Tautological differential forms on moduli spaces of curves

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Chapter 1

Families of curves

The first chapter serves as a preliminary chapter, where we establish some of the notation and theory we will use in Chapter 4. In Section 1.1 we will study families of manifolds, which are roughly speaking collections of manifolds smoothly parametrized by another manifold. Moreover we show why the category of manifolds does not have all fiber products, and show that fiber products of families of manifolds do exist. In Section 1.2 we will study currents, and in particular we will look at the pushforward operator on currents. Afterwards, in Section 1.3, we will be constructing the fiber integral operator along families of manifolds, which is the smooth analogue of the pushforward operator on currents. Finally, in Section 1.4 we will discuss hermitian vector bundles, and construct some hermitian line bundles that appear canonically on families of curves. The theory discussed in this section will allow us to construct tautological forms on families of curves and find relations amongst them, which will be done in Chapter 4.

In this thesis the terms 'manifold' and 'smooth manifold' mean the same thing: a locally ringed space that is second-countable, Hausdorff, and locally isomorphic to the Euclidean space $\mathbb{R}^n$ (for some $n \geq 0$) with its sheaf of smooth real-valued functions. Equivalently, a manifold is a second-countable Hausdorff topological space equipped with a smooth atlas. While we do not require that manifolds are connected, we do assume that all manifolds are equidimensional. This is merely for our convenience; most theory immediately generalizes to nonequidimensional manifolds by reducing to equidimensional components.

Likewise, complex manifolds are always assumed to be second-countable, Hausdorff, and equidimensional.

We will assume that the reader is familiar with the elementary theory of manifolds as in [Lee03] and [dRha84]. For the reader's convenience we will be repeating some definitions.

1.1 Families of manifolds

In this section, we consider submersions: morphisms of manifolds whose fibers are, again, manifolds. Moreover, we will define oriented submersions. These are submersions whose fibers are equipped with orientations that vary continuously.

1.1.1 Submersions

Let $f : X \to Y$ be a morphism of manifolds. Recall that f is a *submersion* if for all points $x \in X$ the associated map of tangent spaces $df_x : T_{X,x} \to T_{Y,f(x)}$ is surjective. So submersions are the analytic analogue to the smooth morphisms in the setting of algebraic geometry.

Example 1.1.1. If X and Y are two manifolds, then the projection $p_2 : X \times Y \to Y$ is a submersion.

By the Constant Rank Theorem ([Lee03, Theorem 4.12]) a submersion locally looks like a projection $\mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$. More precisely, if $f : X \to Y$ is a submersion, then for each point $x \in X$ we can construct a commutative diagram

$$\begin{array}{ccc} U & \hookrightarrow & \mathbb{R}^r \times \mathbb{R}^n \\ \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle p_2} \\ V & \hookrightarrow & \mathbb{R}^n \end{array}$$

where $n = \dim(Y)$ and $r + n = \dim(X)$, where $U \subseteq X$ and $V \subseteq Y$ are open neighborhoods of x and $f(x)$, respectively, such that $f(U) \subseteq V$, and where the horizontal arrows are open immersions of manifolds.

Note that in the above situation the fibers of f are locally isomorphic to $\mathbb{R}^r$. Indeed, the fibers of a submersion $f : X \to Y$ are properly embedded submanifolds of X; see [Lee03, Theorem 5.12]. In other words, the fibers of a submersion $f : X \to Y$ form a family of manifolds parametrized by the points of Y. By using [Lee03, Theorem 5.29] one can easily show that for any $y \in Y$ the fiber $X_y = f^{-1}(y)$ is the fiber product in the category of manifolds of the morphisms $\{y\} \to Y$ and $f : X \to Y$:

$$X_y = X \times_Y \{y\}.$$

In Section 1.1.2 we will look further into fiber products in the category of manifolds.

Definition 1.1.2. A *family of manifolds* is a surjective submersion of manifolds $f : X \to Y$. A *family of compact/connected/... manifolds* is a family of manifolds whose fibers are all compact/connected/... manifolds.

Lemma 1.1.3. Let $f : X \to Y$ be a family of compact connected manifolds. Then f is a proper map.

Proof. First let $f : X \to Y$ be any continuous map of topological spaces. We define an equivalence relation $\sim$ on X, where two points x_1, x_2 are equivalent if and only if $f(x_1) = f(x_2) =: y$ and the points x_1, x_2 lie in the same connected component of the fiber $f^{-1}(y)$. The *component decomposition* is the decomposition of f into the continuous maps $X \to X/\sim \to Y$. By [WD79, B.III] the quotient map $X \to X/\sim$ is closed if X and Y are Hausdorff, X is locally compact, and all connected components of all fibers of f are compact.

This is clearly the case if $f : X \to Y$ is a family of connected compact manifolds. Moreover, in that case, the map $X/\sim \to Y$ is a bijection. As f is a surjective submersion, it is itself a quotient map ([Lee03, Proposition 4.28]). It follows that the map $X/\sim \to Y$ is a homeomorphism, and hence f is closed.

We find that f is a closed map with compact fibers, and therefore f is proper ([Lee03, A.53]). □

Ehresmann's fibration theorem [Ehr52; Voi02, Theorem 9.3] states that proper submersions with a contractible base are, in fact, trivial fiber bundles.

Theorem 1.1.4 (Ehresmann)**.** Let $f : X \to Y$ be a proper submersion of manifolds, and assume that Y is contractible. Then f is a trivial smooth fiber bundle. □

As every manifold can be covered with contractible opens, we immediately obtain the following.

Corollary 1.1.5. Let $f : X \to Y$ be a proper submersion of manifolds. Then f is a smooth fiber bundle. □

In particular we find that families of compact connected manifolds are smooth fiber bundles.

We will be using the following lemma later.

Lemma 1.1.6. Let $f : X \to Y$ be a submersion of manifolds. Then the pullback operator on differential forms

$$f^* : A^*(Y) \to A^*(X)$$

is injective.

Proof. Let $x \in X$ be a point. As f is a submersion, the tangent map $df : T_{X,x} \to T_{Y,f(x)}$ is surjective, and dually, the cotangent map $T^*_{Y,f(x)} \to T^*_{X,x}$ is injective. Taking exterior algebras yields the pullback map

$$\bigwedge T^*_{Y,f(x)} \to \bigwedge T^*_{X,x},$$

which is injective, too. As differential forms on Y and X are sections of the bundles $\bigwedge T^*_Y$ and $\bigwedge T^*_X$, respectively, the lemma follows. □

1.1.2 Fiber products of manifolds

The category **Man** of manifolds is not as well-behaved as, say, the category of schemes. For instance, the category **Man** does not have all fiber products. In this section we will show that a fiber product of two morphisms of manifolds does exist if one of the morphisms is a submersion.

Proposition 1.1.7. Let $f : X \to S$ and $g : Y \to S$ be morphisms of manifolds. If f is a submersion, then the fiber product $X \times_S Y$ exists in **Man**. The underlying topological space of $X \times_S Y$ is the fiber product of the underlying topological spaces of X, Y, and S. The induced morphism $X \times_S Y \to Y$ is again a submersion.

We will prove this proposition later in this section. Before we prove this proposition we will look at some properties a fiber product should satisfy if it exists, and study some cases in which fiber products of manifolds do not exist or behave unexpectedly.

Suppose that S is a manifold, and let $f : X \to S$ and $g : Y \to S$ be two morphisms of manifolds. Suppose, moreover, that the fiber product $X \times_S Y$ exists in the corresponding category of manifolds. As the set of points of any manifold can be identified with the set of morphisms from the one-point manifold to that manifold, one deduces that the set of points of $X \times_S Y$ is the fiber product in the category of sets:

$$|X \times_S Y| = |X| \times_{|S|} |Y| \text{ in } \mathbf{Set}.$$

Moreover, let T be the fiber product of f and g in the category **Top** of topological spaces. By the universal property of the fiber product there exists a continuous map $X \times_S Y \to T$. As the sets of points underlying T and $X \times_S Y$ both equal the fiber product in the category of sets, this map is moreover a bijection. We conclude that $X \times_S Y$ and T have the same underlying sets, and the topology on $X \times_S Y$ is stronger than the topology on T. The following example shows that this topology can be strictly stronger.

Example 1.1.8. Consider the morphism of manifolds

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} \exp(-1/x^2)\sin(2\pi/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Moreover, let $g : \{0\} \to \mathbb{R}$ be the inclusion. The topological fiber product of these two morphisms is then simply the subspace

$$f^{-1}(0) = \{0\} \cup \{1/n : n \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{R}.$$

This space, however, is not locally connected, so it cannot be (the topological space underlying) the fiber product in the category of manifolds.
In fact, one easily checks that the space $f^{-1}(0)$ equipped with the discrete topology is the fiber product of f and g in the category of manifolds. Its underlying topology is strictly stronger than the subspace topology on $f^{-1}(0) \subseteq \mathbb{R}$.

The following example shows a case in which a fiber product does not exist at all.

Example 1.1.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the morphism $(x, y) \mapsto xy$, and let $g : \{0\} \to \mathbb{R}$ be the inclusion. Suppose that the fiber product F of these two morphisms exists. The underlying set is

$$\{(x,y) \in \mathbb{R}^2 : xy = 0\},$$

and the underlying topology is stronger than the subspace topology. Moreover, one can construct using the universal property a morphism $\mathbb{R} \to F : x \mapsto (x,0)$, which shows that the subset of F consisting of the x-axis is in fact homeomorphic to $\mathbb{R}$, and similarly for the y-axis; it follows that the topology on F in fact equals the subspace topology. However, this can never be the topology of a manifold. Indeed, if $U \subseteq F$ is an open neighborhood of the origin, then removing the origin from U breaks U into at least four connected components. In particular such an U can never be homeomorphic to a ball in $\mathbb{R}^n$. We must conclude that the fiber product of f and g does not exist in the category of manifolds.

Let $f : X \to S$ and $g : Y \to S$ be two morphisms of manifolds. We say that f and g are *transversal* if for all $x \in X$ and $y \in Y$ with $f(x) = g(y) =: s$ the linear map $df_x + dg_y : T_{X,x} \oplus T_{Y,y} \to T_{S,s}$ is surjective. The following lemma shows that the fiber product of f and g exists if f and g are transversal.

Lemma 1.1.10. Let $f : X \to S$ and $g : Y \to S$ be morphisms of manifolds, and assume that f and g are transversal. Then the subset

$$X \times_S Y = \{(x,y) \in X \times Y : f(x) = g(y)\} \subseteq X \times Y$$

has the structure of an embedded submanifold of dimension $\dim X + \dim Y - \dim S$, and this submanifold is the fiber product of f and g in the category of smooth manifolds. Its underlying topological space is the topological fiber product of f and g.
If $(x,y) \in X \times_S Y$ is any point, then the tangent space to $X \times_S Y$ at (x,y) is the subspace of the tangent space to $X \times Y$ at (x,y) given by

$$T_{X\times_S Y,(x,y)} = \{(v,w) \in T_{X\times Y,(x,y)} \cong T_{X,x} \times T_{Y,y} : df_x(v) = dg_y(w)\}.$$

Proof. Consider the morphism

$$h : X \times Y \to S \times S : h(x,y) = (f(x), g(y)).$$

As f and g are transversal, it is straightforward to prove that h is transversal to the inclusion map of the diagonal $\Delta \subseteq S \times S$. From [Lee03, Theorem 6.30] it follows that

$$F := h^{-1}(\Delta) = \{(x,y) \in X \times Y : f(x) = g(y)\}$$

is an embedded submanifold of $X \times Y$ whose codimension equals the dimension of S, and from [Lee03, Theorem 5.29] one deduces that F satisfies the universal property of the fiber product of f and g. Moreover, $F \subseteq X \times Y$ with its subspace topology is also the topological fiber product.

Let $(x,y) \in F$ be a point with $s := f(x) = g(y)$, and consider the linear map

$$\phi : T_{X\times Y,(x,y)} \cong T_{X,x} \times T_{Y,y} \to T_{S,s} : (v,w) \mapsto df_x(v) - dg_y(w).$$

As f and g are transversal, it follows that ϕ is surjective. The natural diagram

$$\begin{array}{ccc} T_{F,(x,y)} & \xrightarrow{dp_1} & T_{X,x} \\ \big\downarrow{\scriptstyle dp_2} & & \big\downarrow{\scriptstyle df} \\ T_{Y,y} & \xrightarrow{dg} & T_{S,s} \end{array} \tag{1.1.11}$$

commutes, so the tangent space $T_{F,(x,y)} \subseteq T_{X\times Y,(x,y)}$ to F at (x,y) is contained in the kernel of ϕ. By comparing dimensions we find that $T_{F,(x,y)} = \ker \phi$. □

The lemma allows us to prove Proposition 1.1.7.

Proof of Proposition 1.1.7. As f is a submersion, it is transversal to g. The fiber product $F = X \times_S Y$ therefore exists by Lemma 1.1.10. By chasing through diagram 1.1.11 one finds that the tangent map dp_2 is surjective, so p_2 is a submersion. □

Example 1.1.12. Let Y be a manifold, and let $V \subseteq Y$ be an open submanifold. Let $f : V \to Y$ denote the inclusion. Then f is a submersion. If $g : X \to Y$ is a morphism of manifolds, then the fiber product of f and g is isomorphic to the open submanifold

$$V \times_Y X = g^{-1}(V) \subseteq X.$$

Example 1.1.13. If $f : X \to Y$ is a submersion, $y \in Y$ is a point, and $g : \{y\} \to Y$ is the inclusion, then the fiber product of f and g is the fiber $X_y = f^{-1}(y)$ of f over y.

Example 1.1.14. Let X and Z be manifolds, and consider the projection $p_2 : Z \times X \to X$, which is a submersion. If $g : Y \to X$ is any other morphism, then the fiber product of p_2 and g is $Z \times Y$:

$$\begin{array}{ccc} Z \times Y & \xrightarrow{\mathrm{id}_Z \times g} & Z \times X \\ \big\downarrow{\scriptstyle p_2} & \square & \big\downarrow{\scriptstyle p_2} \\ Y & \xrightarrow{g} & X. \end{array}$$

1.1.3 Oriented submersions

In this section we will define what it means for a submersion to have oriented fibers. Of course, we want to impose some continuity criterion on such orientations. For example, if we consider the Möbius strip as a fiber bundle over S^1 with fibers homeomorphic to $(0,1)$, it is intuitively clear that all these fibers can be given an orientation, but these orientations can never vary continuously over the base S^1.

Recall that giving an orientation of a manifold X is equivalent to giving an orientation of its tangent bundle T_X. This leads to the following definition.

