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Tautological differential forms on moduli spaces of curves

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Chapter 3

Marked graphs

In this chapter we will study r-marked graphs, where $r \geq 0$ is an integer. These are finite graphs of which r of the vertices are labeled with the integers $1, \ldots, r$. We define categories of r-marked graphs, show that these categories have pushouts, and construct pushforward and pullback functors between these categories.

A contracted r-marked graph is an r-marked graph whose vertices have a sufficiently high degree, and any r-marked graph can be turned into a contracted r-marked graph by means of certain contraction operations. It turns out that for each $r \geq 0$ and each $\chi \in \mathbb{Z}$ there are only finitely many isomorphism classes of contracted r-marked graphs of characteristic χ. We will describe an algorithm to compute the number of isomorphism classes.

The reason we are interested in r-marked graphs is that they provide us with a combinatorial framework that can be used to work with tautological differential forms. Fix an integer $g > 1$. In Chapter 4, we will describe a method of assigning to each r-marked graph Γ a tautological differential form $\alpha_\Gamma \in R^*(\mathcal{C}_g^r)$. It turns out that there is an interaction between r-marked graphs and tautological forms on $\mathcal{C}_g^r$, where taking pushouts corresponds to taking wedge products, and taking pushforwards and pullbacks of graphs corresponds to taking pullbacks and fiber integrals of forms.

Moreover, it turns out that *all* tautological differential forms on $\mathcal{C}_g^r$ arise from contracted r-marked graphs. We can give upper bounds to the dimensions of spaces of tautological forms by computing the number of marked graphs. In short, the combinatorial heavy lifting will be done in this chapter, and we use the results from this chapter to bound dimensions of spaces of tautological forms in Chapter 4.

3.1 The category of r-marked graphs

In this thesis, a *graph* is a pair (V, E), consisting of a finite set V of *vertices*, and a finite multiset E of *edges* consisting of unordered pairs (multisets of cardinality 2) of elements of V. If $e \in E$ is an edge, its two elements are called the *endpoints* of e. If these endpoints are the same, we call e a *loop*. The *degree* of a vertex $v \in V$, denoted $\deg v$, is the number of times v occurs as an endpoint of an edge of E; that is: the multiplicity of v in the multiset sum of all edges $e \in E$. In particular, we see that each loop contributes 2 to the degree of the vertex it is based on.

In short, we assume our graphs to be finite and undirected, and our graphs are allowed to have multiple edges and loops. Moreover, our graphs do not necessarily have to be connected.

If $\Gamma = (V, E)$ is a graph, then the *(Euler) characteristic* of Γ is defined as

$$\chi(\Gamma) = |V| - |E|.$$

The Euler characteristic is additive on disjoint unions of graphs.

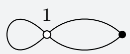
Let $r \geq 0$ be an integer. An *r-marked graph* (V, E, m) is a graph $\Gamma = (V, E)$ equipped with a *marking* m; that is: an injective map $m : \{1, \dots, r\} \to V$. So a marked graph can be seen as a graph of which r vertices are labeled $1, \dots, r$. An *unmarked graph* is a 0-marked graph, which is the same as an 'ordinary' graph.

Let $\Gamma = (V, E, m)$ be an r-marked graph. A vertex $v \in V$ is *marked* if it is in the image of m, and *unmarked* otherwise. We have a partition of V in a subset V_+ of marked vertices and a subset V_- of unmarked vertices.

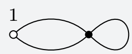
Let $\Gamma = (V, E, m)$ and $\Gamma' = (V', E', m')$ be two r-marked graphs. A *morphism of r-marked graphs* $f : \Gamma \to \Gamma'$ is a pair of maps $(f_{\mathrm{v}} : V \to V', f_{\mathrm{e}} : E \to E')$, such that f_{v} respects the r-marking (that is: $f_{\mathrm{v}} \circ m = m'$), and such that for each edge $e \in E$ with endpoints v, w, the edge $f_{\mathrm{e}}(e) \in E'$ has endpoints $f_{\mathrm{v}}(v)$ and $f_{\mathrm{v}}(w)$.

We obtain a category $\mathcal{G}_r$ of r-marked graphs. Two r-marked graphs Γ and Γ' are isomorphic if and only if there exists a bijection on vertices that respects the markings of Γ and Γ', such that for each pair of vertices v, w of Γ the number of edges between v and w equals the number of edges between the corresponding vertices of Γ'.

Example 3.1.1. The following two 1-marked graphs are not isomorphic:



and



Indeed, the marked vertex in the leftmost graph has degree 4, while the marked vertex in the rightmost graph has degree 2.
The corresponding 0-marked graphs, obtained by 'forgetting' the 1-markings, are isomorphic.

The following construction will return in the next sections. Assume that $\Gamma = (V, E)$ is a graph, and let $f : V \to V'$ be a map of finite sets. The *graph induced (from Γ) by f*, notation Γ_f, is the graph (V', E') with set of vertices equal to V', and with edges

$$E' = \{\{f(v_1), f(v_2)\} : \{v_1, v_2\} \in E\}.$$

Notice that in particular we have $|E| = |E'|$.

The characteristic of Γ_f equals

$$\chi(\Gamma_f) = \chi(\Gamma) + |V'| - |V|.$$

If $v' \in V'$ is a vertex in V', its degree is given by:

$$\deg(v') = \sum_{v \in f^{-1}(v')} \deg(v).$$

3.2 Gluing marked graphs

In this section we define a binary operation $\sqcup_r$ on the category of r-marked graphs $\mathcal{G}_r$. It turns out that $\sqcup_r$ is the coproduct in the category $\mathcal{G}_r$. We define the binary operation $\sqcup_r$ on two r-marked graphs Γ, Γ' by gluing their marked vertices pairwise. More precisely, we proceed as follows.

