpha = h \wedge p_1^* e^A \wedge p_2^* e^A \in \mathcal{R}^6(\mathcal{C}_2^2).$$

This form is the tautological form associated to the 2-marked graph

$\Gamma = \Big($  $\Big).$

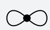
The projection $\mathcal{C}_g^2 \to \mathcal{M}_g$ is the tautological morphism associated to the map $\emptyset \to \{1,2\}$. Therefore, by Proposition 4.5.4, the fiber integral of α along this projection is the tautological form associated to the graph

$\phi^*\Gamma = \Big($  $\Big)$

We can compute the tautological form associated to $\phi^*\Gamma$ by contracting this graph: we have:

$$\alpha_{\phi^*\Gamma} = \alpha_{\tilde{\varrho}_2(\phi^*\Gamma)} = \alpha_{\varrho(\phi^*\Gamma)},$$

and $\varrho(\phi^*\Gamma)$ is the contracted graph

$\varrho(\phi^*\Gamma) = \Big($  $\Big)$

The tautological form associated to this graph is e_1^A, and we find:

$$\int_{\mathcal{C}_2^2 \to \mathcal{M}_2} h \wedge p_1^* e^A \wedge p_2^* e^A = e_1^A.$$

By repeating this procedure for all the 10 tautological forms we found earlier, we obtain the following identity:

$$-8\nu - 12 e_1^A = 0 \in A^2(\mathcal{M}_g).$$

The identity we obtain in Example 4.10.5 can be derived directly from [dJon16]. Proposition 9.1 of loc. cit. gives an identity of 2-forms on $\mathcal{M}_g$

$$e_1^J - e_1^A = \frac{2g-2}{2g+1} \cdot \frac{\partial\bar{\partial}\varphi}{\pi\sqrt{-1}},$$

where e_1^J is a 2-form on $\mathcal{M}_g$ that vanishes on the locus of hyperelliptic curves in $\mathcal{M}_g$ (by loc. cit., Proposition 10.7). Every curve of genus 2 is hyperelliptic, so for $g = 2$ we obtain the following relation:

$$-e_1^A = \frac{2}{5} \cdot \frac{\partial\bar{\partial}\varphi}{\pi\sqrt{-1}} = \frac{2}{5} \cdot (\nu - e_1^A),$$

from which one easily derives the identity found in Example 4.10.5.

4.11 Computations in higher degrees and genera

In Example 4.10.5 we used Corollary 4.10.2 and Corollary 4.10.4 to obtain a relation among tautological forms in $\mathcal{R}^2(\mathcal{M}_2)$. This was relatively easy, as we only needed to work with the third power of ω, and hence needed to compute the fiber integral of 'only' 10 differential forms. Of course, if we want to construct similar relations in higher genera, or in higher degrees, it quickly becomes infeasible to do this by hand. In this section, we describe an algorithm for finding relations among generators of spaces $\mathcal{R}^{2d}(\mathcal{C}_g^s)$, and provide some example computations.

4

Recall that in Section 4.7 we have constructed a surjective graded homomorphism $\mathbb{R}[\mathrm{CG}(r)] \to \mathcal{R}^{2*}(\mathcal{C}_g^r)$. We will denote this morphism by α_r. Computations in the ring $\mathbb{R}[\mathrm{CG}(r)]$ can be carried out effectively. We will be using the following lemma to construct elements in the kernel of α_r.

Lemma 4.11.1. Let $r \geq 2$ be an integer. For $1 \leq i, j \leq r$ let Γ_{ij} be the r-marked graph with no unmarked vertices and a single edge between the vertices marked i and j; notice that Γ_{ij} is contracted as it has no unmarked vertices, and notice that $\Gamma_{ij} = \Gamma_{ji}$. Consider the polynomial ring

$$\mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}],$$

and define $x_r = -x_1 - \cdots - x_{r-1}$. Now define the polynomial

$$W_r = \sum_{i,j=1}^{r} \Gamma_{ij} \cdot x_i x_j \in \mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}].$$

Then W_r^{g+1} lies in the kernel of the homomorphism

$$\bar{\alpha}_r : \mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}] \to \mathcal{R}^{2*}(\mathcal{C}_g^r)[x_1, \ldots, x_{r-1}]$$

induced by α_r. In particular, all coefficients of W_r^{g+1} lie in the kernel of α_r.