Let $f : X \to Y$ be a submersion of manifolds. The *relative tangent bundle* $T_f = T_{X/Y}$ on X is the kernel of the surjective morphism $df : T_X \to f^*T_Y$ of vector bundles on X. Note that for each $y \in Y$ the restriction of $T_{X/Y}$ to the fiber $X_y = f^{-1}(y)$ equals the tangent bundle T_{X_y}.

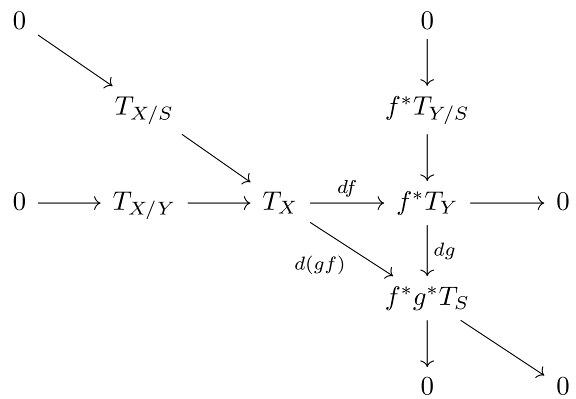
Definition 1.1.15. Let $f : X \to Y$ be a submersion of manifolds. An *orientation* of f (or an orientation of the fibers of f) is an orientation of the relative tangent bundle $T_{X/Y}$. An *oriented submersion* is a submersion together with an orientation.

Note that giving an orientation of the vector bundle $T_{X/Y}$ is equivalent to giving an orientation of its determinant line bundle $\det T_{X/Y} = \bigwedge^r T_{X/Y}$, where r denotes the rank of $T_{X/Y}$.

Example 1.1.16. For any manifold X giving an orientation of X is equivalent to giving an orientation of the morphism $X \to \{*\}$.

Example 1.1.17. Let F be an oriented manifold, let Y be any manifold, and define $X = F \times Y$. The projections $p_1 : X \to F$ and $p_2 : X \to Y$ induce an isomorphism $T_X \xrightarrow{\sim} p_1^*T_F \times p_2^*T_Y$, and hence an isomorphism $T_{X/Y} \xrightarrow{\sim} p_1^*T_F$. The orientation of F, therefore, induces an orientation of p_2.

Let $f : X \to Y$ and $g : Y \to S$ be two oriented submersions, with fibers of dimension r and s, respectively. The composition $gf : X \to S$ can be equipped with a canonical orientation, as follows. By using the exact sequences that define the vector bundles of f, g, and gf, we get a commutative diagram as follows:



from which we extract an exact sequence of vector bundles on X:

$$0 \to T_{X/Y} \to T_{X/S} \xrightarrow{df} f^*T_{Y/S} \to 0.$$

Choose any splitting of this exact sequence. This is always possible: we can construct a Riemannian metric on $T_{X/S}$ by using partitions of unity, and then the tangent map df restricts to an isomorphism of the orthogonal complement of $T_{X/Y} \subseteq T_{X/S}$ with $f^*T_{Y/S}$. Note that such a splitting is not canonical. In any case, we obtain an isomorphism

$$T_{X/Y} \oplus f^*T_{Y/S} \xrightarrow{\sim} T_{X/S},$$

and by taking determinants we get an isomorphism of line bundles

$$\det T_{X/Y} \otimes f^* \det T_{Y/S} \xrightarrow{\sim} \det T_{X/S}.$$

This latter isomorphism *is* canonical: it does not depend on the earlier choice of a splitting. The orientations of f and g, therefore, canonically define an orientation of gf.

Example 1.1.18. Let F_1 and F_2 be oriented manifolds, and let S be any manifold. Define $X = F_1 \times F_2 \times S$ and $Y = F_2 \times S$, and let $f : X \to Y$ and $g : Y \to S$ be the obvious projections. The orientations of F_1 and F_2 induce orientations of f and g (see Example 1.1.17) and hence an orientation of the projection $gf : F_1 \times F_2 \times S \to S$. This orientation agrees with the orientation of gf induced by the product orientation of $F_1 \times F_2$.

Example 1.1.19. Consider the Möbius strip $M \to S^1$ as a fiber bundle with fiber $(-1, 1)$. The submersion $M \to S^1$ does not have an orientation. Indeed, choose an orientation of S^1, and hence of the submersion $S^1 \to \{*\}$. Any orientation of the submersion $M \to S^1$ would induce an orientation of the composition $M \to S^1 \to \{*\}$, and this would yield an orientation of M. As the Möbius strip is not orientable, this cannot happen.

Example 1.1.20. Assume we have a cartesian diagram of manifolds

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

with f a submersion. Note that f' is again a submersion. The differential map $dh : T_{X'} \to h^*T_X$ restricts to an isomorphism of relative tangent bundles $T_{X'/S'} \xrightarrow{\sim} h^*T_{X/S}$. In particular, any orientation of f induces an orientation of f' in a natural way.

1.1.4 Holomorphic submersions and families of curves

A complex manifold (of dimension n) is a Hausdorff second-countable locally ringed space that is locally isomorphic to the space $\mathbb{C}^n$ with its sheaf of holomorphic functions. Equivalently, a complex manifold is a Hausdorff second-countable topological space together with an atlas of charts to opens in $\mathbb{C}^n$ whose transition functions are all holomorphic. One-dimensional complex manifolds are also called *Riemann surfaces.* As $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and biholomorphic maps are diffeomorphisms, we find that every complex manifold has an underlying structure of a smooth manifold, whose (*real*) dimension is twice the dimension of the complex manifold. In this section, we will study holomorphic submersions: morphisms of complex manifolds whose underlying morphism of smooth manifolds is a submersion. It turns out that the fibers of holomorphic submersions are complex manifolds, allowing us to define families of complex manifolds.

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Two morphisms $f : X \to S$ and $g : Y \to S$ of complex manifolds are *transversal* if the underlying morphisms of smooth manifolds are transversal. Analogous to Lemma 1.1.10 we can prove that if f and g are transversal, then the subset

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

is a complex submanifold of $X \times Y$, and it is the fiber product of f and g in the category of complex manifolds. See [FG02, Chapter IV.1] for more details.

A *(holomorphic) submersion* $f : X \to Y$ of complex manifolds is a morphism of complex manifolds, such that the underlying morphism of smooth manifolds is a submersion. If $f : X \to S$ is a holomorphic submersion, then for every morphism $g : Y \to S$ of complex manifolds the fiber product $X \times_S Y$ in the category of complex manifolds exists. In particular, the fibers of a submersion are again complex manifolds: for $s \in S$, the fiber of f above s is

$$X_s = f^{-1}(s) = X \times_S \{s\}.$$

Recall that the space $\mathbb{C}^n$ is endowed with a canonical orientation. Consider the holomorphic coordinates $z_1, \ldots, z_n$ and the corresponding smooth coordinates $x_1, y_1, \ldots, x_n, y_n$ with $z_k = x_k + \sqrt{-1}y_k$. Then the canonical orientation of $\mathbb{C}^n$ is given by

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Biholomorphic maps are orientation-preserving, and it follows that each complex manifold comes with a canonical orientation. The fibers of a holomorphic submersion $f : X \to Y$ are complex manifolds, so they all have a canonical orientation. By passing to coordinate charts one easily checks that these orientations define an orientation of f, called the *canonical orientation*, and this orientation is compatible with the canonical orientations of X and Y.

Definition 1.1.21. A *curve* is a compact connected Riemann surface. A *family of curves* (of genus g) is a surjective holomorphic submersion whose fibers are curves (of genus g).

It follows immediately that families of curves (of genus g) are stable under base change. This makes it possible to talk about moduli spaces of genus g curves, which we will do in Chapter 2.

Note that, by Corollary 1.1.5, the morphism of smooth manifolds underlying a family of curves is a smooth fiber bundle. However, a family of curves is not locally trivial if we consider the complex structure of its fibers. Consider the following example.

Example 1.1.22. Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half space

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

The group $\mathbb{Z}^2$ acts on $\mathbb{C} \times \mathbb{H}$ by

$$(a, b) \cdot (z, \tau) = (z + a + b\tau, \tau).$$

Let $E = (\mathbb{C} \times \mathbb{H})/\mathbb{Z}^2$. This is a complex manifold, the projection $\mathbb{C} \times \mathbb{H} \to \mathbb{H}$ induces a morphism $f : E \to \mathbb{H}$, and this morphism is a family of curves of genus 1. For $\tau \in \mathbb{H}$ the fiber E_τ is the complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. The (nonholomorphic!) diffeomorphism

$$\mathbb{C} \times \mathbb{H} \to \mathbb{C} \times \mathbb{H} : (z, \tau) \mapsto (\mathrm{Re}(z) + \mathrm{Im}(z) \cdot \tau, \tau)$$

induces an isomorphism of smooth fiber bundles

$$\begin{array}{ccc} E_i \times \mathbb{H} & \xrightarrow{\ \sim\ } & E \\ & {\scriptstyle p_2}\searrow \qquad \swarrow{\scriptstyle f} & \\ & \mathbb{H} & \end{array}$$

where $E_i = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ denotes the fiber of $i = \sqrt{-1}$ under f.
It follows that, as a submersion of smooth manifolds, f is a trivial smooth fiber bundle, with fibers diffeomorphic to the torus. However, the fibers of f are not all mutually isomorphic as complex manifolds. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

and the fibers over two points $\tau_1, \tau_2 \in \mathbb{H}$ are isomorphic as complex manifolds if and only if τ_1 and τ_2 lie in the same orbit of this action. See, for example, [Hai11, §1].

1.2 Currents

1.2.1 Currents on manifolds

Definition 1.2.1. Let $U \subseteq \mathbb{R}^n$ be an open subspace, and consider the set $A_c^*(U)$ of smooth forms on U with compact support. A *current* on U is then a $\mathbb{R}$-linear form

$$T : A_c^*(U) \to \mathbb{R}$$

that is continuous in the following sense: if $K \subseteq U$ is a compact subset, and $\{\omega_i\}_{i \geq 0}$ is a sequence of forms in $A_c^*(U)$ whose supports are all contained in K, such that all the coefficient functions and all their partial derivatives converge uniformly to 0, then $T(\omega_i)$ converges to zero.

In general, currents can be defined as follows.

Definition 1.2.2. Let X be a smooth manifold. A *current* on X is an $\mathbb{R}$-linear form

$$T : A_c^*(X) \to \mathbb{R}$$

such that for every open $U \subseteq X$ and every isomorphism φ of U with an open subset $U' \subseteq \mathbb{R}^n$ the composition

$$A_c^*(U') \xrightarrow{\varphi^*} A_c^*(U) \to A_c^*(X) \xrightarrow{T} \mathbb{R}$$

is a current on U'. Here the middle arrow denotes extension by zero.

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The $\mathbb{R}$-vector space of currents on X is denoted by $D^*(X)$.

A current T is said to have *degree* p if $T(\omega) = 0$ for all differential q-forms with $q \neq n - p$. We denote by $D^p(X) \subseteq D^*(X)$ the subspace of degree p currents. We have a decomposition

$$D^*(X) = \bigoplus_{p \geq 0} D^p(X).$$

For each open subset $U \subseteq X$ we have an inclusion $A_c^*(U) \subseteq A_c^*(X)$ and hence a restriction map $D^*(X) \to D^*(U) : T \mapsto T|_U$. The resulting presheaf D^* of currents on X is a sheaf. For each $T \in D^*(X)$ the support $\operatorname{Supp} T$ of T is the complement of the largest open subset $U \subseteq X$ for which $T|_U = 0$. Denote by $D_c^*(X) \subseteq D^*(X)$ the subspace of currents whose support is compact.

Let $T : A_c^*(X) \to \mathbb{R}$ be a current, and let $\omega \in A^*(X)$ be a smooth form such that $\operatorname{Supp} T \cap \operatorname{Supp} \omega$ is compact. If $\{\chi_i\}_{i \in I}$ is any partition of unity of X with compact supports, then the sum

$$T(\omega) := \sum_{i \in I} T(\chi_i \omega)$$

has only finitely many nonzero terms and hence converges. It is straightforward to show that $T(\omega)$ does not depend on the chosen partition of unity. We therefore see that T extends to a linear form on the space of smooth forms ω for which $\operatorname{Supp} T \cap \operatorname{Supp} \omega$ is compact. In particular, for all $T \in D_c^*(X)$ the compactly supported current T extends to a linear form $A^*(X) \to \mathbb{R}$.

Example 1.2.3. Let X be a manifold, and let $Z \subseteq X$ be a closed oriented submanifold of codimension p. The *Dirac delta current* $\delta_Z \in D^p(X)$ associated to Z is the current defined by

$$\delta_Z(\alpha) = \int_Z \alpha|_Z \quad \text{for all } \alpha \in A_c^*(X).$$

The support of δ_Z equals Z. In particular, if Z is compact, the current δ_Z extends

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to a linear form on $A^*(X)$, given by the same integral formula for all $\alpha \in A^*(X)$.

Example 1.2.4. Suppose that X is an oriented manifold. If $\alpha \in A^*(X)$ is a smooth form, we define a current $[\alpha] \in D^*(X)$ as follows:

$$[\alpha](\beta) = \int_X \alpha \wedge \beta \quad \text{for all } \beta \in A_c^*(X).$$

We get an injective map $A^*(X) \to D^*(X)$. If α has degree p, then so does $[\alpha]$. The support of $[\alpha]$ equals the support of α. We therefore also obtain an injective map $A_c^*(X) \to D_c^*(X)$.

Example 1.2.5. If f is a locally integrable function on an n-dimensional oriented manifold X, then f induces a current $[f] \in D^0(X)$ given by

$$[f](\beta) = \int_X f \cdot \beta \quad \text{for all } \beta \in A_c^n(X).$$

We call a current T on an oriented manifold *smooth* if it is of the form $T = [\alpha]$ for some smooth differential form α.

Recall the exterior derivative d on the space of differential forms on X. Dually, we have a linear operator

$$b : D^*(X) \to D^*(X) : T \mapsto Td = (\omega \mapsto T(d\omega)).$$

Using Stokes' theorem, one easily proves that, given an oriented manifold X and a smooth p-form α on X, one has:

$$b[\alpha] = (-1)^{p+1}[d\alpha].$$

We define the *exterior derivative* on currents to be the linear operator d on $D^*(X)$ defined by

$$d = (-1)^{p+1} b$$

for every degree p current T. We obtain the identity

$$d[\alpha] = [d\alpha]$$

for every oriented manifold X and every smooth form α on X.

