Tautological differential forms on moduli spaces of curves

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Chapter 2

The moduli space of genus g curves

Moduli spaces can be thought of as spaces that parametrize objects of a certain type. For example, if V is a complex vector space and $k \geq 0$ an integer, the Grassmannian $G_{k,V}$ is a complex manifold whose underlying set is the set of k-dimensional linear subspaces of V, so $G_{k,V}$ parametrizes k-linear subspaces of V. There is a universal vector bundle $E_{k,V} \to G_{k,V}$ that induces every other family of k-dimensional subspaces via base change. By studying this universal family we can make statements that are valid 'universally' among families of k-dimensional subspaces of V. In Section 2.1 we will see which cohomology classes occur universally among such families.

This thesis is aimed at the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ and the universal family $\mathcal{C}_g \to \mathcal{M}_g$. Unfortunately, such a fine moduli space does not exist in the category of complex manifolds. In Section 2.2 we will see why the existence of nontrivial automorphisms on genus g curves prevents the existence of a fine moduli space for genus g curves.

Riemann surfaces were first studied by Riemann [Rie51; Rie57] in the context of multi-valued functions on the complex plane. Riemann already knew heuristically that a compact Riemann surface of genus $g \geq 2$ depends on $3g - 3$ parameters, or in a more modern terminology, that the moduli space $\mathcal{M}_g$ should be $(3g-3)$-dimensional. Teichmüller [Tei44] made this statement more formal. He realized that it is impossible to endow the moduli space of genus g curves $\mathcal{M}_g$ with a well-behaved complex structure, as this space has certain singularities. He therefore constructed a covering $\mathcal{T}_g$ of $\mathcal{M}_g$ whose points are isomorphism classes of genus g curves with Teichmüller structure, and endowed this space with the structure of a complex manifold. Moreover, Teichmüller constructed a family $\mathcal{X}_g \to \mathcal{T}_g$ that is universal in the sense that any other family of genus g curves with Teichmüller structure can be obtained from this universal family by base change. He remarked that the complex manifold $\mathcal{T}_g$ is $(3g-3)$-dimensional, and hence gave a formal meaning to Riemann's heuristic argument.

In a series of 10 talks at Henri Cartan's seminar, Grothendieck [Gro60] reformulated Teichmüller's results in a language of algebraic geometry. More precisely, he proved that the functor from complex analytic spaces to sets mapping a complex analytic space S to the set of isomorphism classes of families of genus g curves with Teichmüller structure over S is representable by the Teichmüller space $\mathcal{T}_g$. This means that $\mathcal{T}_g$ is the fine moduli space for families of genus g curves with Teichmüller structure. In Section 2.3 we will discuss the Teichmüller space $\mathcal{T}_g$.

Another approach at tackling the moduli space $\mathcal{M}_g$ was made by Deligne and Mumford [DM69]. They view $\mathcal{M}_g$ as a *stack*, rather than a complex manifold. This is the approach we will also be taking in this thesis. This approach will give us a moduli space $\mathcal{M}_g$ and a universal family $\mathcal{C}_g \to \mathcal{M}_g$. Although these are not honest complex manifolds, some theory about complex manifolds generalizes to stacks. For example, it is still possible to define differential forms and hermitian line bundles on stacks, which we will do in Sections 2.5 and 2.6.

As it turns out, it is possible to understand these differential forms and hermitian vector bundles without understanding much about the underlying stacks at all. Any reader who is not comfortable with (or interested in) using stacks can read Section 2.1 for a motivation, and afterwards read Proposition 2.5.10 and Example 2.6.1 to get some intuition for working with differential forms and vector bundles on moduli stacks.

In Chapter 4 we will often treat $\mathcal{M}_g$ and $\mathcal{C}_g$ as if they were honest complex manifolds. In such cases, the reader should understand that there is an argument being made 'behind the scenes': the given statements hold universally for families of genus g curves, and hence on the moduli stacks themselves.

In this section we will be working over the category **CMan** of complex manifolds. We also fix a Grothendieck topology on **CMan**, where a collection of morphisms $\{X_i \to X\}$ is a covering if and only if all these morphisms are open immersions and their images cover X. It now makes sense to talk about stacks over **CMan**. In this entire section every stack is assumed to be over **CMan**, so we will often abbreviate 'stack over **CMan**' to 'stack'.

2.1 Motivating example: the Grassmannian manifold

To motivate our study of families and moduli spaces of curves, we first look at a simpler and better understood example of a moduli space, the Grassmannian. We will see that studying moduli spaces can yield information on properties that hold universally on the families they classify. We refer to [BT82, §23] and [GH94, §1.5] for a more detailed treatment of the material in this section.

Definition 2.1.1. Let V be a complex vector space. A *family of k-dimensional subspaces of V* over a complex manifold S is a holomorphic sub-vector bundle $f : E \to S$ of the trivial vector bundle $V \times S \to S$, such that every fiber of f is k-dimensional.

For instance, consider the complex manifold $S = \mathbb{P}^n_{\mathbb{C}}$ whose points correspond to lines through the origin in $\mathbb{C}^{n+1}$. Let $E \to S$ be the subbundle of $\mathbb{C}^{n+1} \times S \to S$

whose fiber over a point $s \in S$ is the line in $\mathbb{C}^{n+1}$ that corresponds to s. Then $E \to S$ is a family of one-dimensional subspaces of $\mathbb{C}^{n+1}$.

More generally, for all $k \geq 0$ and all complex vector spaces V, we can consider the *Grassmannian* manifold $G_{k,V}$. Its underlying set is the set of k-dimensional subspaces of V:

$$|G_{k,V}| = \{W \subseteq V : \dim(W) = k\}$$

For example, the Grassmannian $G_{1,\mathbb{C}^{n+1}}$ is the projective space $\mathbb{P}^n_{\mathbb{C}}$. Recall that the complex structure on $\mathbb{P}^n_{\mathbb{C}}$ is constructed by gluing affine charts; the complex structure on general Grassmannians is constructed in a similar way.

The Grassmannian $G_{k,V}$ comes with a canonical family of k-dimensional subspaces of V. It is the subbundle $u : E_{k,V} \to G_{k,V}$ of the trivial bundle $V \times G_{k,V} \to G_{k,V}$ whose fiber over a point in $G_{k,V}$ equals the corresponding k-dimensional subspace of V.

Now let us assume that $f : E \to S$ is any family of k-dimensional subspaces. Then associated to f we have a morphism $\Phi_f : S \to G_{k,V}$, which maps any point $s \in S$ to the fiber $E_s \in G_{k,V}$. Moreover, the bundle $E \to S$ is the pullback of the canonical bundle $E_{k,V} \to G_{k,V}$ along the morphism Φ_f:

$$\begin{array}{ccc} E & \longrightarrow & E_{k,V} \\ \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle u} \\ S & \xrightarrow{\Phi_f} & G_{k,V}. \end{array}$$

It follows that the family $u : E_{k,V} \to G_{k,V}$ induces every other family $f : E \to S$ by pullback along a *unique* morphism $\Phi_f : S \to G_{k,V}$. We therefore call $u : E_{k,V} \to G_{k,V}$ the *universal* family of k-dimensional subspaces of V. We say that $G_{k,V}$ is a *fine moduli space* for k-dimensional subspaces of V.

Suppose that $f : E \to S$ is any family of k-dimensional subspaces of V. Associated to f we have some cohomology classes on S, the Chern classes

$$c_1(E), \ldots, c_k(E) \in H^*(S).$$

Moreover, we have a vector bundle Q over S defined by the following exact sequence

$$0 \to E \to V \times S \to Q \to 0.$$

Associated to Q we have some more cohomology classes on S:

$$c_1(Q), \ldots, c_{n-k}(Q) \in H^*(S),$$

where $n = \dim(V)$. These classes have the following relation:

$$(1 + c_1(E) + \cdots + c_k(E))(1 + c_1(Q) + \cdots + c_{n-k}(Q)) = 1.$$

Moreover, these cohomology classes behave well with respect to base change: if $g : T \to S$ is any morphism, and $g^*E \to T$ is the pullback of $E \to S$ along g, then $g^*Q \to T$ is the quotient bundle associated to $g^*E \to T$, and we have equalities

$$c_i(g^*E) = g^*c_i(E) \quad \text{and} \quad c_i(g^*Q) = g^*c_i(Q).$$

In particular we have for each family $f : E \to S$ of k-dimensional subspaces of V equalities of Chern classes

$$c_1(E) = \Phi_f^* c_1(E_{k,V}) \quad \text{and} \quad c_1(Q) = \Phi_f^* c_1(Q_{k,V}),$$

where $Q_{k,V}$ is the universal quotient bundle on $G_{k,V}$ defined by the exact sequence

$$0 \to E_{k,V} \to V \times G_{k,V} \to Q_{k,V} \to 0.$$

It follows that these Chern classes are in some sense *universal* on families of subspaces, and the relation we found among them is a *universal* relation. One might wonder if there are any more such universal classes or relations on families of subspaces. We can answer this question by studying the cohomology ring of the Grassmannian. Indeed, any cohomology class on the Grassmannian yields a cohomology class on the base of every family of subspaces $f : E \to S$ via pullback along Φ_f. Conversely, every universal class on bases of families of subspaces gives in particular a class on the base of the bundle $u : E_{k,V} \to G_{k,V}$. The cohomology of the Grassmannian is

$$H^*(G_{k,V}) = \frac{\mathbb{Z}[c_1(E), \ldots, c_k(E), c_1(Q), \ldots, c_{n-k}(Q)]}{((1 + c_1(E) + \cdots + c_k(E))(1 + c_1(Q) + \cdots + c_{n-k}(Q)) - 1)},$$

where $E = E_{k,V}$ and $Q = Q_{k,V}$ is the associated quotient bundle. In particular, it follows that there are no further cohomology classes or relations that are universal on families of subspaces.

Similarly, we can study other types of objects, such as Chow classes or differential forms, that are universal on families of subspaces simply by studying these objects on the Grassmannian.

The main takeaway from this section is the following.

> *Making statements about (objects on) moduli spaces is equivalent to making statements that hold universally among the families these moduli spaces classify.*

2.2 Fine moduli spaces

In Section 2.1 we constructed the Grassmannian $G_{k,V}$ that parametrizes k-dimensional subspaces of a complex vector space V, together with a universal family $u : E_{k,V} \to G_{k,V}$ that induces every other family of k-dimensional subspaces of V via base change. We called $G_{k,V}$ a *fine moduli space* for families of k-dimensional subspaces of V. In this section we will generalize this discussion. We will start with some abstract nonsense, and then apply this to some concrete examples, such as the Grassmannian we studied in Section 2.1.

Definition 2.2.1. Let $\mathbf{C}$ be a category, let $\mathbf{Set}$ denote the category of sets, and let $F : \mathbf{C} \to \mathbf{Set}$ be a contravariant functor. A *representation* of F consists of an object $\mathcal{M}$ of $\mathbf{C}$ together with a natural isomorphism τ from F to the functor of points $\mathcal{M}(-) = \mathrm{Hom}_{\mathbf{C}}(-, \mathcal{M})$. In this case, we say that $\mathcal{M}$ is a *fine moduli space* for F.