Proof. Set $w_r = \alpha_r(W_r)$. By Corollary 4.10.2 we then have for all $m_1, \dots, m_{r-1} \in \mathbb{Z}$:

$$w_r(m_1, \dots, m_{r-1}) = -2f_m^* \omega_0 \in \mathcal{R}^2(\mathcal{C}_g^r)$$

where m denotes the tuple $(m_1, \dots, m_{r-1}, -m_1 - \cdots - m_{r-1})$. By Proposition 4.10.4 we then see that w_r^{g+1} vanishes on $\mathbb{Z}^{r-1}$, which implies that it must be the zero polynomial. □

Let $s \leq r$ be an integer, and consider the inclusion map $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$. From Diagram 4.7.2 and Proposition 4.5.4 we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{R}[\mathrm{CG}(r)] & \xrightarrow{\alpha_r} & \mathcal{R}^{2*}(\mathcal{C}_g^r) \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle \int_{f\phi}} \\ \mathbb{R}[\mathrm{G}(s)] & \longrightarrow & \mathcal{R}^{2*}(\mathcal{C}_g^s) \\ {\scriptstyle \tilde{\varrho}_g}\downarrow & \nearrow_{\alpha_s} & \\ \mathbb{R}[\mathrm{CG}(s)] & & \end{array} \tag{4.11.2}$$

In particular, we may pass the coefficients of W_r^{g+1} through the homomorphism $\tilde{\varrho}_g \circ \phi^*$ to obtain elements in the kernel of α_s.

Example 4.11.3. Let $r = 2, s = 0, g = 2$. We have:

$$W_r = \Gamma_{11}x_1^2 + \Gamma_{12}x_1x_2 + \Gamma_{21}x_2x_1 + \Gamma_{22}x_2^2 = (\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})x_1^2,$$

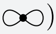
and

$$W_r^3 = (\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})^3 x_1^6.$$

From Lemma 4.11.1 we find that $(\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})^3$ lies in the kernel of α_r. In other words: we have the following identity of tautological forms on $\mathcal{C}_2^2$:

$$(p_1^* e^A - 2h + p_2^* e^A)^3 = 0,$$

which was already clear from Proposition 4.3.6 and Proposition 4.10.4. A computation by hand shows:

$(\tilde{\varrho}_g \circ \phi_*)((\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})^3) = -12$ () $- 8$ () $\in \mathrm{Ker}(\alpha_s)$,

and applying α_s to this element of $\mathbb{R}[\mathrm{CG}(s)]$ then yields the identity

$$-12e_1^A - 8\nu = 0 \in \mathcal{R}^2(\mathcal{M}_2)$$

we found in Example 4.10.5.

Another trick we can use is the following. Instead of viewing the genus g as a constant, we view it as a variable. In Diagram 4.11.2 we replace the base ring $\mathbb{R}$

in the left column by the polynomial ring $\mathbb{R}[g']$. We thus obtain for all $g \geq 2$ a commutative diagram

$$\begin{array}{ccc} \mathbb{R}[g'][\mathrm{CG}(r)] & \xrightarrow{\alpha_r} & \mathcal{R}^{2*}(\mathcal{C}_g^r) \\ \big\downarrow{\scriptstyle \phi^*} & & \big\downarrow{\scriptstyle \int_{f\phi}} \\ \mathbb{R}[g'][\mathrm{G}(s)] & \longrightarrow & \mathcal{R}^{2*}(\mathcal{C}_g^s) \\ \big\downarrow{\scriptstyle \tilde{\varrho}} & \nearrow{\scriptstyle \alpha_s} & \\ \mathbb{R}[g'][\mathrm{CG}(s)] & & \end{array} \tag{4.11.4}$$

Here $\tilde{\varrho}$ denotes the unique morphism of $\mathbb{R}[g']$-algebras that maps a graph $\Gamma \in \mathrm{G}(s)$ to $\lambda_\Gamma \cdot \varrho(\Gamma)$, where λ_Γ is given by

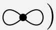
$$\lambda_\Gamma = 0^a(2 - 2g')^b \in \mathbb{R}[g']$$

with a and b the number of connected components of Γ with no marked vertices of characteristic 1 and 0, respectively. The maps from left to right in diagram 4.11.4 are the unique extensions of the corresponding maps in diagram 4.11.2 that map g' to g.

Our algorithm for finding elements in the kernel of α_s is as follows. We pick $r \geq 2$, $0 \leq s \leq r$, and $G \geq 2$. We then compute $W_r^{G+1} \in \mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}]$. Then for each of the coefficients $c \in \mathbb{R}[\mathrm{CG}(r)]$ of W_r^{G+1}, compute $\tilde{\varrho}(\phi^* c)$. The resulting element is in the kernel of the homomorphism $\mathbb{R}[g'][\mathrm{CG}(s)] \to \mathcal{R}^{2*}(\mathcal{C}_g^s)$ for all $2 \leq g \leq G$.

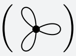
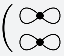
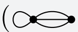
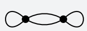
An implementation of this algorithm in Sage is provided in [vdLug21]. We will list some results for low values of r, s, G.

Example 4.11.5. If we set $r = 2$, $s = 0$, and $G = 4$, we find that for all $2 \leq g \leq 4$, the following element lies in the kernel of the map $\mathbb{R}[\mathrm{CG}(0)] \to \mathcal{R}^2(\mathcal{M}_g)$:

$-3(g-4)(g-3)(g-1)$ () $- 2(g-4)(g-3)$ () .

If $g = 2$ we retrieve Example 4.11.3. For $g = 3$ and $g = 4$ we retrieve nothing at all, as the above vector vanishes.