1.2.2 Pushforwards of currents

Recall that we can pull back differential forms along a morphism $f : X \to Y$ of manifolds. Dually, it is possible to push forward some currents along this morphism. Suppose that T is a current on X, and assume that the composition $\operatorname{Supp} T \hookrightarrow X \to Y$ is a proper map. If ω is any compactly supported form on Y, then

$$\operatorname{Supp}(f^*\omega) \cap \operatorname{Supp}(T) \subseteq f^{-1}(\operatorname{Supp}(\omega)) \cap \operatorname{Supp}(T)$$

is compact, and $T(f^*\omega)$ is well-defined.

Definition 1.2.6. Let $f : X \to Y$ be a morphism of manifolds, and let $T \in D^*(X)$ be a current on X. If the composition $\mathrm{Supp}(T) \to X \to Y$ is a proper map, we define the pushforward $f_*T \in D^*(Y)$ of T along f to be the current on Y given by

$$f_*T(\omega) = T(f^*\omega) \quad \text{for all } \omega \in A_c^*(Y).$$

If T is a current of degree p, then f_*T is a current of degree $p-(\dim X - \dim Y)$. We obtain a pushforward map

$$f_* : D_f^*(X) \to D^{*+\dim Y - \dim X}(Y),$$

where $D_f^*(X)$ denotes the set of currents on X whose support is proper over Y.

Example 1.2.7. Let us return to Example 1.2.3, where X is a smooth manifold and $Z \subseteq X$ is a closed oriented submanifold of codimension p. It follows that the Dirac delta current δ_Z on X can be given as the pushforward of the smooth current [1] on Z:

$$\delta_Z = i_*[1]$$

where $i : Z \to X$ is the inclusion morphism.

It follows that the pushforward of a smooth current is not necessarily smooth anymore. For instance, consider the inclusion of a point $\{x\} \to X$ into a manifold with positive dimension. Then the current δ_x on X is the pushforward of the smooth form [1] along the inclusion $\{x\} \to X$, but δ_x is itself not smooth: there are no smooth forms on X whose support equals $\{x\}$. If we make an additional assumption that our morphism is a submersion, then pushforwards of smooth currents along this morphism are again smooth.

Theorem 1.2.8. Let $f : X \to Y$ be an oriented submersion of oriented manifolds. If $T = [\alpha]$ is a smooth current on X whose support is proper over Y, then f_*T is a smooth current on Y.

We will postpone the proof of this theorem until Section 1.3. In this section we will find that $f_*[\alpha]$ is the current associated to the smooth form $\int_f \alpha$ on Y obtained by integrating α along the fibers of f.

Let $f : X \to Y$ be a morphism of manifolds. As d commutes with the pullback map $f^* : A^*(Y) \to A^*(X)$, it follows that the pushforward $f_* : D^*(X) \to D^*(Y)$ commutes with the operator b. Therefore, we obtain for every current $T \in D^*(X)$ whose support is proper over Y:

$$f_* dT = (-1)^{\dim(X)-\dim(Y)} d f_* T.$$

1.2.3 Currents on complex manifolds

We can obtain complex-valued currents on a smooth manifold X by tensoring the space of currents $D^*(X)$ with the complex numbers:

$$D^*(X;\mathbb{C}) = D^*(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

Such a current with complex coefficients can be written uniquely as $T = T_1 + \sqrt{-1} \cdot T_2$, with T_1 and T_2 currents with real coefficients. If ω is a complex-valued differential form on X with compact support, then $\omega = \omega_1 + \sqrt{-1} \cdot \omega_2$ for some real-valued differential forms ω_1, ω_2 with compact support, and

$$T(\omega) = (T_1(\omega_1) - T_2(\omega_2)) + \sqrt{-1} \cdot (T_1(\omega_2) + T_2(\omega_1)).$$

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Suppose now that X is a complex manifold of (complex) dimension n. Recall that the space of complex-valued differential forms has a decomposition

$$A^*(X; \mathbb{C}) = \bigoplus_{p,q \geq 0} A^{p,q}(X).$$

Dually, the space of complex-valued currents has a decomposition

$$D^*(X; \mathbb{C}) = \bigoplus_{p,q \geq 0} D^{p,q}(X),$$

where $D^{p,q}(X)$ is dual to $A_c^{n-p,n-q}(X)$. A current $T \in D^*(X; \mathbb{C})$ is a (p, q)-*current* if it is an element of $D^{p,q}(X)$, which is the case if and only if

$$T(\omega) = 0 \quad \text{for all } \omega \in A_c^{r,s}(X; \mathbb{C}) \text{ with } (p + r, q + s) \neq (n, n).$$

Proposition 1.2.9. Let $f : X \to Y$ be a morphism of complex manifolds, and let $T \in D_f^*(X; \mathbb{C})$ be a complex-valued current on X whose support is proper over Y. If T is a (p, q)-current, then f_*T is a $(p - r, q - r)$-current, where $r = \dim(X) - \dim(Y)$.

Proof. Write $n = \dim(X)$ and $m = \dim(Y)$, so $r = n - m$. Let $\omega \in A_c^{s,t}(Y)$ with $(p - r + s, q - r + t) \neq (m, m)$. We need to prove that $(f_*T)(\omega) = 0$. We have:

$$(f_*T)(\omega) = T(f^*\omega),$$

and as T is a (p, q)-current, $f^*\omega$ is an (s, t)-form, and $(p+s, q+t) \neq (m+r, m+r) = (n, n)$, it follows that $T(f^*\omega) = 0$. □

Recall, moreover, that each complex manifold X has a canonical orientation, so the notion of smooth currents exists on such a complex manifold. The inclusion $A^*(X; \mathbb{C}) \to D^*(X; \mathbb{C})$ restricts to inclusions

$$A^{p,q}(X) \to D^{p,q}(X).$$

In particular, it holds that a differential form on X is a (p, q)-form if and only if the associated current $[\alpha]$ is a (p, q)-current. This observation allows us to provide an easy proof for Proposition 1.3.19.

Finally, notice that the Dolbeault operators can be generalized to a setting of currents: for each p-current T and each smooth form α with compact support we set

$$(\partial T)(\alpha) = (-1)^{p+1} T(\partial\alpha) \quad \text{and} \quad (\overline{\partial} T)(\alpha) = (-1)^{p+1} T(\overline{\partial}\alpha).$$

Note that ∂ maps (p, q)-currents to $(p + 1, q)$-currents, and $\overline{\partial}$ maps (p, q)-currents to $(p, q + 1)$-currents. We have $d = \partial + \overline{\partial}$, and $\partial^2 = \overline{\partial}^2 = 0$.

1.3 Integration along fibers

In this section we will introduce the fiber integral operator along oriented submersions of manifolds. See also [Sto70, Appendix II].

If $f : X \to S$ is a submersion of manifolds, we denote by

$$A^*_f(X) \subseteq A^*(X)$$

the ideal consisting of those differential forms ω on X whose support is proper over S (that is, the composition $\mathrm{Supp}(\omega) \to X \to S$ is a proper map). Notice that the restrictions of such a form to the fibers of f are compactly supported differential forms. The following definition, therefore, makes sense.

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Definition 1.3.1. Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. A *fiber integral* (along f) is a linear map

$$\int_f : A^*_f(X) \to A^*(S)$$

that satisfies the following properties:

1. For any k-form $\omega \in A^k_f(X)$ with $k < r$ we have
$$\int_f \omega = 0.$$
2. For any r-form $\omega \in A^r_f(X)$ the fiber integral $\int_f \omega$ is a 0-form (so a smooth function) on S given by
$$\left(\int_f \omega\right)(s) = \int_{X_s} \omega|_{X_s} \quad \text{for all } s \in S.$$
3. $\int_f$ satisfies the *projection formula*: for all $\omega \in A^*_f(X)$ and all $\eta \in A^*(S)$ we have:
$$\int_f (\omega \wedge f^*\eta) = \left(\int_f \omega\right) \wedge \eta.$$

It turns out that, in fact, these defining properties uniquely determine a linear map $A^*_f(X) \to A^*(S)$.

Theorem 1.3.2. Let $f : X \to S$ be an oriented submersion. There exists a unique fiber integral along f.

This theorem allows us to refer to this linear map as *the* fiber integral along f. We will prove this theorem later in this section.

The fiber integral generalizes the integral operator on compactly supported smooth forms on manifolds.

Example 1.3.3. Let X be an oriented manifold. Let $f : X \to \{*\}$ be the associated oriented submersion. Note that $A^*_f(X) = A^*_c(X)$. The integral operator $\int_X : A^*_c(X) \to \mathbb{R} = A^*(\{*\})$ is a (and hence *the*) fiber integral along f.

Note that we have defined the fiber integral along $f : X \to S$ only for forms ω on X whose support is proper over the base S. A priori, it might seem sufficient for such a smooth form ω to have fiberwise compact support, which is a weaker condition than having proper support over the base. However, the forms we obtain in this way need not be smooth. This is demonstrated by the following example.

Example 1.3.4. Let $b : \mathbb{R} \to \mathbb{R}$ be a *bump function*: b is smooth, we have $b > 0$ on the interval $(-1, 1)$, and $b = 0$ outside this interval. Moreover, we normalize b in such a way that

$$\int_{\mathbb{R}} b(x)dx = 1.$$

Now consider the following smooth 1-form on $\mathbb{R} \times \mathbb{R}$:

$$\omega := yb(xy)dx.$$

Let p be the projection $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ mapping (x, y) to y. One easily checks that the restriction of ω to each fiber of p is compactly supported. However, the function

$$\left(\int_p \omega\right)(y) = \int_{x\in\mathbb{R}} yb(xy)dx = \begin{cases} -1 & \text{if } y < 0 \\ 0 & \text{if } y = 0 \\ 1 & \text{if } y > 0 \end{cases}$$

on $\mathbb{R}$ is not continuous at $y = 0$.
Indeed, the support of ω is not proper over $\mathbb{R}$: it is the closed subset

$$\operatorname{Supp}\omega = \overline{\{(x, y) : y \neq 0, |xy| < 1\}} \subset \mathbb{R} \times \mathbb{R},$$

and its fiber over the point $0 \in \mathbb{R}$ is noncompact, as this fiber equals $\mathbb{R}$.

1.3.1 First properties of the fiber integral

Before proving Theorem 1.3.2, we will deduce some properties of fiber integrals. Fix an oriented submersion $f : X \to S$, and let $\int_f : A^*_f(X) \to A^*(S)$ be a fiber integral along f.

If a form $\omega \in A^*_f(X)$ is zero on $f^{-1}(V)$ for some open $V \subseteq S$, then it is reasonable to expect the fiber integral $\int_f \omega$ to be zero on V. This is indeed the case, as the following proposition shows.

Proposition 1.3.5. For every $\omega \in A^*_f(X)$ we have:

$$\operatorname{Supp}\left(\int_f \omega\right) \subseteq f[\operatorname{Supp}\omega].$$

Proof. Note that $f[\operatorname{Supp}\omega] \subseteq S$ is closed as the composition $\operatorname{Supp}\omega \hookrightarrow X \to S$ is proper. Construct a smooth function $\chi : S \to \mathbb{R}$ with $\chi^{-1}(0) = f[\operatorname{Supp}\omega]$ (cf. [Lee03, Theorem 2.29]). As $\chi \equiv 0$ on $f[\operatorname{Supp}\omega]$, we have $f^*\chi \equiv 0$ on $\operatorname{Supp}\omega$. Now apply the projection formula:

$$\chi \cdot \int_f \omega = \int_f \omega \wedge f^*\chi = \int_f 0 = 0.$$

The support of $\int_f \omega$ must therefore be obtained in $\chi^{-1}(0) = f[\operatorname{Supp}\omega]$. □

The fiber integral is a linear map and therefore commutes with finite sums. A stronger statement holds.

Proposition 1.3.6. Let $\{\omega_i\}_{i\in I}$ be a family of forms in $A^*_f(X)$. Assume that the collection

$$\{f[\operatorname{Supp}\omega_i]\}_{i\in I}$$

is locally finite in S. Then the collection

$$\{\operatorname{Supp}\omega_i\}_{i\in I}$$

is locally finite in X, the sum $\sum_{i\in I}\omega_i$ has proper support over S, and we have an equality of forms on S:

$$\int_f \sum_{i\in I}\omega_i = \sum_{i\in I}\int_f \omega_i.$$

Proof. Let $K \subseteq X$ be compact. If $i \in I$ is such that $K \cap \operatorname{Supp}\omega_i$ is nonempty, then $f[K] \cap f[\operatorname{Supp}\omega_i]$ is nonempty. By assumption there can only be finitely many such $i \in I$. We conclude that $\{\operatorname{Supp}\omega_i\}_{i\in I}$ is locally finite. It follows that $\bigcup_{i\in I}\operatorname{Supp}\omega_i$ is closed in X, and it contains the support of $\sum_{i\in I}\omega_i$.

Now let $L \subseteq S$ be compact. Then there are only finitely many $i \in I$ for which $L \cap f[\operatorname{Supp}\omega_i]$ is nonempty. Therefore $f^{-1}(L) \cap \operatorname{Supp}\omega_i$ is empty for all but finitely many $i \in I$. We have:

$$f^{-1}(L) \cap \operatorname{Supp}\left(\sum_{i\in I}\omega_i\right) \subseteq \bigcup_{i\in I}(f^{-1}(L) \cap \operatorname{Supp}\omega_i).$$

The right hand side of this equation is compact: $f^{-1}(L) \cap \operatorname{Supp}\omega_i$ is compact for all $i \in I$ and nonempty for finitely many $i \in I$. We conclude that $f^{-1}(L) \cap \operatorname{Supp}\left(\sum_{i\in I}\omega_i\right)$ is compact, too, and $\sum_{i\in I}\omega_i$ has proper support over S.

Let $V \subseteq S$ be a relatively compact open subset, and let L denote its closure in S. Take a smooth function $\chi : S \to \mathbb{R}$ such that $\chi|_L \equiv 1$ and such that $\operatorname{Supp}\chi$ is compact (see [dRha84, Corollary 1]). If $i \in I$ is such that $\omega_i \wedge f^*\chi$ is nonzero, then $\operatorname{Supp}\omega_i \cap f^{-1}(\operatorname{Supp}\chi)$ is nonempty, and hence $f[\operatorname{Supp}\omega_i] \cap \operatorname{Supp}\chi$ is nonempty. By assumption, there can only be finitely many such $i \in I$. So $\omega_i \wedge f^*\chi = 0$ for all but finitely many $i \in I$. We can therefore exchange sum and integral as follows:

$$\chi \cdot \int_f \sum_{i \in I} \omega_i = \int_f \sum_{i \in I} (\omega_i \wedge f^*\chi) = \sum_{i \in I} \int_f (\omega_i \wedge f^*\chi) = \chi \cdot \sum_{i \in I} \int_f \omega_i,$$

where the first and last equalities follow from the projection formula. It follows that the restrictions of $\int_f \sum_{i \in I} \omega_i$ and $\sum_{i \in I} \int_f \omega_i$ to V are equal. As S can be covered by such relatively compact opens, the desired result follows. □

1.3.2 Construction of the fiber integral

In this section we will prove Theorem 1.3.2. We will first prove this theorem in the case where the base S is such that the vector bundle A^1_S is free, and then extend to the general case by gluing.

