Scaling limits in algebra, geometry, and probability

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Chapter 4

On the determinantal process associated to spanning trees[1]

4.1 Discrete determinantal processes

Let E be a finite or countable set and $X \subset E$ be a random subset of E. Then, one can associate X to a random variable $\tilde{X}$ with values in $\{0,1\}^E$ where $\tilde{X}(e) = 1$ if $e \in X$. The random variable $\tilde{X}$ is called a point process. A point process $\tilde{X}$ is called *determinantal* (abbreviated to the Determinantal Point Process, or DPP) if there exists a function $\mathbb{K}\colon E \times E \to \mathbb{C}$ such that for any finite collection of distinct points $\{e_1, \ldots, e_n\} \in E$, the probability $\mathbb{P}(e_1, \ldots, e_n \in X)$ is given by the following determinant:

$$\mathbb{P}(e_1, \ldots, e_n \in X) = \mathbb{P}(\tilde{X}(e_i) = 1, i = 1, \ldots, n) = \det\big[\mathbb{K}(e_i, e_j)\big]_{i,j=1}^{n}. \quad (4.1)$$

The function $\mathbb{K}$ is called the *correlation kernel* of the determinantal point process $\tilde{X}$.

Analogously, a probability measure $\mathbb{P}$ on $\{0,1\}^E$ is called determinantal if there exists a function $\mathbb{K} : E \times E \to \mathbb{C}$ such that (4.1) is valid for all $\{e_1, \ldots, e_n\} \in E$ where the points e_i are all distinct. A natural question is to determine which functions $\mathbb{K}$ are correlation kernels, i.e., for which $\mathbb{K}$ the expressions in (4.1) define a probability measure on $\{0,1\}^E$. To address to this question we recall the three most common ways to define determinantal measures and their correlation kernels:

1. **Fourier transform:** Suppose $E = \mathbb{Z}^d$ and $f : \mathbb{T}^d \to [0,1]$ is an L^2-

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integrable function. Define the correlation kernel $\mathbb{K}$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ by

$$\mathbb{K}(n,m) = \widehat{f}(n-m), \quad n,m \in \mathbb{Z}^d,$$

where $\widehat{f}(k)$ is the k-th Fourier coefficient of f given by

$$\widehat{f}(k) = \int_{\mathbb{T}^d} f(\theta) e^{-2\pi i \langle k,\theta\rangle} d\theta, k \in \mathbb{Z}^d.$$

The kernel $\mathbb{K}$ defines a stationary translation invariant determinantal measure on $\mathbb{Z}^d$ [72].

2. **Projection:** Consider the Hilbert space $\mathcal{H} = \ell^2(E)$ with the scalar product $\langle \cdot, \cdot \rangle$. Note that $\mathcal{H}$ is generated by the indicator functions $\{\mathbf{1}_n, n \in E\}$ given by

$$\mathbf{1}_n(m) = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Suppose H is a closed subspace of $\mathcal{H}$. Denote by P_H the orthogonal projection operator onto H. Then the kernel $\mathbb{K}(n,m) = \langle P_H \mathbf{1}_n, \mathbf{1}_m \rangle$ defines a determinantal measure.

3. **Positive contraction:** Again, consider the Hilbert space $\ell^2(E)$, and suppose $Q : \ell^2(E) \to \ell^2(E)$ is a positive contraction, i.e.,

$$0 \leqslant \langle Qu, u \rangle \leqslant \langle u, u \rangle \quad \forall u \in \ell^2(E).$$

Then the kernel $\mathbb{K}(n,m) = \langle Q\mathbf{1}_n, \mathbf{1}_m \rangle$, $n,m \in E$, defines a determinantal measure on $\{0,1\}^E$.

Note that here is a natural bijection between the DPPs generated by projections (2) and those generated by positive contractions (3). Clearly, any projection operator is a positive contraction, so, $(2) \subset (3)$. In the other direction, consider a positive contraction A that acts on $\ell^2(E)$; then, the operator

$$\widehat{A} := \begin{pmatrix} A & \sqrt{A(I-A)} \\ \sqrt{A(I-A)} & I-A \end{pmatrix}$$

is a projection operator on $\ell^2(E_1) \oplus \ell^2(E_2)$ where $E_1 = E_2 = E$ since $\widehat{A}$ is idempotent ($\widehat{A}^2 = \widehat{A}$) and self-adjoint ($\widehat{A}^* = \widehat{A}$). Moreover, $\widehat{A}$ is a dilation of A; in other words, $P_{\ell^2(E_1)} \widehat{A} u = Au$ [73] and $P_{\ell^2(E_2)} \widehat{A} u = (I-A)u$. We have shown that each positive contraction A generates a projection operator.

The third class of examples (3) also contains the first class of examples (1). Indeed, since $\ell^2(\mathbb{Z}^d) \cong L^2(\mathbb{T}^d)$ via the Fourier transform, the operator $\widehat{Q} : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ which acts by the formula $\widehat{Q}h = f \cdot h$, is also a positive contraction on $L^2(\mathbb{T}^d)$ since $f : \mathbb{T}^d \to [0,1]$. It follows that the adjoint operator Q on $\ell^2(\mathbb{Z}^d)$ is also a positive contraction on $\ell^2(\mathbb{Z}^d)$.

In the current work we focus on two primary examples of discrete determinantal point processes which are *Uniform Spanning Trees process* and *Uniform Spanning Forest process.*

Let $G = (V, E)$ be a finite connected graph. A *spanning tree* $T \subset E$ is a subset of edges of G such that the graph (V, T) is connected and does not have cycles. Denote by $\mathcal{T}(G)$ the set of all spanning trees of G. Since G is finite, $|\mathcal{T}(G)| < \infty$, and by a well-known Kirkhoff's Matrix Tree theorem, $|\mathcal{T}(G)|$ is equal to the determinant of the (reduced) Laplacian matrix of G. Let us consider the random variable T assuming values in $\mathcal{T}(G)$ with uniform probabilities. Burton and Pemantle [16] have proved the *Transfer Current Theorem,* showing that the edges of uniformly chosen spanning trees form a determinantal process.

Theorem 4.1.1 ([16])**.** Let $G = (V, E)$ be a finite graph. Denote by $\mathbb{P}$ be the uniform spanning tree measure on G which we view as a probability measure on $\{0,1\}^E$. This measure describes the probability that edges $e_1, \dots, e_n$ are present in a random spanning tree which is chosen uniformly. Choose an arbitrary orientation of the edges of the graph G and let $\mathbb{K}(e_i, e_j)$ be equal to the expected signed number of crossings of $e_i = \overrightarrow{xy}$ by a simple random walk on G started at s and stopped when it hits t with $e_j = \overrightarrow{st}$. Then, the uniform spanning tree measure on G is given by the determinant

$$\mathbb{P}(e_1, \dots, e_n \in T) = \det\big[\mathbb{K}(e_i, e_j)\big]_{i,j=1}^n, \tag{4.2}$$

i.e., $\mathbb{P}$ is a determinantal probability measure.

In [89] Pemantle considered uniform spanning trees on certain infinite graphs, namely, on integer lattices $\mathbb{Z}^d$ with $d \geqslant 2$. He showed that the uniform measures $\mathbb{P}_n$ on spanning trees of finite boxes $B_n = [-n, n]^d \cap \mathbb{Z}^d$ converge weakly as $n \to \infty$ to the limiting measure $\mathbb{P}$. If $d \leqslant 4$ the limiting measure $\mathbb{P}$ is concentrated on spanning trees; otherwise, it concentrates on *spanning forests,* i.e., on collections of disjoint spanning trees that span the whole lattice. This limiting measure is referred to as *uniform spanning tree measure* or *uniform spanning forest measure,* respectively.

As discussed above, Burton and Pemantle [16] established the determinantal structure of uniform measures on spanning trees of finite graphs, c.f, Theorem 4.1.1. They also showed that the determinantal structure is present on some infinite graphs, namely, on $\mathbb{Z}^d$ lattices and some $\mathbb{Z}^d$-periodic graphs:

Theorem 4.1.2 ([16])**.** Let $G = (V, E)$ be a $\mathbb{Z}^d$-periodic, D-regular, connected graph. Denote by $\mathbb{P}$ be the uniform spanning forest (USF) measure on G which is viewed as a probability measure on $\{0, 1\}^E$. Then $\mathbb{P}$ is determinantal with the correlation kernel $\mathbb{K}$ given in Theorem 4.1.1.

Let us illustrate the computation of the correlation kernel that features in the two theorems discussed above:

Example 4.1.3. In order to compute the correlation kernel $\mathbb{K}$ for the USF measure on $\mathbb{Z}^d$ explicitly one can use the Green's function $g\colon \mathbb{Z}^d \to \mathbb{R}$ of the simple random walk. The expression of the Green's function depends on the dimension d: for $d = 2$, it is given by

$$\begin{aligned} g(n) &= \sum_{k\geqslant 0} \Big(\mathbb{P}(X_k = n|X_0 = 0) - \mathbb{P}(X_k = 0|X_0 = 0)\Big) \\ &= \int_{\mathbb{T}^2} \frac{e^{-2\pi i\langle n,\theta\rangle} - 1}{1 - \frac{1}{2}\cos 2\pi\theta_1 - \frac{1}{2}\cos 2\pi\theta_2} d\theta, \end{aligned}$$

and for $d \geqslant 3$ it is given by

$$g(n) = \sum_{k\geqslant 0} \mathbb{P}(X_k = n|X_0 = 0) = \int_{\mathbb{T}^d} \frac{e^{-2\pi i\langle n,\theta\rangle}}{1 - \frac{1}{d}\sum_{j=1}^{d}\cos 2\pi\theta_j} d\theta.$$

Then [16, Theorem 4.2], [71, Proposition 10.15], the kernel $\mathbb{K}$ calculated on the edges $e = \overrightarrow{xy}$ and $\tilde{e} = \overrightarrow{zw}$ is given by the expression

$$\mathbb{K}(e, \tilde{e}) = \frac{1}{2d}\Big[g(z - x) - g(z - y) - g(w - x) + g(w - y)\Big]. \tag{4.3}$$

Note that explicit computation of the Green's function for an arbitrary graph G is not trivial, e.g., even for $\mathbb{Z}^2$ one has to modify the standard definition to accommodate for non-integrability of the denominator. Fortunately, in $\mathbb{Z}^2$ the required modification is straightforward, but even for simple graphs the calculation of the Green's function becomes cumbersome. Therefore, it is natural to consider other methods to calculate the correlation kernel $\mathbb{K}$. In the following Section we recall a method presented in [12] that calculates $\mathbb{K}$ as a projection in a certain Hilbert space.

4.1.1 Hilbert space approach

In [12], Benjamini et al. further developed the electrical network approach that we discussed above and obtained a description of the determinantal structures of the USF measures on general finite and infinite graphs in terms of certain projection operators.

Let us first consider a finite connected undirected graph $G = (V, E)$. For each undirected edge $[e] \in E$ select an arbitrary orientation denoted by e; then, we denote the reversed orientation by $-e$. We obtain a collection or oriented edges $\overline{E}$ containing each edge from E with two possible orientations. In other words, $\overline{E} = \cup_{[e]\in E}\{e, -e\}$. Throughout the paper, we will use the following notation. If the collection of edges is denoted by a capital letter $E, F, \ldots$, then the edges are assumed to be undirected; and for the set of all possible directed graphs we will use $\overline{E}, \overline{F}, \ldots$. For $e \in \overline{E}$ denote by $o(e)$ the origin of e and by $t(e)$ the terminus of e. For the reversed edge $-e$ we naturally have $o(-e) = t(e)$ and $t(-e) = o(e)$.

Denote by $\ell^2(V)$ the real Hilbert space of functions $f\colon V \to \mathbb{R}$ with the standard inner product given by

$$\langle f_1, f_2\rangle = \sum_{v\in V} f_1(v) f_2(v). \tag{4.4}$$

Denote by $\ell^2_-(\overline{E})$ the space of antisymmetric (i.e., $\varphi(-e) = -\varphi(e)$) real functions on $\overline{E}$ with the standard inner product given by

$$\langle \varphi_1, \varphi_2\rangle = \frac{1}{2}\sum_{e\in\overline{E}} \varphi_1(e)\varphi_2(e). \tag{4.5}$$

Define the coboundary operator $d\colon \ell^2(V) \to \ell^2_-(\overline{E})$ and the divergence operator $d^*\colon \ell^2_-(\overline{E}) \to \ell^2(V)$ as

$$df(e) = f(o(e)) - f(t(e)), \quad d^*\theta(v) = \sum_{o(e)=v} \theta(e). \tag{4.6}$$

It is easy to check that d and d^* are adjoint: $\langle df, \theta\rangle = \langle f, d^*\theta\rangle$ for all $f \in \ell^2(V)$ and $\theta \in \ell^2_-(\overline{E})$. Consider a subspace $\operatorname{Im} d$ defined as

$$\operatorname{Im} d = d\,\ell^2(V) = \text{span}\{\, d\,\mathbb{1}_v\}_{v\in V} \subset \ell^2_-(\overline{E}).$$

Finally, denote by $P_{\overline{\operatorname{Im} d}}$ the orthogonal projection from $\ell^2_-(\overline{E})$ onto the closed subspace $\overline{\operatorname{Im} d}$.