Let $\Gamma = (V, E, m)$ and $\Gamma' = (V', E', m')$ be two r-marked graphs, and let $\Gamma \sqcup \Gamma' = (V \sqcup V', E + E')$ denote the disjoint union of the underlying (unmarked) graphs. Consider the set V'' defined by the pushout diagram

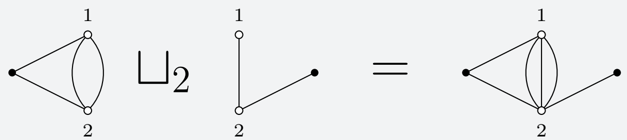
$$\begin{tikzcd} \{1, \ldots, r\} \arrow[r, "m"] \arrow[d, "m'"'] & V \arrow[d] \\ V' \arrow[r] \arrow[ur, phantom, "\lrcorner"] & V''. \end{tikzcd} \tag{3.2.1}$$

In other words, V'' is the set $(V \sqcup V')/\sim$, where $\sim$ is the smallest equivalence relation on $V \sqcup V'$ such that $m(i) \sim m'(i)$ for all $i \in \{1, \ldots, r\}$. Note, moreover, that the map $m'' : \{1, \ldots, r\} \to V''$ induced by the above diagram is injective, since m and m' are injective.

Definition 3.2.2. The r-marked graph $\Gamma \sqcup_r \Gamma'$ is the graph induced from the disjoint union $\Gamma \sqcup \Gamma'$ by the natural map $V \sqcup V' \to V''$, endowed with the r-marking $m'' : \{1, \ldots, r\} \to V''$ obtained from pushout diagram 3.2.1.

In short: we take the disjoint union of Γ and Γ', and identify the vertices in V and V' whose markings are equal.

Example 3.2.3. The following picture illustrates the operation $\sqcup_2$ on $\mathcal{G}_2$.



Suppose that Γ has u unmarked vertices and e edges, and that Γ' has u' unmarked vertices and e' edges. It follows that $\Gamma \sqcup_r \Gamma'$ has $u + u'$ unmarked vertices and $e + e'$ edges. Therefore, the characteristic of $\Gamma \sqcup_r \Gamma'$ is given by

$$\chi(\Gamma \sqcup_r \Gamma') = \chi(\Gamma) + \chi(\Gamma') - r. \tag{3.2.4}$$

The set of vertices of $\Gamma \sqcup_r \Gamma'$ is the pushout of the maps m and m'. It is therefore straightforward to verify the following.

Proposition 3.2.5. For each pair of r-marked graphs Γ_1, Γ_2, the graph $\Gamma_1 \sqcup_r \Gamma_2$ is the coproduct of Γ_1 and Γ_2 in the category $\mathcal{G}_r$. □

We find that the operator $\sqcup_0$ on $\mathcal{G}_0$ is simply the disjoint union. On $\mathcal{G}_1$ the operator $\sqcup_1$ is the wedge sum.

Proposition 3.2.6. The category $\mathcal{G}_r$ has all finite coproducts.

Proof. The graph consisting of r marked vertices, no unmarked vertices, and no edges is the initial object of $\mathcal{G}_r$. As $\mathcal{G}_r$ has an initial object and all binary coproducts, it follows that $\mathcal{G}_r$ has all finite coproducts. □

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3.3 Pushforward maps on marked graphs

Let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be a map of sets. We will define a pushforward functor $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$. Given a graph $\Gamma \in \mathcal{G}_s$, the pushforward $\phi_*\Gamma$ is obtained from Γ by replacing the s marked vertices by r marked vertices, as follows.

Let $\Gamma = (V, E, m)$ be an s-marked graph. Consider the pushout diagram (of sets)

$$\begin{tikzcd} \{1, \ldots, s\} \arrow[r, "m"] \arrow[d, "\phi"] \arrow[dr, phantom, "\lrcorner" very near end] & V \arrow[d, "\phi_V"] \\ \{1, \ldots, r\} \arrow[r, "m'"'] & V'. \end{tikzcd} \tag{3.3.1}$$

As m is injective, it follows that m' must be injective.

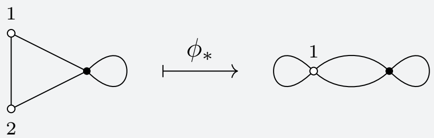
We define $\phi_*\Gamma$ to be the graph (V', E', m'), where (V', E') is the graph induced from (V, E) by ϕ_V, and m' is the map defined in diagram 3.3.1. Notice that ϕ_V then induces a bijection between the unmarked vertices of Γ and $\phi_*\Gamma$.

Alternatively, we can construct the graph $\phi_*\Gamma$ as follows: first, we construct a graph Γ'' by adding r vertices $v_1'', \ldots, v_r''$ to Γ. Next, we define the equivalence relation $\sim$ on the set of vertices V'' to be the smallest equivalence relation such that $v_i \sim v''_{\phi(i)}$ for all $i \in \{1, \ldots, s\}$, where $v_1, \ldots, v_s$ are the marked vertices of Γ. Then $\phi_*\Gamma$ is the graph quotient $\Gamma''/\sim$.

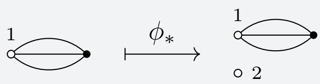
The characteristic of $\phi_*\Gamma$ is given by

$$\chi(\phi_*\Gamma) = \chi(\Gamma) - s + r.$$

Example 3.3.2. Let $\phi : \{1, 2\} \to \{1\}$ be the unique map. The pushforward $\phi_* : \mathcal{G}_2 \to \mathcal{G}_1$ identifies the two marked points of a 2-marked graph. The following picture illustrates taking the pushforward $\phi_*\Gamma$ of a graph $\Gamma \in \mathcal{G}_2$:



Example 3.3.3. Consider the map $\phi : \{1\} \to \{1, 2\}$ that maps 1 to 1. The pushforward map ϕ_* adds a second marked vertex to any 1-marked graph Γ. This second marked vertex is not the endpoint of any edge.



Moreover, if $f : \Gamma_1 \to \Gamma_2$ is a morphism of s-marked graphs, we obtain an induced morphism of r-marked graphs $\phi_* f : \phi_*\Gamma_1 \to \phi_*\Gamma_2$, via the universal property of the pushout diagram 3.3.1. We obtain a covariant functor

$$\phi_* : \mathcal{G}_s \to \mathcal{G}_r.$$

It is not hard to see that the pushforward functor is well-behaved with respect to compositions.

Proposition 3.3.4. Let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ and $\psi : \{1, \ldots, t\} \to \{1, \ldots, s\}$ be maps. Then the functors $\phi_*\psi_*$ and $(\phi\psi)_*$ from $\mathcal{G}_t$ to $\mathcal{G}_r$ are naturally isomorphic. □

The pushforward operator is compatible with gluing.