Let X be a manifold, and let $r \geq 0$ be an integer. We denote by $A^{\leq r}_X$ the subbundle of the vector bundle of differential forms A^*_X given by

$$A^{\leq r}_X = \bigoplus_{k=0}^{r} A^k_X.$$

Sections of $A^{\leq r}_X$, therefore, are finite sums of differential forms of degree at most r. Similarly, for a submersion $f : X \to S$ we define

$$A^{\leq r}_f(X) = \bigoplus_{k=0}^{r} A^k_f(X) = A^{\leq r}(X) \cap A^*_f(X) \subseteq A^*_f(X).$$

Lemma 1.3.7. Let $f : X \to S$ be a submersion whose nonempty fibers have dimension r. Assume that the vector bundle A^1_S is free: there are 1-forms $\eta_1, \ldots, \eta_n$ on S such that

$$A^1_S = A^0_S \cdot \eta_1 \oplus \cdots \oplus A^0_S \cdot \eta_n.$$

For each subset $I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_p$ define

$$\eta_I = \eta_{i_1} \wedge \cdots \wedge \eta_{i_p}.$$

Then each form $\omega \in A^*(X)$ can be written as

$$\omega = \sum_{I \subseteq \{1, \ldots, n\}} \omega_I \wedge f^*\eta_I$$

where each ω_I is an element of $A^{\leq r}(X)$.

If moreover ω has proper support over S, then we may assume each ω_I has proper support over S, too.

In order to deduce the second part of the above lemma from the first part, we will be using bump functions on X with proper support over the base S. The existence of such bump functions is guaranteed by the following lemma.

Lemma 1.3.8. Let $f : X \to S$ be a morphism of manifolds, and let $P \subseteq X$ be a closed subset such that $f|_P : P \to S$ is a proper map. Then there exists a smooth function χ on X with proper support over S such that $\chi|_P \equiv 1$.

Proof. We will construct an open subset $U \subseteq X$ such that $P \subseteq U$ and such that the closure $\bar{U}$ is proper over S. We can then let χ be any bump function for P with support in U (cf. [Lee03, Proposition 2.25]).

As S is locally compact and paracompact, there exists an open covering $S = \bigcup_{i\in I} V_i$ such that the collection $\{V_i\}_{i\in I}$ (and hence $\{\bar{V}_i\}_{i\in I}$) is locally finite and such that the closure $\bar{V}_i \subseteq S$ is compact for all $i \in I$.

For all $i \in I$ the set $f^{-1}(\bar{V}_i) \cap P$ is compact. As X is locally compact and paracompact, there exists an open $W_i \subseteq X$ that contains $f^{-1}(\bar{V}_i) \cap P$, such that the closure $\bar{W}_i$ is compact.

Now define $U_i = f^{-1}(\bar{V}_i) \cap W_i$. The collection $\{U_i\}_{i\in I}$ is easily seen to be locally finite, and the same therefore holds for the collection $\{\bar{U}_i\}_{i\in I}$. Define $U = \bigcup_{i\in I} U_i$. We have $\bar{U} = \bigcup_{i\in I} \bar{U}_i$, and we claim that $\bar{U} \to S$ is proper. If $K \subseteq S$ is compact, then

$$\begin{aligned} f^{-1}(K) \cap \bar{U} = f^{-1}(K) \cap \bigcup_{i\in I} \bar{U}_i &= \bigcup_{i\in I} (f^{-1}(K) \cap \overline{f^{-1}(\bar{V}_i) \cap W_i}) \\ &\subseteq \bigcup_{I\in I} f^{-1}(K \cap \bar{V}_i) \cap \bar{W}_i. \end{aligned}$$

As $\{\bar{V}_i\}_{i\in I}$ is locally finite, the intersection $K \cap \bar{V}_i$ is empty for all but finitely many $i \in I$. The union $\bigcup_{i\in I} f^{-1}(K \cap \bar{V}_i) \cap \bar{W}_i$, therefore, is a finite union of compact sets, and hence compact. We find that $f^{-1}(K) \cap \bar{U}$ is compact, so $\bar{U}$ is proper over S. □

Proof of Lemma 1.3.7. We have an equality of vector bundles

$$A_S^* = \bigwedge A_S^1 = \bigoplus_I A_S^0 \cdot \eta_I$$

with I ranging over all subsets of $\{1, \dots, n\}$.

Consider the subbundles $A_X^{\leq r}$ and $f^* A_S$ of the vector bundle A_X^* on X. Taking wedge products induces a morphism of vector bundles

$$A_X^{\leq r} \otimes f^* A_S^* \to A_X^*.$$

We claim that this morphism is surjective. This can be checked locally. As f locally looks like a projection $p_2 : F \times S \to S$ it suffices to prove surjectivity in the case that f is such a projection. In that case the canonical morphism

$$p_1^* A_F^* \otimes p_2^* A_S^* \to A_{F\times S}^*$$

is an isomorphism, and it factors as

$$p_1^* A_F^* \otimes p_2^* A_S^* \to A_{F\times S}^{\leq r} \otimes p_2^* A_S^* \to A_{F\times S}^*$$

since $p_1^* A_F^*$ is a subbundle of $A_{F\times S}^{\leq r}$. We see, in particular, that the canonical morphism $A_{F\times S}^{\leq r} \otimes p_2^* A_S^* \to A_{F\times S}^*$ is surjective, proving our claim.

As we are working with vector bundles over smooth manifolds, taking global sections is exact and commutes with tensor products. We therefore obtain a surjective map

$$A^{\leq r}(X) \otimes_{A^0(X)} \Gamma(f^* A_S^*) \to A^*(X)$$

induced by the wedge product. As A_S^* is free and generated by η_I, the $A^0(X)$-module $\Gamma(f^* A_S)$ is free and generated by $f^*\eta_I$. We therefore find that every element of $A^*(X)$ can be written as described in the statement of the lemma.

Suppose, moreover, that $\omega \in A_f^*(X)$ has proper support over S. Write $\omega = \sum_I \omega_I \wedge f^*\eta_I$. By Lemma 1.3.8 there is a smooth function χ on X with $\chi \equiv 1$ on $\operatorname{Supp}\omega$ such that $\operatorname{Supp}\chi$ is proper over S. We have

$$\omega = \chi\omega = \sum_I (\chi\omega_I) \wedge f^*\eta_I.$$

Note that the support of each $\chi\omega_I$ is proper over S. □

Lemma 1.3.9. Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. Let $\omega \in A_f^{\leq r}(X)$, and consider the following function on S:

$$T(\omega) : S \to \mathbb{R} : s \mapsto \int_{X_s} \omega|_{X_s}.$$

Then $T(\omega)$ is smooth. In particular we obtain a linear map $T : A_f^{\leq r}(X) \to A^0(S)$.

Proof. As f locally looks like a projection $f : F \times S \to S$ with $F \subseteq \mathbb{R}^r$ and $S \subseteq \mathbb{R}^n$ open subsets, we may use partitions of unity to restrict to the case where f is such a projection and $\omega \in A^{\leq r}(F \times S)$ has compact support. In this specific case smoothness follows from the dominated convergence theorem. □

Using Lemma 1.3.7 we can now show that there exists a unique fiber integral along a submersion $X \to S$ if the base S is such that A_S^1 is free.

Lemma 1.3.10. Let $f : X \to S$ be an oriented submersion. Assume that A_S^1 is free. Then there exists a unique fiber integral operator $\int_f : A_f^*(X) \to A^*(S)$.

Proof. Let r be the dimension of the nonempty fibers of f. Let $T : A_f^{\leq r}(X) \to A^0(S)$ denote the map defined in Lemma 1.3.9. By Lemma 1.3.7 and the defining properties of the fiber integral, a fiber integral $\int_f$, if it exists, is uniquely determined by the identity

$$\int_f \omega \wedge f^*\eta = T(\omega) \cdot \eta \quad \text{for all } \omega \in A_f^{\leq r}(X), \eta \in A^*(S).$$

In order to prove existence of the fiber integral, choose 1-forms $\eta_1, \ldots, \eta_n$ on S such that

$$A_S^1 = A_S^0 \cdot \eta_1 \oplus \cdots \oplus A_S^0 \cdot \eta_n.$$

If $\omega \in A_f^*(X)$ is any form, we can write $\omega = \sum_I \omega_I \wedge f^*\eta_I$ with $\omega_I \in A_f^{\leq r}(X)$ for all $I \subseteq \{1, \ldots, n\}$. We wish to define

$$\int_f \omega = \sum_I T(\omega_I) \cdot \eta_I \in A^*(S).$$

Of course we need to verify this does not depend on the choice of the forms ω_I. We claim that if $\omega_I \in A^{\leq r}(X)$ are such that $\sum_{i \in I} \omega_I \wedge f^*\eta_I = 0$, then the degree r part of each restriction $\omega_I|_{X_s}$ vanishes. In that case we have $T(\omega_I) = 0$ for all I, and the operator $\int_f$ is clearly well-defined.

Suppose that $\omega_I \in A^{\leq r}(X)$ are such that $\sum_I \omega_I \wedge f^*\eta_I = 0$. As the claim can be checked locally on X, and as submersions locally look like projections, we may reduce to the case where f is a projection $f : X = F \times S \to S$. Moreover, we may shrink F such that A_F^1 is free. Let $\xi_1, \ldots, \xi_r \in A^1(F)$ be 1-forms such that

$$A_F^1 = A_F^0 \cdot \xi_1 \oplus \cdots \oplus A_F^0 \cdot \xi_r.$$

We then find that $A^*(F \times S)$ is the free $A^0(F \times S)$-module generated by the forms

$$g^*\xi_J \wedge f^*\eta_K,$$

where J and K range over all subsets of $\{1, \ldots, r\}$ and $\{1, \ldots, n\}$, respectively. As each ω_I is an element of $A^{\leq r}(F \times S)$, there are (unique) smooth functions α_{IJK} on $F \times S$ such that

$$\omega_I = \sum_{\substack{J,K \\ |J|+|K| \leq r}} \alpha_{IJK} \cdot g^*\xi_J \wedge f^*\eta_K.$$

Again J and K range over the subsets of $\{1, \ldots, r\}$ and $\{1, \ldots, n\}$. We thus find:

$$0 = \sum_I \omega_I \wedge f^*\eta_I = \sum_{\substack{I,J,K \\ |J|+|K| \leq r}} \alpha_{IJK} \cdot g^*\xi_J \wedge f^*\eta_K \wedge f^*\eta_I.$$

Let $R = \{1, \ldots, r\}$, and take the $(g^*\xi_R \wedge f^*\eta_I)$-part of the above sum to obtain:

$$\alpha_{IR\emptyset} \cdot g^*\xi_R = 0,$$

so $\alpha_{IR\emptyset} = 0$. Restricting ω_I to a fiber F of f yields the form

$$\sum_J \alpha_{IJ\emptyset}|_F \cdot \xi_J.$$

Its degree r part is $\alpha_{IR\emptyset} \cdot \xi_R = 0$, which proves the claim.

We thus obtain a well-defined linear map $\int_f : A_f^*(X) \to A^*(S)$. It is straightforward to verify that it satisfies the defining properties of the fiber integral. □

The following lemma will allow us to generalize Lemma 1.3.10 to the setting of arbitrary oriented submersions by gluing.

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Lemma 1.3.11. Let $f : X \to S$ be an oriented submersion, and assume that A^1_S is free. Let $V \subseteq S$ be any open subset, and define $U = f^{-1}(V)$. Let $f' : U \to V$ denote the restriction of f to U. For all $\omega \in A^*_f(X)$ we have $\omega|_U \in A^*_{f'}(U)$, and the following equality holds:

$$\left(\int_f \omega\right)\bigg|_V = \int_{f'} \omega|_U.$$

Note that in this setting f' is a submersion, too, and the orientation of f induces an orientation of f'. As both A^1_S and A^1_V are free, fiber integrals along f and f' exist and are unique.

Proof. If $K \subseteq V$ is compact, then

$$(f')^{-1}(K) \cap \operatorname{Supp}(\omega|_U) = f^{-1}(K) \cap \operatorname{Supp}(\omega) \cap U = f^{-1}(K) \cap \operatorname{Supp}(\omega)$$

is compact, too, so $\omega|_U \in A^*_{f'}(U)$.

By Lemma 1.3.7 it suffices to prove the given identity for $\omega \in A^*_f(X)$ of the form $\omega = \omega' \wedge f^*\eta$, with $\omega' \in A^{\leq r}_{\overline{f}}(X)$ and $\eta \in A^*(S)$. In this case the given identity follows directly from the defining properties of the fiber integrals. ☐

We can now prove Theorem 1.3.2.

Proof. (Proof of Theorem 1.3.2) Let $\mathcal{B}$ be the collection consisting of all opens $V \subseteq S$ for which A^1_V is free. Note that $\mathcal{B}$ is a basis for the topology of S.

We will first prove that a fiber integral $\int_f$ along f, if it exists, is necessarily unique. Let $\omega \in A^*_f(X)$. First assume that there exists some $V \in \mathcal{B}$ such that $\operatorname{Supp}\omega \subseteq f^{-1}(V)$. Write $U = f^{-1}(V)$. Lemma 1.3.7 implies that there are forms $\omega'_1, \ldots, \omega'_t \in A^{\leq r}_{\overline{f}}(U)$ and $\eta'_1, \ldots, \eta'_t \in A^*(V)$ such that

$$\omega|_V = \sum_{i=1}^{t} \omega'_i \wedge f^*\eta'_i.$$

Let $\chi \in A^0(X)$ be a bump function for $\operatorname{Supp}\omega$ supported in U, and, likewise, let $\psi \in A^0(S)$ be a bump function for $f[\operatorname{Supp}\omega]$ supported in V. For each $1 \leq i \leq t$ let $\omega_i \in A^{\leq r}_{\overline{f}}(X)$ be the extension by zero of $\chi|_U \cdot \omega'_i \in A^{\leq r}_{\overline{f}}(U)$, and let $\eta_i \in A^*(S)$ be the extension by zero of $\psi|_V \cdot \eta'_i$. We then find:

$$\omega = \sum_{i=1}^{t} \omega_i \wedge f^*\eta_i,$$

from which we deduce that $\int_f \omega$ is uniquely determined by the defining properties of fiber integrals.