Example 4.11.6. If we set $r = 2$, $s = 0$ and $G = 3$, we obtain the following element in the kernel of $\mathbb{R}[\mathrm{CG}(0)] \to \mathcal{R}^{2*}(\mathcal{M}_g)$ for all $g \in \{2, 3\}$:

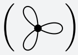
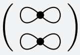
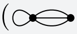
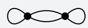
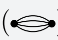
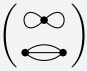
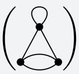
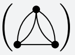
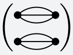
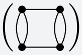
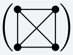
$-8g$ () $+ 3$ () $- 32$ () $+ 24$ () $+ 8$ ()

In terms of differential forms we obtain the following relation among tautological forms in $\mathcal{R}^4(\mathcal{M}_g)$ for $g \in \{2, 3\}$:

$$-8g \cdot e_2^A + 3(e_1^A)^2 - 32 \int (p_1^* e^A \wedge h^3) + 24 \int (p_1^* e^A \wedge p_2^* e^A \wedge h^2) + 8 \int h^4 = 0$$

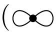
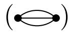
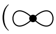
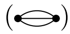
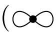
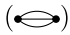
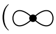
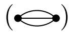
where the integral symbol denotes fiber integration along the map $\mathcal{C}_g^2 \to \mathcal{M}_g$.

Example 4.11.7. If we set $r = 4$, $s = 0$ and $G = 5$, we obtain another element in $\mathbb{R}[\mathrm{CG}(0)]$ that involves all the 11 contracted unmarked graphs of characteristic -2. For $2 \le g \le 5$, the following element lies in the kernel of the map $\mathbb{R}[\mathrm{CG}(0)] \to \mathcal{R}^4(\mathcal{M}_g)$.

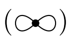
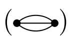
$- 8(g-4)(4g^2-20g+3)$ () $+ (9g^2-87g+201)$ ()
$- 32(g-5)(4g-15)$ () $+ (72g^2-696g+1608)$ ()
$+ 24(g-5)(g-4)$ () $- 4(g-2)$ () $- 48(g-2)$ ()
$- 48(g-5)$ () $- 4$ () $- 72$ () $- 48$ () .

Our algorithm takes integers r, s, G and gives relations in $\mathcal{R}^{2d}(\mathcal{C}_g^s)$, where $d = G + 1 - r + s$. It is interesting to observe what happens to these relations when we fix s and d and let G (and hence $r = G + 1 + s - d$) increase.

For example, fixing $s = 0$, $d = G + 1 - r + s = 1$ and running our algorithm with G increasing from 2 to 5 yields the following elements of $\mathbb{R}[g][\mathrm{CG}(0)]$:

G	vectors in $\mathbb{R}[g][\mathrm{CG}(0)]$
2	$(-3(g-1)$ () $- 2$ ())
3	$(g-3)\,(-3(g-1)$ () $- 2$ ())
4	$(g-4)(g-3)\,(-3(g-1)$ () $- 2$ ())
5	$(g-5)(g-4)(g-3)\,(-3(g-1)$ () $- 2$ ())

The pattern is clear: it seems that for $G \ge 2$ the following vector is obtained in $\mathbb{R}[g][\mathrm{CG}(0,-1)]$:

$$\left(\prod_{k=3}^{G}(g-k)\right) \cdot (-3(g-1)$$ () $- 2$ ()) .

In particular, the only value of g for which this vector yields a nontrivial relation in the ring of tautological forms would then be $g = 2$.

A similar phenomenon occurs if we increase G in examples 4.11.6 and 4.11.7. This suggests that relations (or at least, relations found using ω_0) among elements of $\mathcal{R}^*(\mathcal{M}_g)$ for low values of g vanish if we let g increase. These observations prompt the following question.

Question 4.11.8. Suppose we are given integers $r \ge 0$ and $d \ge 0$. Does there exist a $g_0 \ge 2$ such that for all $g \ge g_0$ the linear map

$$\mathbb{R}[\mathrm{CG}(r, r-d)] \to \mathcal{R}^{2d}(\mathcal{C}_g^r)$$

is an isomorphism? Is there an expression for g_0 in terms of r and d?

Recall that an analogue in rings of tautological classes is given by Mumford's conjecture (proved in [MW07]) that for any $d > 0$ there exists a $g_0 \geq 2$ such that the map $\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to RH^*(\mathcal{M}_g)$ is an isomorphism in degree d for all $g \geq g_0$.

Theorem 4.8.2 moreover states that the above question can be answered with 'yes' if $d = 1$. In this case, we have:

$$\dim \mathbb{R}[\mathrm{CG}(r, r-1)] = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2$$

by Example 3.7.2, and for all $g \geq g_0 = 3$ we have:

$$\dim \mathcal{R}^2(\mathcal{C}_g^r) = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2$$

by Theorem 4.8.2, and the linear map

$$\mathbb{R}[\mathrm{CG}(r, r-d)] \to \mathcal{R}(\mathcal{C}_g^r)$$

is therefore an isomorphism, as it is a surjective map between vector spaces of the same dimension. It is moreover interesting to see that in this case the value $g_0 = 3$ does not depend on r.