Theorem 4.1.4 ([12])**.** Let $\mathbb{P}$ be the uniform spanning tree measure for $G = (V, E)$. Then $\mathbb{P}$ is determinantal with

$$\mathbb{P}(e_1, \ldots, e_k \in T) = \det[(P_{\overline{\operatorname{Im} d}} \mathbb{1}_{e_i}, \mathbb{1}_{e_j})]_{i,j=1,\ldots,k},$$

where for $e \in \overline{E}$, $\mathbb{1}_e \in \ell^2_-(\overline{E})$ is an antisymmetric indicator function of the directed edge e (i.e., $\mathbb{1}_e(e) = 1, \mathbb{1}_e(-e) = 1$ and $\mathbb{1}_e(j) = 0$ for all $j \in \overline{E}$ such that $j \neq e, -e$).

As was mentioned above, Pemantle [89] constructed a USF measure $\mathbb{P}$ on $\mathbb{Z}^d$ by considering weak limits of uniform spanning tree measures on finite graphs. This approach can be extended to a larger class of infinite connected locally finite graphs $G = (V, E)$. Consider a sequence of finite subsets of vertices $\{V_n\}$ of V such that $V_1 \subset V_2 \subset \ldots$ and $\cup_1^\infty V_n = V$. Using the sequence $\{V_n\}$ we can define two sequences of finite graphs converging to $G = (V, E)$. Firstly, let $G_n^F = (V_n, E_n)$ be the graph obtained by deleting all vertices in V which lie outside of V_n (free boundary conditions), and secondly, let $G_n^W = (V_n \cup \{\varnothing\}, \tilde{E}_n)$ be the graph obtained by contracting all vertices in V which lie outside of V_n into one vertex $\varnothing$ (wired boundary conditions). Denote by $\mathbb{P}_n^F$ and $\mathbb{P}_n^W$ the corresponding uniform spanning tree measures on $G_n^F = (V_n, E_n)$ and $G_n^W = (V_n \sqcup \{\varnothing\}, \tilde{E}_n)$, respectively. The measures $\mathbb{P}_n^F$ and $\mathbb{P}_n^W$ have the following monotonicity property: for a fixed collection of edges B and for all sufficiently large n, one has

$$\mathbb{P}_n^F(B \subseteq T) \geqslant \mathbb{P}_{n+1}^F(B \subseteq T), \quad \text{and} \quad \mathbb{P}_n^W(B \subseteq T) \leqslant \mathbb{P}_{n+1}^W(B \subseteq T),$$

which allows one to define limiting measures

$$\mathbb{P}^F = \lim_{n\to\infty} \mathbb{P}_n^F, \quad \mathbb{P}^W = \lim_{n\to\infty} \mathbb{P}_n^W,$$

called the *free* and the *wired* uniform spanning forest measures, respectively. In general, the measures $\mathbb{P}^F$ and $\mathbb{P}^W$ do no necessarily coincide.

It turns out that both measures $\mathbb{P}^F$ and $\mathbb{P}^W$ are determinantal with kernels similar to that in Theorem 4.1.4. To describe the corresponding kernels we introduce additional notations. Let us call *stars* the functions of the form $d\mathbb{1}_e$ where $\mathbb{1}_e$ is an indicator function of an oriented edge $e \in \overline{E}$; denote by $\bigstar = \overline{d\ell^2(V)}$ the closed infinite-dimensional subspace of $\ell^2_-(\overline{E})$ spanned by stars [12]. If a collection of oriented edges $(e_1, \ldots, e_n)$ forms an oriented cycle, then a function $\sum_{i=1}^n \mathbb{1}_{e_i}$ is called a *cycle.* Cycles span a subspace of $\ell^2_-(\overline{E})$ denoted by $\lozenge$. The subspaces $\bigstar$ and $\lozenge$ of $\ell^2_-(\overline{E})$ are orthogonal. In [12] it is shown that $\mathbb{P}^W = \mathbb{P}^F$ if and only if $\bigstar \oplus \lozenge = \ell^2_-(\overline{E})$.

Remark 4.1.5. Let G be a graph equipped with an action of an abelian finitely-generated group Γ (see Sections 3 and 4 below). Then $\mathbb{P}^F = \mathbb{P}^W$ [33].

Theorem 4.1.6 ([12])**.** For an infinite connected graph G, the uniform spanning forest measures $\mathbb{P}^F$ and $\mathbb{P}^W$ corresponding to the free and the wired boundary conditions, are determinantal:

$$\mathbb{P}^W(e_1, \ldots, e_k \in T) = \det[\langle P_\bigstar \mathbb{1}_{e_i}, \mathbb{1}_{e_j}\rangle]_{1\leqslant i,j\leqslant k},$$

$$\mathbb{P}^F(e_1, \ldots, e_k \in T) = \det[\langle P_\Diamond^\perp \mathbb{1}_{e_i}, \mathbb{1}_{e_j}\rangle]_{1\leqslant i,j\leqslant k},$$

where $P_\bigstar$ is an orthogonal projection on the closure of $\bigstar$ and $P_\Diamond^\perp$ is an orthogonal projection on the closure of the orthogonal complement of $\Diamond$.

Remark 4.1.7. Note that both $\mathbb{P}^W(e_1, \ldots, e_k \in T)$ and $\mathbb{P}^F(e_1, \ldots, e_k \in T)$ do not depend on the choice of orientation.

Theorem 4.1.6 shows that the correlation kernel can be expressed in terms of projection operators. However, finding explicit expressions of projections in infinite-dimensional Hilbert spaces is a challenging task. In [71, Chapter 4] one can find computations of correlation kernels for lattices $\mathbb{Z}^d$ using graph Laplacians; this proof requires a number of technical steps. The same construction can, in principle, be used for other graphs, however, the proofs become more complex, in particular, for graphs where the simple random walk is recurrent.

In the following section we show how to compute a projection operator $P_\bigstar$ of Theorem 4.1.6 explicitly for graphs with abelian symmetry groups.

4.2 Projection operator

We start the discussion with a well-known fact on computation of projections in finite-dimensional spaces. Let us consider a k-dimensional subspace $W \subset \mathbb{R}^n$ with $k < n$, and let P_W be the orthogonal projection from $\mathbb{R}^n$ onto W. Suppose W is spanned by linearly independent vectors $w_1, \ldots, w_k$.

Denote by A the $n \times k$ matrix, made of column vectors $w_1, \ldots, w_k$:

$$A = [w_1; \ldots; w_k] \in \mathbb{R}^{n\times k}.$$

Since the vectors w_i are linearly independent, the $k \times k$ matrix A^TA is invertible. A standard exercise in Linear Algebra shows that the projection operator P_W is given by the matrix product

$$P_W = A(A^TA)^{-1}A^T \tag{4.7}$$

In view of Theorem 4.1.6, one would be interested whether a similar expression is true in greater generality. The following result is a natural generalization of (4.7) to the infinite-dimensional separable Hilbert spaces.

Theorem 4.2.1. Suppose X, Y are separable Hilbert spaces, and $A\colon X \to Y$ is a bounded linear operator. Let $A^*\colon Y \to X$ be the adjoint operator of A and let $P_A : Y \to Y$ be the orthogonal projection onto the closed subspace $\overline{\operatorname{Im} A} \subset Y$. Then

$$\lim_{\epsilon \to 0} A(A^*A + \epsilon I)^{-1}A^* = P_A,$$

where $I : X \to X$ is the identity operator, and the convergence is understood in the strong operator topology.

We start the proof of Theorem 4.2.1 with the following proposition:

Proposition 4.2.2. Under conditions of Theorem 4.2.1, for any $\epsilon > 0$ the operators $P(\epsilon) : Y \to Y$ given by

$$\mathbb{P}(\epsilon) = A(A^*A + \epsilon I)^{-1}A^*,$$

are well-defined and have uniformly bounded norms.

Proof. The operator $(A^*A+\epsilon I)^{-1}$ is non-negative and self-adjoint, and therefore, has a unique non-negative self-adjoint square root $B(\epsilon) = (A^*A + \epsilon I)^{-1/2}$. Let $C(\epsilon) = AB(\epsilon) = A(A^*A + \epsilon I)^{-1/2}$, then $C^*(\epsilon) = B(\epsilon)A^*$, and since $P(\epsilon) = C(\epsilon)C^*(\epsilon)$, one has

$$\begin{aligned} ||\mathbb{P}(\epsilon)|| &= ||C(\epsilon)C^*(\epsilon)|| = ||C^*(\epsilon)C(\epsilon)|| \\ &= ||(A^*A + \epsilon I)^{-1/2}A^*A(A^*A + \epsilon I)^{-1/2}||, \end{aligned} \tag{4.8}$$

therefore, $||\mathbb{P}(\epsilon)|| = ||g(A^*A)||$ where $g(x) = \frac{x}{x+\epsilon}, \epsilon > 0$. Note that the operator A^*A is bounded and positive-definite: therefore, its spectral radius $\lambda_\infty(A^*A)$ coincides with the norm of A^*A. The spectral mapping theorem implies that $||\mathbb{P}(\epsilon)|| = ||g(A^*A)|| = g(\lambda_\infty(A^*A)) < 1$; one concludes that $||\mathbb{P}(\epsilon)|| < 1$ for every $\epsilon > 0$. □

Remark 4.2.3. The operator $P(\epsilon), \epsilon > 0$ is a positive contraction; for discrete (finite or countable) Y it defines a determinantal point process with the correlation kernel $\mathbb{K}_\epsilon(e_i, e_j) = \langle P_\epsilon \mathbb{1}_{e_i}, \mathbb{1}_{e_j}\rangle, e_i, e_j \in Y$.

We present the reader with two proofs of Theorem 4.2.1:

Analytic proof of Theorem 4.2.1. Let us start by showing that $\lim_{\epsilon\to 0}\mathbb{P}(\epsilon)Ax = Ax$ for every $x \in X$. Applying $\mathbb{P}(\epsilon)$ to Ax, we see that

$$\begin{aligned}\mathbb{P}(\epsilon)Ax &= A(A^*A+\epsilon I)^{-1}A^*Ax = A(A^*A+\epsilon I)^{-1}[(A^*A+\epsilon I)-\epsilon I]x \\ &= Ax - \epsilon A(A^*A+\epsilon I)^{-1}x.\end{aligned} \tag{4.9}$$

However, the norm of an operator $\epsilon A(A^*A+\epsilon I)^{-1}$ can be bounded as follows:

$$\begin{aligned}||\epsilon A(A^*A+\epsilon I)^{-1}|| &= ||\epsilon^{1/2}A(A^*A+\epsilon I)^{-1/2}\epsilon^{1/2}(A^*A+\epsilon I)^{-1/2}|| \\ &\leqslant \epsilon^{1/2}||A(A^*A+\epsilon I)^{-1/2}|| \cdot ||\epsilon^{1/2}(A^*A+\epsilon I)^{-1/2}||.\end{aligned} \tag{4.10}$$

The norm of the operator $C(\epsilon) = A(A^*A+\epsilon I)^{-1/2}$ is bounded. Indeed, since $C^*(\epsilon)C(\epsilon) = P(\epsilon)$, one has $||C(\epsilon)||^2 = ||C^*(\epsilon)C(\epsilon)|| = ||\mathbb{P}(\epsilon)|| \leqslant 2$ by Proposition 4.2.2. The norm of the operator $\epsilon^{1/2}(A^*A+\epsilon I)^{-1/2}$ is bounded by 1. Therefore, $||\epsilon A(A^*A+\epsilon I)^{-1}|| \leqslant \epsilon^{1/2}\cdot||C(\epsilon)||\cdot 1 \leqslant \sqrt{2\epsilon} \to 0$ as $\epsilon \to 0$. Thus, by (4.9)

$$\mathbb{P}(\epsilon)Ax - Ax = \epsilon A(A^*A+\epsilon I)^{-1}x, \text{ with } ||\epsilon A(A^*A+\epsilon I)^{-1}|| \leqslant \sqrt{2\epsilon}$$

Hence, $\mathbb{P}(\epsilon)Ax \to Ax$, and thus $\mathbb{P}(\epsilon) \to I$ on $\overline{\mathrm{Im}A}$. Moreover, $\mathbb{P}(\epsilon) = 0$ on $(\overline{\mathrm{Im}A})^{\perp}$. Therefore, one concludes that

$$\lim_{\epsilon\to 0} A(A^*A+\epsilon I)^{-1}A^* = \mathrm{Proj}_{\overline{\mathrm{Im}A}} = P_A.$$

□

Spectral proof of Theorem 4.2.1. Note that $Y = \overline{\mathrm{Im}A} \oplus (\overline{\mathrm{Im}A})^{\perp}$. Let us consider a sequence of operators $A(A^*A+\epsilon I)^{-1}A^*Ax$ and prove that it converges to Ax.

Each self-adjoint operator $K\colon H \to H$ on a separable Hilbert space H admits a sum representation $K = \sum_{m\in\mathbb{N}} q_m\langle x, w_m\rangle w_m$ where $\{w_m\}_{m\in\mathbb{N}}$ is a countable orthonormal basis of H and $\{q_m\}_{m\in\mathbb{N}}$ is the spectrum of K. Note that if $K = A^*A$ for some operator A then its spectrum is non-negative and real. Then, one can define a family of spectral projections $\{E_{\lambda_n}\}_{n\in[0,\infty)}$ by

$$E_{\lambda_n}(K)(x) = \sum_{q_m\in(-\infty,\lambda_n]} q_m\langle x, w_m\rangle w_m.$$

This family is called the resolution of unity of the operator K; it implies that $K = \int_{-\infty}^{\infty}\lambda dE_\lambda(K)$. If K is non-negative, clearly, $K = \int_0^{\infty}\lambda dE_\lambda(K)$.