Proposition 3.3.5. Let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be a map. Let Γ and Γ' be two s-marked graphs. Then there is a canonical isomorphism of graphs

$$\phi_*(\Gamma \sqcup_s \Gamma') \simeq \phi_*(\Gamma) \sqcup_r \phi_*(\Gamma').$$

Proof. Let V and V' denote the sets of vertices of Γ and Γ', respectively. The set V_1 of vertices of $\phi_*(\Gamma \sqcup_s \Gamma')$ can be obtained by repeatedly taking pushouts:

$$V_1 = (V \sqcup_{\{1,\ldots,s\}} V') \sqcup_{\{1,\ldots,s\}} \{1, \ldots, r\}$$

The same holds for the set V_2 of vertices of $\phi_*(\Gamma) \sqcup_r \phi_*(\Gamma')$:

$$V_2 = (V \sqcup_{\{1,\ldots,s\}} \{1, \ldots, r\}) \sqcup_{\{1,\ldots,r\}} (V' \sqcup_{\{1,\ldots,s\}} \{1, \ldots, r\}),$$

By the universal property of pushouts these sets are canonically isomorphic. It is straightforward to verify that the canonical isomorphism induces a bijection on edges. □

3.4 Pullback maps on marked graphs

The next operation we will consider is a pullback operation. Let

$$\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$$

be an *injective* map. Then we define a *pullback* functor

$$\phi^* : \mathcal{G}_r \to \mathcal{G}_s$$

as follows. For any r-marked graph $\Gamma = (V, E, m)$ we simply define $\phi^*\Gamma$ by pre-composing the marking m with the injection ϕ:

$$\phi^*(\Gamma) = (V, E, m \circ \phi).$$

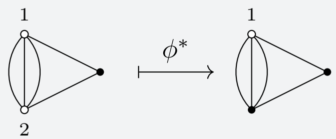
So all the pullback functor does is maybe re-ordering and possibly forgetting some markings of vertices.

It follows that

$$\chi(\phi^*\Gamma) = \chi(\Gamma).$$

If $f : \Gamma_1 \to \Gamma_2$ is a morphism of r-marked graphs, then f induces a morphism $\phi^* f : \phi^*\Gamma_1 \to \phi^*\Gamma_2$ in a natural way. It is straightforward to verify that ϕ^* is a functor from $\mathcal{G}_r \to \mathcal{G}_s$.

Example 3.4.1. Let $\phi : \{1\} \to \{1, 2\}$ be the inclusion. The pullback $\phi^* : \mathcal{G}_2 \to \mathcal{G}_1$ takes a 2-marked graph and turns it into a 1-marked graph by forgetting the marking of the second marked point.



Similarly to the pushforward, it is easy to see that the pullback is well-behaved with respect to compositions.

Proposition 3.4.2. Let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ and $\psi : \{1, \ldots, t\} \to \{1, \ldots, s\}$ be injective maps. Then the functors $\psi^*\phi^*$ and $(\phi\psi)^*$ from $\mathcal{G}_r$ to $\mathcal{G}_t$ are equal. $\square$

There is an adjointness between the pushforward and pullback functor. Contrary to what the terms ‘pushforward’ and ‘pullback’ might suggest to a geometer, the pushforward functor is *left* adjoint to the pullback. To ease our minds, we recall that the (left adjoint) pushforward functor does pushouts on sets of vertices, and the (right adjoint) pullback functor is a functor that *forgets* some of the markings.

Proposition 3.4.3. If $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ is an injective map, the push-forward functor $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$ is left adjoint to the pullback functor $\phi^* : \mathcal{G}_r \to \mathcal{G}_s$.

Proof. The statement follows from the universal property of the pushout diagram 3.3.1, together with the fact that the pushforward and pullback maps do not alter the sets of edges. □

3.5 The monoid of r-marked graphs

In the previous sections we have defined the categories of marked graphs, showed that these categories have coproducts, and constructed pushforward and pullback functors between these categories. Later on, we will try to classify such r-marked graphs, or rather a specific subset of *contracted* r-marked graphs. In Chapter 4 we will use this classification of r-marked graphs in order to classify tautological differential forms. For these purposes, categories of marked graphs are too big; we only need to find all r-marked graphs up to isomorphism.

Definition 3.5.1. Let $r \geq 0$ be an integer. Then we denote by $\mathrm{G}(r)$ the set of isomorphism classes of r-marked graphs. If $\chi \in \mathbb{Z}$ is another integer, we let $\mathrm{G}(r, \chi) \subseteq \mathrm{G}(r)$ denote the subset of the isomorphism classes of r-marked graphs with characteristic χ. Lastly, if $u \geq 0$ is an integer, the subset $\mathrm{G}(r, \chi, u) \subseteq \mathrm{G}(r, \chi)$ denotes the subset of the classes of the graphs that have u unmarked vertices.

Of course, we need to be a bit careful here and remark that this definition of $\mathrm{G}(r)$ does not yield a set under the ZFC axioms, as almost every element of $\mathrm{G}(r)$ will be a proper class. However, it is straightforward to construct a set S of r-marked graphs such that every r-marked graph is isomorphic to one in S, and we can then view $G(r)$ as the quotient set $S/{\cong}$.

Given isomorphisms of r-marked graphs $\Gamma_1 \cong \Gamma_2$ and $\Gamma_1' \cong \Gamma_2'$, there exists a natural isomorphism $(\Gamma_1 \sqcup_r \Gamma_1') \cong (\Gamma_2 \sqcup_r \Gamma_2')$. This implies that the operator $\sqcup_r$ defines a binary operation on the set $\mathrm{G}(r)$. We immediately obtain the following proposition.

Proposition 3.5.2. Equipping the set $\mathrm{G}(r)$ with the binary operation $\sqcup_r$ yields a commutative monoid, whose identity element is (the class of) the r-marked graph with no edges and no unmarked vertices. □

It follows from Equation 3.2.4 that the map

$$\chi_r : \mathrm{G}(r) \to \mathbb{Z} : \Gamma \mapsto \chi(\Gamma) - r$$

is a homomorphism of monoids.

Suppose, now, that $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ is a map of sets. This map defines a pushforward functor on marked graphs $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$. This functor induces a map $\phi_* : \mathrm{G}(s) \to \mathrm{G}(r)$. The following proposition follows directly from Proposition 3.3.5.