It follows from Yoneda's lemma that a representation is unique up to a unique isomorphism. In particular a fine moduli space is unique up to isomorphism.

Assume that F is representable, and fix a representation $\tau : F \xrightarrow{\sim} \mathcal{M}(-)$. Then under the bijection $\tau_{\mathcal{M}} : F(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}(\mathcal{M})$, the identity $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ corresponds to an element $u \in F(\mathcal{M})$. We call this element the *universal element.*

Let $\alpha : S \to \mathcal{M}$ be a morphism. We obtain a commutative diagram of sets:

$$\begin{array}{ccc} F(\mathcal{M}) & \xrightarrow[\sim]{\tau_{\mathcal{M}}} & \mathcal{M}(\mathcal{M}) \\ \big\downarrow{\scriptstyle \alpha^*} & & \big\downarrow{\scriptstyle -\circ\alpha} \\ F(S) & \xrightarrow[\sim]{\tau_S} & \mathcal{M}(S), \end{array}$$

where we denote by α^* the map $F(\alpha) : F(\mathcal{M}) \to F(S)$ induced by $\alpha : S \to \mathcal{M}$. By chasing through this diagram we find that $\tau_S(\alpha^* u) = \alpha$. In particular, for every $f \in F(S)$ there exists a *unique* morphism $\Phi_f : S \to \mathcal{M}$ (namely $\Phi_f = \tau_S(f)$) for which $\Phi_f^* u = f$. In other words: for every object S of $\mathbf{C}$ and every element $f \in F(S)$ we can obtain f from the universal element u by pulling u back along a unique morphism $\Phi_f : S \to \mathcal{M}$.

For example, let k be a nonnegative integer, and let V be a complex vector space. Consider the following contravariant functor from the category of complex manifolds to the category of sets:

$$\begin{aligned} F : \mathbf{CMan} &\to \mathbf{Set} \\ S &\mapsto \{\text{families } E \to S \text{ of } k\text{-dimensional subspaces of } V\}. \end{aligned}$$

A morphism of complex manifolds $T \to S$ is mapped to the pullback operator that transforms families over S into families over T. We claim that the Grassmannian $G_{k,V}$ is a fine moduli space for F. Indeed, for any complex manifold S we define a map

$$\tau_S : F(S) \to G_{k,V}(S) = \mathrm{Hom}(S, G_{k,V})$$

that sends a family $f : E \to S$ to the morphism $\Phi_f : S \to G_{k,V}$ given by $\Phi_f(s) = f^{-1}(s) \subseteq V$ for all $s \in S$. Notice that τ_S is in fact a bijection. The maps τ_S induce a natural isomorphism $\tau : F \to G_{k,V}(-)$.

The universal element of the functor F is the universal family $u : E_{k,V} \to G_{k,V}$. Indeed, under the bijection $\tau_{G_{k,V}} : F(G_{k,V}) \to G_{k,V}(G_{k,V})$ this family is mapped to the identity $G_{k,V} \to G_{k,V}$. It follows once again that every family $E \to S$ of k-dimensional subspaces of V can be obtained from the universal family by taking its pullback along a unique morphism $\Phi_f : S \to G_{k,V}$.

Analogous to the Grassmannian we would like to construct a moduli space $\mathcal{M}_g$ that classifies genus g curves for a fixed integer $g \geq 0$. As there are too many genus g curves to fit into a set, we cannot expect the points of $\mathcal{M}_g$ to correspond bijectively with genus g curves. Our next best bet is to try to construct a moduli space $\mathcal{M}_g$ whose points correspond to *isomorphism classes* of genus g curves. We proceed as follows.

Two families $f : X \to S$ and $f' : X' \to S$ are *isomorphic* if there exists an isomorphism $g : X \to X'$ with $f' \circ g = f$. We can then consider the following functor:

$$\begin{aligned} F : \mathbf{CMan} &\to \mathbf{Set} \\ S &\mapsto \{\text{families } X \to S \text{ of genus } g \text{ curves}\}/\cong . \end{aligned}$$

Let us assume that F is representable by a complex manifold $\mathcal{M}_g$. Under the bijection $F(\mathcal{M}_g) \xrightarrow{\sim} \mathcal{M}_g(\mathcal{M}_g)$ the identity $\mathcal{M}_g \to \mathcal{M}_g$ corresponds to a universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ of genus g curves. The bijection $F(*) \xrightarrow{\sim} \mathcal{M}_g(*)$ gives us a bijective correspondence between the points of $\mathcal{M}_g$ and isomorphism classes of genus g curves. If $f : X \to S$ is a family of genus g curves, then the associated morphism $\Phi_f : S \to \mathcal{M}_g$ maps a point $s \in S$ to the point of $\mathcal{M}_g$ that corresponds to the isomorphism class of the curve X_s, and we obtain a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{C}_g \\ \downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle p} \\ S & \xrightarrow[\Phi_f]{} & \mathcal{M}_g. \end{array}$$

Now let $f : X \to S$ be an *isotrivial* family of genus g curves. That is, the fibers of f are pairwise isomorphic. Then the induced morphism $\Phi_f : S \to \mathcal{M}_g$ is constant, and factors over the singleton manifold. We obtain the following diagram with cartesian squares:

$$\begin{array}{ccccc} X & \longrightarrow & C & \longrightarrow & \mathcal{C}_g \\ \downarrow{\scriptstyle f} & \square & \downarrow & \square & \downarrow{\scriptstyle p} \\ S & \longrightarrow & \{*\} & \longrightarrow & \mathcal{M}_g \end{array} \qquad \Phi_f = (S \to \{*\} \to \mathcal{M}_g)$$

where C is a genus g curve that is isomorphic to the fibers of f. We therefore see that f is a trivial family: it is isomorphic to the family $C \times S \to S$.

So the existence of a moduli space of genus g curves would imply that every isotrivial family of genus g curves is trivial. However, as the next proposition states, it is possible to construct nontrivial isotrivial families, and thus show that a fine moduli space $\mathcal{M}_g$ cannot exist.

Proposition 2.2.2. Let $g \geq 0$ be an integer. There exists an isotrivial family of genus g curves which is not trivial. In particular, there is no fine moduli space $\mathcal{M}_g$ of genus g curves in the category of complex manifolds.

Let us first prove this proposition in the case $g = 0$.

Proof for $g = 0$. First, assume that $g = 0$. Consider the projective plane $\mathbb{P}^2$ and fix a point $x \in \mathbb{P}^2$, and blow up the plane at this point. In other words, the set S

of lines through x can be identified with $\mathbb{P}^1$ and hence be given the structure of a complex manifold, and we consider the complex manifold

$$X = \{(y, \ell) \in \mathbb{P}^2 \times S : y \in \ell\}.$$

The morphism $f : X \to S : (y, \ell) \mapsto \ell$ is an isotrivial family of genus 0 curves: we have $f^{-1}(\ell) \cong \ell \cong \mathbb{P}^1$ for all $\ell \in S$. However, we claim that f is not a trivial family. If it were, it would have to be isomorphic to $\mathbb{P}^1 \times S \cong \mathbb{P}^1 \times \mathbb{P}^1$. One can show, for instance by using intersection theory, that this is not the case. The exceptional locus

$$E = \{(x, \ell) : \ell \in S\} \subseteq X$$

is a prime divisor of X with self-intersection -1 ([Har77, Proposition V.3.1]), whereas $\mathbb{P}^1 \times \mathbb{P}^1$ can be shown not to have any such prime divisors ([Har77, Example V.1.4.3]). □

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For genus $g > 0$ we can construct nontrivial isotrivial families by taking a genus g curve C with a nontrivial automorphism and using this automorphism to 'twist' C. Compare this to the construction of the Möbius strip by twisting a line segment onto itself. We will finish the proof of this proposition in the next section.

2.3 Mapping class groups and Teichmüller structures

In this section we will prove Proposition 2.2.2, proving that there is no fine moduli space for genus g curves. The problem here is that genus g curves admit 'too many' automorphisms, allowing us to twist trivial families into nontrivial isotrivial families. Teichmüller [Tei44] realized this and added extra structures (which we now call *Teichmüller structures*) to the curves we are trying to classify. He thus obtained a universal family $\mathcal{X}_g \to \mathcal{T}_g$ of genus g curves with Teichmüller structures. Grothendieck [Gro60] was able to rephrase Teichmüller's results in a language of algebraic geometry. We will first discuss the results from Teichmüller and Grothendieck, and finish the section by proving that a fine moduli space $\mathcal{M}_g$ does not exist in the category of complex manifolds.

For a more detailed treatment of the material in this section we refer to Grothendieck [Gro60]; see also [AJP16] for a survey of Grothendieck's work on Teichmüller theory.

If X', X are two topological spaces, we denote by $I(X', X)$ the set of homeomorphisms $X' \to X$ modulo homotopy. If $X' = X$ then composition induces a group structure on $I(X, X)$; the resulting group is called the *mapping class group* of X and denoted $\mathrm{MCG}(X)$. In general $\mathrm{MCG}(X)$ acts from the *left*[1] on $I(X', X)$ by composition.

[1]In fact, Grothendieck considers the *right* action of $\mathrm{MCG}(X)$ on $I(X, X')$. In the proof of Lemma 2.3.1 we will be using the left action of the mapping class group to construct a monodromy representation. As taking inverses yields a bijective correspondence between left and right actions we lose no information if we consider left actions instead.

Assume, now, that $f : X \to S$ is a fiber bundle whose fiber F is a finite simplicial complex. The disjoint union of sets

$$\bigsqcup_{s \in S} I(X_s, F)$$

has a natural structure of a MCG(F)-covering space over S. We denote this topological space by $\mathcal{R}(X/S)$. For fixed F and S the assignment $X \mapsto \mathcal{R}(X/S)$ is functorial. Moreover it is compatible with base changes of fiber bundles: for a continuous map $T \to S$ we have

$$\mathcal{R}(X \times_S T/T) = \mathcal{R}(X/S) \times_S T.$$

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Similarly, if F is a compact connected oriented manifold, then we can consider the group $\mathrm{MCG}^+(F)$ of F consisting of homotopy classes of *orientation-preserving* homeomorphisms $F \to F$, which is a subgroup of MCG(F) of index at most 2. In this context the group MCG(F) is often called the *extended mapping class group* of F, and $\mathrm{MCG}^+(F)$ is the *mapping class group* of F. If $X \to S$ is an oriented fiber bundle with fiber F, then analogous to the MCG(F)-covering $\mathcal{R}(X/S)$ of S we construct an $\mathrm{MCG}^+(F)$-covering $\mathcal{P}(X/S)$ of S, whose fiber over a point $s \in S$ consists of the homotopy classes of orientation-preserving homeomorphisms $X_s \to F$.