Denote by $\mathrm{Spec} A^*A \subset \mathbb{R}$ the spectrum of the operator A^*A (note that the spectrum of a self-adjoint operator is real). Then, the spectrum of the inverse operator $\mathrm{Spec}(A^*A)^{-1}$ on its range $\mathrm{Im} A^*A$ is equal to $(\mathrm{Spec} A^*A \backslash \{0\})^{-1}$. Moreover, if w is an eigenvector of A^*A with a positive eigenvalue λ_e then w is an eigenvector of $(A^*A+\epsilon I)^{-1}$ with an eigenvalue $\frac{1}{\lambda+\epsilon}$. We conclude that

$$\mathrm{Spec}(A^*A+\epsilon I)^{-1}A^*A = \left\{\frac{\lambda}{\lambda+\epsilon}\right\}_{\lambda \in \mathrm{Spec} A^*A}.$$

By the property of the resolution of unity it follows that on $(\ker A^*A)^{\perp}$

$$(A^*A+\epsilon)^{-1}A^*A = \int_{(0,\infty)} \frac{\lambda}{\lambda+\epsilon} dE_\lambda(A^*A).$$

If $x \in \ker A^*A$ then $(A^*A+\epsilon)^{-1}A^*A(x) = 0$. Therefore, on the whole H_1,

$$A(A^*A+\epsilon)^{-1}A^*A = A\int_{(0,\infty)} \frac{\lambda}{\lambda+\epsilon} dE_\lambda(A^*A).$$

Note that $\ker A^*A = \ker(A^*A)^{-1} = \ker A$. We leave the proof of this fact as an exercise for the reader. Then, $A(E_{\{0\}}(A^*A)) = 0$; so,

$$A(A^*A+\epsilon)^{-1}A^*A = A\int_{[0,\infty)} \frac{\lambda}{\lambda+\epsilon} dE_\lambda(A^*A).$$

When $\epsilon \to 0$, the expression $\frac{\lambda}{\lambda+\epsilon} \to 1$, so

$$\lim_{\epsilon\to 0} A(A^*A+\epsilon)^{-1}A^*A = A\int_0^\infty dE_\lambda(A^*A).$$

By the definition of the resolution of unity, $\int_0^\infty dE_\lambda(A^*A) = I$. Then, for $y = Ax+z, x \in H_1, z \in (\mathrm{Im} A)^\perp$, it holds that

$$\lim_{\epsilon\to 0} A(A^*A+\epsilon)^{-1}A^*y = Ax.$$

It follows that for any $y \in H_2$,

$$\lim_{\epsilon\to 0} A(A^*A+\epsilon)^{-1}A^*y = P_A y,$$

where $P_A(y)$ is a projection of y onto $\overline{\mathrm{Im}(A)}$.

□

In the following sections, we will apply Theorem 4.2.1 to computation of projections in relation to Theorem 4.1.6. Namely, we will show that for infinite graphs with *symmetry* one can obtain explicit expressions of correlation kernels.

4.3 Graphs with abelian symmetries

In this section we discuss infinite graphs that possess sufficiently rich symmetry groups, for instance, periodic graphs. Consider a connected unoriented graph $G = (V, E)$ where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of unoriented edges. We remind the reader of the notation used in Section 1: $\overline{E}$ is the set of oriented edges: each unoriented edge $[e] \in E$ is included with two possible orientations $e, -e$ into $\overline{E}$; $o(e)$ and $t(e)$ are the origin and the terminus of $e \in \overline{E}$, respectively.

Definition 4.3.1. A mapping $g : G \to G$ consists of a pair of mappings $g_V : V \to V$ and $g_{\overline{E}} : \overline{E} \to \overline{E}$. The mapping $g : G \to G$ is called a *graph automorphism* if

1. $o(g_{\overline{E}}(e)) = g_V(o(e))$, i.e., the origin of the edge is mapped onto the origin of the edge,

2. $t(g_{\overline{E}}(e)) = g_V(t(e))$, i.e., the terminus of the edge is mapped onto the terminus of the edge,

3. $-g_{\overline{E}}(e) = g_{\overline{E}}(-e)$ $(e \in \overline{E})$, i.e., the image of the reverse of an edge is the reverse of the image of the edge.

It is easy to check that due to the conditions above, any graph automorphism $g : G \to G$ naturally induces a mapping on undirected edges $g_E : E \to E$ on E by $g_E([e]) := [g_{\overline{E}}(e)]$

Let Γ be a discrete countable group which acts on a graph $G = (V, E)$ by graph automorphisms:

$$\Gamma \ni \gamma \mapsto g^\gamma = (g^\gamma_V, g^\gamma_E) \in \mathsf{Aut}(G).$$

Definition 4.3.2. A pair $(G_0 = (V_0, E_0), \pi : G \to G_0)$ is called *a quotient of* G *with respect to* Γ denoted by G/Γ if for every $v_0 \in V_0$, $\pi^{-1}(v_0) = \{g^\gamma_V(v) | \gamma \in \Gamma, v \in \pi^{-1}(v_0)\}_\gamma := \Gamma v$ and for every $e_0 \in E_0$, $\pi^{-1}(e_0) = \{g^\gamma_E(e) | \gamma \in \Gamma, e \in \pi^{-1}(e_0)\} := \Gamma e$. In other words, π is a covering map of G_0, i.e., for every $\gamma \in \Gamma$ one has $\pi \circ g^\gamma = \pi$.

From now on, we will simplify the notation and use the same letter γ to denote the graph automorphism g^γ corresponding to $\gamma \in \Gamma$.

Let $G = (V, E)$ be an infinite graph and let Γ be a finitely generated abelian group of automorphisms of G that satisfies the following conditions:

1. The group Γ acts freely on G: namely, $\gamma v \neq v$ and $\gamma e \neq \pm e$ unless $\gamma = \mathsf{id}$.
2. The quotient graph G/Γ is a finite graph $G_0 = (V_0, E_0)$.

We would need one more algebraic object associated with a countable additive group Γ : namely, the group ring $\mathbb{Z}\Gamma$, which we view as the ring of Laurent polynomials in variable $\boldsymbol{x}$. This ring is formed by all expressions of the form

$$f = \sum_{\gamma \in \Gamma} f_\gamma \cdot \boldsymbol{x}^\gamma,$$

where $f_\gamma \in \mathbb{Z}$ for all $\gamma \in \Gamma$ are such that the set $\{\gamma \in \Gamma : f_\gamma \neq 0\}$ is finite. For the Laurent polynomial $f = \sum_{\gamma \in \Gamma} f_\gamma \cdot \boldsymbol{x}^\gamma$, the adjoint of f is defined as $f^* = \sum_{\gamma \in \Gamma} f_\gamma \cdot \boldsymbol{x}^{-\gamma}$. The sum of two polynomials $f = \sum_{\gamma \in \Gamma} f_\gamma \cdot \boldsymbol{x}^\gamma$ and $h = \sum_{\gamma \in \Gamma} h_\gamma \cdot \boldsymbol{x}^\gamma$ is defined as $f + h = \sum_{\gamma \in \Gamma}(f_\gamma + h_\gamma) \cdot \boldsymbol{x}^\gamma$, and the product is defined as $f \cdot h = \sum_{g \in \Gamma}(f \cdot h)_\gamma \cdot \boldsymbol{x}^\gamma$ with

$$(f \cdot h)_\gamma = \sum_{\gamma' \in \Gamma} f_{\gamma - \gamma'} h_{\gamma'}.$$

We will write 1 for $\boldsymbol{x}^0$.

Let $\mathscr{F} = (\mathscr{F}_V, \mathscr{F}_{\overline{E}})$ be an arbitrary fundamental set of G with respect to the Γ-action. It means that

$$V = \coprod_{\gamma \in \Gamma} \gamma(\mathscr{F}_V), \quad \overline{E} = \coprod_{\gamma \in \Gamma} \gamma(\mathscr{F}_{\overline{E}}).$$

Let us fix an orientation $\mathscr{F}_{E^o}$ on $\mathscr{F}_E$: note that it defines an orientation E^0 on E. Take $\pi \colon G \to \mathscr{F}$ as a covering map. For any $e \in \mathscr{F}_{E^o}$ fix $v, v' \in \mathscr{F}_V, e' \in \overline{E}$ and $\gamma \in \Gamma$ (γ can be $\mathbf{1}$) such that $\pi(e') = e$, $o(e') = v \in \mathscr{F}_V$ and $t(e') = \gamma v' \in V$ (naturally, $\mathscr{F}_{\overline{E}} = \mathscr{F}_{E^o} \sqcup -\mathscr{F}_{E^o}$). Let us define an $|E_0| \times |V_0|$ "edge-vertex" incidence matrix $\partial_{\mathscr{F}} = (\partial_{\mathscr{F}_{E^o}}(e, v))$ where $e \in \mathscr{F}_{E^o}, v \in \mathscr{F}_V$ and its adjoint $\partial^*_{\mathscr{F}} = (\partial^*_{\mathscr{F}}(v, e))$ where $v \in \mathscr{F}_V, e \in \mathscr{F}_{E^o}$, by the formulas

$$\begin{aligned} \partial_{\mathscr{F}}(e, v) &= \mathbb{1}_v(o(e')) - \sum_{\gamma \in \Gamma \colon \gamma v = t(e')} \boldsymbol{x}^\gamma, \\ \partial^*_{\mathscr{F}}(v, e) &= \mathbb{1}_v(o(e')) - \sum_{\gamma \in \Gamma \colon \gamma v = t(e')} \boldsymbol{x}^{-\gamma}. \end{aligned} \tag{4.11}$$

Remark 4.3.3. The size of the sets $\mathscr{F}_V, \mathscr{F}_{E^o}$ does not depend on $\mathscr{F}$; therefore, the size of the matrices $\partial_{\mathscr{F}}, \partial^*_{\mathscr{F}}$ also does not depend on the choice of $\mathscr{F}$. The set $\mathscr{F}_V$ contains exactly one element of each orbit of Γ acting on

V; the same holds for $\mathscr{F}_{E^o}$ with respect to E^0. One can check that the expression $\mathbb{1}_v(o(e')) - \sum_{\gamma\in\Gamma\colon\, \gamma v = t(e')} \boldsymbol{x}^\gamma$ depends only on the orbits containing v and e, but not explicitly on the choice of elements; therefore, the matrices $\partial_{\mathscr{F}}, \partial^*_{\mathscr{F}}$ depend on $\mathscr{F}$ only up to reordering of elements in $\mathscr{F}_V$ and $\mathscr{F}_{E^o}$. Therefore, hereafter we sometimes omit the index $\mathscr{F}$ and simply write ∂, ∂^*.

Definition 4.3.4. The graph Laplacian of $\mathscr{F}$ is a $|V_0| \times |V_0|$ matrix with elements in $\mathbb{Z}\Gamma$ given by

$$\Delta_{\mathscr{F}} = \partial^*_{\mathscr{F}} \partial_{\mathscr{F}}.$$

The entries of $\Delta_{\mathscr{F}}$ are given by

$$\begin{aligned}\Delta_{\mathscr{F}}(i,j) &= \deg v_i \mathbb{1}_{v_i}(v_j) \\ &\quad - \sum_{e\in\mathscr{F}_{E^o}, \gamma\in\Gamma} \boldsymbol{x}^\gamma \left(\mathbb{1}_{v_i}(o(e))\mathbb{1}_{\gamma v_j}(t(e)) + \mathbb{1}_{v_j}(o(e))\mathbb{1}_{-\gamma v_i}(t(e))\right) \\ &= \deg v_i \mathbb{1}_{v_i}(v_j) - \sum_{\gamma\in\Gamma\colon\, v_i\sim\gamma v_j} \boldsymbol{x}^\gamma.\end{aligned}$$

The choice of another fundamental domain $\mathscr{F}$ will correspond to the change of basis of matrix $\Delta_{\mathscr{F}}$ and, therefore, the determinant $\det \Delta_{\mathscr{F}}$ does not depend on the choice of $\mathscr{F}$.

4.3.1 Examples

In order to demonstrate the computation of the introduced notions, we will now consider a number of simple examples.

Example 4.3.5 ($\mathbb{Z}^2$-lattice, Figure 4.1)**.** For the lattice $\mathbb{Z}^2$ the quotient graph consists of one vertex and two loops corresponding to two basis vectors in $\mathbb{Z}^2$. Thus, $\mathscr{F}_V = (0,0)$ and $\mathscr{F}_{E^o} = \{e_1 = \{(0,0),(1,0)\}, e_2 = \{(0,0),(0,1)\}\}$.