Proposition 3.5.3. For every map $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$, the map $G(s) \to G(r)$ induced by the functor $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$ is a homomorphism of monoids. $\square$

If ϕ is an injective map, we also have a pullback functor $\phi^* : \mathcal{G}_r \to \mathcal{G}_s$. The induced map on monoids $\mathrm{G}(r) \to \mathrm{G}(s)$, however, is almost never a homomorphism. For instance, consider the identity element $\Gamma \in \mathrm{G}(r)$, which is the r-marked graph with no edges and no unmarked vertices. Then $\phi^*\Gamma$ is an s-marked graph with $(r - s)$ unmarked vertices, so this graph is not the identity element of $\mathrm{G}(s)$ unless $r = s$.

3.6 Contracted graphs

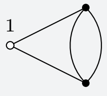
In Chapter 4 we will fix an integer $g > 1$, and associate to any r-marked graph Γ a differential form α_Γ on the moduli stack $\mathcal{C}_g^r$. It will turn out that the form α_Γ is invariant under certain contraction operations on these r-marked graphs. This will allow us to restrict ourselves to studying differential forms associated to graphs which cannot be contracted any further. We will study such graphs in this section.

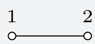
Definition 3.6.1. Let Γ be an r-marked graph. Then Γ is *contracted* if all its unmarked vertices have degree at least 3, and each unmarked vertex of degree 3 is incident to three distinct edges.
We denote by $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$ the subset of isomorphism classes of contracted r-marked graphs.

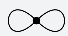
Equivalently, a marked graph is contracted if each unmarked vertex has degree at least 3, and there are no loops at any of the unmarked vertices of degree 3.

Notice that, by construction of the binary operation $\sqcup_r$, the subset $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$ is in fact a submonoid.

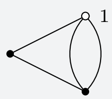
Example 3.6.2. The following marked graphs are contracted.

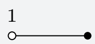


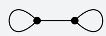




The following marked graphs are not contracted.







If a graph is not contracted, we can attempt to turn this graph into a contracted graph by altering the problematic vertices.

Definition 3.6.3. Let $\Gamma = (V, E, m)$ be an r-marked graph, and let $v \in V$ be an unmarked vertex such that $\deg(v) \leq 2$, or such that $\deg(v) = 3$ and v is incident to a loop. The graph obtained from Γ by *contracting* v is an r-marked graph Γ' defined by the following operation:

0. If $\deg v = 0$, remove v;
1. If $\deg v = 1$, remove v and the unique edge incident to v;
2. If $\deg v = 2$, *smooth out* the vertex v; that is:
   (a) If v is incident to two distinct edges, whose other endpoints w, w' are distinct, remove v and these two edges, and add an edge between w and w';
   (b) If v is incident to two distinct edges, whose second endpoint is the same vertex w, remove v and these two edges, and add a loop at w;
   (c) If v is incident to a single loop, remove v and this loop;
3. If $\deg v = 3$, and w is the other endpoint of the non-loop edge incident to v, remove v, this edge, and the loop at v, and add a loop at w.

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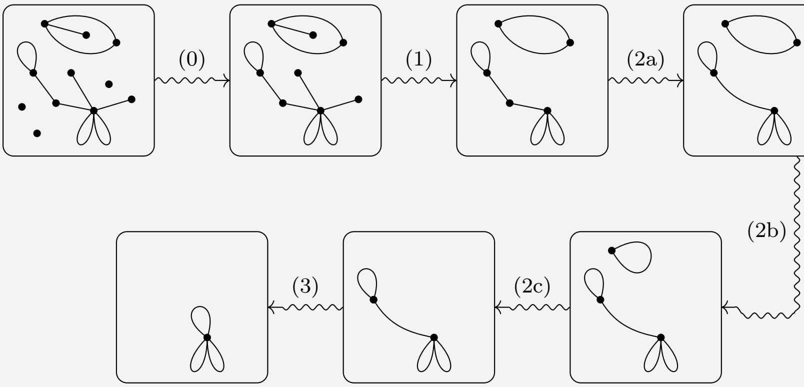
It follows that the vertex set of the graph obtained from Γ by contracting v is equal to $V \setminus \{v\}$.

Example 3.6.4. The following example illustrates all of the graph manipulations listed in the above definition. The graph is unmarked.



If we are given an r-marked graph Γ, we can always reduce Γ to a contracted r-marked graph by applying a finite amount of graph contractions. We hence obtain a map

$$\varrho : \mathrm{G}(r) \to \mathrm{CG}(r),$$

and this map is a retraction of the inclusion $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$. Moreover, as the contraction operations only apply to unmarked vertices, it follows that the contraction map commutes with gluing of r-marked graphs:

$$\varrho(\Gamma_1 \sqcup_r \Gamma_2) = \varrho(\Gamma_1) \sqcup_r \varrho(\Gamma_2).$$

In other words: $\mathrm{CG}(r)$ is a submonoid of $\mathrm{G}(r)$ and the contraction map $\varrho : \mathrm{G}(r) \to \mathrm{CG}(r)$ is a homomorphism of monoids.

3.7 Counting contracted graphs

The number of contracted r-marked graphs is infinite. For instance, for $r = 0$, and $e \geq 2$, we can consider the unmarked graph consisting of 1 vertex and e loops. This example gives us an infinite family of contracted unmarked graphs. However, a finiteness result does hold if we only consider r-marked graphs of a fixed characteristic.

Theorem 3.7.1. Let $r \geq 0$, and $\chi \in \mathbb{Z}$. There are, up to isomorphism, only finitely many contracted r-marked graphs of characteristic χ. These graphs have at most $2(r - \chi)$ unmarked vertices.

