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Tautological differential forms on moduli spaces of curves

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Introduction

Background

Let $g \geq 2$. Mumford [Mum83] set up a Chow ring on the moduli space $\mathcal{M}_g$ of genus g curves (and on its Deligne–Mumford compactification $\overline{\mathcal{M}_g}$). He modeled this construction after the enumerative geometry of the Grassmannians, where canonical elements of the Chow ring of a Grassmannian are given by the Chern classes of the universal vector bundle and the universal quotient bundle. He defined canonical classes κ_i ($i \geq 0$) in the Chow ring of $\mathcal{M}_g$ by taking pushforwards of powers of the first Chern class of the relative cotangent bundle $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ of the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$. The *tautological ring* $R^*(\mathcal{M}_g)$ is then defined as the subring of the Chow ring of $\mathcal{M}_g$ generated by these κ-classes. Mumford proved that all Chern classes of the Hodge bundle $p_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ of the universal family lie in the tautological ring $\mathcal{M}_g$.

Moreover, Mumford proved that the tautological ring $R^*(\mathcal{M}_g)$ is generated by the classes $\kappa_1, \dots, \kappa_{g-2}$. Looijenga [Loo95] then proved that the tautological ring $R^*(\mathcal{M}_g)$ vanishes in degrees higher than $g-2$, and that $R^{g-2}(\mathcal{M}_g)$ is at most one-dimensional, and spanned by κ_{g-2}. Faber [Fab97] then proved that κ_{g-2} is nonzero, and hence that $R^{g-2}(\mathcal{M}_g)$ is one-dimensional. He then conjectured in [Fab99] that $R^*(\mathcal{M}_g)$ is a *Gorenstein algebra*: that is, for all $0 \leq d \leq g-2$ the pairing

$$R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q},$$

induced by multiplication in the Chow ring, is perfect. Faber's conjecture has been verified for all $g \leq 23$ (see [Fab13]), but for $g = 24$ not enough relations have yet been found to verify the conjecture.

Rather than fixing $g \geq 2$, one could study the behavior of the cohomology of $\mathcal{M}_g$ as g tends to infinity. Mumford conjectured in [Mum83] that the homomorphism $\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to H^*(\mathcal{M}_g, \mathbb{Q})$ is an isomorphism in degrees $\leq k$, where k tends to infinity as g tends to infinity. In other words: the only cohomology classes that occur on $\mathcal{M}_g$ for all $g \geq 2$ are the κ-classes, and there are no nontrivial relations among these classes that hold for arbitrary values of g. Harer [Har85] showed that for all $k \geq 0$ there is a g for which we have isomorphisms

$$H^k(\mathcal{M}_g; \mathbb{Q}) \cong H^k(\mathcal{M}_{g+1}; \mathbb{Q}) \cong H^k(\mathcal{M}_{g+2}; \mathbb{Q}) \cong \dots.$$

In other words: for each $k \geq 0$ the kth cohomology of $\mathcal{M}_g$ stabilizes as g tends to infinity. Miller [Mil86] and Morita [Mor87] showed that the homomorphism

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to H^*(\mathcal{M}_g; \mathbb{Q})$$

is *injective* in degrees $\leq k$, where k tends to infinity as g tends to infinity. Finally, Mumford's conjecture was proved by Madsen and Weiss in [MW07].

The moduli space $\mathcal{M}_g$ has also been studied using analytical tools. Faltings [Fal84] assigns to each admissible line bundle L on a Riemann surface X a hermitian metric on the determinant of cohomology $\det R\Gamma(L)$. By comparing these metrics with a canonical metric on the line bundle $O(-\Theta)$ on the Jacobian, Faltings obtains an invariant $\delta(X)$ of the Riemann surface X. This invariant gives rise to a function $\delta_g : \mathcal{M}_g \to \mathbb{R}$. Hain and Reed [HR04] construct a natural metric on the biextension line bundle on $\mathcal{M}_g$. This line bundle is isomorphic to the $(8g+4)$th power of the Hodge bundle; by comparing metrics they obtain another invariant $\beta_g : \mathcal{M}_g \to \mathbb{R}$, defined up to a constant. Another invariant $a_g = 2\pi\varphi_g : \mathcal{M}_g \to \mathbb{R}$ was found by Kawazumi [Kaw08; Kaw09] and Zhang [Zha10] in different contexts. Kawazumi constructed two differential 2-forms e^J, e^A on $\mathcal{C}_g$ that both represent the class of the relative tangent bundle of the universal family $\mathcal{C}_g \to \mathcal{M}_g$. He shows that these forms are not equal but are related by the identity

$$e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g+1)} \partial\bar{\partial} a_g.$$

De Jong [dJon13; dJon16] showed that the invariants of Faltings, Hain–Reed, and Kawazumi–Zhang are linearly dependent.