In particular, if $f : \mathcal{C} \to S$ is a family of genus g curves, then f is also a fiber bundle whose fiber is the compact oriented surface Σ_g of genus g, and Grothendieck calls the $\mathrm{MCG}^+(\Sigma_g)$-covering $\mathcal{P}(\mathcal{C}/S) \to S$ the *Teichmüller covering* of S. The topological space $\mathcal{P}(\mathcal{C}/S)$ obtains the structure of a complex manifold: it is the unique structure for which $\mathcal{P}(\mathcal{C}/S) \to S$ is locally an isomorphism. A *Teichmüller structure* on the family f is a section of the Teichmüller covering $\mathcal{P}(\mathcal{C}/S) \to S$. In other words: giving a Teichmüller structure on f is equivalent to giving a homotopy class of a homeomorphism $\Sigma_g \xrightarrow{\sim} \mathcal{C}_s$ for each $s \in S$, such that these classes 'vary continuously' with s.

Adding Teichmüller structures *rigidifies* the genus g curves we are working with. More precisely: families of genus g curves with Teichmüller structure do not admit nontrivial automorphisms. From this Grothendieck then deduces that the functor

$$S \mapsto \{\text{families of genus } g \text{ curves } f : \mathcal{C} \to S \text{ with Teichmüller structure}\}/\cong$$

is representable. Let $\mathcal{T}_g$ be a representing object; we hence obtain a universal family $\mathcal{X}_g \to \mathcal{T}_g$ of genus g curves with Teichmüller structure. Grothendieck, moreover, remarks that $\mathcal{T}_g$ is homeomorphic to a ball.

Notice, moreover, that there is a natural action of the mapping class group $\Gamma_g = \mathrm{MCG}^+(\Sigma_g)$ on the Teichmüller space $\mathcal{T}_g$. The set of orbits of this action is in bijective correspondence with the set of isomorphism classes of genus g curves. We may therefore view the quotient $M_g = \mathcal{T}_g/\Gamma_g$ as the moduli space of genus g curves. The quotient M_g, however, does not obtain the structure of a complex manifold.

In the remainder of this section we will finish the proof of Proposition 2.2.2. We will be using the following lemma to construct an nontrivial isotrivial family of genus g curves.

Lemma 2.3.1. Let Y and F be topological spaces, and let G be a discrete group. Assume that G acts from the left on Y and F, and assume moreover that the action of G on Y is a covering space action (as defined in [Ful95, Section 1.3]). Note that the actions of G on Y and F induce a G-action on $Y \times F$. Define $S = Y/G$ and $X = (Y \times F)/G$. Then the quotient map $p : Y \to S$ is a covering, and the induced map

$$f : X \to S$$

is a fiber bundle with fiber F. If, moreover, Y is path-connected and the homomorphism $G \to \mathrm{MCG}(F)$ induced by the G-action on F is nontrivial, then the fiber bundle f is nontrivial.

Proof. As the action of G on Y is a covering space action, the quotient map $Y \to Y/G = S$ is a covering map. Moreover, it is straightforward to prove that f is a fiber bundle with fiber F.

Fix points $y \in Y$, $z \in F$, and set $s = p(y)$ and let $x \in X$ be the image of (y, z) under the quotient map $Y \times F \to X$. Note that the composition

$$F \xrightarrow{\sim} \{y\} \times F \hookrightarrow Y \times F \twoheadrightarrow X$$

induces a homeomorphism $F \xrightarrow{\sim} X_s = f^{-1}(s)$; we denote the inverse of this homeomorphism by $\varphi : X_s \xrightarrow{\sim} F$.

The monodromy representation of the pointed G-covering $p : (Y, y) \to (S, s)$ induces a homomorphism

$$\rho : \pi_1(S, s) \to G$$

(c.f. [Ful95, §14a]); it is uniquely determined by the property that $\rho(\alpha) \cdot y = y * \alpha$, where $*$ denotes the monodromy right action of $\pi_1(S, s)$ on the fiber Y_s.

Likewise, the fiber bundle f induces a $\mathrm{MCG}(F)$-covering $\mathcal{R}(X/S) \to S$. The homeomorphism φ induces a point in $\mathcal{R}(X/S)$ over s, and the monodromy representation yields a homomorphism

$$\rho' : \pi_1(S, s) \to \mathrm{MCG}(F).$$

It is now a routine exercise to prove that these two monodromy representations are compatible in the following sense. The action of G on F induces a homomorphism $G \to \mathrm{MCG}(F)$, and the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(S, s) & \xrightarrow{\quad\rho\quad} & G \\ & {\scriptstyle \rho'}\searrow \quad \swarrow & \\ & \mathrm{MCG}(F) & \end{array}$$

Assume that f is a trivial fiber bundle. Then $\mathcal{R}(X/S) \to S$ is a trivial covering, and the monodromy representation $\pi_1(S, s) \to \mathrm{MCG}(F)$ is trivial. If moreover Y

is path-connected then the monodromy representation $\pi_1(S, s) \to G$ is surjective, which implies that the homomorphism $G \to \mathrm{MCG}(F)$ must be trivial. □

Proof of Proposition 2.2.2 with $g > 0$. Assume that $g > 0$. Let C denote the (hyper)elliptic curve of genus g given by the equation

$$y^2 = \prod_{\lambda=0}^{2g+1} (x - \lambda).$$

Consider the involution $\sigma : C \to C$ given by $(x, y) \mapsto (x, -y)$. Note that σ induces a $\mathbb{Z}$-action on C:

$$\mathbb{Z} \to \mathrm{Aut}(C) : 1 \mapsto \sigma$$

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and hence a homomorphism $\mathbb{Z} \to \mathrm{MCG}(C)$ that maps 1 to the class of σ. We claim that the class $[\sigma] \in \mathrm{MCG}(C)$ is nontrivial. In that case, we can consider the $\mathbb{Z}$-covering

$$f : \mathbb{C} \to \mathbb{C}^\times : z \mapsto \exp(2\pi i z),$$

and use Lemma 2.3.1 to construct a nontrivial isotrivial family of genus g curves with fiber C, finishing the proof of Proposition 2.2.2. Note that since f and σ are holomorphic the family obtained from 2.3.1 is a holomorphic fiber bundle.

The involution σ acts as multiplication with -1 on the first (singular) homology group $H_1(C) \cong \mathbb{Z}^{2g}$. Indeed, the group $H_1(C)$ is generated by classes of the form $[\gamma_1 - \gamma_2]$, where γ_1, γ_2 are the two lifts of a path in $\mathbb{P}^1$ between two branch points of the morphism $C \to \mathbb{P}^1 : (x, y) \mapsto x$. The involution σ, then, permutes γ_1 and γ_2, and therefore acts as multiplication by -1 on these classes and hence the whole group. In particular the action of σ on $H_1(C)$ is nontrivial. The automorphism σ, therefore, is not homotopic to the identity id_C, and its class in $\mathrm{MCG}(C)$ is nontrivial. □

2.4 Stacks

As we have seen in Section 2.2 there is no fine moduli space for genus g curves. The reason is the existence of nontrivial automorphisms that we can exploit to ‘twist’ trivial families into nontrivial isotrivial families. This is a common reason for nonexistence of a fine moduli space for many types of families. We can fix the problem in multiple ways.

One way is to impose extra structure on the objects we classify, as we have seen in Section 2.3. Adding Teichmüller structures to our curves annihilates any nontrivial automorphisms, and a fine moduli space for curves with Teichmüller structure exists.

Another way to circumvent the nonexistence of a fine moduli space for genus g curves is by enlarging our category of complex manifolds by introducing *stacks* over the category of complex manifolds, as was done by Deligne and Mumford [DM69]. For an introduction to stacks we refer to [Fan01], a more thorough treatment is given in [FGI+05]. We also refer to [BX11] and [Hei05] for a treatment of stacks in the context of manifolds.

Roughly speaking, a *stack* (over **CMan**) is a category $\mathcal{M}$ equipped with a functor $F : \mathcal{M} \to \mathbf{CMan}$ that allows base changes, gluing of isomorphisms, and gluing of objects.

For instance, consider the category $\mathcal{M}_g$, whose objects are families $f : \mathcal{C} \to S$ of genus g curves, and whose morphisms $(f' : \mathcal{C}' \to S') \to (f : \mathcal{C} \to S)$ are cartesian diagrams

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ {\scriptstyle f'}\big\downarrow & \square & \big\downarrow{\scriptstyle f} \\ S' & \longrightarrow & S. \end{array}$$

Moreover, consider the functor $F : \mathcal{M}_g \to \mathbf{CMan}$ that maps a family $f : \mathcal{C} \to S$ to its base S.

We observe the following properties:

- If $f : \mathcal{C} \to S$ is a family of genus g curves, and $h : S' \to S$ is any morphism of complex manifolds, then we can take the *base change* $\mathcal{C} \times_S S' \to S'$ of f along h, and this is again a family of genus g curves;
- We can *glue isomorphisms* of families. Let S be a complex manifold with an open covering $S = \bigcup_{i \in I} S_i$, and let $f : \mathcal{C} \to S$ and $f' : \mathcal{C}' \to S$ be families of genus g curves. If we are given isomorphisms between the restrictions of f and f' to S_i for each $i \in I$, and these isomorphisms are compatible on overlaps, then we may glue them to obtain an isomorphism between the families f and f'.
- We can *glue objects* of $\mathcal{M}_g$. If we are given a complex manifold S, an open covering $S = \bigcup_{i \in I} S_i$, for each $i \in I$ a family $f_i : \mathcal{C}_i \to S_i$ of genus g curves, and appropriate gluing data, then we can glue these families together to obtain a family $\mathcal{C} \to S$ of genus g curves.

These three properties ensure that the category $\mathcal{M}_g$ with the functor $\mathcal{M}_g \to \mathbf{CMan}$ is a stack.

Definition 2.4.1. The stack $\mathcal{M}_g$ is the *stack of (families of) genus g curves.*

We can view complex manifolds as stacks, too, as the following example demonstrates.

Example 2.4.2. Let S be a complex manifold. Consider the category $[S]$. Objects of $[S]$ are morphisms $f : T \to S$ of complex manifolds. Morphisms $(f : T \to S) \to (f' : T' \to S)$ in $[S]$ are morphisms of complex manifolds $g : T \to T'$ that satisfy $f' \circ g = f$. We fix the functor $[S] \to \mathbf{CMan}$ that maps an object $(f : T \to S)$ of $[S]$ to the complex manifold T. This functor gives $[S]$ the structure of a stack.
Let S' be another complex manifold. If $f : S' \to S$ is a morphism of complex manifolds, then composition with f yields a functor $[f] : [S'] \to [S]$, and this

functor is a morphism of stacks: the diagram

$$\begin{array}{ccc} [S'] & \xrightarrow{\quad [f] \quad} & [S] \\ & \searrow \qquad \swarrow & \\ & \mathbf{CMan} & \end{array}$$

is commutative.
Conversely, if $F : [S'] \to [S]$ is a morphism of stacks, then $F(\mathrm{id}_{S'})$ is a morphism $S' \to S$ of complex manifolds.
One checks that these two operations are inverses, and we therefore see that morphisms of complex manifolds $S' \to S$ correspond bijectively with morphisms of stacks $[S'] \to [S]$.
We will often identify a complex manifold S with its associated stack $[S]$.

Stacks form a 2-category. This means that the morphisms between any two stacks form a category rather than a set. In other words: the category of stacks consists of objects, morphisms, and morphisms between morphisms (which are called 2-morphisms).