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Proof. Let Γ be a contracted r-marked graph of characteristic χ, and let u denote its number of unmarked vertices, and e its number of edges. We have:

$$\chi = r + u - e.$$

Moreover, every unmarked vertex has degree at least 3. It follows that

$$2e = \sum_{v \in \Gamma} \deg(v) \geq 3u.$$

After substituting $e = r + u - \chi$, we find:

$$u \leq 2r - 2\chi,$$

and hence

$$e = r + u - \chi \leq 3r - 3\chi.$$

We have obtained upper bounds for the number of vertices and edges of Γ, and a simple combinatorial argument then shows that there can only be finitely many graphs of this form up to isomorphism. $\square$

We let $\mathrm{CG}(r, \chi) \subseteq \mathrm{CG}(r)$ denote the set of equivalence classes of contracted r-marked graphs of characteristic χ. By Theorem 3.7.1 the cardinality of this set is finite. Moreover, we denote by $\mathrm{CG}(r, \chi, u) \subseteq \mathrm{CG}(r, \chi)$ the set of isomorphism classes of graphs with u unmarked vertices. Theorem 3.7.1 yields:

$$\mathrm{CG}(r, \chi) = \bigsqcup_{u=0}^{2(r-\chi)} \mathrm{CG}(r, \chi, u).$$

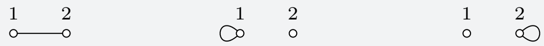
Example 3.7.2. In this example we will compute the set $\mathrm{CG}(r, r-1)$ for all

$r \geq 0$. By Theorem 3.7.1 we have:

$$\mathrm{CG}(r, r-1) = \mathrm{CG}(r, r-1, 0) \sqcup \mathrm{CG}(r, r-1, 1) \sqcup \mathrm{CG}(r, r-1, 2).$$

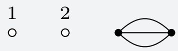
Note that for each graph $\Gamma \in \mathrm{CG}(r, r-1, u)$ with r marked vertices, u unmarked vertices, and e edges, one has $\chi(\Gamma) = r + u - e = r - 1$, so $e = u + 1$.

- Every graph $\Gamma \in \mathrm{CG}(r, r-1, 0)$ has no unmarked vertices and one edge (possibly a loop) between two (not necessarily distinct) marked vertices. It follows that there are $\frac{1}{2}r(r+1)$ such graphs.



- Every graph $\Gamma \in \mathrm{CG}(r, r-1, 1)$ has two edges and one unmarked vertex. As Γ is contracted and contains only two distinct edges, its unmarked vertex has degree at least 4, so the two edges of Γ must be two loops based at the unmarked vertex. We find that $\mathrm{CG}(r, r-1, 1)$ only has one element.

- Every graph $\Gamma \in \mathrm{CG}(r, r-1, 2)$ has three edges and two unmarked vertices. As each unmarked vertex must have degree ≥ 3 and there are only three edges in Γ, it follows that the degree of both unmarked vertices equals 3, and that they are both incident to each of the three edges of Γ. Therefore $\mathrm{CG}(r, r-1, 2)$ has only one element.



We find that the number of elements of $\mathrm{CG}(r, r-1)$ equals

$$|\mathrm{CG}(r, r-1)| = \tfrac{1}{2}r(r+1) + 1 + 1 = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2.$$

In general the set $\mathrm{CG}(r, \chi, u)$ can be computed with an algorithm as follows:

Algorithm 3.7.3. The following (pseudocode) algorithm computes all isomorphism classes of contracted r-marked graphs with u unmarked vertices and characteristic χ.

```
def compute_CG(r,χ,u):
    L = ∅  # list of graphs
    e = r + u - χ  # number of edges
    Γ = (r-marked graph with u unmarked vertices, no edges)
    P = {unordered pairs of vertices of Γ}  # possible edges
    # loop over all possible configurations of edges:
    for E in {multisets of size e with elements from P}:
```

```
        ΓE = Γ equipped with edges from E
        # check if ΓE is contracted, and if ΓE was not found before:
        if ΓE.is_contracted() and ΓE ≇ Γ2 for all Γ2 in L:
            # if so, add Γ to the list
            L.add(ΓE)
    return L
```

Moreover, Theorem 3.7.1 implies that we can compute the set $\mathrm{CG}(r, \chi)$ in finite time by computing $\mathrm{CG}(r, \chi, u)$ for $0 \leq u \leq 2(r-\chi)$ and taking their disjoint union.

Of course, the combinatorial complexity of constructing graphs, checking if they are contracted, and checking if they are isomorphic to any of the graphs we found before, will become worse and worse if r increases and if χ decreases. In Section 3.8 we will show that for all $d \geq 0$ the size of $\mathrm{CG}(r, r-d)$ in terms of r can be expressed as a polynomial of degree $2d$; so if we are given d, we can compute a formula for $|\mathrm{CG}(r, r-d)|$ for all $r \geq 0$ in finite time. In Section 3.9 we will then compute these polynomials for $d \leq 4$ by computing its first values and applying Lagrange interpolation.

3

3.8 The size of $\mathrm{CG}(r, r-d)$ in terms of r

In this section we will prove that, given an integer $d \geq 0$, one can compute a closed formula for the number of contracted r-marked graphs of characteristic $r-d$ for all $r \geq 0$. This will prove useful in Chapter 4, as it allows us to compute upper bounds for the dimensions of spaces of tautological forms on $\mathcal{C}_g^r$ that do not depend on the genus g.

Theorem 3.8.1. Let $d \in \mathbb{Z}$ be an integer. If d is negative, $\mathrm{CG}(r, r-d)$ is empty for all $r \geq 0$. If $d \geq 0$, then there exists a polynomial $f_d \in \mathbb{Q}[x]$ of degree $2d$ such that

$$|\mathrm{CG}(r, r-d)| = f_d(r) \quad \text{for all } r \geq 0.$$

The leading coefficient of f_d is $1/(2^d \cdot d!)$.

In fact, the following stronger theorem directly implies Theorem 3.8.1.

Theorem 3.8.2. Let $d \in \mathbb{Z}$ and $u \geq 0$ be integers. If $2d - u$ is negative, then $\mathrm{CG}(r, r-d, u)$ is empty for all $r \geq 0$. If $2d - u \geq 0$, then there exists a polynomial $f_{d,u} \in \mathbb{Q}[x]$ of degree at most $2d - u$ such that

$$|\mathrm{CG}(r, r-d, u)| = f_{d,u}(r) \quad \text{for all } r \geq 0.$$

If $u = 0$, then $f_{d,0}$ has degree $2d$, and the leading coefficient of $f_{d,0}$ is $1/(2^d \cdot d!)$.

To see that Theorem 3.8.2 implies Theorem 3.8.1, notice that

$$|\mathrm{CG}(r, r-d)| = \sum_{u=0}^{\infty} |\mathrm{CG}(r, r-d, u)| = \sum_{u=0}^{2d} |\mathrm{CG}(r, r-d, u)| = \sum_{u=0}^{2d} f_{d,u}(r),$$

where the middle equality follows from Theorem 3.7.1.

It remains to prove Theorem 3.8.2. The crucial observation here is that after r becomes large enough, no 'new' contracted graphs appear, and the only contracted graphs that do appear come from lower values of r. This is implied by the following lemmas.