In general, let $S = \bigcup_{i\in I} V_i$ be an open covering with $V_i \in \mathcal{B}$ for all $i \in I$, and let $\{\chi_i\}_{i\in I}$ be a partition of unity subordinate to this open covering. By Proposition 1.3.6 we then have for each $\omega \in A_f^*(X)$:

$$\int_f \omega = \int_f \sum_{i\in I} f^*\chi_i \cdot \omega = \sum_{i\in I} \int_f f^*\chi_i \cdot \omega.$$

As the support of each $f^*\chi_i \cdot \omega$ is contained in $f^{-1}(V_i)$ we find that each fiber integral $\int_f f^*\chi_i \cdot \omega$ is uniquely determined, and the same therefore holds for $\int_f \omega$.

We will now construct a fiber integral along f by gluing. Consider the sheaf $f_*A_f^*$ on S given by

$$f_*A_f^*(V) = A_f^*(f^{-1}(V)).$$

We will construct a sheaf morphism $\int_f : f_*A_f^* \to A_S^*$, which in particular induces a linear map $A_f^*(X) \to A^*(S)$. Lemma 1.3.10 implies that for every $V \in \mathcal{B}$ we have a unique fiber integral operator

$$\int_{f|_{f^{-1}(V)}} : A_f^*(f^{-1}(V)) \to A^*(S).$$

Lemma 1.3.11 states that these fiber integrals are compatible with restrictions and hence define a sheaf morphism on the basis $\mathcal{B}$ and therefore a sheaf morphism $f_*A_f^* \to A_S^*$. We obtain a linear map on global sections $\int_f : A_f^*(X) \to A^*(S)$. Note that it is uniquely determined by the following property: for each open $V \in \mathcal{B}$ and each form $\omega \in A_f^*(X)$, we have

$$\left(\int_f \omega\right)\Big|_V = \int_{f|_{f^{-1}(V)}} \omega|_{f^{-1}(V)}.$$

We can prove that $\int_f$ is in fact a fiber integral by verifying its defining properties locally. □

Remark 1.3.12. We have constructed the fiber integral along the oriented submersion $f : X \to S$ by 'gluing' fiber integrals along the induced submersions $f^{-1}(V) \to V$ for each open $V \subseteq S$ with A_V^1 free. Note that, in particular, this implies that the fiber integral is well-behaved with respect to restrictions to open subsets. More precisely, a stronger version of Lemma 1.3.11 holds: we no longer need to assume that A_S^1 is free.

Another approach of constructing the fiber integral is provided by Stoll [Sto70]; we will briefly sketch the construction here. Let $f : X \to S$ be a submersion, and let $r, q \geq 0$. For each point $s \in S$ Stoll then obtains a canonical linear map

$$A^{q+r}(X) \to A^r(X_s) \otimes_{\mathbb{R}} \bigwedge^q T_{S,s}^*,$$

where $X_s = f^{-1}(s)$ and where $T_{S,s}^*$ is the fiber of the cotangent bundle A_S^1 at s. If f is the projection $\mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ and if we let $x_1, \ldots, x_r$ and $y_1, \ldots, y_n$ denote the standard coordinates of $\mathbb{R}^r$ and $\mathbb{R}^n$, respectively, then this linear map is given by

$$\alpha \cdot dx_I \wedge dy_J \mapsto \begin{cases} (\alpha|_{X_s} \cdot dx_I) \otimes dy_J & \text{if } |I| = r \\ 0 & \text{if } |I| < r. \end{cases}$$

In general, the above linear map restricts to a linear map

$$A_f^{q+r}(X) \to A_c^r(X_s) \otimes_{\mathbb{R}} \bigwedge^q T_{S,s}^*,$$

and composing this linear map with the integral operator $\int_{X_s} : A_c^r(X_s) \to \mathbb{R}$ finally yields a linear map

$$A_f^{q+r}(X) \to \bigwedge^q T_{S,s}^*.$$

By repeating this construction for each $s \in S$ we obtain a function that maps forms $\omega \in A_f^{r+q}(X)$ to sections of the sheaf A_S^q. Stoll then shows that these sections are smooth, and that the operator we obtain satisfies all properties one would expect of the fiber integral. It is straightforward to prove that Stoll's fiber integral matches the one defined in this section.

1.3.3 More properties of the fiber integral

In this section we will prove some more properties of the fiber integral $\int_f$ along an oriented submersion f.

Let r denote the dimension of the nonempty fibers of f. The fiber integral $\int_f$ maps k-forms with $k < r$ to zero, and it maps r-forms to smooth functions. More generally, we can show that the fiber integral maps k-forms to $(k-r)$-forms for all $k \geq 0$ (where it is understood that $A^{k-r}(S) = \{0\}$ if $k - r < 0$).

Proposition 1.3.13. Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. For all $k \geq 0$ and all $\omega \in A_f^k(X)$ we have $\int_f \omega \in A^{k-r}(S)$, where $A^l(S) = 0$ for all $l < 0$.

Proof. For $k \geq 0$ we obtain a linear map $A_f^k(X) \to A^{k-r}(S)$ by composition:

$$A_f^k(X) \hookrightarrow A_f^*(X) \xrightarrow{\int_f} A^*(S) = \bigoplus_{l \in \mathbb{Z}} A^l(S) \twoheadrightarrow A^{k-r}(S).$$

Taking the direct sum over all $k \geq 0$ yields a linear map

$$A_f^*(X) \to A^{*-r}(S).$$

It is straightforward to verify that this linear map is again a fiber integral, so it must in fact be equal to $\int_f$ by Theorem 1.3.2. □

In Section 1.3.2 we have seen that the fiber integral can be computed locally on the base. This fact can be used to prove the following base change formula.

Proposition 1.3.14 (Base change formula)**.** Suppose we have a cartesian diagram of manifolds

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

where f (and hence f') is an oriented submersion. For every form $\omega \in A^*_f(X)$, we have $h^*\omega \in A^*_{f'}(X')$, and the following identity holds:

$$\int_{f'} h^*\omega = g^* \int_f \omega.$$

Proof. A map between locally compact Hausdorff spaces is proper if and only if it is universally closed. The map $\operatorname{Supp}\omega \to S$, therefore, is universally closed, and hence the same holds for the map $h^{-1}(\operatorname{Supp}\omega) \to S'$, which is therefore proper. As $\operatorname{Supp} h^*\omega \subseteq h^{-1}(\operatorname{Supp}\omega)$ is a closed subset it follows that the induced morphism $\operatorname{Supp} h^*\omega \to S'$ is proper, so $h^*\omega \in A^*_{f'}(X')$.

As fiber integrals can be computed locally on the base, it suffices to prove the given identity in the case where A^1_S is free. In this case we may use Lemma 1.3.7 to reduce to the case where ω is of the form $\omega' \wedge f^*\eta$ with $\omega' \in A^{\leq r}_f(X)$ and $\eta \in A^*(S)$. Now the given identity follows immediately from the defining properties of the fiber integral. $\square$

Lemma 1.3.15. Let $f : X \to S$ be an oriented submersion. Let $U \subseteq X$ be an open subset, and consider the induced submersion $f|_U : U \to S$. If $\omega \in A^*_f(X)$ is such that $\operatorname{Supp}\omega \subseteq U$, then

$$\int_{f|_U} \omega|_U = \int_f \omega \in A^*(S).$$

Proof. We may assume that A^1_S is free. Let r denote the dimension of the nonempty fibers of f. By Lemma 1.3.7 we can write

$$\omega = \sum_{i=1}^{t} \omega_i \wedge f^*\eta_i$$

with $\omega_i \in A^{\leq r}_f(X)$ and $\eta_i \in A^*(S)$. By using a bump function for $\operatorname{Supp}\omega$ with support in U we may assume that $\operatorname{Supp}\omega_i \subseteq U$ for all $1 \leq i \leq t$. The lemma now follows immediately from the defining properties of the fiber integral. $\square$

Lemma 1.3.16. Let X and S be oriented manifolds, and let $f : X \to S$ be an oriented submersion. Assume that all orientations are compatible: we assume that the given orientation of X matches the orientation induced by the composition of the oriented submersions $X \xrightarrow{f} S \to \{*\}$. Then for all $\omega \in A_c^*(X)$ we have:

$$\int_X \omega = \int_S \int_f \omega.$$

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Proof. By the base change formula and Lemma 1.3.15 we may use partitions of unity to reduce to the case where f is a projection of the form $\mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$. In this specific case the above identity follows from Fubini's theorem. □

As a corollary we obtain Theorem 1.2.8.

Proof of Theorem 1.2.8. Let $\alpha \in A_f^*(X)$. An application of Lemma 1.3.16 and the projection formula then gives:

$$f_*[\alpha] = \left[\int_f \alpha\right].$$ □

Proposition 1.3.17. Let $f : X \to Y$ and $g : Y \to S$ be oriented submersions. Note that the composition $gf : X \to S$ can be endowed with a canonical orientation. For each $\omega \in A_{gf}^*(X)$ we have:

$$\omega \in A_f^*(X), \quad \int_f \omega \in A_g^*(Y), \quad \text{and} \quad \int_g \int_f \omega = \int_{gf} \omega.$$

Proof. Let $K \subseteq Y$ be compact. Then

$$\operatorname{Supp} \omega \cap f^{-1}(K) \subseteq \operatorname{Supp} \omega \cap (gf)^{-1}(f[K])$$

is a closed subset of a compact set and hence compact. We find that $\omega \in A_f^*(X)$.

Similarly, for $L \subseteq S$ compact, we see that

$$\operatorname{Supp}\left(\int_f \omega\right) \cap g^{-1}(L) \subseteq f[\operatorname{Supp} \omega] \cap g^{-1}(L) = f[\operatorname{Supp} \omega \cap (gf)^{-1}(L)]$$

is a closed subset of a compact set and therefore compact, so $\int_f \omega \in A_g^*(X)$.

In order to prove the given identity, first assume that S is orientable, and fix an orientation of S. Then the orientations of S, f, and g induce orientations of X and Y. We have an equality of smooth currents:

$$\left[\int_{gf} \omega\right] = (gf)_*[\omega] = g_*(f_*[\omega]) = g_*\left[\int_f \omega\right] = \left[\int_g \int_f \omega\right]$$

and hence an equality of the underlying differential forms.

If S is not orientable, denote by $\pi_S : \tilde{S} \to S$ its orientation double cover. Write $\tilde{X} = X \times_S \tilde{S}$ and $\tilde{Y} = Y \times_S \tilde{S}$. We obtain a commutative diagram with cartesian squares:

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \xrightarrow{\tilde{g}} & \tilde{S} \\ \Big\downarrow{\scriptstyle \pi_X} \ \square & & \Big\downarrow{\scriptstyle \pi_Y} \ \square & & \Big\downarrow{\scriptstyle \pi_S} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & S \end{array}$$

and $\tilde{f}$, $\tilde{g}$, and $\tilde{g}\tilde{f}$ are oriented submersions. We apply the base change formula, and the fact that the proposition holds if the base is orientable:

$$\pi_S^* \int_{gf} \omega = \int_{\tilde{g}\tilde{f}} \pi_X^* \omega = \int_{\tilde{g}} \int_{\tilde{f}} \pi_X^* \omega = \int_{\tilde{g}} \pi_Y^* \int_f \omega = \pi_S^* \int_g \int_f \omega.$$

As π_S is a submersion, its associated pullback operator on differential forms is injective, and the desired result follows. □

Proposition 1.3.18 (Relative Stokes' Theorem)**.** Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. For each $\omega \in A_f^*(X)$ we have:

$$\int_f d\omega = (-1)^r d \int_f \omega.$$

Proof. Assume that S is orientable; picking an orientation of S yields an orientation of X. Then we obtain the following equality of smooth currents for each $\omega \in A_f^p(X)$:

$$\left[\int_f d\omega\right] = f_*[d\omega] = f_* d[\omega] = (-1)^r df_*[\omega] = (-1)^r d \left[\int_f \omega\right] = \left[(-1)^r d \int_f \omega\right].$$

The underlying smooth forms on S are, therefore, equal.

By using orientation double covers as in the proof of Proposition 1.3.17 it is possible to extend the proof to the case where S is not orientable. □

1.3.4 The fiber integral along holomorphic submersions

In this section we will study the fiber integral in the setting where the manifolds are complex and the submersion is a holomorphic map.

Assume that X and S are complex manifolds, and that $f : X \to S$ is a holomorphic submersion. Recall that we have decompositions

$$A^*(X; \mathbb{C}) = \bigoplus_{p,q \geq 0} A^{p,q}(X) \quad \text{and} \quad A^*(S; \mathbb{C}) = \bigoplus_{p,q \geq 0} A^{p,q}(S).$$

The fiber integral $\int_f$ respects these decompositions.

Proposition 1.3.19. Let $f : X \to S$ be a holomorphic submersion of complex manifolds whose nonempty fibers have (complex) dimension r. For each (p, q)-form

ω on X with proper support over S, the form $\int_f \omega$ is a $(p-r, q-r)$-form on S.

Proof. As ω is a (p,q)-form, the current $[\omega]$ is a (p,q)-current. By Proposition 1.2.9 the pushforward $f_*[\omega]$ is a $(p-r, q-r)$-current on S, and as $f_*[\omega] = [\int_f \omega]$ it follows that $\int_f \omega$ is a $(p-r, q-r)$-form. □

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A version of the relative Stokes' Theorem 1.3.18 exists for the Dolbeault operators

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X) \quad \text{and} \quad \overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X).$$

Proposition 1.3.20. Let $f : X \to S$ be a holomorphic submersion of complex manifolds. For each smooth complex-valued form ω on X with proper support over S the following identities hold:

$$\int_f \partial\omega = \partial \int_f \omega \quad \text{and} \quad \int_f \overline{\partial}\omega = \overline{\partial} \int_f \omega.$$

Proof. We may assume that ω is a form of type (p,q) for some $p, q \geq 0$. Let r denote the (complex) dimension of the nonempty fibers of f. By Proposition 1.3.18 we have

$$\int_f \partial\omega + \int_f \overline{\partial}\omega = \int_f d\omega = d \int_f \omega = \partial \int_f \omega + \overline{\partial} \int_f \omega.$$

It follows from Proposition 1.3.19 that $\int_f \partial\omega$ and $\partial \int_f \omega$ are both $(p+1-r, q-r)$-forms, whereas $\int_f \overline{\partial}\omega$ and $\overline{\partial} \int_f \omega$ are both $(p-r, q+1-r)$-forms. The desired equalities follow. □

1.4 Hermitian line bundles on families of curves

In this section we will construct several line bundles that appear canonically on families of genus g curves. Moreover we will equip these line bundles with canonical hermitian metrics. By using the Deligne pairing we will be able to exhibit some canonical isometries between these hermitian line bundles. These canonical isometries will be used in Chapter 4 to prove certain equalities in rings of tautological differential forms.