Let $F : \mathcal{M} \to$ **CMan** be a stack and let S be a complex manifold. We denote by $\mathcal{M}_S$ the subcategory of $\mathcal{M}$ whose objects are those objects x of $\mathcal{M}$ that satisfy $F(x) = S$, and whose morphisms are those morphisms f of $\mathcal{M}$ that satisfy $F(f) = \mathrm{id}_S$. The 2-Yoneda lemma [SP, Tag 004B] states that there is an equivalence of categories

$$\mathbf{Hom}([S], \mathcal{M}) \xrightarrow{\sim} \mathcal{M}_S$$

given by $F \mapsto F(\mathrm{id}_S)$.

Consider the stack $\mathcal{M}_g$ of families of genus g curves, and let S be a complex manifold. Then $(\mathcal{M}_g)_S$ is the category of genus g curves over S, and the 2-Yoneda lemma gives us an equivalence of categories

$$\mathbf{Hom}([S], \mathcal{M}_g) \xrightarrow{\sim} (\mathcal{M}_g)_S.$$

So morphisms $[S] \to \mathcal{M}_g$ induce families of genus g curves over S. An inverse of Yoneda's equivalence is found as follows: to a family $f : \mathcal{C} \to S$ of genus g curves we associate the functor $\Phi_f : [S] \to \mathcal{M}_g$ given by $(T \to S) \mapsto (f_T : \mathcal{C} \times_S T \to T)$.

Example 2.4.3. Let $\mathcal{C}_g$ be the category whose objects are pairs (f, σ) where $f : \mathcal{C} \to S$ is a family of genus g curves and $\sigma : S \to \mathcal{C}$ is a section of f. Morphisms $(f', \sigma') \to (f, \sigma)$ in $\mathcal{C}_g$ are cartesian diagrams of the form

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{h'} & \mathcal{C} \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{h} & S \end{array}$$

such that $h' \circ \sigma' = \sigma \circ h$. As families with sections are well-behaved with respect to base changes and gluing, it follows that the functor

$$\mathcal{C}_g \to \mathbf{CMan} : (f : \mathcal{C} \to S, \sigma : S \to \mathcal{C}) \mapsto S$$

gives $\mathcal{C}_g$ the structure of a stack over **CMan**. There is a morphism of stacks $p : \mathcal{C}_g \to \mathcal{M}_g$ that forgets the sections.
Let $f : \mathcal{C} \to S$ be a family of genus g curves. The 2-Yoneda lemma implies that f corresponds to a morphism of stacks $\Phi_f : [S] \to \mathcal{M}_g$. We obtain a morphism $\Psi_f : [\mathcal{C}] \to \mathcal{C}_g$ as follows. An object of $[\mathcal{C}]$ is a morphism $g : T \to \mathcal{C}$ of complex manifolds. The functor Ψ_f then maps g to the family $\mathcal{C} \times_S T \to T$ with the section $(g, \mathrm{id}_T) : T \to \mathcal{C} \times_S T$. We obtain a diagram of stacks

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Psi_f} & \mathcal{C}_g \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow[\Phi_f]{} & \mathcal{M}_g \end{array}$$

and this diagram 2-commutes: there is a 2-isomorphism between the two induced morphisms $\mathcal{C} \to \mathcal{M}_g$. In fact, the diagram induces a representation of the fiber product $S \times_{\mathcal{M}_g} \mathcal{C}_g$ by $\mathcal{C}$. We see that the morphism of stacks $p : \mathcal{C}_g \to \mathcal{M}_g$ behaves like a universal family of genus g curves.

Definition 2.4.4. The *universal family of genus g curves* is the morphism of stacks $p : \mathcal{C}_g \to \mathcal{M}_g$ defined in Example 2.4.3.

Recall that a morphism of stacks $f : \mathcal{X} \to \mathcal{S}$ is *representable* if for each complex manifold and each morphism of stacks $S \to \mathcal{S}$ the fiber product $\mathcal{X} \times_{\mathcal{S}} S$ is again representable by a complex manifold. Equivalently, for each morphism of stacks $\Phi : S \to \mathcal{S}$ there exists a 2-cartesian diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow{\scriptstyle f} \\ S & \xrightarrow[\Phi]{} & \mathcal{S} \end{array}$$

where X is a complex manifold. We say that the morphism $\mathcal{X} \to \mathcal{S}$ is a *submersion* if it is representable and for each cartesian diagram of the above form the morphism of complex manifolds $X \to S$ is a submersion. Analogously, any property of morphisms of complex manifolds that is stable under base change can be generalized to morphisms of stacks. It follows from the discussion in Example 2.4.3 that the universal family of genus g curves $\mathcal{C}_g \to \mathcal{M}_g$ is, indeed, a family of genus g curves.

Example 2.4.5. This example generalizes Example 2.4.3. Let $r \geq 0$ be an integer, and define the category $\mathcal{C}_g^r$ as follows. Objects of $\mathcal{C}_g^r$ are tuples $(f, \sigma_1, \dots, \sigma_r)$ where f is a family of genus g curves, and $\sigma_1, \dots, \sigma_r$ are sections of f. Morphisms are cartesian diagrams of families compatible with the sections. The functor $\mathcal{C}_g^r \to$ **CMan** that maps a tuple $(f, \sigma_1, \dots, \sigma_r)$ to the base of f gives $\mathcal{C}_g^r$ the structure of a stack.
Let $f : \mathcal{C} \to S$ be a family of curves. Let $\mathcal{C}^r$ denote the r-fold fiber product

$$\mathcal{C}^r = \mathcal{C} \times_S \cdots \times_S \mathcal{C}.$$

and for $i = 1, \dots, r$ let $p_i : \mathcal{C}^r \to \mathcal{C}$ denote the projection onto the ith coordinate. Then f induces a morphism of stacks $\Psi_f^r : [\mathcal{C}^r] \to \mathcal{C}_g^r$ as follows. An object of $[\mathcal{C}^r]$ is a morphism of manifolds $g : T \to \mathcal{C}^r$. Such a morphism induces a family $f_T : \mathcal{C} \times_S T \to T$, together with r sections σ_i given by $\sigma_i = (p_i \circ g, \mathrm{id}_T) : T \to \mathcal{C} \times_S T$. The functor Ψ_f^r maps g to the object $(f_T, \sigma_1, \dots, \sigma_r)$ of $\mathcal{C}_g^r$. Moreover, the morphism Ψ_f^r, together with the morphism $\mathcal{C}^r \to S$ induced by f, gives rise to a representation by $\mathcal{C}^r$ of the fiber product $\mathcal{C}_g^r \times_{\mathcal{M}_g} S$:

$$\begin{array}{ccc} \mathcal{C}^r & \xrightarrow{\Psi_f^r} & \mathcal{C}_g^r \\ \downarrow & \square & \downarrow \\ S & \xrightarrow[\Phi_f]{} & \mathcal{M}_g \end{array}$$

Here the morphism $\mathcal{C}_g^r \to \mathcal{M}_g$ simply forgets all sections.

The stack $\mathcal{C}_g^r$ defined in Example 2.4.5 is the r-fold fiber product

$$\mathcal{C}_g^r = \mathcal{C}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{C}_g.$$

Note that $\mathcal{C}_g^1 = \mathcal{C}_g$, and $\mathcal{C}_g^0 = \mathcal{M}_g$.

Let $f : \mathcal{C} \to S$ be a family of genus g curves. For each integer $r \geq 0$ denote by $\mathcal{C}^r$ the r-fold fiber product

$$\mathcal{C}^r = \mathcal{C} \times_S \cdots \times_S \mathcal{C}.$$

Let $r, s \geq 0$ be integers, and let $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$ be a map of sets. Then we define a morphism f^ϕ of complex manifolds:

$$f^\phi : \mathcal{C}^r \to \mathcal{C}^s : (x_1, \dots, x_r) \mapsto (x_{\phi(1)}, \dots, x_{\phi(s)}).$$

In other words, f^ϕ permutes, forgets, and repeats coordinates of the fiber product $\mathcal{C}^r$. Note that, if $s = 0$, then f^ϕ is the morphism $\mathcal{C}^r \to S$ induced by f.

This construction can be generalized to the universal family $f : \mathcal{C}_g \to \mathcal{M}_g$ as follows. To ϕ we associate a functor $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$:

$$f^\phi : (f, \sigma_1, \dots, \sigma_r) \mapsto (f, \sigma_{\phi(1)}, \dots, \sigma_{\phi(s)}).$$

Morphisms in $\mathcal{C}_g^r$ and $\mathcal{C}_g^s$ are cartesian diagrams of families; these are left in place by the functor f^ϕ. It is easy to verify that f^ϕ is a morphism of stacks.

Definition 2.4.6. A *tautological map* is a morphism of stacks of the form $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$, with $r, s \geq 0$ integers and ϕ a map $\{1, \ldots, s\} \to \{1, \ldots, r\}$.

Let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be a tautological map associated to a map $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$. Moreover, let $h : \mathcal{C} \to S$ be a family of curves of genus g. Then the diagram

$$\begin{array}{ccc} \mathcal{C}^r & \xrightarrow{\Psi_h^r} & \mathcal{C}_g^r \\ {\scriptstyle h^\phi}\downarrow & \square & \downarrow{\scriptstyle f^\phi} \\ \mathcal{C}^s & \xrightarrow[\Psi_h^s]{} & \mathcal{C}_g^s \end{array} \tag{2.4.7}$$

is cartesian.

Let $f : \mathcal{C} \to S$ be a family of genus g curves, let $r, s, t, u \geq 0$ be integers, and consider the following commutative diagram of sets and the associated commutative diagram of complex manifolds:

$$\begin{array}{ccc} \{1, \ldots, t\} & \xleftarrow{\psi} & \{1, \ldots, r\} \\ {\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \phi} \\ \{1, \ldots, u\} & \xleftarrow[\chi]{} & \{1, \ldots, s\} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C}^t & \xrightarrow{f^\psi} & \mathcal{C}^r \\ {\scriptstyle f^\eta}\downarrow & & \downarrow{\scriptstyle f^\phi} \\ \mathcal{C}^u & \xrightarrow[f^\chi]{} & \mathcal{C}^s \end{array}$$

Using the Yoneda lemma it is straightforward to show that if the leftmost diagram is a pushout diagram in the category of sets, then the rightmost diagram is a cartesian diagram. Similarly, if the leftmost diagram is a pushout diagram, the associated diagram of tautological maps between stacks

$$\begin{array}{ccc} \mathcal{C}_g^t & \xrightarrow{f^\psi} & \mathcal{C}_g^r \\ {\scriptstyle f^\eta}\downarrow & & \downarrow{\scriptstyle f^\phi} \\ \mathcal{C}_g^u & \xrightarrow[f^\chi]{} & \mathcal{C}_g^s \end{array}$$

is cartesian.

Lemma 2.4.8. If $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ is injective, then for each family $f : \mathcal{C} \to S$ of genus g curves the induced morphism $f^\phi : \mathcal{C}^r \to \mathcal{C}^s$ is a submersion. Likewise, if ϕ is injective, then the associated tautological map $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ is a submersion.