Lemma 3.8.3. Let $\Gamma \in \mathrm{CG}(r, r-d, u)$ be a contracted graph. The number of marked vertices that have positive degree is at most $2d-u$. In particular, if $r > 2d-u$, there are marked vertices in Γ of degree 0.

Proof. Let $\Gamma \in \mathrm{CG}(r, r-d, u)$. Denote by r_+ the number of marked vertices with positive degree. As Γ is contracted, it follows that each unmarked vertex has degree at least 3. We obtain:

$$2e = \sum_{v \in \Gamma} \deg(v) \geq r_+ + 3u.$$

The characteristic of Γ equals $r-d$, so we have:

$$r - d = \chi = r + u - e.$$

Substituting $e = u + d$ into the above inequality yields $r_+ \leq 2d - u$. If $r > 2d-u$ we must conclude that $r_+ < r$, so Γ has a marked vertex of degree 0. $\square$

Lemma 3.8.4. Let $0 \leq s \leq r$ be integers, and let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be an injection. Set $C = \{1, \ldots, r\} \setminus \mathrm{Im}(\phi)$. Recall that the pushforward functor induces a map

$$\phi_* : \mathrm{G}(s) \to \mathrm{G}(r).$$

This map is injective, and its image consists of those classes of r-marked graphs whose ith marked vertex has degree 0 for all $i \in C$.

Proof. Let $\Gamma = (V, E, m)$ be an s-marked graph, and let $\Gamma' = \phi_*\Gamma = (V', E', m')$. The following diagram is a pushout diagram

$$\begin{tikzcd}
\{1, \ldots, s\} \arrow[r, "m"] \arrow[d, "\phi"] \arrow[dr, phantom, "\lrcorner", very near end] & V \arrow[d, "\phi_{\mathrm{v}}"] \\
\{1, \ldots, r\} \arrow[r, "m'"'] & V'.
\end{tikzcd}$$

As m and ϕ are injective, it follows that:

$$\mathrm{Im}(\phi_{\mathrm{v}}) \cap \mathrm{Im}(m') = \mathrm{Im}(\phi_{\mathrm{v}} \circ m) = \mathrm{Im}(m' \circ \phi).$$

If $v' = m'(i)$ is a marked vertex of Γ' of positive degree, then v' must lie in the image of ϕ_{v}, as the endpoints of every edge in Γ' lie in the image of ϕ_{v}, by construction. As $v' = m'(i) \in \mathrm{Im}(\phi_{\mathrm{v}}) \cap \mathrm{Im}(m')$, we see that $i \in \mathrm{Im}(\phi)$, so $i \notin C$. In other words: for each $i \in C$ the marked vertex $m'(i)$ has degree 0.

Assume, now, that $\Gamma' = (V', E', m')$ is an r-marked graph such that the vertex $m'(i)$ has degree 0 for all $i \in C$. We construct an s-marked graph $\Gamma = (V, E, m)$ as follows: we let $V = V' \setminus m'(C)$, $E' = E$, and $m = m' \circ \phi$. It follows that this construction is inverse to the construction done in the first paragraph of this proof. This observation hinges on the fact that the following diagram is a pushout diagram.

$$\begin{tikzcd} \{1, \ldots, s\} \arrow[r, "m' \circ \phi"] \arrow[d, "\phi"] & V' \setminus m(C) \arrow[d, hook] \\ \{1, \ldots, r\} \arrow[r, "m'"'] \arrow[ur, phantom, "\lrcorner"] & V'. \end{tikzcd}$$

It follows that ϕ_* induces a bijection from classes of s-marked graphs to classes of r-marked graphs whose ith marked vertex has degree 0 for all $i \in C$. □

3

Lemma 3.8.5. Let $0 \leq s \leq r$ be integers, and let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be an injection. The pushforward functor ϕ_* induces an injective map

$$\phi_* : \mathrm{CG}(s) \to \mathrm{CG}(r),$$

and hence injective maps for $d \geq 0$

$$\phi_* : \mathrm{CG}(s, s-d) \to \mathrm{CG}(r, r-d),$$

and for $d, u \geq 0$

$$\phi_* : \mathrm{CG}(s, s-d, u) \to \mathrm{CG}(r, r-d, u),$$

The images of these maps consist of the classes of those graphs whose ith marked vertex has degree 0 for all $i \in \{1, \ldots, r\} \setminus \mathrm{Im}(\phi)$.

Proof. The constructions made in the proof of Lemma 3.8.4 (deleting degree 0 marked vertices, taking pushforwards) do not affect unmarked vertices. Therefore it follows that an s-marked graph Γ is contracted if and only if $\phi_*\Gamma$ is contracted. As we observed in Section 3.3, the pushforward operator increases the characteristic of a graph by $r - s$, so ϕ_* maps graphs of characteristic $s - d$ to graphs of characteristic $r - d$. The number of unmarked vertices remains the same. □

We are almost ready to prove Theorem 3.8.2. In the proof of this theorem we will use a combinatorial argument to show that $|\mathrm{CG}(r, r-d, u)|$ can be expressed as a polynomial in r of degree at most $2d - u$. More precisely, we will show that $|\mathrm{CG}(r, r-d, u)|$ is fully determined by the values it takes for $0 \leq r \leq 2d - u$, and given by a recurrence relation. We will then apply the following recurrence relation for polynomials to see that $|\mathrm{CG}(r, r-d, u)|$ can be expressed as a polynomial in r.

Lemma 3.8.6. Let R be a commutative ring, let $r \geq 0$, and let $f \in R[x]$ a polynomial of degree less than r. Then

$$\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(x+k)=0.$$

Proof. The forward difference operator $\Delta : R[x] \to R[x]$ maps a polynomial f to $f(x+1)-f(x)$. This map is R-linear, and maps polynomials of degree $\leq d$ to polynomials of degree $\leq d-1$. It follows that Δ^r annihilates all polynomials of degree less than r.

Moreover, one can prove using an inductive argument the identity

$$\Delta^r f = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(x+k),$$

for all $f \in R[x]$ and all $r \geq 0$. If $\deg f < r$ the desired identity follows. ☐

3

Proof of Theorem 3.8.2. If $u > 2d$, then there are no contracted r-marked graphs of characteristic $r-d$; this follows from Theorem 3.7.1.