1.4.1 The Poincaré bundle

In this section we will define the Poincaré bundle on families of complex tori. We refer to [BL04], [BHdJ18] for more details.

Let T be a complex torus, and let $T^\vee = \mathrm{Pic}^0(T)$ be its dual torus. A *Poincaré bundle* on the product $T \times T^\vee$ is a line bundle $\mathcal{P}$ that satisfies the following properties:

- For each class $[L] \in T^\vee = \mathrm{Pic}^0(T)$ the pullback of $\mathcal{P}$ along the inclusion $T \to T \times T^\vee : x \mapsto (x, [L])$ is isomorphic to L;
- The pullback of $\mathcal{P}$ along the zero section $\nu_0 : T^\vee \to T \times T^\vee$ is trivial.

A Poincaré bundle always exists, and is unique up to isomorphism. A *rigidified Poincaré bundle* is a Poincaré bundle $\mathcal{P}$ together with a *rigidification*, which is an isomorphism $\nu_0^*\mathcal{P} \xrightarrow{\sim} O_{T^\vee}$. A rigidified Poincaré bundle exists and is unique up to a unique isomorphism.

Next, let $f : \mathcal{T} \to S$ be a family of complex tori. We may construct the *dual family* $f^\vee : \mathcal{T}^\vee \to S$, whose fiber $(\mathcal{T}^\vee)_s$ above a point $s \in S$ is the torus $(\mathcal{T}_s)^\vee$ dual to the torus $\mathcal{T}_s = f^{-1}(s)$. The fiber product $\mathcal{T} \times_S \mathcal{T}^\vee$ admits a *rigidified Poincaré bundle* $\mathcal{P}$, which is a line bundle $\mathcal{P}$ together with an isomorphism $\nu_0^*\mathcal{P} \xrightarrow{\sim} O_{\mathcal{T}^\vee}$ with $\nu_0 : \mathcal{T}^\vee \to \mathcal{T} \times_S \mathcal{T}^\vee$ the zero section, such that for each $s \in S$ the restriction of $\mathcal{P}$ to the fiber $(\mathcal{T} \times_S \mathcal{T}^\vee)_s = \mathcal{T}_s \times \mathcal{T}_s^\vee$ is the rigidified Poincaré bundle on that fiber. The rigidified Poincaré bundle on $\mathcal{T} \times_S \mathcal{T}^\vee$ is unique up to a unique isomorphism.

Let $f : \mathcal{C} \to S$ be a family of curves. Associated to the family f is the Jacobian family $\mathcal{J} \to S$, which is a family of complex tori whose fiber over a point $s \in S$ is the Jacobian $\mathrm{Jac}(\mathcal{C}_s)$ of the curve $\mathcal{C}_s = f^{-1}(s)$. The canonical principal polarizations of the fibers of the Jacobian family give rise to a morphism $\lambda : \mathcal{J} \to \mathcal{J}^\vee$ of families over S. We will denote by $\mathcal{P}_\lambda$ the line bundle on $\mathcal{J} \times_S \mathcal{J}$ obtained by pulling back the Poincaré bundle $\mathcal{P}$ along the morphism $\mathrm{id}_{\mathcal{J}} \times \lambda : \mathcal{J} \times_S \mathcal{J} \to \mathcal{J} \times_S \mathcal{J}^\vee$. Moreover, pulling back $\mathcal{P}_\lambda$ along the diagonal morphism $\mathcal{J} \to \mathcal{J} \times_S \mathcal{J}$ yields a line bundle on $\mathcal{J}$, the *canonical bundle* on $\mathcal{J}$, which we will denote by $\mathcal{B}$.

Recall the following: if T is a complex torus, and L is a line bundle on T, then L induces a morphism

$$\varphi_L : T \to T^\vee : x \mapsto [t_x^* L \otimes L^{\otimes -1}] \in T^\vee = \mathrm{Pic}^0(T).$$

A line bundle L on the Jacobian $J = \mathrm{Jac}(C)$ is called *polarizing* if the associated polarization $\varphi_L : J \to J^\vee$ is a multiple of the canonical polarization $\lambda : J \to J^\vee$. Equivalently, L is polarizing if and only if its first Chern class is a multiple of the Chern class $c_1(O(\Theta))$ of any theta divisor Θ on J.

Proposition 1.4.1. Let C be a curve. The canonical bundle $\mathcal{B}$ on the Jacobian $\mathrm{Jac}(C)$ is polarizing.

We will use the following lemma.

Lemma 1.4.2. Let T be a complex torus, and let $\lambda : T \to T^\vee$ be any homomorphism of complex tori. Let L be the pullback of the Poincaré bundle $\mathcal{P}$ along the morphism $(\mathrm{id}_T, \lambda) : T \to T \times T^\vee$. Then we have an equality of homomorphisms

$$\varphi_L = \lambda + \lambda^\vee \circ \kappa : T \to T^\vee,$$

where $\kappa : T \to T^{\vee\vee}$ is the canonical isomorphism.

Proof. It is a routine exercise to verify that the following diagram is commutative.

$$
\begin{array}{ccccc}
T & \xrightarrow{\varphi_L} & T^\vee & \xleftarrow{\mathrm{id}_{T^\vee} + \lambda^\vee \kappa} & T^\vee \times T \\
\Big\downarrow{\scriptstyle (\mathrm{id},\lambda)} & & \Big\uparrow{\scriptstyle (\mathrm{id},\lambda)^\vee} \quad \nwarrow^{\mathrm{id}_{T^\vee} + \lambda^\vee} & & \Big\downarrow{\scriptstyle \mathrm{id}_{T^\vee} \times \kappa} \\
T \times T^\vee & \xrightarrow{\varphi_{\mathcal{P}}} & (T \times T^\vee)^\vee & \xrightarrow{\sim} & T^\vee \times T^{\vee\vee}
\end{array}
$$

Moreover, the composition

$$T \times T^\vee \xrightarrow{\varphi_{\mathcal{P}}} (T \times T^\vee)^\vee \xrightarrow{\sim} T^\vee \times T^{\vee\vee} \xrightarrow{\mathrm{id}_{T^\vee} \times \kappa^{-1}} T^\vee \times T$$

is the isomorphism $T \times T^\vee \to T^\vee \times T$ that swaps coordinates; see, for instance, [BL04, Exercise 2.16]. The desired result now follows. □

Proof of Proposition 1.4.1. Let Θ be any theta divisor on $\mathrm{Jac}(C)$, and let $\lambda = \varphi_{O(\Theta)}$ be the canonical principal polarization of $\mathrm{Jac}(C)$. By Lemma 1.4.2 and [BL04, Corollary 2.4.6(c)] we have:

$$\varphi_{\mathcal{B}} = \lambda + \lambda^\vee \circ \kappa = 2\lambda,$$

so $c_1(\mathcal{B}) = 2c_1(O(\Theta))$. □

1.4.2 The Deligne pairing

This section serves to introduce the Deligne pairing, which is a pairing associated to a family $f : \mathcal{C} \to S$ of curves that maps a pair of line bundles on $\mathcal{C}$ to a line bundle on S. The Deligne pairing will be used to construct isomorphisms between line bundles that appear canonically on families of curves. We refer to [Del87], [ACG11] for a more detailed treatment.

Let C be a curve. Suppose that f is a nonzero meromorphic function on C, and let $D = \sum_{x \in C} n_x \cdot x$ be a divisor of C such that $\mathrm{div}\, f$ and D are disjoint. We then define

$$f[D] := \prod_{x \in C} f(x)^{n_x} \in \mathbb{C} \setminus \{0\}.$$

To any two line bundles L, M on C we assign a vector space $\langle L, M \rangle$ as follows. Denote by V the complex vector space whose basis consists of pairs (l, m), where l and m are nonzero meromorphic sections of L and M respectively whose divisors are disjoint. The vector space $\langle L, M \rangle$, then, is the quotient of V modulo the subspace spanned by the relations

$$(fl, m) - f[\mathrm{div}\, m] \cdot (l, m) \quad \text{and} \quad (l, gm) - g[\mathrm{div}\, l] \cdot (l, m),$$

where f and g are meromorphic functions on C, such that $\mathrm{div}\, f$ and $\mathrm{div}\, m$ are disjoint, and $\mathrm{div}\, g$ and $\mathrm{div}\, l$ are disjoint. The image of a pair $(l, m) \in V$ under the quotient map $V \to \langle L, M \rangle$ is denoted by $\langle l, m \rangle$. Using Weil reciprocity one can show that $\langle L, M\rangle$ is a one-dimensional vector space. We call the vector space $\langle L, M\rangle$ the *Deligne pairing* of L and M.

We can generalize the Deligne pairing to families as follows. Assume that $f : \mathcal{C} \to S$ is a family of curves, and let L, M be holomorphic line bundles on $\mathcal{C}$. Then L and M induce a holomorphic line bundle $\langle L, M\rangle_f$ on S. The fiber of $\langle L, M\rangle_f$ at a point $s \in S$ is the Deligne pairing $\langle L_s, M_s\rangle$ of the restrictions of L and M to the curve $\mathcal{C}_s$. If $U \subseteq S$ is an open subset, and l and m are nonzero meromorphic sections of $L|_{f^{-1}(U)}$ and $M|_{f^{-1}(U)}$ whose divisors are disjoint and do not contain any of the fibers of f, then

$$\langle l, m\rangle : s \mapsto \langle l(s), m(s)\rangle$$

is a generating section of $\langle L, M\rangle_f|_U$. We will often omit the subscript and write $\langle L, M\rangle$ instead of $\langle L, M\rangle_f$ if the morphism f is clear from the context.

For holomorphic line bundles L, L_1, L_2, M on $\mathcal{C}$ we have canonical isomorphisms

$$\begin{aligned} \langle L, M\rangle &\xrightarrow{\sim} \langle M, L\rangle : & \langle l, m\rangle &\mapsto \langle m, l\rangle \\ \langle L_1, M\rangle \otimes \langle L_2, M\rangle &\xrightarrow{\sim} \langle L_1 \otimes L_2, M\rangle : & \langle l_1, m\rangle \otimes \langle l_2, m\rangle &\mapsto \langle l_1 \otimes l_2, m\rangle \\ \langle O_{\mathcal{C}}, M\rangle &\xrightarrow{\sim} O_S : & \langle 1, m\rangle &\mapsto 1 \\ \left\langle L^{\otimes -1}, M\right\rangle &\xrightarrow{\sim} \langle L, M\rangle^{\otimes -1} : & \left\langle l^{\otimes -1}, m\right\rangle &\mapsto \langle l, m\rangle^{\otimes -1} \end{aligned}$$

where for every nonzero vector v in a one-dimensional vector space V the vector $v^{\otimes -1} \in V^{\otimes -1} = V^\vee$ denotes the vector dual to v.

Isomorphisms $L_1 \xrightarrow{\sim} L_2$ and $M_1 \xrightarrow{\sim} M_2$ induce isomorphisms $\langle L_1, M\rangle \xrightarrow{\sim} \langle L_2, M\rangle$ and $\langle L, M_1\rangle \xrightarrow{\sim} \langle L, M_2\rangle$, respectively. If $\sigma : S \to \mathcal{C}$ is a section of f, we denote by $O(\sigma)$ the line bundle on $\mathcal{C}$ associated to the divisor $\sigma[S] \subseteq \mathcal{C}$. We have a canonical isomorphism

$$\langle O(\sigma), L\rangle \xrightarrow{\sim} \sigma^* L : \langle 1, l\rangle \mapsto \sigma^* l.$$

Moreover, suppose that the degree of the restriction of L to each fiber of f equals d. Then for each line bundle N on S we have a canonical isomorphism

$$\langle L, f^* N\rangle \xrightarrow{\sim} N^{\otimes d} : \langle l, f^* n\rangle \mapsto n^{\otimes d}.$$

Finally, the Deligne pairing is well-behaved with respect to base change. More precisely: if we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{h} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

where f (and hence f') is a family of curves, and L and M are line bundles on $\mathcal{C}$, then there is a canonical isomorphism

$$g^* \langle L, M\rangle_f \xrightarrow{\sim} \langle h^* L, h^* M\rangle_{f'}.$$

The Deligne pairing and the Poincaré bundle are related as follows. We say that a line bundle L on $\mathcal{C}$ has *relative degree 0* (with respect to $f:\mathcal{C}\to S$) if its restriction to each fiber of f has degree 0. If L is a line bundle on $\mathcal{C}$ of relative degree 0, then L induces a section $[L]: S\to\mathcal{J}$ of the Jacobian family $\mathcal{J}\to S$. The morphism $[L]$ maps a point $s\in S$ to the class of the restriction $L|_{\mathcal{C}_s}$ in $\mathcal{J}_s=\mathrm{Jac}(\mathcal{C}_s)$.

Proposition 1.4.3 ([Mor85]). Let L and M be line bundles on $\mathcal{C}$ of relative degree 0. Then there is a canonical isomorphism

$$\langle L, M\rangle \xrightarrow{\sim} ([L],[M])^*\mathcal{P}_\lambda^{\otimes -1}.$$

In particular, for any line bundle L on $\mathcal{C}$ with relative degree 0 one has a canonical isomorphism

$$\langle L, L\rangle \simeq ([L],[L])^*\mathcal{P}_\lambda^{\otimes -1} = [L]^*\mathcal{B}^{\otimes -1}.$$ □

Let $\omega=\omega_{\mathcal{C}/S}$ denote the relative holomorphic cotangent bundle of f. This is a line bundle on $\mathcal{C}$ whose restriction to each fiber $\mathcal{C}_s$ is the sheaf $\Omega^1_{\mathcal{C}_s}$ of holomorphic 1-forms on $\mathcal{C}_s$. It can be obtained as follows. Let $T^{1,0}_{\mathcal{C}}$ and $T^{1,0}_S$ denote the holomorphic tangent bundles of $\mathcal{C}$ and S. The tangent map $df: T^{1,0}_{\mathcal{C}}\to f^*T^{1,0}_S$ is surjective since f is a submersion, and its kernel is a line bundle $T^{1,0}_{\mathcal{C}/S}$, the *relative holomorphic tangent bundle* of f. Its dual is the line bundle $\omega_{\mathcal{C}/S}$. In other words: as f is a submersion, we can view the pullback $f^*\Omega^1_S$ along f of the bundle Ω^1_S of holomorphic 1-forms as a subbundle of $\Omega^1_{\mathcal{C}}$, and $\omega_{\mathcal{C}/S}$ is the quotient bundle $\Omega^1_X/f^*\Omega^1_S$. There is a canonical isomorphism of line bundles on $\mathcal{C}$:

$$\Delta^*O(\Delta)\xrightarrow{\sim}\omega^{\otimes -1},$$

where $\Delta:\mathcal{C}\to\mathcal{C}\times_S\mathcal{C}$ is the diagonal morphism.