Therefore,

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1 - \boldsymbol{x}^{(1,0)} \\ 1 - \boldsymbol{x}^{(0,1)} \end{pmatrix}, \quad \partial^*_{\mathscr{F}} = \begin{pmatrix} 1 - \boldsymbol{x}^{-(1,0)}, & 1 - \boldsymbol{x}^{-(0,1)} \end{pmatrix}$$

If we take $x_1 = \boldsymbol{x}^{(1,0)}$ and $x_2 = \boldsymbol{x}^{(0,1)}$, then for any $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, we can write $\boldsymbol{x}^{\mathbf{n}}$ as $x_1^{n_1} x_2^{n_2}$ (multi-index notation), and then the incidence matrices

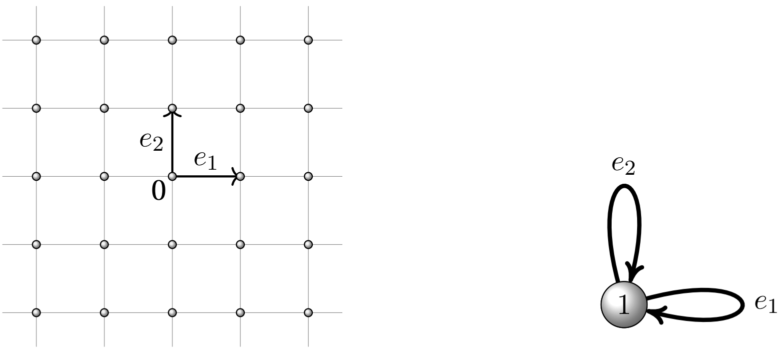

Figure 4.1: $\mathbb{Z}^2$-lattice and the corresponding quotient graph

can be rewritten as

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1-x_1 \\ 1-x_2 \end{pmatrix}, \quad \partial^*_{\mathscr{F}} = \begin{pmatrix} 1-x_1^{-1} & 1-x_2^{-1} \end{pmatrix}.$$

Therefore, the Laplacian is

$$\Delta_{\mathscr{F}} = \partial^*_{\mathscr{F}}\partial_{\mathscr{F}} = \sum_{i=1}^{2}(1-x_i^{-1})(1-x_i) = 4-(x_1+x_1^{-1}+x_2+x_2^{-1}).$$

Example 4.3.6 (Ladder graph, Figure 4.2)**.** We consider the ladder graph $G=(V,E)$ where $V=\mathbb{Z}\times B$ with $B=\{1,2\}$, hence $V=\{(k,1),(k,2):k\in\mathbb{Z}\}$ and $E^o=\{e_{k,p}, p=1,2,3, k\in\mathbb{Z}\}$. Here for $k\in\mathbb{Z}$, the edges are

$$e_{k,1}=\{(k,1),(k+1,1)\},\ e_{k,2}=\{(k,2),(k+1,2)\},\ e_{k,3}=\{(k,1),(k,2)\}.$$

Modulo the $\mathbb{Z}$-symmetry, one has $\mathscr{F}_V=\{1,2\}$ and $\mathscr{F}_{E^o}=\{e_1,e_2,e_3\}$.

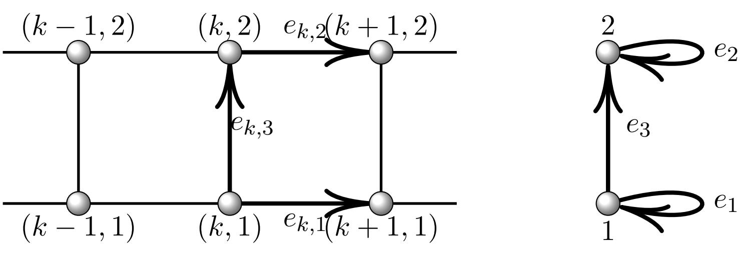

Figure 4.2: Ladder graph and its quotient.

Therefore,

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1-x & 0 \\ 0 & 1-x \\ 1 & -1 \end{pmatrix},\ \partial^*_{\mathscr{F}} = \begin{pmatrix} 1-x^{-1} & 0 & 1 \\ 0 & 1-x^{-1} & -1 \end{pmatrix},$$

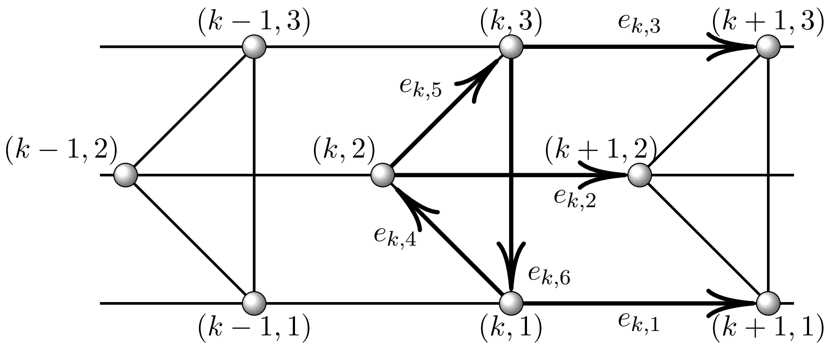

Figure 4.3: A three-ladder graph

$$\text{and } \Delta_{\mathscr{F}} = \begin{pmatrix} 3-x-x^{-1} & -1 \\ -1 & 3-x-x^{-1} \end{pmatrix}.$$

Example 4.3.7 (Three-ladder graph, Figure 4.3)**.** We consider the three-ladder graph $G=(V,E)$ (see Figure 4.3) where $V=\mathbb{Z}\times B$ with $B=\{1,2,3\}$ and $E^o=\{(e_{k,p}, p=\{1,2,3\}, k\in\mathbb{Z}\}$ where

$$e_{k,1}=\{(k,1),(k+1,1)\}, e_{k,2}=\{(k,2),(k+1,2)\}, e_{k,3}=\{(k,3),(k+1,3)\},$$

$$e_{k,4}=\{(k,1),(k,2)\}, e_{k,5}=\{(k,2),(k,3)\}, e_{k,6}=\{(k,3),(k,1)\}.$$

Let $\mathscr{F}^1_V=\{(0,1),(0,2),(0,3)\}$ and $\mathscr{F}^1_{E^o}=\{e_{0,1},e_{0,2},e_{0,3},e_{0,4},e_{0,5},e_{0,6}\}$. Then (see Figure 4.4)

$$\partial_{\mathscr{F}^1} = \begin{bmatrix} 1-x & 0 & 0 \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

$$\partial^*_{\mathscr{F}^1} = \begin{bmatrix} 1-x^{-1} & 0 & 0 & 1 & 0 & -1 \\ 0 & 1-x^{-1} & 0 & -1 & 1 & 0 \\ 0 & 0 & 1-x^{-1} & 0 & -1 & 1 \end{bmatrix},$$

and

$$\Delta_{\mathscr{F}^1} = \begin{bmatrix} 4-x-x^{-1} & -1 & -1 \\ -1 & 4-x-x^{-1} & -1 \\ -1 & -1 & 4-x-x^{-1} \end{bmatrix}.$$

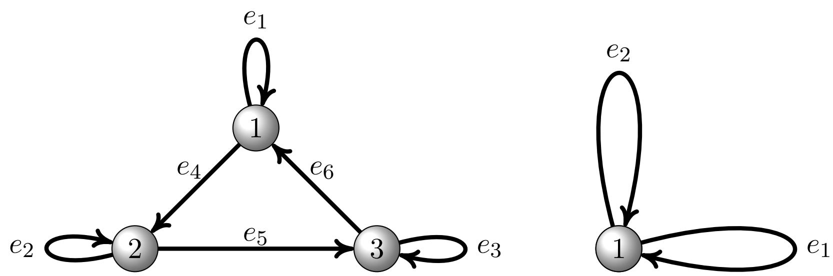

Figure 4.4: $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ quotients of the three-ladder graph

The three-ladder graph also admits an action of a group $\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$: let $\mathscr{F}_V^2 = \{(0,1)\}$ and $\mathscr{F}_{E^o}^2 = \{e_{0,1}, e_{0,4}\}$ (see Figure 4.4). Elements of $\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ can be written as (n,k), $n\in\mathbb{Z}$ and $k\in\mathbb{Z}/3\mathbb{Z}$.

Let $x_1 = x^{(1,0)}$, $x_2 = x^{(0,1)}$. Then $x^{(n,k)} = x_1^n x_2^k$, and

$$\partial_{\mathscr{F}^2} = \begin{bmatrix} 1-x_1 \\ 1-x_2 \end{bmatrix}, \quad \partial^*_{\mathscr{F}^2} = \begin{bmatrix} 1-x_1^{-1} & 1-x_2^{-1} \end{bmatrix}, \text{ and } \Delta_{\mathscr{F}} = 4-x_1-x_1^{-1}-x_2-x_2^{-1}.$$

Example 4.3.8 (Triangular graph, Figure 4.5)**.** We consider the triangular graph $G=(V,E)$ where $V=\mathbb{Z}\times B$ with $B=\{1,2\}$ and $E^o=\{e_{k,p}, p = 1,2,3, k\in\mathbb{Z}\}$ where for $k\in\mathbb{Z}$,

$$e_{k,1} = \{(k,2),(k+1,2)\},\ e_{k,2} = \{(k,2),(k,1)\},\ e_{k,3} = \{(k,1),(k+1,2)\}.$$

Let $\mathscr{F}_V = \{(0,1),(0,2)\}$ and $\mathscr{F}_{E^o} = \{e_{0,1}, e_{0,2}, e_{0,3}\}$ (see Figure 4.5). Denote by x the horizontal translation by $(1,0)$. Then,

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1-x^{-1} & 1 & x^{-1} \\ 0 & -1 & -1 \end{pmatrix}, \quad \Delta_{\mathscr{F}} = \begin{pmatrix} 4-x-x^{-1} & -1-x^{-1} \\ -1-x & 2 \end{pmatrix}.$$

Example 4.3.9 (Kagome lattice, Figure 4.6)**.**

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1 & -1 & 0 & 1 & -y^{-1} & 0 \\ -1 & 0 & -x & -x & 0 & -y^{-1} \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$\Delta_{\mathscr{F}} = \begin{pmatrix} 4 & -1-x^{-1} & -1-y^{-1} \\ -1-x & 4 & -x-y^{-1} \\ -1-y & -x^{-1}-y & 4 \end{pmatrix}$$

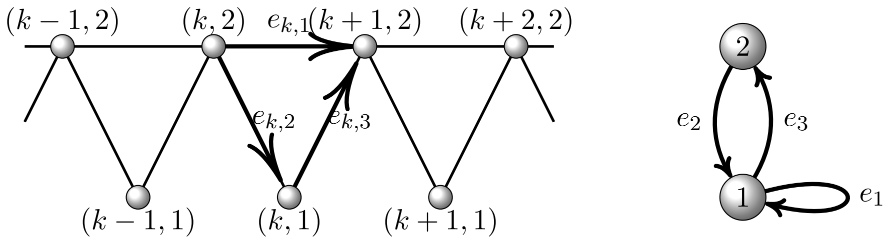

Figure 4.5: A triangular graph with $\mathbb{Z}$ action and the corresponding quotient graph

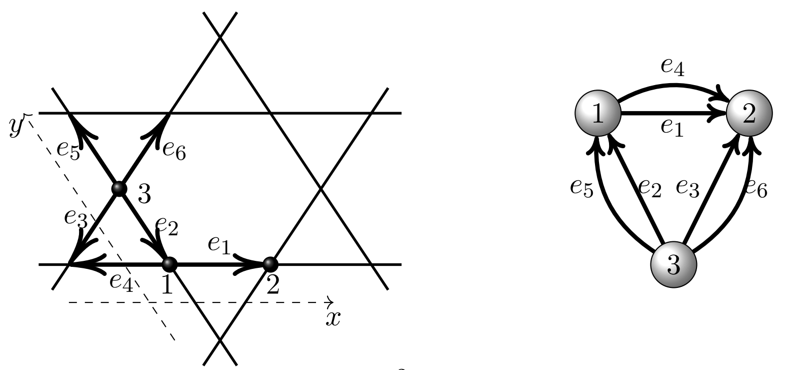

Figure 4.6: A Kagome lattice with $\mathbb{Z}^2$ action and the corresponding quotient graph

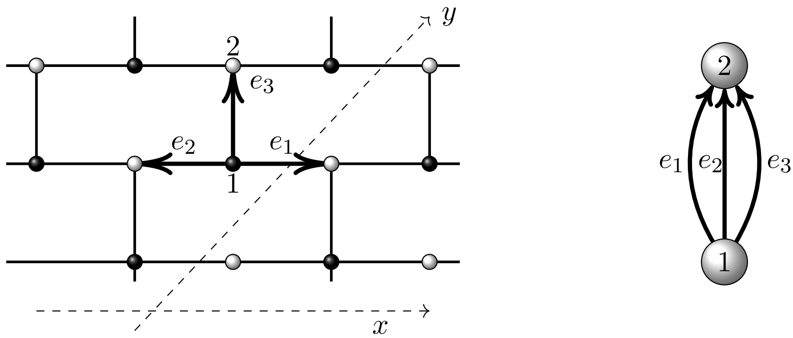

Figure 4.7: A hexagonal lattice with $\mathbb{Z}^2$ action and the corresponding quotient graph

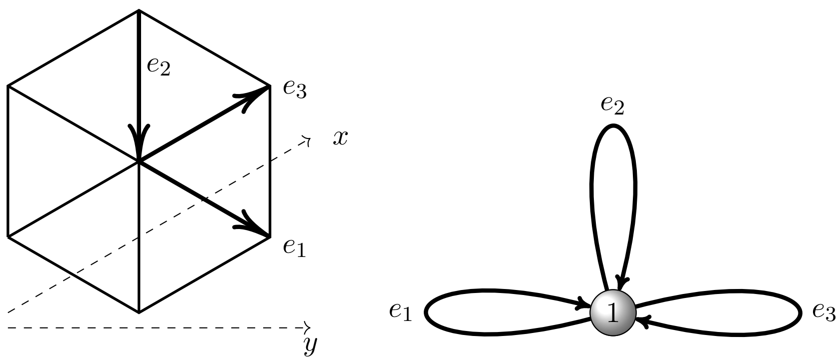

Figure 4.8: A triangular lattice with $\mathbb{Z}^2$ action and the corresponding quotient graph

Example 4.3.10 (Hexagonal lattice, Figure 4.7)**.**

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1 & 1 & 1 \\ -x^{-1}y & -1 & -y \end{pmatrix}, \quad \Delta_{\mathscr{F}} = \begin{pmatrix} 3 & -1-y^{-1}-xy^{-1} \\ -1-y-x^{-1}y & 3 \end{pmatrix}$$

Example 4.3.11 (Triangular lattice, Figure 4.8)**.**

$$\partial_{\mathscr{F}} = \begin{pmatrix} 1 & 1 & 1 \\ -x^{-1}y & -1 & -y \end{pmatrix}, \quad \Delta_{\mathscr{F}} = \begin{pmatrix} 3 & -1-y^{-1}-xy^{-1} \\ -1-y-x^{-1}y & 3 \end{pmatrix}$$

4.4 Explicit expressions of correlation kernels

In this section we show how to explicitly compute the correlation kernel using Theorem 4.2.1. Our method applies to infinite connected graphs with a finitely generated abelian symmetry group Γ. The graphs are assumed to be locally finite, i.e., for any $v \in V$, $\deg v < \infty$.