Assume, from now on, that $u \leq 2d$. For each subset $A \subseteq \{1,\dots,r\}$ of cardinality k, we define a subset

$$S_A \subseteq \mathrm{CG}(r, r-d, u)$$

as follows: we let $\phi_A : \{1,\dots,r-k\} \to \{1,\dots,r\}$ denote the increasing map with image $\{1,\dots,r\} \setminus A$. Then S_A is the image of the (injective!) pushforward map

$$\phi_{A,*} : \mathrm{CG}(r-k, r-k-d, u) \to \mathrm{CG}(r, r-d, u).$$

It follows from Lemma 3.8.5 that a graph $\Gamma \in \mathrm{CG}(r, r-d, u)$ lies in S_A if and only if for all $i \in A$ the vertex $m(i)$ has degree 0. We therefore have, for subsets $A_1,\dots,A_m \subseteq \{1,\dots,r\}$:

$$S_{A_1} \cap \dots \cap S_{A_m} = S_{A_1 \cup \dots \cup A_m}.$$

If $r > 2d-u$, it follows from Lemma 3.8.3 that at least one of the marked vertices of each graph in $\mathrm{CG}(r, r-d, u)$ has degree 0. We can therefore write $\mathrm{CG}(r, r-d, u)$ as:

$$\mathrm{CG}(r, r-d, u) = \bigcup_{i=1}^{r} S_{\{i\}}.$$

The inclusion–exclusion principle then gives, for all $r > 2d - u$:

$$\begin{aligned}|\mathrm{CG}(r, r-d, u)| &= \left|\bigcup_{i=1}^{r} S_{\{i\}}\right| \\ &= \sum_{k=1}^{r}(-1)^{k+1}\left(\sum_{\substack{A\subseteq\{1,\dots,r\}\\ |A|=k}} |S_A|\right) \\ &= \sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k}|\mathrm{CG}(r-k, r-k-d, u)|.\end{aligned}$$

So, if we let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ denote the function

$$g(r) = |\mathrm{CG}(r, r-d, u)|,$$

we see that $g(r)$ is determined by its values in $0, \dots, 2d-u$ and the recursive formula

$$g(r) = \sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k} g(r-k).$$

Let $f_{d,u}$ be the unique polynomial in $\mathbb{Q}[x]$ of degree at most $2d-u$ that satisfies $f_{d,u}(r) = g(r)$ for all $0 \leq r \leq 2d-u$. Using Lemma 3.8.6 one can show that $f_{d,u}$ satisfies the same recurrence relation as g does, and hence we conclude that $f_{d,u}(r) = g(r)$ for all $r \geq 0$.

Assume now that $u = 0$. Then $f_{d,0}(r)$ counts the number of contracted r-marked graphs with no unmarked vertices of characteristic $r-d$ (so the number of edges equals d). Every graph with no unmarked vertices is contracted, so $f_{d,0}(r)$ simply counts the number of ways we can put d edges in an r-marked graph with no unmarked vertices. There are up to ordering $\frac{1}{2}r(r+1)$ pairs of not necessarily distinct vertices in such a graph. Therefore, $f_{d,0}(r)$ equals the number of multisets of cardinality d with elements taken from a set of cardinality $\frac{1}{2}r(r+1)$. It follows that

$$f_{d,0}(r) = \left(\!\!\binom{\frac{1}{2}r(r+1)}{d}\!\!\right) = \binom{\frac{1}{2}r(r+1)+d-1}{d}.$$

After expanding the binomial coefficient, we see that $f_{d,0}(r)$ is a degree $2d$ polynomial whose leading coefficient equals $1/(2^d \cdot d!)$. □

Remark 3.8.7. The polynomials $f_{d,u} \in \mathbb{Q}[x]$ are integer-valued. The coefficients of $f_{d,u}$, however, are not. By the theory of integer-valued polynomials (see, for instance, [CC16]), we can write

$$f_{d,u} = \sum_{k=0}^{2d-u} c_k \binom{x}{k},$$

where $c_0, \ldots, c_{2d-u}$ are *integers*, defined recursively by the following formula:

$$c_k = f_{d,u}(k) - \sum_{j=0}^{k-1} c_j \binom{k}{i}.$$

3.9 Computing closed formulas for $|\mathrm{CG}(r, r-d)|$

In the previous section we proved that given nonnegative integers d, u there is a polynomial $f_{d,u}$ of degree at most $2d - u$ such that for all $r \geq 0$ one has

$$|\mathrm{CG}(r, r-d, u)| = f_{d,u}(r).$$

By Theorem 3.7.1 we then find for all $d, r \geq 0$:

$$|\mathrm{CG}(r, r-d)| = f_d(r) := \sum_{u=0}^{2d} f_{d,u}(r).$$

In this section we will use Algorithm 3.7.3 to compute the polynomial f_d for low values of d. An implementation of this algorithm in Python 3 (along with numerous optimizations to the 'naive' Algorithm 3.7.3) can be found in [vdLug21].

Using the algorithm, we can compute the polynomials $f_{d,u}$ and f_d for all $d \leq 4$ and $u \geq 0$.

- $d < 0$: By Theorem 3.8.2 we have $f_{d,u} = 0$ for all $u \geq 0$, and hence $f_d = 0$.
- $d = 0$: Theorem 3.8.2 implies that $f_{0,u} = 0$ for all $u > 0$. We have $f_{0,0} = 1$ since there is a unique r-marked graph with no unmarked vertices and no edges, and this graph is automatically contracted. We obtain $f_0 = 1$.
- $d = 1$: Using our algorithm, we find the following values for $f_{1,u}(r)$ for $r + u \leq 2$:

r	$f_{1,0}(r)$	$f_{1,1}(r)$	$f_{1,2}(r)$
0	0	1	1
1	1	1	
2	3		

  Lagrange interpolation gives us the following expressions for the polynomials $f_{1,u}$:

$$f_{1,0} = \tfrac{1}{2}r^2 + \tfrac{1}{2}r$$
$$f_{1,1} = 1$$
$$f_{1,2} = 1$$

  By summing these polynomials, we find

$$f_1 = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2.$$

  Note that this agrees with the formula for $\mathrm{CG}(r, r-1)$ we computed by hand in Example 3.7.2.