Proof. Assume, first, that $s = 0$. Let $f : \mathcal{C} \to S$ be a family of genus g curves. Now f^ϕ is the morphism $\mathcal{C}^r \to S$. As submersions are stable under compositions and base changes, the morphism $\mathcal{C}^r \to S$ is a submersion.

As $s = 0$ we have $\mathcal{C}_g^s = \mathcal{M}_g$. We wish to prove that the morphism $\mathcal{C}_g^r \to \mathcal{M}_g$ is a submersion. Let S be a complex manifold and let $\Phi : S \to \mathcal{M}_g$ be a morphism. Then Φ corresponds to a family $f : \mathcal{C} \to S$ of genus g curves, and the fiber product

$\mathcal{C}_g^r \times_{\mathcal{M}_g} S$ is represented by $\mathcal{C}^r$. The induced morphism of complex manifolds $\mathcal{C}^r \to S$ is the tautological morphism associated to ϕ. As we have seen, this morphism is a submersion of complex manifolds. We may therefore conclude that the tautological map $\mathcal{C}_g^r \to \mathcal{M}_g$ is a submersion.

More generally, let $s \geq 0$ be any integer, and choose an injective map $\eta : \{1, \dots, r-s\} \to \{1, \dots, r\}$ whose image is disjoint from the image of ϕ. Let $f : \mathcal{C} \to S$ be a family of genus g curves. We obtain a pushout diagram of sets, and an associated cartesian diagram of complex manifolds:

$$\begin{array}{ccc} \{1,\dots,r\} & \xleftarrow{\psi} & \{1,\dots,r-s\} \\ \phi\uparrow & & \uparrow \\ \{1,\dots,s\} & \longleftarrow & \emptyset \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C}^r & \xrightarrow{f^\psi} & \mathcal{C}^{r-s} \\ f^\phi\downarrow & & \downarrow \\ \mathcal{C}^s & \longrightarrow & S \end{array}$$

The morphism $\mathcal{C}^{r-s} \to S$ is a submersion by the first part of this proof, so f^ϕ must be a submersion, too, as submersions are stable under base change.

Analogously, the tautological map $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ can be written as the base change of the submersion $\mathcal{C}_g^{r-s} \to \mathcal{M}_g$ by some tautological map $\mathcal{C}_g^s \to \mathcal{M}_g$, and therefore f^ϕ is a submersion. □

Remark 2.4.9. As we already saw in Section 2.3 the Teichmüller space $\mathcal{T}_g$ is closely related to the moduli space of genus g curves. The mapping class group $\Gamma_g = \mathrm{MCG}^+(\Sigma_g)$ of the compact oriented genus g surface Σ_g acts on $\mathcal{T}_g$, and the points in the quotient $\mathcal{T}_g/\Gamma_g$ are in bijective correspondence with isomorphism classes of genus g curves. This quotient, however, does not admit the structure of a complex manifold.
Instead of looking at the topological quotient $\mathcal{T}_g/\Gamma_g$, one could consider the *quotient stack*

$$[\mathcal{T}_g/\Gamma_g],$$

which is defined (in a more general setting of a complex Lie group acting on a complex manifold) as follows. Objects of $[\mathcal{T}_g/\Gamma_g]$ are pairs of morphisms $(P \to S, P \to \mathcal{T}_g)$, where the morphism $P \to S$ is a Γ_g-covering and the morphism $P \to \mathcal{T}_g$ is Γ_g-equivariant. Morphisms are cartesian diagrams of Γ_g-coverings compatible with the equivariant morphisms to $\mathcal{T}_g$. The functor mapping $(P \to S, P \to \mathcal{T}_g) \mapsto S$ gives $[\mathcal{T}_g/\Gamma_g]$ the structure of a stack.
Suppose, now, that $f : \mathcal{C} \to S$ is a family of genus g curves. Recall from Section 2.3 that we obtain a Γ_g-covering $\mathcal{P}(\mathcal{C}/S) \to S$. Points of $\mathcal{P}(\mathcal{C}/S)$ over $s \in S$ are Teichmüller structures on $\mathcal{C}_s$, so we obtain a canonical morphism

$$\mathcal{P}(\mathcal{C}/S) \to \mathcal{T}_g,$$

and this morphism is clearly Γ_g-equivariant.
We hence obtain a canonical morphism of stacks

$$\mathcal{M}_g \to [\mathcal{T}_g/\Gamma_g],$$

We leave it to the reader to verify that this is an isomorphism of stacks and to construct an inverse.

2.5 Differential forms on stacks

2.5.1 Differential forms

Differential forms on complex manifolds can be pulled back along morphisms of manifolds. Moreover, it is possible to glue differential forms along open coverings. As differential forms are well-behaved with respect to pullbacks and gluing, it makes sense to construct a stack of differential forms.

Indeed, let us consider the category A^* whose objects are simply differential forms on *any* complex manifold. If η and ω are some differential forms on complex manifolds T and S, respectively, then the morphisms $\eta \to \omega$ in A^* are precisely those morphisms $f : T \to S$ of the underlying manifolds for which $f^*\omega = \eta$. We consider the functor $A^* \to \mathbf{CMan}$ that maps a differential form to its underlying complex manifold. Note that this functor is faithful. It is not difficult to verify that this functor makes A^* a stack over **CMan**.

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Let S be a complex manifold, and let $\Phi : [S] \to A^*$ be a morphism of stacks. Then $\Phi(\mathrm{id}_S)$ is a differential form on S. Conversely, given a differential form ω on S we can define a morphism $\Phi_\omega : [S] \to A^*$ of stacks that maps a morphism $f : T \to S$ of complex manifolds (that is, an object of $[S]$) to the differential form $f^*\omega$ on T. These constructions are inverses; we see therefore that differential forms on S correspond one-to-one with morphisms of stacks $[S] \to A^*$. This legitimizes the following definition.

Definition 2.5.1. Let $\mathcal{X}$ be a stack over **CMan**. A *differential form on* $\mathcal{X}$ is a morphism of stacks $\mathcal{X} \to A^*$.

Notice that, by the above discussion, differential forms on a complex manifold S correspond canonically to differential forms on the underlying stack $[S]$. In other words: differential forms on stacks generalize differential forms on complex manifolds. From now on, we may identify the differential forms on a complex manifold S with the differential forms on the associated stack $[S]$.

Let $\mathcal{X}$ be a stack, and denote by $\pi : \mathcal{X} \to \mathbf{CMan}$ the corresponding functor. As the functor $A^* \to \mathbf{CMan}$ is faithful, any morphism of stacks $\mathcal{X} \to A^*$ over **CMan** is uniquely determined by its action on the objects of $\mathcal{X}$. Giving a differential form ω on $\mathcal{X}$ is, therefore, equivalent to giving for each object x of $\mathcal{X}$ a differential form $\omega(x)$ on the complex manifold $\pi(x)$, such that for each morphism $f : x \to y$ in $\mathcal{X}$ we have the equality $\pi(f)^*\omega(y) = \omega(x)$ of differential forms on $\pi(x)$.

Recall that stacks form a 2-category, so morphisms between two stacks do not form a set but a category. For an arbitrary stack $\mathcal{X}$ we obtain a category (and not a set) $A^*(\mathcal{X})$ of differential forms on $\mathcal{X}$. Fortunately, it is easy to verify that there are no 2-morphisms between two differential forms on any given stack, apart from identity morphisms. So $A^*(\mathcal{X})$ is a discrete category; differential forms on $\mathcal{X}$ form a class. If $\mathcal{X} = [S]$ is the stack associated to a complex manifold, then the objects of $A^*([S])$ are in bijective correspondence with differential forms on S. We can therefore view the discrete category $A^*([S])$ as a set by identifying its objects with the elements of $A^*(S)$.

2.5.2 Pullbacks

Suppose that $f : S' \to S$ is a morphism of complex manifolds, and let $[f] : [S'] \to [S]$ denote the associated morphism of stacks. Let $\omega \in A^*(S)$ be a differential form on S. Recall that differential forms on a complex manifold correspond bijectively to differential forms on the associated stack. If ω corresponds to the morphism of stacks $\Phi_\omega : [S] \to A^*$, then the pullback $f^*\omega$ corresponds to the composition $\Phi_\omega \circ [f] : [S'] \to A^*$. So it makes sense to define pullbacks of differential forms along morphisms of stacks as follows.

Definition 2.5.2. Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism of stacks. Given a differential form $\omega : \mathcal{X} \to A^*$ we define the *pullback* of ω along f to be the differential form $f^*\omega := \omega \circ f : \mathcal{X}' \to A^*$ on $\mathcal{X}'$.

This definition generalizes the definition of pullbacks of differential forms on complex manifolds.

Example 2.5.3. The following example allows us to switch seamlessly between evaluating differential forms on objects of stacks and taking pullbacks of differential forms.
Let $\mathcal{X}$ be a stack with a differential form $\omega : \mathcal{X} \to A^*$. Let X be any complex manifold. Recall the 2-Yoneda equivalence

$$\mathbf{Hom}(X, \mathcal{X}) \xrightarrow{\sim} \mathcal{X}_X : \Phi \mapsto \Phi(\mathrm{id}_X).$$

Let $\Phi : X \to \mathcal{X}$ be a morphism of stacks and let x be an object of $\mathcal{X}_X$. If $\Phi(\mathrm{id}_X) \cong x$ in $\mathcal{X}_X$, then we have an equality

$$\Phi^*\omega = \omega(x) \in A^*(X).$$

The following observation is useful when working with differential forms on stacks. Let $f, g : \mathcal{X}' \to \mathcal{X}$ be two morphisms of stacks, and let $\omega : \mathcal{X} \to A^*$ be a differential form on $\mathcal{X}$. Assume that there exists a 2-isomorphism between f and g. Then the compositions $\omega \circ f$ and $\omega \circ g : \mathcal{X}' \to A^*$ are 2-isomorphic as well. As there are no nontrivial 2-isomorphisms between differential forms, it follows that $f^*\omega = g^*\omega$.

The following is a generalization of Lemma 1.1.6.

Lemma 2.5.4. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks. Then the functor $f^* : A^*(\mathcal{S}) \to A^*(\mathcal{X})$ is injective.

Proof. Let ω and η be two differential forms on $\mathcal{S}$ such that $f^*\omega = f^*\eta$. In order to prove that $\omega = \eta$, it suffices to show that these functors evaluate equally on all objects of $\mathcal{S}$. Let s be an object of $\mathcal{S}$ over the complex manifold S. By using the

2-Yoneda lemma, we can construct a 2-cartesian diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & \mathcal{X} \\ \downarrow{\scriptstyle f_S} & \square & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\Phi} & \mathcal{S} \end{array}$$

such that X is a complex manifold, and such that $\Phi(\mathrm{id}_S) \cong s$ in $\mathcal{S}$. We now have the following equality of differential forms on X:

$$f_S^*(\omega(s)) = f_S^*\Phi^*\omega = \Psi^* f^*\omega = \Psi^* f^*\eta = f_S^*\Phi^*\eta = f_S^*(\eta(s)).$$

As f_S is a submersion, we deduce from Lemma 1.1.6 that $\omega(s) = \eta(s)$. □

2

2.5.3 Fiber integrals

Now, let us generalize taking fiber integrals to the setting of stacks. Recall that a morphism $\mathcal{X} \to \mathcal{S}$ of stacks is a *submersion* if it is representable and a submersion. That is, for each complex manifold S and each morphism $\Phi : S \to \mathcal{S}$ there exists a 2-cartesian diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & \mathcal{X} \\ \downarrow{\scriptstyle f_S} & \square & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\Phi} & \mathcal{S} \end{array} \tag{2.5.5}$$

where X is a complex manifold, and the morphism $f_S : X \to S$ is a submersion of complex manifolds.