4.4.1 Duality and Fourier transform

Suppose Γ is a countable finitely generated abelian group. Then Γ is a finite product of infinite and finite cyclic groups: for some $d \geqslant 1$, and $k_1, \dots, k_m \in \mathbb{N}$, we have

$$\Gamma = \mathbb{Z}^d \times \mathbb{Z}_{k_1} \times \dots \mathbb{Z}_{k_m}.$$

Let us denote by $\Gamma_0 = \mathbb{Z}_{k_1} \times \dots \mathbb{Z}_{k_m}$ the finite torsion part of Γ (note that the representation of the torsion part is not unique). The elements of Γ will be written as $\boldsymbol{n} = (n, \ell)$, where $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, $\ell = (\ell_1, \dots, \ell_m) \in \Gamma_0$, with $0 \leqslant l_j < k_j$ for all $j = 1, \dots, m$. The dual group $\widehat{\Gamma}$ is also easy to describe:

$$\widehat{\Gamma} = \mathbb{T}^d \times \mathbb{Z}_{k_1} \times \dots \mathbb{Z}_{k_m},$$

where $\mathbb{T}^d = [0,1)^d$. We denote elements of $\widehat{\Gamma}$ by $\boldsymbol{\theta} = (\theta, \varphi)$, $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, and $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_j \in \mathbb{T}$ are such that $k_j \varphi_j = 0$. The corresponding character χ is then

$$\chi_{\boldsymbol{\theta}}(\boldsymbol{n}) = \chi_{(\theta,\varphi)}(n, \ell) = \exp\Big(2\pi i\big(\langle n, \theta\rangle + \langle \ell, \varphi\rangle\big)\Big),$$

$$\langle n, \theta\rangle = \sum_{t=1}^{d} n_t \theta_t, \quad \langle \ell, \varphi\rangle = \sum_{s=1}^{m} \ell_s \varphi_s.$$

We equip $\widehat{\Gamma}$ with the normalized Haar measure $\boldsymbol{\lambda} = \lambda \times \rho$ which is a product of the Lebesgue measure $\lambda = d\theta$ on $\mathbb{T}^d$ and the uniform probability measure ρ on $\widehat{\Gamma}_0 \cong \Gamma_0$.

Consider an unoriented graph $G = (V, E)$, and suppose Γ acts on G by graph automorphisms freely and such that the quotient $G/\Gamma = G_0 = (V_0, E_0)$ is finite. Then, any $v \in V$ can be presented as $v = \mathbf{n} \cdot v_0$ where $v_0 \in V_0$ and $\mathbf{n} \in \Gamma$. Fix an arbitrary orientation E_0^0 for edges in E_0, and let $\overline{E_0} = E_0^0 \sqcup -E_0^0$.

As earlier, consider the Hilbert space $\ell^2(V)$ of square-summable functions on the vertices V equipped with the scalar product

$$\langle f_1, f_2\rangle = \sum_{v \in V} f_1(v)\overline{f_2(v)} = \sum_{v_0 \in V_0} \sum_{\boldsymbol{n} \in \Gamma} f_1(\boldsymbol{n} v_0)\overline{f_2(\boldsymbol{n} v_0)},$$

and the Hilbert space $\ell^2_-(\overline{E})$ of square-summable antisymmetric functions on the oriented edges $\overline{E}$ equipped with a scalar product

$$\langle g_1, g_2\rangle = \frac{1}{2} \sum_{e \in \overline{E}} g_1(e)\overline{g_2(e)}.$$

Definition 4.4.1. A Fourier transform of $f \in \ell^2(V)$ is a function $\mathbb{F}f\colon \widehat{\Gamma} \to \mathbb{C}^{|V_0|}$ given by

$$\mathbb{F}f(\boldsymbol{\theta}, v_0) = \sum_{\boldsymbol{n}\in\Gamma} f(\boldsymbol{n}v_0)\chi_{\boldsymbol{n}}(\boldsymbol{\theta}) = \sum_{(n,l)\in\Gamma} f\big((n,l)v_0\big)e^{2\pi i(\langle n,\theta\rangle + \langle l,\varphi\rangle)}.$$

Note that for $f \in \ell^2(V)$ one has $\mathbb{F}f \in L^2(\widehat{\Gamma}, \mathbb{C}^{|V_0|})$. The space $L^2(\widehat{\Gamma}, \mathbb{C}^{|V_0|})$ is equipped with the inner product

$$\begin{aligned}\langle F_1, F_2\rangle &= \sum_{v_0\in V_0} \int_{\widehat{\Gamma}} F_1(\boldsymbol{\theta}, v_0)\overline{F_2(\boldsymbol{\theta}, v_0)}\boldsymbol{\lambda}(d\boldsymbol{\theta}) \\ &= \sum_{v_0\in V_0, \varphi\in\Gamma_0} \frac{1}{|\Gamma_0|}\int_{\mathbb{T}^d} F_1((\theta,\varphi), v_0)\overline{F_2((\theta,\varphi), v_0)}\lambda(d\theta),\end{aligned}$$

where $\boldsymbol{\lambda}(d\boldsymbol{\theta})$ is the normalized Haar measure on $\widehat{\Gamma}$ and $\lambda(d\theta)$ is the normalised Lebesgue measure on $\mathbb{T}^d$.

Definition 4.4.2. The inverse Fourier transform $\mathbb{F}^{-1}$ of $\widehat{f} \in L^2(\widehat{\Gamma}, \mathbb{C}^{|V_0|})$ is given by

$$\begin{aligned}\mathbb{F}^{-1}\widehat{f}(\boldsymbol{n}, v_0) &= \int_{\widehat{\Gamma}} \widehat{f}(\boldsymbol{\theta}, v_0)\chi_{\boldsymbol{n}}^{-1}(\boldsymbol{\theta})\boldsymbol{\lambda}(d\boldsymbol{\theta}) \\ &= \frac{1}{|\Gamma_0|}\sum_{\varphi\in\Gamma_0}\int_{\mathbb{T}^d} \widehat{f}\big((\theta,\varphi),\ v_0\big)e^{-2\pi i(\langle n,\theta\rangle + \langle l,\varphi\rangle)}\lambda(d\theta).\end{aligned}$$

Note that since for each $v \in V$, there are unique $v_0 \in V_0, \mathbf{n} \in \Gamma$ such that $v = \mathbf{n}v_0$, we can view $\mathbb{F}^{-1}\widehat{f}$ as a function on V:

$$\mathbb{F}^{-1}\widehat{f}(\mathbf{n}v_0) = \mathbb{F}^{-1}\widehat{f}(\mathbf{n}, v_0).$$

Suppose $f \in \mathbb{Z}\Gamma$, $f = \sum_{\mathbf{n}\in\Gamma} f_{\mathbf{n}} \cdot \boldsymbol{x}^{\mathbf{n}}$, we define the Fourier transform of the Laurent polynomial f as

$$\widehat{f}(\boldsymbol{\theta}) = \sum_{\mathbf{n}\in\Gamma} f_{\mathbf{n}}\chi_{\boldsymbol{\theta}}(\mathbf{n}),$$

and similarly for matrices: if $Q = (Q_{jk})$ with $Q_{jk} \in \mathbb{Z}\Gamma$, then

$$\widehat{Q}(\boldsymbol{\theta}) = (\widehat{Q}_{jk}(\boldsymbol{\theta})).$$

Theorem 4.4.3. Let $G = (V, E)$ be a locally finite graph, and Γ is a finitely generated abelian group acting freely on G by graph automorphisms, and such that $G_0 = (V_0, E_0) = G/\Gamma$ is a finite graph. Consider the following $|E_0| \times |E_0|$ matrix valued function on $\widehat{\Gamma}$

$$K(\boldsymbol{\theta}) = \lim_{\epsilon \to 0} D(\boldsymbol{\theta})\Big(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta}) + \epsilon I\Big)^{-1} D^*(\boldsymbol{\theta}),$$

where $D(\boldsymbol{\theta}), D^*(\boldsymbol{\theta})$ are the Fourier transforms of the matrices ∂, ∂^*, respectively (as it was explained before, the dependence on the choice of $\mathscr{F}$ only influences the ordering of the basis vectors of these matrices).

The kernel $\mathbb{K}(e_1, e_2)$, $e_1, e_2 \in E$, of the determinantal USF measure $\mathbb{P}_W$ can be computed as follows. There exist unique $\mathbf{n}_i \in \Gamma$, $e_i^0 \in E_0$, such that $e_i = \mathbf{n}_i e_i^0$, $i = 1, 2$. Then

$$\mathbb{K}(e_1, e_2) = \int_{\widehat{\Gamma}} \chi_{\boldsymbol{\theta}}(\mathbf{n}_1 - \mathbf{n}_2) K(\boldsymbol{\theta})(e_1^0, e_2^0) \boldsymbol{\lambda}(d\boldsymbol{\theta}),$$

where $K(\boldsymbol{\theta})(e_1^0, e_2^0)$ is the (e_1^0, e_2^0)-element of $K(\boldsymbol{\theta})$.

Remark 4.4.4. Consider a family of operators

$$K_\epsilon(\boldsymbol{\theta}) = D(\boldsymbol{\theta})\Big(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta}) + \epsilon I\Big)^{-1} D^*(\boldsymbol{\theta}).$$

By construction, $\lim_{\epsilon \to 0} K_\epsilon(\boldsymbol{\theta}) = K(\boldsymbol{\theta})$. Then, [4]

$$\int_{\widehat{\Gamma}} \langle K_\epsilon(\boldsymbol{\theta}) \mathbb{F} \mathbb{1}_{e_i}, \mathbb{F} \mathbb{1}_{e_j} \rangle d\boldsymbol{\lambda}(d\boldsymbol{\theta}) = H(e_i, e_j),$$

where $H(e_i, e_j)$ is a Green's function of a random walk on G conditioned to be killed at any $v \in V$ with probability ϵ. Moreover, the correlation kernel $\mathbb{K}_\epsilon = (e_1, e_2) = \int_{\widehat{\Gamma}} \chi_{\boldsymbol{\theta}}(\mathbf{n}_1 - \mathbf{n}_2) K_\epsilon(\boldsymbol{\theta})(e_1^0, e_2^0) \boldsymbol{\lambda}(d\boldsymbol{\theta})$ corresponds to a spanning forest determinantal point process on G with an added cemetery state [4].

Remark 4.4.5. By analogy with $d : \ell^2(V) \to \ell^2_-(E)$ and $d^* : \ell^2_-(E) \to \ell^2(V)$, it is natural to view $D(\boldsymbol{\theta}), D^*(\boldsymbol{\theta})$ as linear operators: $D(\boldsymbol{\theta}) : \mathbb{C}^{|V_0|} \to \mathbb{C}^{|E_0|}$ and $D^*(\boldsymbol{\theta}) : \mathbb{C}^{|E_0|} \to \mathbb{C}^{|V_0|}$.

Before we turn to the proof of Theorem 4.4.3, let us give the two examples demonstrating the application of the theorem.