Consider the fiber product $\mathcal{C}^2=\mathcal{C}\times_S\mathcal{C}$ and let $p_1,p_2:\mathcal{C}^2\to\mathcal{C}$ be the two projections. Notice that p_1 and p_2 are families of genus g curves. Consider the following line bundle on $\mathcal{C}^2$:

$$O((2g-2)\Delta)\otimes p_2^*\omega^{\otimes -1}.$$

This line bundle has relative degree 0 with respect to p_1 and hence induces a morphism

$$\kappa := \mathcal{C}\to\mathcal{J} : x\mapsto [O((2g-2)x)\otimes\omega^{\otimes -1}]\in \mathrm{Jac}(\mathcal{C}_{f(x)}),$$

and by Proposition 1.4.3 there is a canonical isomorphism of line bundles on $\mathcal{C}$:

$$\left\langle O((2g-2)\Delta)\otimes p_2^*\omega^{\otimes -1}, O((2g-2)\Delta)\otimes p_2^*\omega^{\otimes -1}\right\rangle_{p_1}\xrightarrow{\sim}\kappa^*\mathcal{B}^{\otimes -1}.$$

By the bilinearity of the Deligne pairing, the left hand side of this isomorphism is canonically isomorphic to

$$\langle O(\Delta),O(\Delta)\rangle_{p_1}^{\otimes(2g-2)^2}\otimes\langle O(\Delta),p_2^*\omega\rangle_{p_1}^{\otimes -2(2g-2)}\otimes\langle p_2^*\omega,p_2^*\omega\rangle_{p_1}.$$

Recall that there are canonical isomorphisms

$$\langle O(\Delta), O(\Delta)\rangle_{p_1} \xrightarrow{\sim} \Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1} \quad \text{and} \quad \langle O(\Delta), p_2^*\omega\rangle_{p_1} \xrightarrow{\sim} \Delta^* p_2^*\omega = \omega,$$

and as the Deligne pairing is well-behaved with respect to base change

$$\langle p_2^*\omega, p_2^*\omega\rangle_{p_1} \xrightarrow{\sim} f^*\langle \omega, \omega\rangle_f.$$

By piecing all canonical isomorphisms together we obtain the following result.

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Proposition 1.4.4. Let $f : \mathcal{C} \to S$ be a family of curves, and let $\mathcal{J} \to S$ be the corresponding Jacobian family. Let $\kappa : \mathcal{C} \to \mathcal{J}$ be the morphism that maps a point $x \in \mathcal{C}_s$ to the class $[O((2g-2)x) \otimes \omega^{\otimes -1}] \in \mathrm{Jac}(\mathcal{C}_s)$. Then we have a canonical isomorphism

$$\kappa^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} \omega^{\otimes -2g(2g-2)} \otimes f^*\langle \omega, \omega\rangle_f. \qquad \square$$

Likewise, we may consider the two sections $\sigma_1, \sigma_2 : \mathcal{C}^2 \to \mathcal{C}^3$ of the projection $p_{12} : \mathcal{C}^3 \to \mathcal{C}^2$ given by

$$\sigma_i : \mathcal{C}^2 \to \mathcal{C}^3 : (x_1, x_2) \mapsto (x_1, x_2, x_i).$$

These sections induce a line bundle $O(\sigma_2 - \sigma_1) = O(\sigma_2) \otimes O(\sigma_1)^{\otimes -1}$ on $\mathcal{C}^3$ with relative degree 0 with respect to p_{12}. Analogous to Proposition 1.4.4 we obtain the following identity.

Proposition 1.4.5. Let $f : \mathcal{C} \to S$ be a family of curves, and let $\mathcal{J} \to S$ be the corresponding Jacobian family. Let $\delta : \mathcal{C}^2 \to \mathcal{J}$ denote the morphism that maps a pair $(x, y) \in \mathcal{C}_s^2$ to the class $[O(y - x)] \in \mathrm{Jac}(\mathcal{C}_s)$. Then we have a canonical isomorphism

$$\delta^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2}. \qquad \square$$

By applying Proposition 1.4.5 we moreover obtain a canonical isomorphism

$$\left\langle \delta^* \mathcal{B}^{\otimes -1}, \delta^* \mathcal{B}^{\otimes -1} \right\rangle_{p_1} \xrightarrow{\sim} \omega^{\otimes 4g} \otimes f^*\langle \omega, \omega\rangle_f. \tag{1.4.6}$$

Combining this canonical isomorphism with the one from 1.4.4 finally yields yet another canonical isomorphism

$$\left\langle \delta^* \mathcal{B}^{\otimes -1}, \delta^* \mathcal{B}^{\otimes -1} \right\rangle_{p_1} \otimes \kappa^* \mathcal{B} \xrightarrow{\sim} \omega^{\otimes 4g^2}. \tag{1.4.7}$$

1.4.3 Hermitian metrics, first Chern form

Let X be a complex manifold, and let E be a holomorphic vector bundle on X. A *hermitian metric* on E consists of a hermitian inner product $\langle \cdot, \cdot \rangle_x$ on each fiber E_x of E, such that these inner products vary smoothly with $x \in X$: if $U \subseteq X$ is open and $\sigma, \tau \in E(U)$ are sections, then the function

$$\langle \sigma, \tau \rangle : U \to \mathbb{C} : x \mapsto \langle \sigma(x), \tau(x)\rangle_x$$

is smooth. A *hermitian vector bundle* is a vector bundle equipped with a hermitian metric.

If σ is a holomorphic section of a hermitian vector bundle E, its norm is the real-valued function $\|\sigma\|$ on X given by

$$\langle\sigma\rangle(x) = \langle\sigma(x), \sigma(x)\rangle_x^{1/2}.$$

Conversely, if L is a holomorphic *line* bundle on X, with a generating section σ, and $f : X \to \mathbb{R}_{>0}$ is a positive-valued smooth function, then there exists a unique hermitian metric on L for which $\|\sigma\| = f$.

If V and W are two complex vector spaces with hermitian inner products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, the tensor product $V \otimes W$ has a canonical hermitian inner product $\langle\cdot,\cdot\rangle_{V\otimes W}$ given by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle_{V\otimes W} = \langle v_1, v_2\rangle_V \cdot \langle w_1, w_2\rangle_W.$$

More generally, the tensor product of two hermitian vector bundles is canonically a hermitian vector bundle.

Likewise, for any hermitian line bundle L on X, the dual line bundle $L^{\otimes -1}$ has a unique hermitian metric for which the canonical isomorphism $L \otimes L^{\otimes -1} \xrightarrow{\sim} O_X$ is an isometry, where O_X is endowed with the canonical metric $\|1\| = 1$.

Let L be a line bundle on the complex manifold X with a hermitian metric $\|\cdot\|$. The *first Chern form* $c_1(L)$ is the differential $(1,1)$-form on X defined locally by

$$c_1(L, \|\cdot\|) = \frac{\partial\bar{\partial}}{2\pi i} \log\|\sigma\|^2$$

where σ is any local generating section of L. As $\partial\bar{\partial}\log|f| = 0$ for every holomorphic function f, it follows that the first Chern form is well-defined. It is a closed real differential form whose De Rham cohomology class matches the first Chern class of the line bundle L under the isomorphism $H^2_{\mathrm{dR}}(X) \xrightarrow{\sim} H^2(X;\mathbb{R})$; see [GH94, Chapter 1.1]. If the metric $\|\cdot\|$ is clear from the context we will often omit it from our notation and write $c_1(L)$ instead of $c_1(L, \|\cdot\|)$.

The operator mapping a hermitian line bundle to its first Chern form is linear: if L_1, L_2 are two hermitian line bundles on X, then

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in A^2(X).$$

Theorem 1.4.8 (Poincaré–Lelong formula, [GH94, p. 388])**.** Let L be a hermitian line bundle on a complex manifold X and let s be a meromorphic section of L. Then we have an equality of 2-currents on X:

$$\frac{\partial\bar{\partial}}{\pi\sqrt{-1}}[\log\|s\|] + \delta_{\operatorname{div} s} = c_1(L, \|\cdot\|). \qquad \square$$

Note that the Poincaré–Lelong formula is stated in [GH94] in a less general setting, where L is assumed to be the trivial bundle with its canonical metric $\|1\| = 1$, and s is assumed to be a holomorphic function on X. By working locally one easily deduces Theorem 1.4.8 from the version in [GH94].

Lemma 1.4.9. Let C be a curve and let L be a hermitian line bundle on C. Then

$$\int_C c_1(L) = \deg L.$$

Proof. Take any nonzero meromorphic section s of L. By using the Poincaré–Lelong formula and evaluating the resulting 2-currents in the form $1 \in A^0_c(C)$ we obtain the desired result. □

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The Poincaré–Lelong formula implies the following lemma, which will be used in Chapter 4.

Lemma 1.4.10. Let $f : \mathcal{C} \to S$ be a family of curves, and let L be a hermitian line bundle on $\mathcal{C}$. Then the fiber integral of the 2-form $c_1(L)$ along f is the function $S \to \mathbb{R}$ given by

$$\left(\int_f c_1(L)\right)(s) = \deg L|_{\mathcal{C}_s} \quad \text{for all } s \in S.$$

In particular, the induced function $\int_f c_1(L) : S \to \mathbb{Z} : s \mapsto \deg L|_{\mathcal{C}_s}$ is locally constant.

Proof. As $c_1(L)$ is a smooth real 2-form and f is a proper submersion with fibers of real dimension 2, the fiber integral $\int_f c_1(L)$ is a smooth function on S given by

$$\left(\int_f c_1(L)\right)(s) = \int_{\mathcal{C}_s} c_1(L)|_{\mathcal{C}_s} = \int_{\mathcal{C}_s} c_1(L|_{\mathcal{C}_s}) = \deg L|_{\mathcal{C}_s},$$

where the final equality follows from Lemma 1.4.9. This fiber integral is a smooth integer-valued function, so it is in particular locally constant. □

1.4.4 Admissible metrics

Line bundles on curves and complex tori admit many hermitian forms. In this section we restrict our study to *admissible* hermitian metrics. We will follow [Mor85].

Let T be a complex torus, and let L be a line bundle on T. A hermitian metric on L is *admissible* if its first Chern form is translation-invariant. The admissible metrics on O_T are exactly the constant metrics. Every line bundle L admits admissible metrics, and any two such metrics are equal up to a multiplicative constant. If L_1 and L_2 are endowed with admissible metric, then the induced metric on $L_1 \otimes L_2$ is admissible, too. If $f : T' \to T$ is a morphism of complex tori, and L is a line bundle on T with an admissible metric, then the induced metric on f^*L is admissible. See [Mor85, §3] for more details.

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Example 1.4.11. Let T be a complex torus, and let $\mathcal{P}$ be the rigidified Poincaré bundle on the product torus $T \times T^\vee$. Then any two admissible metrics on $\mathcal{P}$ are equal up to a multiplicative constant. Fixing any admissible metric on $\mathcal{P}$ induces an admissible, and hence constant, metric on $O_{T^\vee}$ via the fixed isomorphism $\nu_0^*\mathcal{P} \xrightarrow{\sim} O_{T^\vee}$. It follows that there exists a unique admissible metric on $\mathcal{P}$ for which the fixed isomorphism $\nu_0^*\mathcal{P} \xrightarrow{\sim} O_{T^\vee}$ is an isometry if we endow $O_{T^\vee}$ with the canonical metric $\|1\| = 1$. We call this metric the *canonical metric* on $\mathcal{P}$. More generally, if $f : \mathcal{T} \to S$ is a family of complex tori, then there is a unique hermitian metric on the rigidified Poincaré bundle $\mathcal{P}$ on $\mathcal{T} \times_S \mathcal{T}^\vee$ whose restriction to each fiber $\mathcal{T}_s \times \mathcal{T}_s^\vee$ above S is the canonical metric on the rigidified Poincaré bundle on $\mathcal{T}_s \times \mathcal{T}_s^\vee$. See [BHdJ18, Proposition 2.8]. This metric, too, is called the *canonical metric* on $\mathcal{P}$.
If $f : \mathcal{C} \to S$ is a family of curves with Jacobian family $\mathcal{J} \to S$, then the canonical metric on the Poincaré bundle $\mathcal{P}$ on $\mathcal{J} \times_S \mathcal{J}^\vee$ and the canonical principal polarization $\lambda : \mathcal{J} \to \mathcal{J}^\vee$ induce canonical metrics on the line bundles $\mathcal{P}_\lambda = (\mathrm{id}, \lambda)^*\mathcal{P}$ and $\mathcal{B} = \Delta^*\mathcal{P}_\lambda$, and these metrics are fiberwise admissible.

Now let C be a curve of genus $g > 0$. The g-dimensional complex vector space $\Omega^1(X)$ of holomorphic 1-forms on C is endowed with an inner product

$$\langle \omega, \eta \rangle = \frac{\sqrt{-1}}{2} \int_C \omega \wedge \bar{\eta} \quad \text{for all } \omega, \eta \in \Omega^1(C).$$

Fix any orthonormal basis $\omega_1, \ldots, \omega_g$ of $\Omega^1(C)$. The *canonical 2-form* of C is the form

$$\mu := \frac{\sqrt{-1}}{2g} \sum_{i=1}^{g} \omega_i \wedge \bar{\omega}_i.$$

The canonical 2-form is a real form that satisfies

$$\int_C \mu = 1.$$

It does not depend on the choice of an orthonormal basis, and by the Riemann-Roch theorem it is a volume form.

The canonical 2-form can also be obtained from the canonical metric on $\mathcal{B}$ as follows: for $x \in C$ any point we have

$$2g\mu = c_1(j_x^*\mathcal{B}) = j_x^*(c_1(\mathcal{B})),$$

with $j_x : C \to J : y \mapsto [O(y - x)]$ the Abel–Jacobi map. Note that the 2-form $j_x^*(c_1(\mathcal{B}))$ does not depend on the choice of $x \in C$, since $c_1(\mathcal{B})$ is translation-invariant.