We must first generalize the notion of differential forms with proper support over the base of a submersion to the setting of stacks. By Proposition 1.3.14 this property is stable under base change, and therefore it makes sense to generalize it as follows.

Definition 2.5.6. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks, and let ω be a differential form on $\mathcal{X}$. We say that ω has *proper support* over $\mathcal{S}$ if for each 2-cartesian diagram of the form 2.5.5 the pullback $\Psi^*\omega \in A^*(X)$ has proper support over S.

It follows from Proposition 1.3.14 that for each submersion $f : X \to S$ of complex manifolds, and each differential form $\omega \in A^*(X)$, the form ω has proper support over S if and only if the corresponding differential form on the stack $[X]$ has proper support over $[S]$.

Now, let us generalize the fiber integral operator along submersions of complex manifolds to the setting of stacks. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks, and let ω be a differential form on $\mathcal{X}$ with proper support over $\mathcal{S}$. Moreover, we denote the (implicitly given) functor $\mathcal{S} \to \mathbf{CMan}$ by π. We will construct a differential form $\int_f \omega$ on $\mathcal{S}$ as follows. Let s be any object of $\mathcal{S}$, and let $S = \pi(s)$. The functor $\int_f \omega$ should assign to s a differential form on S. By applying the 2-Yoneda lemma, we can construct a cartesian diagram of the form 2.5.5, such that $\Phi(\mathrm{id}_S) \cong s$ in $\mathcal{S}_S$. We obtain a differential form on S by pulling back ω along the morphism $\Psi : X \to \mathcal{X}$, and then taking the fiber integral of the resulting form along the submersion $f_S : X \to S$.

The following lemma implies that the resulting form $(\int_f \omega)_s$ on S does not depend on any choices. Moreover, one can show using this lemma that the above construction indeed defines a differential form on $\mathcal{S}$.

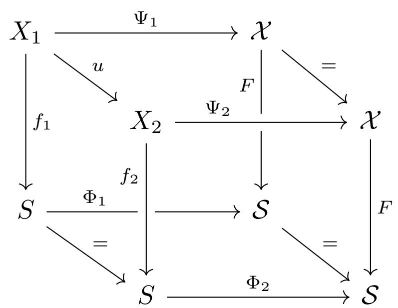
Lemma 2.5.7. Let $F : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks. Assume we have two 2-cartesian diagrams of stacks

$$\begin{array}{ccc} X_1 & \xrightarrow{\Psi_1} & \mathcal{X} \\ {\scriptstyle f_1}\downarrow & \square & \downarrow{\scriptstyle F} \\ S & \xrightarrow[\Phi_1]{} & \mathcal{S} \end{array} \qquad\qquad \begin{array}{ccc} X_2 & \xrightarrow{\Psi_2} & \mathcal{X} \\ {\scriptstyle f_2}\downarrow & \square & \downarrow{\scriptstyle F} \\ S & \xrightarrow[\Phi_2]{} & \mathcal{S} \end{array}$$

where X_1, X_2, S are complex manifolds, and assume that there exists a 2-isomorphism $\Phi_1 \Rightarrow \Phi_2$, or, equivalently, that $\Phi_1(\mathrm{id}_S) \cong \Phi_2(\mathrm{id}_S)$ in $\mathcal{S}_S$. Then for each differential form ω on $\mathcal{X}$ with proper support over $\mathcal{S}$ we have

$$\int_{f_1} \Psi_1^* \omega = \int_{f_2} \Psi_2^* \omega \in A^*(S).$$

Proof. Any 2-isomorphism $\Phi_1 \Rightarrow \Phi_2$ induces a morphism $u : X_1 \to X_2$ that makes the following cube-shaped diagram 2-commute:



See [SP, Tag 02XA]. Of this cube, the front, back, and rightmost face are 2-cartesian, so the same holds for the leftmost face, which is therefore a cartesian square of complex manifolds. By chasing through the above diagram we find the equality

$$\int_{f_1} \Psi_1^* \omega = \int_{f_1} u^* \Psi_2^* \omega = \mathrm{id}_S^* \int_{f_2} \Psi_2^* \omega = \int_{f_2} \Psi_2^* \omega,$$

where the middle equality follows from Proposition 1.3.14. $\square$

The defining property of the fiber integral can therefore be given as follows.

Definition 2.5.8. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks, and let $\omega : \mathcal{X} \to A^*$ be a differential form on $\mathcal{X}$ with proper support over $\mathcal{S}$. The *fiber integral* of ω along f is the unique differential form $\int_f \omega : \mathcal{S} \to A^*$ on $\mathcal{S}$ that satisfies the following property: for each cartesian diagram of the form 2.5.5, one has:

$$\Phi^* \left(\int_f \omega \right) = \left(\int_f \omega \right) (\Phi(\mathrm{id}_S)) = \int_{f_S} \Psi^* \omega \in A^*(S).$$

The properties of the fiber integral, as listed in Section 1.3, can be generalized immediately to the setting of stacks. For example, the base change formula 1.3.14 generalizes as follows.

2

Proposition 2.5.9 (Base change formula for stacks)**.** Consider a 2-cartesian diagram of stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ \mathcal{S}' & \xrightarrow[g]{} & \mathcal{S} \end{array}$$

where f and f' are submersions. If ω is a differential form on $\mathcal{X}$ that has proper support over $\mathcal{S}$, then $h^*\omega$ has proper support over $\mathcal{S}'$, and the following equality holds:

$$g^* \left(\int_f \omega \right) = \int_{f'} h^* \omega.$$

Proof. Suppose that we are given a 2-cartesian diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & \mathcal{X}' \\ \downarrow{\scriptstyle f_S} & \square & \downarrow{\scriptstyle f'} \\ S & \xrightarrow{\Phi} & \mathcal{S}' \end{array}$$

where X and S are complex manifolds. Then the following diagram is 2-cartesian, too:

$$\begin{array}{ccc} X & \xrightarrow{h \circ \Psi} & \mathcal{X} \\ \downarrow{\scriptstyle f_S} & \square & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g \circ \Phi} & \mathcal{S} \end{array}$$

By assumption the pullback $(h \circ \Psi)^*\omega = \Psi^* h^* \omega$ has proper support over S. We deduce that $h^*\omega$ has proper support over $\mathcal{S}'$.

Let s' be any object of $\mathcal{S}'$, and let S be its image under the functor $\mathcal{S}' \to$ **CMan**. Define $s = g(s')$. By the 2-Yoneda lemma, we can construct 2-cartesian diagrams as in the first part of this proof, such that $\Phi(\mathrm{id}_S) \cong s'$ in $\mathcal{S}'_S$. We then have, by definition of the fiber integral along f':

$$\left(\int_{f'} h^* \omega \right) (s') = \int_{f_S} \Psi^* h^* \omega \in A^*(S').$$

Note, moreover, that we have an isomorphism $(g \circ \Phi)(\mathrm{id}_S) \cong g(s') = s$ in $\mathcal{S}_S$. Applying the definition of the fiber integral along f now yields

$$\left(g^* \int_f \omega\right)(s') = \left(\int_f \omega\right)(g(s')) = \left(\int_f \omega\right)(s) = \int_{f_{s'}} (h \circ \Psi)^* \omega \in A^*(S).$$

We find that the two differential forms evaluate equally on objects of $\mathcal{S}'$, so they are equal. □

2.5.4 Differential forms on moduli stacks

Let $p : \mathcal{C}_g \to \mathcal{M}_g$ be the universal family of genus g curves. In this thesis we are mostly interested in differential forms on the moduli stacks $\mathcal{C}_g^r = \mathcal{C}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{C}_g$ for $r \geq 0$, where $\mathcal{C}_g^0 = \mathcal{M}_g$ and $\mathcal{C}_g^1 = \mathcal{C}_g$. In this section we will see that we can often pretend that these stacks are honest complex manifolds, when it comes to studying differential forms on them. In particular, we will be able to view pullbacks and fiber integrals along morphisms between these stacks in an intuitive way.

Let $f : \mathcal{C} \to S$ be a family of genus g curves. Recall that f corresponds to a morphism $\Phi_f : S \to \mathcal{M}_g$, and that for all $r \geq 0$ we have morphisms $\Psi_f^r : \mathcal{C}^r \to \mathcal{C}_g^r$ that make the following diagram cartesian:

$$\begin{array}{ccc} \mathcal{C}^r & \xrightarrow{\Psi_f^r} & \mathcal{C}_g^r \\ \downarrow & \square & \downarrow p \\ S & \xrightarrow{\Phi_f} & \mathcal{M}_g \end{array}$$

Proposition 2.5.10. Let $r \geq 0$ be an integer. Let ω be a differential form on $\mathcal{C}_g^r$. For every family $f : \mathcal{C} \to S$ denote by ω_f the differential form on $\mathcal{C}^r$ obtained by pulling back ω along the canonical morphism $\Psi_f^r : \mathcal{C}^r \to \mathcal{C}_g^r$. The forms ω_f are compatible with base change: if we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow f' & \square & \downarrow f \\ S' & \longrightarrow & S \end{array}$$

where f and f' are families of genus g curves, then the pullback of ω_f along the induced morphism $\mathcal{C}'^r \to \mathcal{C}^r$ equals $\omega_{f'}$.
Conversely, if we are given a differential form $\omega_f \in A^*(\mathcal{C}^r)$ for each family $f : \mathcal{C} \to S$ of genus g curves, and these forms are compatible with base change, then there is a unique differential form ω on $\mathcal{C}_g^r$ such that $(\Psi_f^r)^*\omega = \omega_f$ for each family $f : \mathcal{C} \to S$ of genus g curves.

In other words: differential forms on $\mathcal{M}_g$ are differential forms that occur universally on the bases of families of genus g curves, differential forms on $\mathcal{C}_g$ are differential forms that occur universally on the sources of families of genus g

curves, and analogous statements hold for differential forms on $\mathcal{C}_g^r$ for $r \geq 2$. We will prove Proposition 2.5.10 later in this section.