Example 4.4.6 (Square lattice)**.** Consider the lattice $\mathbb{Z}^d$ with the natural shift action of $\Gamma = \mathbb{Z}^d$. As in Example 4.3.5, one readily checks that the quotient graph G/Γ is a d-bouquet graph consisting of a single vertex and d loops. Since $\mathbb{T}^d = \widehat{\mathbb{Z}^d}$, we thus have

$$\partial = \begin{pmatrix} 1 - x_1 \\ \vdots \\ 1 - x_d \end{pmatrix}, \; D(\boldsymbol{\theta}) = \begin{pmatrix} 1 - e^{2\pi i \theta_1} \\ \vdots \\ 1 - e^{2\pi i \theta_d} \end{pmatrix},$$

and

$$\partial^* = \left(1 - x_1^{-1}, 1 - x_2^{-1}, \ldots, 1 - x_d^{-1}\right),\ D^*(\boldsymbol{\theta}) = (1 - e^{-2\pi i\theta_1}, \ldots, 1 - e^{-2\pi i\theta_d}),$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$. Therefore $D(\boldsymbol{\theta})^* D(\boldsymbol{\theta}) = 2d - 2\sum_{m=1}^{d} \cos(2\pi\theta_m)$, and hence

$$\begin{aligned} K_\epsilon(\boldsymbol{\theta}) &= D(\boldsymbol{\theta})(D(\boldsymbol{\theta})^* D(\boldsymbol{\theta}) + \epsilon I)^{-1} D(\boldsymbol{\theta}) \\ &= \left(\frac{(1 - e^{2\pi i\theta_j})(1 - e^{-2\pi i\theta_k})}{2d + \epsilon - 2\sum_{m=1}^{d} \cos(2\pi\theta_m)} \right)_{j,k=1}^{d}, \end{aligned}$$

and

$$K(\boldsymbol{\theta}) = \lim_{\epsilon \to 0} K_\epsilon(\boldsymbol{\theta}) = \left(\frac{(1 - e^{2\pi i\theta_j})(1 - e^{-2\pi i\theta_k})}{2d - 2\sum_{m=1}^{d} \cos(2\pi\theta_m)} \right)_{j,k=1}^{d}.$$

Consider now two edges $e_1 = (\mathbf{n}_1, \mathbf{n}_1 + q_j)$ and $e_2 = (\mathbf{n}_2, \mathbf{n}_2 + q_k)$, where $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{Z}^d$ and q_j, q_k are the j-the and k-the basis vectors of $\mathbb{Z}^d$, respectively. Then

$$\mathbb{K}(e_1, e_2) = \int_{\mathbb{T}^d} e^{2\pi i\langle \mathbf{n}_1 - \mathbf{n}_2, \boldsymbol{\theta}\rangle} \frac{(1 - e^{2\pi i\theta_j})(1 - e^{-2\pi i\theta_k})}{2d - 2\sum_{m=1}^{d} \cos(2\pi\theta_m)}\, d\boldsymbol{\theta} =$$

$$\int_{\mathbb{T}^d} \frac{e^{2\pi i\langle \mathbf{n}_1 - \mathbf{n}_2, \boldsymbol{\theta}\rangle} - e^{2\pi i\langle \mathbf{n}_1 + q_j - \mathbf{n}_2, \boldsymbol{\theta}\rangle} - e^{2\pi i\langle \mathbf{n}_1 - (\mathbf{n}_2 + q_k), \boldsymbol{\theta}\rangle} + e^{2\pi i\langle \mathbf{n}_1 + q_j - (\mathbf{n}_2 + q_k), \boldsymbol{\theta}\rangle}}{2d - 2\sum_{m=1}^{d} \cos(2\pi\theta_m)}\, d\boldsymbol{\theta}$$

If we let $x = \mathbf{n}_1$, $y = \mathbf{n}_1 + q_j$, $z = \mathbf{n}_2$, $w = \mathbf{n}_2 + q_k$, then we immediately see that we have obtained exactly the same expression as in the Example 4.1.3:

$$\mathbb{K}(e_1, e_2) = \frac{1}{2d}\Big[g(z - x) - g(z - y) - g(w - x) + g(w - y) \Big].$$

Note however, that the method does not have to take into account that for $d = 2$ the simple random walk is recurrent, and hence the Green's function has to be redefined appropriately.

Example 4.4.7 (Three-ladder graph, $\mathbb{Z}$-symmetry, Figure 4.4)**.** Consider an unoriented graph G with a vertex set $V = \{(k, i), k \in \mathbb{Z}, i \in \{1, 2, 3\}\}$ and the set of edges

$$E = \{((k, i), (k \pm 1, i))\} \cup \{((k, 1), (k, 2)), ((k, 2), (k, 3)), ((k, 3), (k, 1))\}.$$

The graph G is called a three-ladder graph (see Figure 4.3).

Define an automorphism $x\colon V \to V$ acting by $x(k, i) = (k + 1, i)$. Clearly, the automorphism x implies the action of $\mathbb{Z}$ on G: consider a quotient graph

$G/\mathbb{Z}$ (see Figure 4.4). We write down the corresponding $D(\theta)$ and $D^*(\theta)$ matrices:

$$D(\theta) = \begin{bmatrix} 1-e^{2\pi i\theta} & 0 & 0 \\ 0 & 1-e^{2\pi i\theta} & 0 \\ 0 & 0 & 1-e^{2\pi i\theta} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

$$D^*(\theta) = \begin{bmatrix} 1-e^{-2\pi i\theta} & 0 & 0 & 1 & 0 & -1 \\ 0 & 1-e^{-2\pi i\theta} & 0 & -1 & 1 & 0 \\ 0 & 0 & 1-e^{--2\pi i\theta} & 0 & -1 & 1 \end{bmatrix}$$

We conclude that

$$D^*(\theta)D(\theta) = \begin{bmatrix} 4-e^{2\pi i\theta}-e^{-2\pi i\theta} & -1 & -1 \\ -1 & 4-e^{2\pi i\theta}-e^{-2\pi i\theta} & -1 \\ -1 & -1 & 4-e^{2\pi i\theta}-e^{-2\pi i\theta} \end{bmatrix}$$

and since $(D^*(\theta)D(\theta))$ is formally invertible,

$$\frac{(e^{2\pi i\theta}-1)^2(e^{4\pi i\theta}-5e^{2\pi i\theta}+1)}{e^{2\pi i\theta}}(D^*(\theta)D(\theta))^{-1} =$$

$$\begin{bmatrix} -e^{4\pi i\theta}+3e^{2\pi i\theta}-1 & e^{2\pi i\theta} & e^{2\pi i\theta} \\ e^{2\pi i\theta} & -e^{4\pi i\theta}+3e^{2\pi i\theta}-1 & e^{2\pi i\theta} \\ e^{2\pi i\theta} & e^{2\pi i\theta} & -e^{4\pi i\theta}+3e^{2\pi i\theta}-1 \end{bmatrix}$$

we do not need to take the limit with respect to ϵ. We finish the calculation by multiplying $K(\theta) = D(\theta)(D^*(\theta)D(\theta))^{-1}D^*(\theta)$:

$$K(\theta) = \frac{1}{5-2\cos(\theta)}\cdot$$

$$\begin{bmatrix} 3-2\cos(2\pi\theta) & 1 & 1 & e^{2\pi i\theta}-1 & 0 & 1-e^{2\pi i\theta} \\ 1 & 3-2\cos(2\pi\theta) & 1 & 1-e^{2\pi i\theta} & e^{2\pi i\theta}-1 & 0 \\ 1 & 1 & 3-2\cos(2\pi\theta) & 0 & 1-e^{2\pi i\theta} & e^{2\pi i\theta}-1 \\ 1-e^{-2\pi i\theta} & e^{-2\pi i\theta}-1 & 0 & 2 & -1 & -1 \\ 0 & 1-e^{-2\pi i\theta} & e^{-2\pi i\theta}-1 & -1 & 2 & -1 \\ e^{-2\pi i\theta}-1 & 0 & 1-e^{-2\pi i\theta} & -1 & -1 & 2 \end{bmatrix}$$

Take $k_1, k_2 \in \Gamma = \mathbb{Z}$. The correlation kernel of two edges $e_{k_1,i}, e_{k_2,j}$ is given by the i,j-th entry of the matrix $\mathbb{K}(k_1,k_1)$ given by the formula

$$\mathbb{K}(k_1,k_2) = \int_0^1 K(\theta)e^{2\pi i(k_1-k_2)\theta}d\theta.$$

Example 4.4.8 (Three-ladder graph, $\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$-symmetry, Figure 4.4)**.** Define an automorphism $y\colon V\to V$ acting by the formula

$$y(k,i)=\begin{cases}(k,i+1), i=1,2\\(k,1), i=3\end{cases}$$

Clearly, y is a cyclic permutation on a set of three elements and it is independent of x. Therefore, x and y generate a $\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$-action on G. The quotient graph is shown on Figure 4.4. The corresponding $D(\boldsymbol{\theta})$ and $D^*(\boldsymbol{\theta})$ are:

$$D(\boldsymbol{\theta})=\begin{bmatrix}1-e^{2\pi i\theta_1}\\1-e^{2\pi i\theta_2}\end{bmatrix},\quad D^*(\boldsymbol{\theta})=\begin{bmatrix}1-e^{-2\pi i\theta_1} & 1-e^{-2\pi i\theta_2}\end{bmatrix}$$

Then, $(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta}))^{-1}=\frac{1}{4-2\cos(2\pi\theta_1)-2\cos(2\pi\theta_2)}$ and $K(\boldsymbol{\theta})$ is given by

$$\frac{1}{4-2\cos(2\pi\theta_1)-2\cos(2\pi\theta_2)}\begin{bmatrix}2-2\cos(2\pi\theta_1) & (1-e^{2\pi i\theta_1})(1-e^{-2\pi i\theta_2})\\(1-e^{-2\pi i\theta_1})(1-e^{2\pi i\theta_2}) & 2-2\cos(2\pi\theta_2)\end{bmatrix}$$

Take two edges $e_1,e_2\in E$ such that $e_1=\boldsymbol{n}e_i^0$ and $e_2=\boldsymbol{m}e_j^0$, $\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ and $e_i^0,e_j^0\in E_0$, then their correlation is given by the (i,j)-th element of the matrix

$$\begin{aligned}\mathbb{K}(\boldsymbol{n},\boldsymbol{m})=&\frac{1}{3}\int_0^1 K(\theta,0)e^{2\pi i(n_1-m_1)\theta}d\theta+\frac{1}{3}\int_0^1 K(\theta,1/3)e^{2\pi i(n_1-m_1)\theta}e^{2/3\pi i(n_2-m_2)}d\theta\\&+\frac{1}{3}\int_0^1 K(\theta,-1/3)e^{2\pi i(n_1-m_1)\theta}e^{-2/3\pi i(n_2-m_2)}d\theta\end{aligned}$$

Proof of Theorem 4.4.3. Applying Theorem 4.2.1 to the coboundary operator $d:\ell^2(V)\to\ell^2_-(\overline{E})$ and its adjoint – the divergence operator – $d^*:\ell^2_-(\overline{E})\to\ell^2(V)$ (c.f., (4.6)):

$$df(e)=f(o(e))-f(t(e)),\quad d^*\theta(v)=\sum_{o(e)=v}\theta(e),$$

we conclude that the operators $P_\epsilon=d(d^*d+\epsilon I)^{-1}d^*$ converge, as $\epsilon\to0$, to the orthogonal projection operator P onto the closure of $\mathsf{Im}\,d$. For two edges $e_1,e_2\in\overline{E}$

$$\begin{aligned}\mathbb{K}(e_1,e_2)&=\lim_{\epsilon\to0}\Big\langle d(d^*d+\epsilon I)^{-1}d^*\mathbb{1}_{e_1},\mathbb{1}_{e_2}\Big\rangle_{\ell^2_-(\overline{E})}\\&=\lim_{\epsilon\to0}\Big\langle (d^*d+\epsilon I)^{-1}d^*\mathbb{1}_{e_1},d^*\mathbb{1}_{e_2}\Big\rangle_{\ell^2(V)}\qquad(4.12)\\&=\lim_{\epsilon\to0}\Big\langle \mathbb{F}[(d^*d+\epsilon I)^{-1}d^*\mathbb{1}_{e_1}],\mathbb{F}[d^*\mathbb{1}_{e_2}]\Big\rangle_{L^2(\widehat{\Gamma},\mathbb{C}^{|V_0|})}.\end{aligned}$$

Now we show how to compute these expressions in the Fourier domain. In order to do it, we use several Lemma's whose proofs we provide in the end of this Subsection.