- $d = 2$: The algorithm produces the following values for $f_{2,u}(r)$ for $r + u \leq 4$:

r	$f_{2,0}(r)$	$f_{2,1}(r)$	$f_{2,2}(r)$	$f_{2,3}(r)$	$f_{2,4}(r)$
0	0	1	4	3	3
1	1	5	8	4	
2	6	13	14		
3	21	26			
4	55				

By interpolation, we find the polynomial equations

$$\begin{aligned} f_{2,0} &= \tfrac{1}{8}r^4 + \tfrac{1}{4}r^3 + \tfrac{3}{8}r^2 + \tfrac{1}{4}r \\ f_{2,1} &= \tfrac{1}{6}r^3 + \tfrac{3}{2}r^2 + \tfrac{7}{3}r + 1 \\ f_{2,2} &= r^2 + 3r + 4 \\ f_{2,3} &= r + 3 \\ f_{2,4} &= 3. \end{aligned}$$

Hence we obtain

$$f_2 = \tfrac{1}{8}r^4 + \tfrac{5}{12}r^3 + \tfrac{23}{8}r^2 + \tfrac{79}{12}r + 11.$$

- $d = 3$: We find the following values for $f_{3,u}(r)$ for $r + u \leq 6$:

r	$f_{3,0}(r)$	$f_{3,1}(r)$	$f_{3,2}(r)$	$f_{3,3}(r)$	$f_{3,4}(r)$	$f_{3,5}(r)$	$f_{3,6}(r)$
0	0	1	7	18	23	15	9
1	1	10	33	49	44	20	
2	10	51	104	106	73		
3	56	176	257	197			
4	220	475	541				
5	680	1086					
6	1771						

Interpolation of the found data yields the following polynomial expressions.

$$\begin{aligned} f_{3,0} &= \tfrac{1}{48}r^6 + \tfrac{1}{16}r^5 + \tfrac{3}{16}r^4 + \tfrac{13}{48}r^3 + \tfrac{7}{24}r^2 + \tfrac{1}{6}r \\ f_{3,1} &= \tfrac{1}{12}r^5 + \tfrac{3}{4}r^4 + \tfrac{25}{12}r^3 + \tfrac{13}{4}r^2 + \tfrac{17}{6}r + 1 \\ f_{3,2} &= \tfrac{1}{2}r^4 + \tfrac{19}{6}r^3 + \tfrac{19}{2}r^2 + \tfrac{77}{6}r + 7 \\ f_{3,3} &= \tfrac{4}{3}r^3 + 9r^2 + \tfrac{62}{3}r + 18 \\ f_{3,4} &= 4r^2 + 17r + 23 \\ f_{3,5} &= 5r + 15 \\ f_{3,6} &= 9 \end{aligned}$$

We therefore find that the number of contracted r-marked graphs of characteristic $r - 3$ is equal to:

$$f_3 = \tfrac{1}{48}r^6 + \tfrac{7}{48}r^5 + \tfrac{23}{16}r^4 + \tfrac{329}{48}r^3 + \tfrac{625}{24}r^2 + \tfrac{117}{2}r + 73$$

- $d = 4$: After a while the algorithm outputs the following values for $f_{4,u}(r)$ for $u + r \leq 8$.

r	$f_{4,0}(r)$	$f_{4,1}(r)$	$f_{4,2}(r)$	$f_{4,3}(r)$	$f_{4,4}(r)$	$f_{4,5}(r)$	$f_{4,6}(r)$	$f_{4,7}(r)$	$f_{4,8}(r)$
0	0	1	11	47	123	172	160	79	32
1	1	16	93	257	425	423	282	105	
2	15	136	496	948	1131	854	443		
3	126	742	1897	2707	2513	1515			
4	715	2971	5756	6485	4916				
5	3060	9542	14 786	13 687					
6	10 626	26 047	33 538						
7	31 465	62 812							
8	82 251								

We therefore obtain the following polynomial expressions for $f_{4,u}(r)$:

$$\begin{aligned}
f_{4,0} &= \tfrac{1}{384}r^8 + \tfrac{1}{96}r^7 + \tfrac{3}{64}r^6 + \tfrac{5}{48}r^5 + \tfrac{27}{128}r^4 + \tfrac{25}{96}r^3 + \tfrac{23}{96}r^2 + \tfrac{1}{8}r \\
f_{4,1} &= \tfrac{1}{48}r^7 + \tfrac{5}{24}r^6 + \tfrac{101}{120}r^5 + \tfrac{9}{4}r^4 + \tfrac{63}{16}r^3 + \tfrac{109}{24}r^2 + \tfrac{16}{5}r + 1 \\
f_{4,2} &= \tfrac{23}{144}r^6 + \tfrac{73}{48}r^5 + \tfrac{1013}{144}r^4 + \tfrac{875}{48}r^3 + \tfrac{1037}{36}r^2 + \tfrac{105}{4}r + 11 \\
f_{4,3} &= \tfrac{3}{4}r^5 + \tfrac{23}{3}r^4 + \tfrac{397}{12}r^3 + \tfrac{229}{3}r^2 + \tfrac{553}{6}r + 47 \\
f_{4,4} &= \tfrac{73}{24}r^4 + \tfrac{325}{12}r^3 + \tfrac{2387}{24}r^2 + \tfrac{2069}{12}r + 123 \\
f_{4,5} &= \tfrac{25}{3}r^3 + 65r^2 + \tfrac{533}{3}r + 172 \\
f_{4,6} &= \tfrac{39}{2}r^2 + \tfrac{205}{2}r + 160 \\
f_{4,7} &= 26r + 79 \\
f_{4,8} &= 32
\end{aligned}$$

By summing these polynomials, we obtain the polynomial f_4 that counts the number of contracted r-marked graphs of characteristic $r - 4$.

$$f_4 = \tfrac{1}{384}r^8 + \tfrac{1}{32}r^7 + \tfrac{239}{576}r^6 + \tfrac{193}{60}r^5 + \tfrac{23275}{1152}r^4 + \tfrac{8729}{96}r^3 + \tfrac{84637}{288}r^2 + \tfrac{24013}{40}r + 625$$

While our algorithm can quickly compute all values for $f_{d,u}(r)$ with $u+r \leq 2d$ for all $d \leq 3$, it takes a very long time in the case $d = 4$ on the same server. We observe the following runtimes:

d	Runtime (s)
1	4.3×10^{-4}
2	2.3×10^{-2}
3	3.6
4	8.2×10^{3}

It seems unlikely that f_5 can be computed in reasonable time without significant improvements to either the algorithm or the hardware.