The proposition also implies that taking pullbacks and fiber integrals along tautological maps works 'as expected'. Indeed, let $r, s \geq 0$ be integers, let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be a map, and let $p^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be the associated tautological map. Recall that we have for each family $f : \mathcal{C} \to S$ an induced 2-cartesian diagram of stacks (2.4.7):

$$\begin{array}{ccc} \mathcal{C}^r & \xrightarrow{\Psi_f^r} & \mathcal{C}_g^r \\ {\scriptstyle f^\phi}\downarrow & \square & \downarrow{\scriptstyle p^\phi} \\ \mathcal{C}^s & \xrightarrow[\Psi_f^s]{} & \mathcal{C}_g^s \end{array}$$

Let ω be a differential form on $\mathcal{C}_g^s$. By the proposition, ω induces for each family $f : \mathcal{C} \to S$ a differential form $\omega_f := (\Psi_f^s)^*\omega$ on $\mathcal{C}^s$. Likewise, the pullback $\omega' := (p^\phi)^*\omega$ on $\mathcal{C}_g^r$ associates to each family $f : \mathcal{C} \to S$ a differential form $\omega'_f := (\Psi_f^r)^*\omega'$ on $\mathcal{C}^r$. But as the above diagram 2-commutes, we simply find that $\omega'_f = (f^\phi)^*\omega_f$ for each family f. So, roughly speaking, under the correspondences of Proposition 2.5.10, taking pullbacks of differential forms along tautological maps works 'as expected'.

An analogous statement can be made for fiber integrals. Suppose ϕ is injective, so p^ϕ is a submersion. Let ω be a differential form on $\mathcal{C}_g^r$, and set $\omega' := \int_{p^\phi} \omega$. By the base change formula we have $(\Psi_f^s)^*\omega' = \int_{f^\phi} \Psi_f^r\omega$. Therefore the fiber integral is compatible with the correspondences of Proposition 2.5.10, too.

These observations will allow us to pretend that moduli stacks behave like honest complex manifolds in Chapter 4 when we are working with differential forms on these stacks.

Another observation we should make is the following. Assume that $g \geq 2$. Recall from Section 2.3 that the stack $\mathcal{T}_g$ of families of genus g curves with Teichmüller structure is representable by a complex manifold. We have, moreover, a morphism of stacks $\mathcal{T}_g \to \mathcal{M}_g$. This is a covering map. Indeed, let S be any complex manifold, and let $\Phi : S \to \mathcal{M}_g$ be any morphism. Then Φ corresponds to a family of curves $f : \mathcal{C} \to S$. Consider the covering $\mathcal{P}(\mathcal{C}/S) \to S$ as defined in Section 2.3, and notice that the base change $\mathcal{P}(\mathcal{C}/S) \times_S \mathcal{C} \to \mathcal{P}(\mathcal{C}/S)$ is a family of genus g curves with a *canonical* Teichmüller structure. We hence obtain a canonical morphism $\mathcal{P}(\mathcal{C}/S) \to \mathcal{T}_g$. The following diagram is 2-commutative:

$$\begin{array}{ccc} \mathcal{P}(\mathcal{C}/S) & \longrightarrow & \mathcal{T}_g \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Phi} & \mathcal{M}_g \end{array}$$

In fact, this diagram is 2-cartesian: it induces a representation of the fiber product $\mathcal{T}_g \times_{\mathcal{M}_g} S$ by the complex manifold $\mathcal{P}(\mathcal{C}/S)$. It follows that the morphism $\mathcal{T}_g \to \mathcal{M}_g$ is representable, and a covering, and in particular a submersion. This implies that the pullback operator

$$A^*(\mathcal{M}_g) \to A^*(\mathcal{T}_g)$$

is injective. We may therefore view $A^*(\mathcal{M}_g)$ as a *subset* of $A^*(\mathcal{T}_g)$.

Analogously, the induced morphism $\mathcal{X}_g^r \to \mathcal{C}_g^r$ is a covering map for all $r \geq 1$, and we obtain inclusions

$$A^*(\mathcal{C}_g^r) \to A^*(\mathcal{X}_g^r).$$

We will finish this section by proving Proposition 2.5.10. We will use the following lemmas.

Lemma 2.5.11. Recall that the objects of the stack $\mathcal{C}_g$ are pairs (f, σ), where f is a family of genus g curves and σ is a section of f. Let $\omega : \mathcal{C}_g \to A^*$ be a differential form. Let $f : \mathcal{C} \to S$ be a family of genus g curves, and let $\Psi_f : \mathcal{C} \to \mathcal{C}_g$ be the canonical morphism. Then we have an equality of differential forms

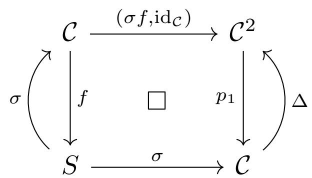
$$\Psi_f^* \omega = \omega(p_1, \Delta) \in A^*(\mathcal{C}),$$

where $p_1 : \mathcal{C}^2 \to \mathcal{C}$ is the projection and $\Delta : \mathcal{C} \to \mathcal{C}^2$ is its diagonal section.
If, moreover, $\sigma : S \to \mathcal{C}$ is a section of f, then

$$\omega(f, \sigma) = \sigma^* \Psi_f^* \omega \in A^*(S).$$

Proof. The morphism of stacks $\Psi_f : [\mathcal{C}] \to \mathcal{C}_g$ maps the canonical object $\mathrm{id}_{\mathcal{C}}$ of $[\mathcal{C}]$ to the pair (p_1, Δ), which proves the first statement.

For the second statement, consider the cartesian diagram with sections



As ω is a functor, we find:

$$\sigma^* \Psi_f^* \omega = \sigma^* \omega(p_1, \Delta) = \omega(f, \sigma).$$

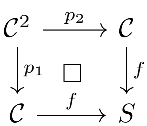
□

Lemma 2.5.12. Assume we are given for each family $f : \mathcal{C} \to S$ of genus g curves a differential form $\omega_f \in A^*(\mathcal{C})$, compatible with base change. Then for each family $f : \mathcal{C} \to S$ we have an equality

$$\Delta^* \omega_{p_1} = \omega_f$$

where $p_1 : \mathcal{C}^2 \to \mathcal{C}$ is the projection and $\Delta : \mathcal{C} \to \mathcal{C}^2$ is its diagonal section.

Proof. Consider the cartesian diagram



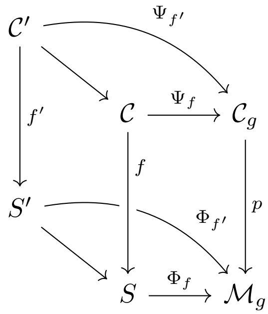
We find: $p_2^*\omega_f = \omega_{p_1}$, and pulling this equality back along the diagonal gives the desired result. □

Proof of Proposition 2.5.10. Verifying Proposition 2.5.10 is straightforward if $r = 0$.

Suppose, now, that $r = 1$. Let $\omega : \mathcal{C}_g \to A^*$ be a differential form, and for each family $f : \mathcal{C} \to S$ set

$$\omega_f := \Psi_f^* \omega \in A^*(\mathcal{C}).$$

If we have a cartesian diagram as in the statement of the proposition, then the following diagram is 2-commutative:



We therefore find that the pullback of ω_f along the morphism $\mathcal{C}' \to \mathcal{C}$ equals $\omega_{f'}$.

Conversely, suppose that we have for each family $f : \mathcal{C} \to S$ a differential form $\omega_f \in A^*(\mathcal{C})$, compatible with base change. We then construct a functor $\omega : \mathcal{C}_g \to A^*$ as follows: ω sends a pair (f, σ), with f a family and σ a section, to the differential form $\sigma^*\omega_f \in A^*(S)$. As the forms ω_f are compatible with base change, this defines a morphism of stacks, so we obtain a differential form ω on $\mathcal{C}_g$.

Lemmas 2.5.11 and 2.5.12 now imply that the two constructions we described above are inverses.

The proof for $r \geq 2$ is very similar and hence omitted. □

Remark 2.5.13. Recall from Remark 2.4.9 that we may view $\mathcal{M}_g$ as the quotient stack $[\mathcal{T}_g/\Gamma_g]$, where Γ_g is the mapping class group of the compact oriented genus g surface Σ_g that acts on the Teichmüller space $\mathcal{T}_g$. Moreover we have a canonical submersion $\mathcal{T}_g \to \mathcal{M}_g$, and the corresponding pullback operator gives an inclusion $A^*(\mathcal{M}_g) \to A^*(\mathcal{T}_g)$. The image of this inclusion consists of the Γ_g-invariant forms on $\mathcal{T}_g$. This gives us yet another way of thinking about differential forms on $\mathcal{M}_g$.

2.6 Hermitian vector bundles on moduli spaces of curves

Vector bundles on complex manifolds are well-behaved: we can take pullbacks of vector bundles, glue vector bundles on open coverings, and glue isomorphisms of vector bundles. It therefore makes sense to construct a stack of vector bundles on complex manifolds, as follows. The *stack of (rank n) vector bundles* $\mathcal{V}_n$ has as its objects holomorphic vector bundles $E \to S$ of rank n, and its morphisms are pullback diagrams of vector bundles

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

The functor $\mathcal{V}_n \to \mathbf{CMan}$ sends a vector bundle $E \to S$ to its base space S.

If $\mathcal{X}$ is any other stack over **CMan**, then a *vector bundle (of rank n)* on $\mathcal{X}$ is a morphism of stacks $\mathcal{X} \to \mathcal{V}_n$. We obtain a category $\mathcal{V}_n(\mathcal{X}) = \mathbf{Hom}(\mathcal{X}, \mathcal{V}_n)$ of rank n vector bundles on $\mathcal{X}$. If S is a complex manifold, the 2-Yoneda lemma gives an equivalence of categories between the category $\mathcal{V}_n([S])$ of vector bundles on the stack $[S]$ and the category $(\mathcal{V}_n)_S$ of vector bundles on the complex manifold S. Note that, unlike in the setting of differential forms, this is not a bijection but 'merely' an equivalence of categories. This is to be expected: the pullback of a vector bundle along a morphism of manifolds is only defined up to a unique isomorphism.

If $f : \mathcal{X}' \to \mathcal{X}$ is a morphism of stacks, and $E : \mathcal{X} \to \mathcal{V}_n$ a rank n vector bundle on $\mathcal{X}$, we can define the *pullback* f^*E to be the rank n vector bundle $E \circ f : \mathcal{X}' \to \mathcal{V}_n$ on $\mathcal{X}'$. We obtain a pullback functor $f^* : \mathcal{V}_n(\mathcal{X}) \to \mathcal{V}_n(\mathcal{X}')$.

Analogously, one can define the *stack of hermitian vector bundles (of rank n)* $\overline{\mathcal{V}}_n$ in a similar way: its objects are hermitian vector bundles of rank n, and its morphisms are base change diagrams that induce isometries on all fibers. A *hermitian vector bundle of rank n* on a stack $\mathcal{X}$ is then a morphism of stacks $\mathcal{X} \to \overline{\mathcal{V}}_n$. For each morphism $f : \mathcal{X}' \to \mathcal{X}$ we obtain a pullback functor $f^* : \overline{\mathcal{V}}_n(\mathcal{X}) \to \overline{\mathcal{V}}_n(\mathcal{X}')$.