Lemma 4.4.9. For an edge $e \in E^0$, $\mathbb{F}(d^*\mathbb{1}_e)(\boldsymbol{\theta}, \cdot)$ is a vector of length $|V_0|$ with entries given by

$$\mathbb{F}(d^*\mathbb{1}_e)(\boldsymbol{\theta}, v_0) = \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=o(e)} \chi_{\mathbf{n}}(\boldsymbol{\theta}) - \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=t(e)} \chi_{\mathbf{n}}(\boldsymbol{\theta}).$$

Lemma 4.4.10. For every $f \in \ell^2(V)$ one has

$$\mathbb{F}[(d^*d + \epsilon I)^{-1} f] = (M(\boldsymbol{\theta}) + \epsilon I)^{-1}\mathbb{F}f,$$

where for each $\boldsymbol{\theta} \in \widehat{\Gamma}$, $M(\boldsymbol{\theta})$ is a $|V_0| \times |V_0|$ matrix with entries

$$M(\boldsymbol{\theta})_{v_j,v_k} = \begin{cases} \deg v_j, & v_j = v_k, \\ -\sum_{\in\Gamma:\boldsymbol{v}_k\sim v_j} \chi_{\boldsymbol{\theta}}(-), & v_j \neq v_k. \end{cases}$$

We introduced the $|E_0|\times|V_0|$-incidence matrix $\partial_{\mathscr{F}} = (\partial_{\mathscr{F}_{E^o}}(e,v))_{e\in\mathscr{F}_{E^o},v\in\mathscr{F}_V}$ and its adjoint $\partial^*_{\mathscr{F}} = (\partial^*_{\mathscr{F}}(e,v))_{e\in\mathscr{F}_{E^o},v\in\mathscr{F}_V}$ as

$$\partial_{\mathscr{F}}(e, v) = \mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \boldsymbol{x}^{\mathbf{n}}, \tag{4.13}$$

$$\partial^*_{\mathscr{F}}(v, e) = \mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \boldsymbol{x}^{-\mathbf{n}}. \tag{4.14}$$

From now on we will drop the index $\mathscr{F}$. Let $D(\boldsymbol{\theta}) = \widehat{\partial}$ and $D^*(\boldsymbol{\theta}) = \widehat{\partial^*}$ be the corresponding Fourier transforms of matrices ∂, ∂^*, i.e., for $v \in \mathscr{F}_V, e \in \mathscr{F}_{E^o}$,

$$D(\boldsymbol{\theta})(e, v) = \mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \chi_{\boldsymbol{\theta}}(\mathbf{n}),$$

$$D^*(\boldsymbol{\theta})(v, e) = \mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \chi_{\boldsymbol{\theta}}(-\mathbf{n}).$$

Then $D^*(\boldsymbol{\theta}) = (D(\boldsymbol{\theta}))^*$. Indeed,

$$\begin{aligned}\widehat{\partial(e,v)} &= \mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \chi_{\boldsymbol{\theta}}(\mathbf{n}) = \mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \overline{\chi_{\boldsymbol{\theta}}(-\mathbf{n})} \\ &= \overline{\mathbb{1}_v(o(e)) - \sum_{\mathbf{n}\in\Gamma:\ \mathbf{n}v=t(e)} \chi_{\boldsymbol{\theta}}(-\mathbf{n})} = \overline{\widehat{\partial}^*(v, e)}.\end{aligned}$$

Furthermore, suppose $v_0 \in \mathscr{F}_V$ and $e \in E^o$. Then there exist unique $e_0 \in \mathscr{F}_{E^o}$ and $\mathbf{n}_0 \in \Gamma$ such that $e = \mathbf{n}_0(e_0)$. Therefore,

$$\begin{aligned}\mathbb{F}(d^*\mathbb{1}_e)(\boldsymbol{\theta}, v_0) &= \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=o(e)} \chi_{\boldsymbol{\theta}}(\mathbf{n}) - \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=t(e)} \chi_{\boldsymbol{\theta}}(\mathbf{n}) \\ &= \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=o(\mathbf{n}_0(e_0))} \chi_{\boldsymbol{\theta}}(\mathbf{n}) - \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=t(\mathbf{n}_0(e_0))} \chi_{\boldsymbol{\theta}}(\mathbf{n}) \\ &= \chi_{\boldsymbol{\theta}}(\mathbf{n}_0)\Big(\mathbb{1}_{v_0}(o(e_0)) - \sum_{\in\Gamma:\boldsymbol{v}_0=t(e_0)} \mathbb{1}_{\boldsymbol{\theta}}()\Big) = \chi_{\boldsymbol{\theta}}(\mathbf{n}_0)D(\boldsymbol{\theta})(e_0, v_0).\end{aligned} \tag{4.15}$$

For the $|V_0| \times |V_0|$ matrix $M(\boldsymbol{\theta})$ of Lemma 4.4.10 we have

Lemma 4.4.11.

$$M(\boldsymbol{\theta}) = D^*(\boldsymbol{\theta})D(\boldsymbol{\theta}). \tag{4.16}$$

Now we are ready to derive the expression for the kernel K. Suppose $e_1, e_2 \in E^0$, and hence there are unique $\mathbf{n}_j \in \Gamma$, $e_j^0 \in \mathscr{F}_{E^o}$, $j = 1, 2$, such that $e_j = \mathbf{n}_j e_j^0$. Then using the auxiliary Lemmas, and the corresponding expressions (4.15), (4.16), we can continue (4.12) as follows: $K(e_1, e_2) =$

$$\begin{aligned}&\lim_{\epsilon\to 0}\Big\langle \mathbb{F}[(d^*d+\epsilon I)^{-1}d^*\mathbb{1}_{e_1}], \mathbb{F}[d^*\mathbb{1}_{e_2}]\Big\rangle_{L^2(\widehat{\Gamma},\mathbb{C}^{|V_0|})} \\ &= \lim_{\epsilon\to 0}\Big\langle (M+\epsilon I)^{-1}\mathbb{F}[d^*\mathbb{1}_{e_1}], \mathbb{F}[d^*\mathbb{1}_{e_2}]\Big\rangle_{L^2(\widehat{\Gamma},\mathbb{C}^{|V_0|})} \\ &= \lim_{\epsilon\to 0}\sum_{v_0\in\mathscr{F}_V}\int_{\widehat{\Gamma}}\Big[\sum_{v_1\in\mathscr{F}_V}(M(\boldsymbol{\theta})+\epsilon I)^{-1}_{v_0,v_1}\mathbb{F}d^*\mathbb{1}_{e_1}(\boldsymbol{\theta}, v_1)\Big]\overline{\Big[\mathbb{F}d^*\mathbb{1}_{e_2}(\boldsymbol{\theta}, v_0)\Big]}\boldsymbol{\lambda}(d\boldsymbol{\theta}) \\ &= \lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\sum_{v_0\in\mathscr{F}_V}\left[\sum_{v_1\in\mathscr{F}_V}(M(\boldsymbol{\theta})+\epsilon I)^{-1}_{v_0,v_1}\chi_{\boldsymbol{\theta}}(\mathbf{n}_1)D(\boldsymbol{\theta})(e_1^0, v_1)\right]\cdot \\ &\quad\cdot\overline{\big[\chi_{\boldsymbol{\theta}}(\mathbf{n}_2)D(\boldsymbol{\theta})(e_2^0, v_0)\big]}\boldsymbol{\lambda}(d\boldsymbol{\theta}) \\ &= \lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\chi_{\boldsymbol{\theta}}(\mathbf{n}_1-\mathbf{n}_2)\cdot \\ &\quad\cdot\left[\sum_{v_1\in\mathscr{F}_V}\sum_{v_0\in\mathscr{F}_V}D(\boldsymbol{\theta})(e_1^0, v_1)(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta})+\epsilon I)^{-1}_{v_0,v_1}D^*(\boldsymbol{\theta})(v_0, e_2^0)\right]\boldsymbol{\lambda}(d\boldsymbol{\theta}).\end{aligned}$$

It is now easy to see that the expression in the square brackets is nothing else but the (e_1^0, e_2^0)-entry of the matrix $\mathbb{K}_\epsilon = D(\boldsymbol{\theta})(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta})+\epsilon I)^{-1}D^*(\boldsymbol{\theta})$. Applying Theorem 4.2.1 again, but now in the finite dimensional situation (c.f., (4.7)), we conclude that the limit

$$K(\boldsymbol{\theta}) = \lim_{\epsilon\to 0} D^*(\boldsymbol{\theta})\Big(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta})+\epsilon I\Big)^{-1}D(\boldsymbol{\theta})$$

exists, and in fact, coincides with $D(\boldsymbol{\theta})(D(\boldsymbol{\theta})^*D(\boldsymbol{\theta}))^{-1}D^*(\boldsymbol{\theta})$ since the matrix $D^*(\boldsymbol{\theta})D(\boldsymbol{\theta})$ is invertible on $W = \mathsf{Im}\ D \subset \mathbb{C}^{|E_0|}$. Moreover, by Proposition 4.2.2, for each $\boldsymbol{\theta}$, the norm of $(D^*(\boldsymbol{\theta})D(\boldsymbol{\theta})+\epsilon I)^{-1}$ is bounded by 2, uniformly in ϵ. Hence, by the Lebesgue dominated convergence theorem

$$\begin{aligned}\mathbb{K}(e_1,e_2) &= \lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\chi_{\boldsymbol{\theta}}(\mathbf{n}_1-\mathbf{n}_2)\mathbb{K}_\epsilon(\boldsymbol{\theta})(e_1^0,e_2^0)\boldsymbol{\lambda}(d\boldsymbol{\theta})\\ &= \int_{\widehat{\Gamma}}\chi_{\boldsymbol{\theta}}(\mathbf{n}_1-\mathbf{n}_2)\Big[\lim_{\epsilon\to 0}\mathbb{K}_\epsilon(\boldsymbol{\theta})(e_1^0,e_2^0)\Big]\boldsymbol{\lambda}(d\boldsymbol{\theta})\\ &= \int_{\widehat{\Gamma}}\chi_{\boldsymbol{\theta}}(\mathbf{n}_1-\mathbf{n}_2)\mathbb{K}(\boldsymbol{\theta})(e_1^0,e_2^0)\boldsymbol{\lambda}(d\boldsymbol{\theta})\end{aligned}$$

This finishes the proof of the Theorem. □

Proof of Lemma 4.4.9. One has

$$d^*\mathbb{1}_e(v) = \begin{cases}1, & v=o(e),\\ -1, & v=t(e),\\ 0, & v\neq o(e),t(e).\end{cases}$$

Therefore,

$$\mathbb{F}(d^*\mathbb{1}_e)(\boldsymbol{\theta},v_0) = \sum_{\mathbf{n}\in\Gamma}(d^*\mathbb{1}_e)(\mathbf{n}v_0)\chi_{\boldsymbol{\theta}}(\mathbf{n}) = \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=o(e)}\chi_{\boldsymbol{\theta}}(\mathbf{n}) - \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_0=t(e)}\chi_{\boldsymbol{\theta}}(\mathbf{n}).$$

□

Proof of Lemma 4.4.10. The Laplacian $\mathcal{L} = d^*d\colon \ell^2(V)\to\ell^2(V)$ is given by

$$(\mathcal{L}f)(v) = \sum_{y\sim v}\Big(f(v)-f(y)\Big) = \deg v\cdot f(v) - \sum_{y\sim v}f(y),$$

where the sum is taken over all vertices $y\in V$ that are connected to v by an edge. Let us now identify the operator $\widehat{\mathcal{L}}\colon L^2(\widehat{\Gamma},\mathbb{C}^{|V_0|})\to L^2(\widehat{\Gamma},\mathbb{C}^{|V_0|})$ such that for all $f\in\ell^2(V)$, one has

$$\mathbb{F}(\mathcal{L}f) = \widehat{\mathcal{L}}(\mathbb{F}f).$$

A simple calculation shows that $\mathbb{F}(\mathcal{L}f)(\boldsymbol{\theta}, v_0) =$

$$
\begin{aligned}
&= \sum_{\boldsymbol{n}\in\Gamma} (\mathcal{L}f)(\boldsymbol{n}v_0)\chi_{\boldsymbol{\theta}}(\boldsymbol{n}) = \sum_{\boldsymbol{n}\in\Gamma} \Big[\deg(\boldsymbol{n}v_0)\cdot f(\boldsymbol{n}v_0) - \sum_{y\sim \boldsymbol{n}v_0} f(y)\Big]\chi_{\boldsymbol{\theta}}(\boldsymbol{n}) \\
&= \deg(v_0)\mathbb{F}f(\boldsymbol{\theta}, v_0) - \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n}) \sum_{\substack{\mathbf{n}'\in\Gamma, v'\in\mathscr{F}_V \\ \mathbf{n}'v'\sim\mathbf{n}v_0}} f(\boldsymbol{n}'v') \\
&= \deg v_0 \mathbb{F}f(\boldsymbol{\theta}, v_0) - \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n}')\chi_{\boldsymbol{\theta}}(\mathbf{n}-\mathbf{n}') \sum_{\substack{\mathbf{n}'\in\Gamma, v'\in\mathscr{F}_V \\ (\mathbf{n}'-\mathbf{n})v'\sim v_0}} f(\boldsymbol{n}'v') \\
&= \deg(v_0)\mathbb{F}f(\boldsymbol{\theta}, v_0) - \sum_{v'\in\mathscr{F}_V} \sum_{\mathbf{n}'\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n}') f(\boldsymbol{n}'v') \Big(\sum_{\mathbf{n}\in\Gamma:\mathbf{n}'v'\sim\mathbf{n}v_0} \chi_{\boldsymbol{\theta}}(\mathbf{n}-\mathbf{n}') \Big) \\
&= \deg(v_0)\mathbb{F}f(\boldsymbol{\theta}, v_0) - \sum_{v'\in\mathscr{F}_V} \sum_{\mathbf{n}'\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n}') f(\boldsymbol{n}'v') \Big(\sum_{\mathbf{n}\in\Gamma:(\mathbf{n}'-\mathbf{n})v'\sim v_0} \chi_{\boldsymbol{\theta}}(\mathbf{n}-\mathbf{n}') \Big) \\
&= \deg(v_0)\mathbb{F}f(\boldsymbol{\theta}, v_0) - \sum_{v'\in\mathscr{F}_V} \sum_{\mathbf{n}'\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n}') f(\boldsymbol{n}'v') \Big(\sum_{\in\Gamma:\boldsymbol{v}'\sim v_0} \chi_{\boldsymbol{\theta}}(-) \Big) \\
&= \deg(v_0)\mathbb{F}f(\boldsymbol{\theta}, v_0) - \sum_{v'\in\mathscr{F}_V} \mathbb{F}f(\boldsymbol{\theta}, v') \Big(\sum_{\in\Gamma:\boldsymbol{v}'\sim v_0} \chi_{\boldsymbol{\theta}}(-) \Big).
\end{aligned}
$$

Hence, for every $\boldsymbol{\theta} \in \widehat{\Gamma}$, one has

$$\mathbb{F}(\mathcal{L} + \epsilon I)f = (M(\boldsymbol{\theta}) + \epsilon I)\mathbb{F}f,$$

where for $V = \{v_1, \ldots, v_m\}$,

$$M(\boldsymbol{\theta})_{v_j,v_k} = \begin{cases} \deg v_j, & v_j = v_k, \\ -\sum_{\in\Gamma:\boldsymbol{v}_k\sim v_j} \chi_{\boldsymbol{\theta}}(-), & v_j \neq v_k. \end{cases}$$

Note that for all $\epsilon > 0$, the matrix $M(\boldsymbol{\theta})+\epsilon I$ is diagonally dominant: $|(M(\boldsymbol{\theta})+\epsilon I)_{v,v}| = \deg v + \epsilon > \sum_{v'\neq v} |M(\boldsymbol{\theta})_{v,v'}|$, and hence is invertible.