Let L be a line bundle on C. A hermitian metric $\|\cdot\|$ on L is *admissible* if its first Chern form $c_1(L, \|\cdot\|)$ is a multiple of the canonical 2-form μ. Note that $\|\cdot\|$ is admissible if and only if $c_1(L, \|\cdot\|) = \deg(L) \cdot \mu$, since $\int_C c_1(L, \|\cdot\|) = \deg L$. The admissible metrics on the trivial bundle O_C are precisely the constant metrics. If

L_1, L_2 are endowed with admissible metrics, then the induced metric on $L_1 \otimes L_2$ is again admissible.

If $x \in C$ is a point, and M is a *polarizing* line bundle on the Jacobian $J = \mathrm{Jac}(C)$ with an admissible metric, then the induced metric on the pullback $j_x^* M$ along the Abel–Jacobi morphism $j_x : C \to J$ is again admissible.

Remark 1.4.12. If the genus of C equals 1, then we may view C both as a curve and a complex torus via the Abel–Jacobi map, and we have two definitions for admissible metrics on line bundles on C. As the canonical 2-form on C is translation-invariant, these definitions agree.

1.4.5 Biadmissible metrics

Let C be a curve of genus g, and let L be a line bundle on the product $C \times C$. A hermitian metric on L is *biadmissible* if its restrictions to the fibers of the projection maps $p_1, p_2 : C \times C \to C$ are all admissible. Any two biadmissible metrics on L are equal up to a positive multiplicative constant, and biadmissible metrics are well-behaved with respect to tensor products. If M is a line bundle on C with an admissible metric, then the induced metrics on $p_1^* M$ and $p_2^* M$ are biadmissible.

Recall from Proposition 1.4.5 that we have a canonical isomorphism of line bundles on $C \times C$:

$$\delta^* \mathcal{B} \xrightarrow{\sim} p_1^* \omega \otimes p_2^* \omega \otimes O(\Delta)^{\otimes 2}.$$

As the canonical bundle $\mathcal{B}$ is polarizing, it is straightforward to verify that the canonical metric on $\delta^* \mathcal{B}$ is biadmissible. If ω is endowed with an admissible metric, then the induced metrics on $p_1^* \omega, p_2^* \omega$ are biadmissible, and the above isomorphism induces a biadmissible metric on $O(\Delta)^{\otimes 2}$ and therefore on $O(\Delta)$. Notice, moreover, that this metric on $O(\Delta)$ is symmetric. We thus obtain a canonical map

$$\{\text{admissible metrics on } \omega\} \to \{\text{biadmissible metrics on } O(\Delta)\}.$$

As this map is compatible with the free and transitive action of the multiplicative group $\mathbb{R}_{>0}$ on both sets, it is a bijection. In order to find the inverse of this bijection, notice that restricting the above isomorphism to the diagonal yields the canonical isomorphism

$$O_C \simeq \Delta^* \delta^* \mathcal{B} \xrightarrow{\sim} \omega^{\otimes 2} \otimes \Delta^* O(\Delta)^{\otimes 2},$$

and the induced metric on $O_C \simeq \Delta^* \delta^* \mathcal{B}$ is the canonical metric $\|1\| = 1$. Every biadmissible metric on $O(\Delta)$, therefore, induces an admissible metric on ω via the canonical isomorphism $\Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1}$.

Let $\|\cdot\|$ be any biadmissible metric on $O(\Delta)$. Taking the norm of the canonical global section 1 of $O(\Delta)$ yields a function $G = \|1\| : C \times C \to \mathbb{R}_{\geq 0}$. The function G has the following properties:

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1. G is smooth and positive-valued outside the diagonal, and vanishes on the diagonal. If z is a local coordinate on an open $U \subseteq C$, then on $U \times U$ the function G can be expressed as

$$G(x, y) = |z(x) - z(y)| \cdot u(x, y)$$

with u a smooth and positive-valued function on $U \times U$.
2. G is symmetric: $G(x, y) = G(y, x)$ for all $x, y \in C$.
3. For each point $x \in X$ we have an equality of 2-currents on C:

$$\frac{\partial\overline{\partial}}{\pi\sqrt{-1}}[\log G(x, \cdot)] = \mu - \delta_x,$$

by the Poincaré–Lelong formula.

Conversely, every function $G : C \times C \to \mathbb{R}_{\geq 0}$ that satisfies these properties determines a biadmissible metric on $O(\Delta)$. According to Arakelov [Ara74] the function

$$C \to \mathbb{R} : x \mapsto \int_{y \in C} \log G(x, y)\mu(y)$$

is constant. The *Arakelov–Green function* G is the unique function that satisfies the above properties and the normalizing condition

$$\int_{y \in C} \log G(x, y)\mu(y) = 0 \quad \text{for all } x \in X.$$

We will call the biadmissible metric on $O(\Delta)$ it determines the *canonical metric* on $O(\Delta)$. The canonical metric on $O(\Delta)$ determines an admissible metric on ω via the canonical isomorphism $\Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1}$, which we will also call the *canonical metric* on ω.

Finally, if $x \in C$ is any point, restricting the canonical metric on $O(\Delta)$ to the fiber $C \times \{x\}$ yields an admissible metric on the line bundle $O(x)$, given by

$$\|1\|(y) = G(x, y).$$

We call this metric the *canonical metric* on $O(x)$. More generally, for $D = \sum_x n_x x$ a divisor on C, we obtain a *canonical metric* on the line bundle $O(D)$ via the canonical isomorphism

$$\bigotimes_{x \in X} O(x)^{\otimes n_x} \xrightarrow{\sim} O(D).$$

1.4.6 Canonical isometries

Let C be a curve with Jacobian J. We have defined canonical metrics on the line bundles $\mathcal{B}$ on J, $O(\Delta)$ on $C \times C$, and ω and $O(D)$ (with D a divisor) on C. In this section we will show that the canonical isomorphisms we obtained in Section 1.4.2 are in fact isometries. The canonical isomorphism

$$\delta^* \mathcal{B} \xrightarrow{\sim} p_1^* \omega \otimes p_2^* \omega \otimes O(\Delta)^{\otimes 2}$$

is an isometry by construction of the metrics on ω and $O(\Delta)$. For the other isomorphisms we will be using the Deligne pairing.

Let L, M be hermitian line bundles on C. To the hermitian metrics on L, M we associate a norm on the vector space $\langle L, M\rangle$:

$$\begin{aligned}\log \|\langle l, m\rangle\| &:= (\log \|m\|)[\operatorname{div}(l)] + [\log \|l\|](c_1(M)) \\ &= (\log \|l\|)[\operatorname{div}(m)] + [\log \|m\|](c_1(L)),\end{aligned}$$

where the second equality can be proved using Stokes' theorem.

More generally, let $f : \mathcal{C} \to S$ be a family of curves, and let L, M be hermitian line bundles on $\mathcal{C}$. The induced metrics on the fibers of the Deligne pairing $\langle L, M\rangle$ induce a hermitian metric on $\langle L, M\rangle$. See also [Del87, §6].

For hermitian line bundles L, L_1, L_2, M on $\mathcal{C}$ the canonical isomorphisms

$$\begin{aligned}\langle L, M\rangle &\xrightarrow{\sim} \langle M, L\rangle \\ \langle L_1, M\rangle \otimes \langle L_2, M\rangle &\xrightarrow{\sim} \langle L_1 \otimes L_2, M\rangle \\ \langle O_{\mathcal{C}}, M\rangle &\xrightarrow{\sim} O_S \\ \left\langle L^{\otimes -1}, M\right\rangle &\xrightarrow{\sim} \langle L, M\rangle^{\otimes -1}\end{aligned}$$

are isometries, where $O_{\mathcal{C}}$ and O_S are endowed with the canonical metrics given by $\|1\| = 1$.

Likewise, if N is a hermitian line bundle on S, and L a hermitian line bundle on $\mathcal{C}$ whose restriction to each fiber of f has degree d, then the canonical isomorphism

$$\langle L, f^*N\rangle \xrightarrow{\sim} N^{\otimes d}$$

is an isometry.

If we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{h} & \mathcal{C} \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

where f and f' are families of curves, and L and M are hermitian line bundles on $\mathcal{C}$, then the canonical isomorphism

$$g^*\langle L, M\rangle_f \xrightarrow{\sim} \langle h^*L, h^*M\rangle_{f'}$$

is an isometry.

Proposition 1.4.13 ([Del87, §6])**.** For any two hermitian line bundles L, M on $\mathcal{C}$ we have

$$c_1(\langle L, M\rangle) = \int_f c_1(L) \wedge c_1(M). \qquad \square$$

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Proposition 1.4.14 ([HdJ15, Corollary 4.2])**.** Let L and M be line bundles on $\mathcal{C}$ of relative degree 0. Then the canonical isomorphism

$$\langle L, M \rangle \xrightarrow{\sim} ([L],[M])^* \mathcal{P}_\lambda^{\otimes -1}$$

from Proposition 1.4.3 is an isometry. □

Finally, assume that C is a curve, let $x \in C$ be a point, and endow $O(x)$ with its canonical metric. Then for each *admissible* line bundle M on C the canonical isomorphism

$$\langle O(x), M \rangle \xrightarrow{\sim} M_x$$

is easily seen to be an isometry. Likewise, if we equip the diagonal bundle $O(\Delta)$ on $C \times C$ with its canonical metric, and if M is a line bundle on C whose restriction to each fiber of p_1 is admissible, then the canonical isomorphism

$$\langle O(\Delta), M \rangle \xrightarrow{\sim} \Delta^* M$$

is an isometry. Later in this section we will generalize this statement to include arbitrary sections of families of curves.

From the above canonical isometries involving the Deligne pairing and the computations of the canonical isomorphisms 1.4.4, 1.4.6, and 1.4.7, we deduce the following result.

Proposition 1.4.15. Let C be a curve with Jacobian J, and endow the line bundles $O(\Delta)$ on $C \times C$, ω on C, and $\mathcal{B}$ on J with their canonical metrics. Then the canonical isomorphisms

$$\begin{aligned}
\kappa^* \mathcal{B}^{\otimes -1} &\xrightarrow{\sim} \omega^{\otimes -2g(2g-2)} \otimes f^* \langle \omega, \omega \rangle_f \\
\delta^* \mathcal{B}^{\otimes -1} &\xrightarrow{\sim} p_1^* \omega^{\otimes -1} \otimes p_2^* \omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2} \\
\left\langle \delta^* \mathcal{B}^{\otimes -1}, \delta^* \mathcal{B}^{\otimes -1} \right\rangle_{p_1} &\xrightarrow{\sim} \omega^{4g} \otimes f^* \langle \omega, \omega \rangle_f \\
\left\langle \delta^* \mathcal{B}^{\otimes -1}, \delta^* \mathcal{B}^{\otimes -1} \right\rangle_{p_1} \otimes \kappa^* \mathcal{B} &\xrightarrow{\sim} \omega^{\otimes 4g^2}
\end{aligned}$$

are isometries. □

Notice that, in particular, the canonical metric on ω can be obtained from the canonical metric on $\mathcal{B}$ by taking pullbacks along δ and κ and Deligne pairings along p_1.

Corollary 1.4.16. Let $f : \mathcal{C} \to S$ be a family of curves, and consider the relative dualizing sheaf $\omega = \omega_{\mathcal{C}/S}$ on $\mathcal{C}$ and the diagonal bundle $O(\Delta)$ on $\mathcal{C} \times_S \mathcal{C}$.
There exists a unique fiberwise admissible hermitian metric on ω whose restriction to each fiber $\mathcal{C}_s$ of f is the canonical metric on $\omega|_{\mathcal{C}_s} = \omega_C$.
Likewise, there exists a unique fiberwise biadmissible hermitian metric on $O(\Delta)$ whose restriction to each fiber $(\mathcal{C} \times_S \mathcal{C})_s = \mathcal{C}_s \times \mathcal{C}_s$ of the morphism $\mathcal{C} \times_S \mathcal{C} \to S$

is the canonical metric on the diagonal bundle on $\mathcal{C}_s \times \mathcal{C}_s$.

As the reader may already expect, we will call these metrics on ω and $O(\Delta)$ the *canonical metrics*.

Proof. Let $\mathcal{J} \to S$ be the Jacobian family of f, and let $\mathcal{B}$ be the Poincaré bundle on $\mathcal{J}$ with its canonical metric. The canonical isomorphism

$$\left\langle \delta^*\mathcal{B}^{\otimes -1}, \delta^*\mathcal{B}^{\otimes -1} \right\rangle_{p_1} \otimes \kappa^*\mathcal{B} \xrightarrow{\sim} \omega^{\otimes 4g^2}$$

then defines a hermitian metric on $\omega^{\otimes 4g^2}$, and hence on ω. By Proposition 1.4.15 the restriction of this hermitian metric to each fiber $\mathcal{C}_s$ of f is equal to the canonical metric on $\omega_{\mathcal{C}_s}$.

Likewise, the hermitian metrics on $\mathcal{B}$ and ω determine a hermitian metric on $O(\Delta)$ via the canonical isomorphism

$$\delta^*\mathcal{B}^{\otimes -1} \xrightarrow{\sim} p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2},$$

and Proposition 1.4.15 ensures that the restriction of this metric to each fiber is indeed the canonical metric. □

Let $f : \mathcal{C} \to S$ be a family of curves, and let $\sigma : S \to \mathcal{C}$ be a section. We will endow the line bundle $O(\sigma) = O(\sigma[S])$ on $\mathcal{C}$ with a canonical metric, as follows. Both squares in the following diagram are cartesian:

$$\begin{array}{ccc}
S & \xrightarrow{\quad\sigma\quad} & \mathcal{C} \\
\downarrow{\scriptstyle\sigma} & \square & \downarrow{\scriptstyle\Delta} \\
\mathcal{C} & \xrightarrow{(\sigma f, \mathrm{id}_{\mathcal{C}})} & \mathcal{C} \times_S \mathcal{C} \\
\downarrow{\scriptstyle f} & \square & \downarrow{\scriptstyle p_1} \\
S & \xrightarrow{\quad\sigma\quad} & \mathcal{C}
\end{array}$$

We therefore have a canonical isomorphism

$$O(\sigma) \xrightarrow{\sim} (\sigma f, \mathrm{id}_{\mathcal{C}})^* O(\Delta),$$

and the canonical metric on $O(\Delta)$ canonically induces a metric on $O(\sigma)$, which we will call the *canonical metric*. Notice that for each point $s \in S$ restricting the line bundle $O(\sigma)$ on $\mathcal{C}$ to the fiber $\mathcal{C}_s$ yields the line bundle $O(\sigma(s))$ with its canonical metric. Moreover, if M is a line bundle on $\mathcal{C}$ with a hermitian metric that is fiberwise (with respect to f) admissible, then the canonical isomorphism

$$\langle O(\sigma), M \rangle \xrightarrow{\sim} \sigma^* M$$

is an isometry.