Analogous to Proposition 2.5.10 we have:

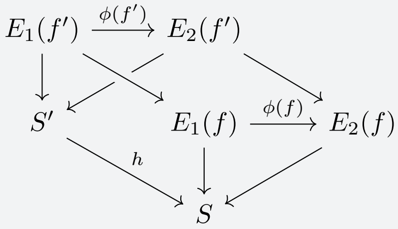
Example 2.6.1. The category $\mathcal{V}_n(\mathcal{M}_g)$ of rank n vector bundles on $\mathcal{M}_g$ has as its objects functors $\mathcal{M}_g \to \mathcal{V}_n$ over **CMan**. That is: a rank n vector bundle E on $\mathcal{M}_g$ assigns to each family $f : \mathcal{C} \to S$ of genus g curves a rank n vector bundle $E(f) \to S$, and to each cartesian square of the form

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{h} & S \end{array}$$

a pullback diagram

$$\begin{array}{ccc} E(f') & \longrightarrow & E(f) \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{h} & S, \end{array}$$

under the obvious compatibility criterion with respect to compositions of cartesian squares.
Let E_1, E_2 be two rank n vector bundles on $\mathcal{M}_g$. A morphism of vector bundles $\phi : E_1 \to E_2$ is a morphism of functors over **CMan**. That is: ϕ assigns to each family of genus g curves a morphism $\phi(f) : E_1(f) \to E_2(f)$ of vector bundles over S, such that for each cartesian diagram of curves as above the following induced diagram is commutative:



Hermitian vector bundles on $\mathcal{M}_g$ can be described analogously.

Analogous to the situation with differential forms, one can show that giving a (hermitian) vector bundle on $\mathcal{C}_g^r$ is equivalent to assigning to every family $f : \mathcal{C} \to S$ of genus g curves a (hermitian) vector bundle on the r-fold fiber product $\mathcal{C}^r$ over S, under the corresponding base change compatibility criterion.

Example 2.6.2. Suppose that $f : \mathcal{C} \to S$ is a family of curves. On the diagonal bundle $O(\Delta)$ we have constructed a canonical hermitian metric in Chapter 1. This construction is stable under base change, so universally we obtain a hermitian line bundle $O(\Delta)$ on $\mathcal{C}_g^2$.

Example 2.6.3. Let $f : \mathcal{C} \to S$ be a family of curves. Recall that we have a canonical isomorphism

$$\Delta^* O(\Delta) \xrightarrow{\sim} \omega_f^{\otimes -1}$$

and this canonical isomorphism induces a canonical metric on the relative cotangent bundle ω.
This construction, too, is compatible with base change. We therefore obtain a canonical hermitian line bundle $\omega = \omega_{\mathcal{C}_g/\mathcal{M}_g}$ on the universal family $\mathcal{C}_g$ of genus g curves.

The *first Chern form* is a differential form c_1 on the stack of hermitian line bundles $\overline{\mathcal{V}}_1$, defined as follows. The functor $c_1 : \overline{\mathcal{V}}_1 \to A^*$ takes a hermitian line bundle $L \to S$ and maps it to the differential form $c_1(L) \in A^*(S)$, where $c_1(L)$ denotes the first Chern form of L on the complex manifold S. This gives a well-defined functor as taking first Chern forms on complex manifolds commutes with taking pullbacks. It follows that for each stack $\mathcal{X}$ and each hermitian line bundle $\mathcal{L}$ on $\mathcal{X}$ we can take the first Chern form of $\mathcal{L}$ by composing with c_1 to obtain a differential form $c_1(\mathcal{L})$ on $\mathcal{X}$. Notice that this construction generalizes taking the first Chern form of a hermitian line bundle on a complex manifold.

Next, we will generalize the construction of the Deligne pairing to the setting of stacks. Suppose that $f : \mathcal{X} \to \mathcal{S}$ is a morphism of stacks, and suppose moreover that f is a family of curves. Let $L, M : \mathcal{X} \to \overline{\mathcal{V}}_1$ be two hermitian line bundles on $\mathcal{C}_g$. We will define a hermitian line bundle $\langle L, M\rangle : \mathcal{S} \to \overline{\mathcal{V}}_1$ as follows. For each object s of $\mathcal{S}$ over the complex manifold S choose a cartesian diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\Psi_s} & \mathcal{X} \\ \big\downarrow{\scriptstyle f_s} & \square & \big\downarrow{\scriptstyle f} \\ S & \xrightarrow{\Phi_s} & \mathcal{S}. \end{array}$$

where X is a complex manifold, and Φ_s is such that $\Phi_s(\mathrm{id}_S) \cong s$ in $\mathcal{S}_S$. Then $L(\Psi_s(\mathrm{id}_X))$ and $M(\Psi_s(\mathrm{id}_X))$ are line bundles on X. Taking the Deligne pairing of these line bundles along the family f_s of curves then yields a line bundle $\langle L(\Psi_s(\mathrm{id}_X)), M(\Psi_s(\mathrm{id}_X))\rangle$ on S. The functor $\langle L, M\rangle : \mathcal{S} \to \overline{\mathcal{V}}_1$ maps s to this line bundle. A morphism in $\mathcal{S}$ is mapped to the canonically induced pullback diagram of corresponding line bundles. Note that the functor $\langle L, M\rangle : \mathcal{S} \to \overline{\mathcal{V}}_1$ does depend on choices, and is only determined up to 2-isomorphism. In other words: the Deligne pairing of L and M is a line bundle on $\mathcal{S}$, defined up to isomorphism.

It follows immediately that Proposition 1.4.13 generalizes to the setting of stacks: we have the following equality of differential forms on $\mathcal{S}$:

$$c_1(\langle L, M\rangle) = \int_f c_1(L) \wedge c_1(M).$$

In particular, the Deligne pairing along tautological submersions $\mathcal{C}_g^{r+1} \to \mathcal{C}_g^r$ behaves as expected.

Example 2.6.4. Consider the diagonal bundle $O(\Delta)$ on $\mathcal{C}_g^2$ and the relative cotangent bundle $\omega = \omega_{\mathcal{C}_g/\mathcal{M}_g}$ on $\mathcal{C}_g$ with their canonical metrics. We have an equality of differential forms on $\mathcal{C}_g$:

$$\int_{p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g} c_1(O(\Delta))^2 = c_1(\langle O(\Delta), O(\Delta)\rangle) = c_1(\Delta^* O(\Delta)) = -c_1(\omega).$$

2.7 The universal Jacobian bundle

The *Jacobian* of the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ is a stack whose objects are pairs (f, σ) where $f : \mathcal{C} \to S$ is a family of genus g curves and $\sigma : S \to \mathcal{J}_{\mathcal{C}/S}$ is a section of the Jacobian family $\mathcal{J}_{\mathcal{C}/S}$ associated to f. Morphisms $(f', \sigma') \to (f, \sigma)$ in $\mathcal{J}_g$ are cartesian diagrams

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \big\downarrow{\scriptstyle f'} & \square & \big\downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

such that the induced diagram

$$\begin{array}{ccc} \mathcal{J}_{\mathcal{C}'/S'} & \longrightarrow & \mathcal{J}_{\mathcal{C}/S} \\ {\scriptstyle\sigma'}\uparrow & & \uparrow{\scriptstyle\sigma} \\ S' & \longrightarrow & S \end{array}$$

is commutative. The functor $\mathcal{J}_g \to \mathbf{CMan}$ that maps a pair (f, σ) to the base of f gives $\mathcal{J}_g$ the structure of a stack. Forgetting sections yields a canonical morphism of stacks $\mathcal{J}_g \to \mathcal{M}_g$.

Recall that every family $f : \mathcal{C} \to S$ of genus g curves gives rise to a morphism of stacks $\Phi_f : S \to \mathcal{M}_g$. It is straightforward to show that the fiber product of stacks $S \times_{\mathcal{M}_g} \mathcal{J}_g$ is then represented by the relative Jacobian $\mathcal{J}_{\mathcal{C}/S}$ of f. More precisely: there is a natural morphism $\mathcal{J}_{\mathcal{C}/S} \to \mathcal{J}_g$, and the following diagram of stacks is 2-cartesian:

$$\begin{array}{ccc} \mathcal{J}_{\mathcal{C}/S} & \longrightarrow & \mathcal{J}_g \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{\Phi_f} & \mathcal{M}_g \end{array}$$

It follows that the morphism of stacks $\mathcal{J}_g \to \mathcal{M}_g$ is a family of complex tori.

The following analogue to Proposition 2.5.10 is easily seen to hold:

Proposition 2.7.1. Let ω be a differential form on $\mathcal{J}_g$. For each family $f : \mathcal{C} \to S$ of genus g curves let $\mathcal{J}_f \to S$ denote the relative Jacobian family of f, and let $\omega_f \in A^*(\mathcal{J}_f)$ denote the pullback of ω along the induced morphism of stacks $\mathcal{J}_f \to \mathcal{J}_g$. The forms ω_f are compatible with base change: for each cartesian diagram

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

with f and f' families of genus g curves, the pullback of ω_f along the induced morphism $\mathcal{J}_{f'} \to \mathcal{J}_f$ equals $\omega_{f'}$.
Conversely, if we are given a differential form $\omega_f \in A^*(\mathcal{J}_f)$ for each family f of genus g curves, satisfying the necessary compatibility conditions under pullbacks, then there is a unique differential form ω on $\mathcal{J}_g$ such that the pullback of ω along the canonical morphism $\mathcal{J}_f \to \mathcal{J}_g$ equals ω_f for each family f of genus g curves. $\square$

Similarly, (hermitian) vector bundles on $\mathcal{J}_g$ can be viewed as (hermitian) vector bundles that occur universally on the relative Jacobians of all families of genus g curves. For instance, we obtain the canonical hermitian line bundle $\mathcal{B}$ on the universal Jacobian bundle $\mathcal{J}_g$. This allows us to generalize the results from Section 1.4 to the universal setting. For instance, we have canonical morphisms of stacks

$$\delta : \mathcal{C}_g^2 \to \mathcal{J}_g \quad \text{and} \quad \kappa : \mathcal{C}_g \to \mathcal{J}_g.$$

Here δ maps a family $f : \mathcal{C} \to S$ with sections $\sigma_1, \sigma_2 : S \to \mathcal{C}$ to the pair (f, σ) in $\mathcal{J}_g$, with σ the section

$$\sigma : S \to \mathcal{J}_f : s \mapsto [O(\sigma_2(s) - \sigma_1(s))] \in \mathrm{Jac}(\mathcal{C}_s).$$

Likewise, the morphism κ maps a family $f : \mathcal{C} \to S$ with section σ to the pair (f, σ) in $\mathcal{J}_g$, where σ is the section

$$\sigma : S \to \mathcal{J}_f : s \mapsto [O((2g-2)\sigma(s)) \otimes \omega^{\otimes -1}] \in \mathrm{Jac}(\mathcal{C}_s).$$

We then have canonical isometries

$$\delta^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} p_1^* \omega^{\otimes -1} \otimes p_2^* \omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2}$$

and

$$\kappa^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} \omega^{-2g(2g-2)} \otimes p^* \langle \omega, \omega \rangle_p.$$

of hermitian vector bundles on $\mathcal{C}_g^2$ and $\mathcal{C}_g$, respectively.