If $g = (\mathcal{L} + \epsilon I)^{-1}f$, then $\mathbb{F}f = \mathbb{F}(\mathcal{L} + \epsilon I)g = (M(\boldsymbol{\theta}) + \epsilon I)\mathbb{F}g$, and hence, $\mathbb{F}(\mathcal{L} + \epsilon I)^{-1}f = \mathbb{F}g = (M(\boldsymbol{\theta}) + \epsilon I)^{-1}\mathbb{F}f$.

□

Proof of Lemma 4.4.11. For the matrix $D(\boldsymbol{\theta})D^*(\boldsymbol{\theta})$, we have

$$
\begin{aligned}
D(\boldsymbol{\theta})D^*(\boldsymbol{\theta})(v_1, v_2) &= \sum_{e\in\mathscr{F}_{E^o}} D(\boldsymbol{\theta})(v_1, e)D^*(\boldsymbol{\theta})(e, v_2) \\
&= \sum_{e\in\mathscr{F}_{E^o}} \left(\mathbb{1}[v_1 = o(e)] - \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n})\mathbb{1}[\mathbf{n}v_1 = t(e)]\right)\cdot \\
&\quad \cdot \left(\mathbb{1}[v_2 = o(e)] - \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(-\boldsymbol{n})\mathbb{1}[\mathbf{n}v_2 = t(e)]\right) \\
&= \sum_{e\in\mathscr{F}_{E^o}} \Big(\mathbb{1}[v_1 = o(e)]\mathbb{1}[v_2 = o(e)] \\
&\quad + \sum_{\mathbf{n},\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n})\mathbb{1}[\mathbf{n}v_1 = t(e)]\chi_{\boldsymbol{\theta}}(-\boldsymbol{m})\mathbb{1}[\boldsymbol{v}_2 = t(e)] \\
&\quad - \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n})\mathbb{1}[\mathbf{n}v_1 = t(e)]\mathbb{1}[v_2 = o(e)] \\
&\quad - \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(-\boldsymbol{n})\mathbb{1}[\mathbf{n}v_2 = t(e)]\mathbb{1}[v_1 = o(e)]\Big) \\
&= |\{e \in \mathscr{F}_{E^o} : o(e) = v_1\}|\mathbb{1}[v_1 = v_2] \\
&\quad + \sum_{e\in\mathscr{F}_{E^o}} \sum_{\mathbf{n},\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n}-)\mathbb{1}[\mathbf{n}v_1 = \boldsymbol{v}_2 = t(e)] \\
&\quad - \sum_{e\in\mathscr{F}_{E^o}} \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n})\mathbb{1}[\mathbf{n}v_1 = t(e), v_2 = o(e)] \\
&\quad - \sum_{e\in\mathscr{F}_{E^o}} \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(-\boldsymbol{n})\mathbb{1}[\mathbf{n}v_2 = t(e), v_1 = o(e)] \\
&= |\{e \in \mathscr{F}_{E^o} : o(e) = v_1\}| \cdot \mathbb{1}[v_1 = v_2] \\
&\quad + |\{e \in E_0^0 : t(e) = v_1\}| \cdot \mathbb{1}[v_1 = v_2] \\
&\quad - \sum_{e\in\mathscr{F}_{E^o}} \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(\boldsymbol{n})\mathbb{1}[\mathbf{n}v_1 = t(e), v_2 = o(e)] \\
&\quad + \sum_{e\in\mathscr{F}_{E^o}} \sum_{\boldsymbol{n}\in\Gamma} \chi_{\boldsymbol{\theta}}(-\boldsymbol{n})\mathbb{1}[\mathbf{n}v_2 = t(e), v_1 = o(e)].
\end{aligned}
$$

It is clear that the term in brackets is zero if $v_1 = v_2$. For $v_1, v_2 \in V_0$, $v_1 \neq v_2$, if $\mathbf{n} \in \Gamma$ is such that $\mathbf{n}v_2 \sim v_1$, i.e., v_1 and $\mathbf{n}v_2$ are connected by an edge, then either there is an edge $e \in \mathscr{F}_{E^o}$ such that $v_1 = o(e)$ and $\mathbf{n}v_2 = t(e)$, or $v_1 = t(e)$ and $\mathbf{n}v_2 = o(e)$, which is equivalent to $-\mathbf{n}v_1 = t(e')$, $v_2 = o(e')$ for some e'. Hence the term in the brackets is given by $\sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_2\sim v_1} \chi_{\boldsymbol{\theta}}(-\mathbf{n})$.

Thus we can conclude that

$$D(\boldsymbol{\theta})D^*(\boldsymbol{\theta})(v_1, v_2) = \deg(v_1)\mathbb{1}[v_1 = v_2] - \sum_{\mathbf{n}\in\Gamma:\mathbf{n}v_2\sim v_1} \chi_{\boldsymbol{\theta}}(-\mathbf{n}) = M(\boldsymbol{\theta})(v_1, v_2).$$

□

4.4.2 Example: calculation of correlations for the three-ladder graph

Consider an unoriented graph G with a vertex set $V = \{v_{1,i}, v_{2,i}, v_{3,i}\}_{i\in\mathbb{Z}}$ and the set of edges

$$E = \{(v_{k,i}, v_{k,i\pm1})\}_{k=\{1,2,3\},i\in\mathbb{Z}} \cup \{(v_{1,i}, v_{2,i}), (v_{2,i}, v_{3,i}), (v_{3,i}, v_{1,i})\}_{i\in\mathbb{Z}}.$$

We call G a 3-ladder graph (see Figure 4.3). This graph has two symmetries groups: $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}_3$. Above we showed how to calculate the corresponding determinantal kernels: for $\mathbb{Z}$ reduction we obtain

$$K_{\mathbb{Z}}(\theta) = \frac{1}{5 - 2\cos(\theta)}.$$

$$\begin{bmatrix} 3-2\cos(2\pi\theta) & 1 & 1 & e^{2\pi i\theta}-1 & 0 & 1-e^{2\pi i\theta} \\ 1 & 3-2\cos(2\pi\theta) & 1 & 1-e^{2\pi i\theta} & e^{2\pi i\theta}-1 & 0 \\ 1 & 1 & 3-2\cos(2\pi\theta) & 0 & 1-e^{2\pi i\theta} & e^{2\pi i\theta}-1 \\ 1-e^{-2\pi i\theta} & e^{-2\pi i\theta}-1 & 0 & 2 & -1 & -1 \\ 0 & 1-e^{-2\pi i\theta} & e^{-2\pi i\theta}-1 & -1 & 2 & -1 \\ e^{-2\pi i\theta}-1 & 0 & 1-e^{-2\pi i\theta} & -1 & -1 & 2 \end{bmatrix}$$

Take $k_1, k_2 \in \Gamma = \mathbb{Z}$. The correlation kernel of two edges $e_{k_1,i}, e_{k_2,j}$ is given by the i,j-th entry of the matrix $\mathbb{K}(k_1, k_1)$ given by the formula

$$\mathbb{K}_{\mathbb{Z}}(k_1, k_2) = \int_0^1 K(\theta)e^{2\pi i(k_1-k_2)\theta}d\theta.$$

For $\mathbb{Z}\times\mathbb{Z}_3$ reduction we obtain that $K_{\mathbb{Z}\times\mathbb{Z}_3}(\boldsymbol{\theta})$ is given by

$$\frac{1}{4-2\cos(2\pi\theta_1)-2\cos(2\pi\theta_2)}\begin{bmatrix} 2-2\cos(2\pi\theta_1) & (1-e^{2\pi i\theta_1})(1-e^{-2\pi i\theta_2}) \\ (1-e^{-2\pi i\theta_1})(1-e^{2\pi i\theta_2}) & 2-2\cos(2\pi\theta_2) \end{bmatrix}$$

Take two edges $e_1, e_2 \in E$ such that $e_1 = \boldsymbol{n}e_i^0$ and $e_2 = \boldsymbol{m}e_j^0$, $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ and $e_i^0, e_j^0 \in E_0$, then their correlation is given by the (i,j)-th element of the matrix

$$\begin{aligned}\mathbb{K}_{\mathbb{Z}\times\mathbb{Z}_3}(\boldsymbol{n},\boldsymbol{m}) =& \frac{1}{3}\int_0^1 K(\theta,0)e^{2\pi i(n_1-m_1)\theta}d\theta \\ &+ \frac{1}{3}\int_0^1 K(\theta,1/3)e^{2\pi i(n_1-m_1)\theta}e^{2/3\pi i(n_2-m_2)}d\theta \\ &+ \frac{1}{3}\int_0^1 K(\theta,-1/3)e^{2\pi i(n_1-m_1)\theta}e^{-2/3\pi i(n_2-m_2)}d\theta\end{aligned}$$

In this subsection we calculate the correlations of pairs of edges using two different correlation kernels and show that the results coincide.

1-1 correlation:

$$\mathbb{K}_{\mathbb{Z}}(0,n) = \int_0^1 \frac{3-2\cos(2\pi\theta)}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

$$\begin{aligned}\mathbb{K}_{\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}}((0,0),(n,0)) &= \int_0^1 \left(\frac{e^{2\pi in\theta}}{3} + \frac{2(2-2\cos(2\pi\theta))}{3(5-2\cos(2\pi\theta))}e^{2\pi in\theta}\right)d\theta \\ &= \int_0^1 \frac{5-2\cos(2\pi\theta)+4-4\cos(2\pi\theta)}{3(5-2\cos(2\pi\theta))}e^{2\pi in\theta}d\theta \\ &= \int_0^1 \frac{3-2\cos(2\pi\theta)}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta\end{aligned}$$

1-2 correlation:

$$\mathbb{K}_{\mathbb{Z}}(0,n) = \int_0^1 \frac{1}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

$$\begin{aligned}&\mathbb{K}_{\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}}((0,0),(n,1)) = \\ &\quad\int_0^1 \Big(\frac{1}{3}e^{2\pi in\theta} + \frac{2-2\cos(2\pi\theta)}{3(5-2\cos(2\pi\theta))}e^{2\pi in\theta}e^{2\pi i/3} \\ &\quad+ \frac{2-2\cos(2\pi\theta)}{3(5-2\cos(2\pi\theta))}e^{2\pi in\theta}e^{-2\pi i/3}\Big) \\ &\quad= \int_0^1 \frac{1}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta\end{aligned}$$

1-4 correlation:

$$\mathbb{K}_{\mathbb{Z}}(0,n) = \int_0^1 \frac{e^{-2\pi i\theta}-1}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

$$\mathbb{K}_{\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}}((0,0),(n,1)) =$$
$$\int_0^1 \frac{1}{3(5-2\cos(2\pi\theta))}(1-e^{-2\pi i\theta})(1-e^{-2\pi i/3})e^{2\pi in\theta}e^{2\pi i/3}+$$
$$+\int_0^1 \frac{1}{3(5-2\cos(2\pi\theta))}(1-e^{-2\pi i\theta})(1-e^{2\pi i/3})e^{2\pi in\theta}e^{-2\pi i/3}$$
$$= \int_0^1 \frac{e^{-2\pi i\theta}-1}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

4-5 correlation:

$$\mathbb{K}_{\mathbb{Z}}(0,n) = \int_0^1 \frac{-1}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

$$\mathbb{K}_{\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}}((0,0),(n,1)) = \int_0^1 \left(\frac{3e^{2\pi in\theta}e^{2\pi i/3}}{3(5-2\cos(2\pi\theta))} + \frac{3e^{2\pi in\theta}e^{-2\pi i/3}}{3(5-2\cos(2\pi\theta))}\right)d\theta$$
$$= \int_0^1 \frac{-1}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

4-4 correlation:

$$\mathbb{K}_{\mathbb{Z}}(0,n) = \int_0^1 \frac{2}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

$$\mathbb{K}_{\mathbb{Z}}((0,0),(n,0)) = \int_0^1 \left(\frac{3e^{2\pi in\theta}}{3(5-2\cos(2\pi\theta))} + \frac{3e^{2\pi in\theta}}{3(5-2\cos(2\pi\theta))}\right)$$
$$= \int_0^1 \frac{2}{5-2\cos(2\pi\theta)}e^{2\pi in\theta}d\theta$$

Remark 4.4.12. The computation of correlation kernels for ladder-like graphs has been presented in [57] (in particular, for the ladder and the 3-ladder graphs, that we also discussed in the present work). In the work mentioned above, it is demonstrated how to compute the correlation kernels using two approaches: the counting approach and the electrical network approach. However, the approach presented in the current work is more universal and, moreover, appeals to simpler mathematical techniques.