Overview

In this thesis we will generate differential 2-forms on the moduli space $\mathcal{M}_g$, the universal family $\mathcal{C}_g \to \mathcal{M}_g$, and higher powers of $\mathcal{C}_g$, by listing various canonical hermitian line bundles on these moduli spaces and taking their first Chern forms. By using these 2-forms we will construct an analytic analogue to the tautological rings, the rings of *tautological differential forms*. We will show that these rings are not 'too big' (i.e. they are finite-dimensional in each degree, and hence finite-dimensional). By using a canonical line bundle on the universal Jacobian bundle $\mathcal{J}_g \to \mathcal{M}_g$ we will moreover be able to compute various relations between tautological differential forms. We will carry out some of these computations in a combinatorial framework based on marked graphs.

In Chapter 1 we will recall the general theory of families of manifolds. We will show that every family of oriented manifolds admits a uniquely defined fiber integral operator. Afterwards we will specialize to holomorphic families of compact Riemann surfaces of a fixed genus. Some hermitian line bundles that appear canonically on such families will be constructed, and we will find some canonical isometries among them.

The theory of moduli spaces is explored in Chapter 2. We will recall that no fine moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g exists in the category of complex manifolds. The problem is that compact Riemann surfaces admit nontrivial automorphisms that can be exploited to construct nontrivial isotrivial families by twisting. One solution to this problem is adding extra structure to our Riemann surfaces, such as Teichmüller structures, thereby annihilating any nontrivial automorphisms. Another solution, by Deligne and Mumford [DM69], is to construct the moduli space $\mathcal{M}_g$ as a stack, rather than a complex manifold. Although $\mathcal{M}_g$ is not a complex manifold anymore, we may still define objects such as differential forms and hermitian line bundles on $\mathcal{M}_g$, and we will discuss why such objects on $\mathcal{M}_g$ can be viewed as objects that occur universally on families of compact Riemann surfaces of genus g.

In Chapter 3 we will take a detour and discuss r-marked graphs, which are graphs for which r vertices are marked with the integers $1, \dots, r$. These graphs give us a combinatorial framework for working with tautological differential forms in Chapter 4. We will discuss contraction operations on marked graphs, and show that there are only finitely many contracted graphs up to isomorphism. In fact, given an integer $d \in \mathbb{Z}$, the number of isomorphism classes of contracted r-marked graphs of characteristic $r - d$ can be expressed as a polynomial in r:

Theorem 3.8.1. Let $d \in \mathbb{Z}$ be an integer. If d is negative, then for any $r \geq 0$ there are no contracted r-marked graphs of characteristic $r - d$.
If $d \geq 0$, then there exists a polynomial $f_d \in \mathbb{Q}[x]$ of degree $2d$ such that the number of isomorphism classes of contracted r-marked graphs of characteristic $r - d$ is equal to $f_d(r)$ for all $r \geq 0$. The leading coefficient of f_d is $1/(2^d \cdot d!)$.

We will compute the polynomial f_d for all $d \leq 4$.

Finally, in Chapter 4, we will construct rings of tautological differential forms on $\mathcal{M}_g$, $\mathcal{C}_g$, and $\mathcal{C}_g^r$ ($r \geq 2$), where $\mathcal{C}_g \to \mathcal{M}_g$ is the universal family of compact Riemann surfaces of genus g. Not every definition of the tautological Chow rings translates immediately to the analytical setting; we will discuss several of these definitions and see which translates best. Next we will give a method of constructing tautological differential forms from marked graphs, and show that tautological forms associated to contracted graphs span the ring of tautological forms, thereby proving that the rings of tautological forms are finite-dimensional in each degree, and hence finite-dimensional:

Theorem 4.6.1. For all $r \geq 0$ and $g \geq 2$, the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is finite-dimensional.

We will fully compute the degree 2 part of the ring of tautological forms on $\mathcal{C}_g^r$ for all $r \geq 0$, and provide an algorithm for computing more relations among tautological differential forms associated to marked graphs.